Title: Hydrodynamics from scalar fields coupled to gravity

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Abstract: In this talk I discuss methods to determine hydrodynamical dispersion relations from an extra-dimensional gravity dual wherein the metric is supported by scalar fields. Such a setup may eventually be used as a model of the strongly coupled plasma created in heavy ion collisions. I examine examples of both the shear and sound modes. The shear mode is analyzed using the black hole membrane paradigm; a calculation of the shear viscosity is reviewed, and then the calculation is extended to the next hydrodynamical order. All results agree with those found using the AdS/CFT prescription. The sound mode is analyzed by examining the quasi-normal modes of the dual black hole. The relevant gauge invariant equations are derived, and a special case solution of these equations is discussed.

Hydrodynamics from scalars coupled to gravity

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12/04/08

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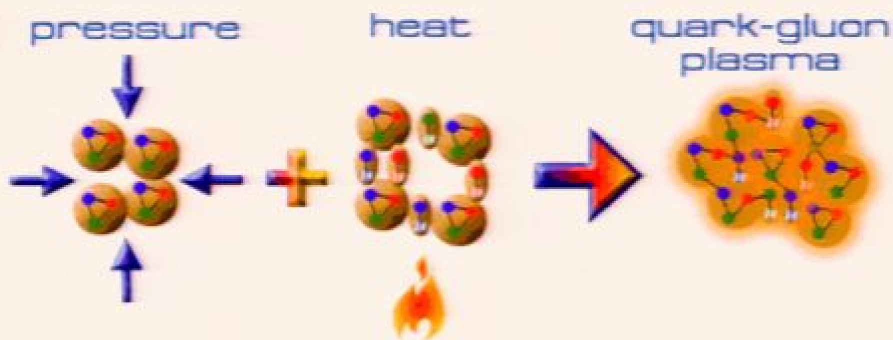
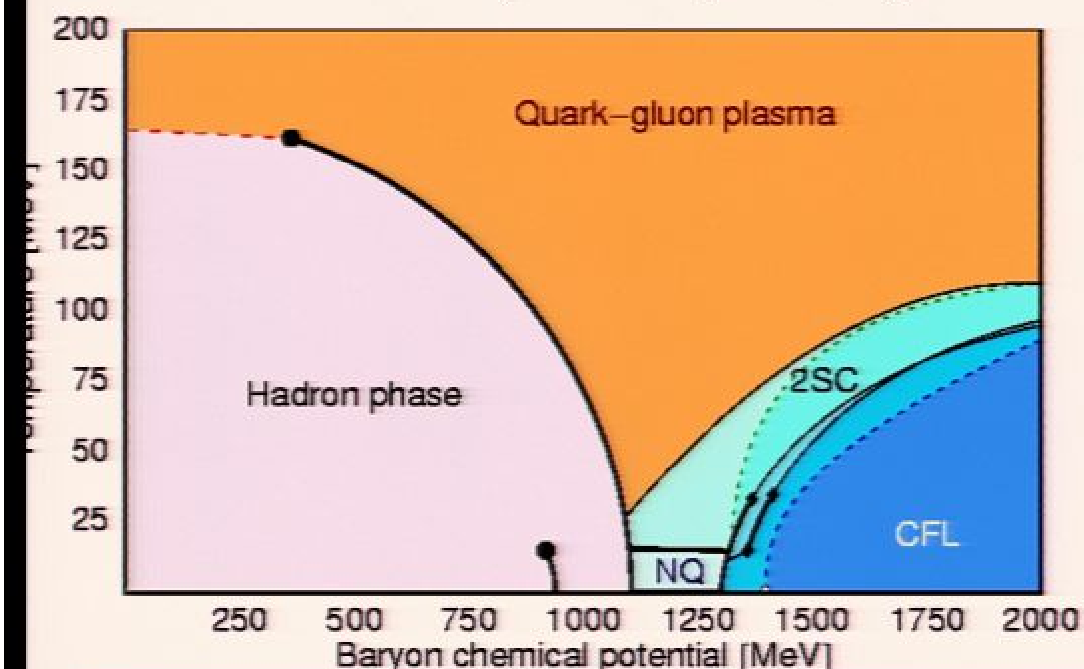
Outline

- Introduction
- Dispersion relations from dual gravitational fluctuations
- Shear Mode
  - Shear Viscosity (KSS 2003)
  - "Shear Relaxation Time"
  - Applications
- Sound Mode
  - Gauge Invariant Equations
  - Special case application
- Works in Progress and Discussion

Quark-Gluon Plasma

- Predominant state of matter shortly after Big Bang
- Formed in heavy ion collisions (RHIC, LHC)





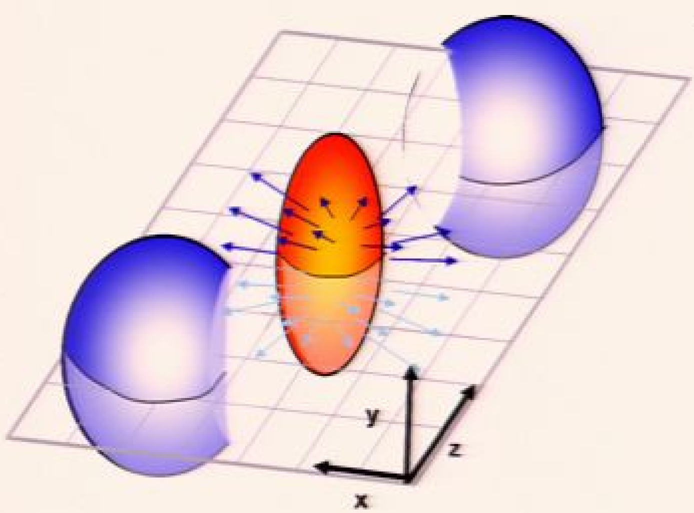
- Strongly coupled
- Well described by hydrodynamics

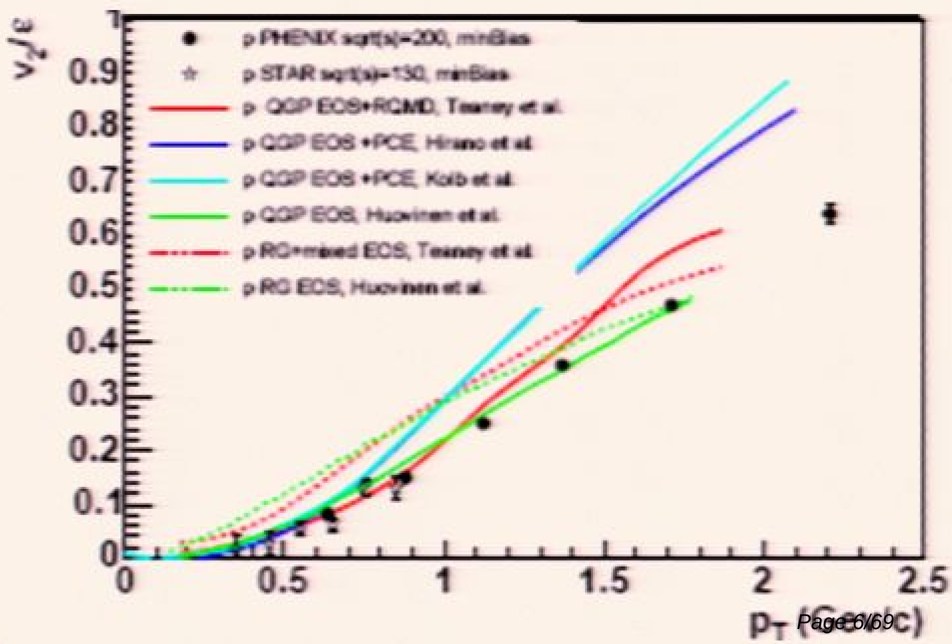
Hydrodynamics (First Order)

Construct Stress Energy Tensor from conserved quantities $T^{0\mu} = \epsilon, \pi^i$

η = shear viscosity
ζ = bulk viscosity
ϵ = energy density
P = pressure
v_s = Speed of Sound

$$
\begin{aligned}
T_{00} &= \epsilon + \delta\epsilon \\
T_{ij} &= \delta_{ij}(P + v_s \delta\epsilon) - \frac{\eta}{\epsilon + P}\left(\partial_i \pi_j + \partial_j \pi_i - \frac{2}{3}\delta_{ij}\partial_k \pi^k\right) - \frac{\zeta}{\epsilon + P}\delta_{ij}\partial_k \pi^k
\end{aligned}
$$





$$\frac{d^2N}{d\phi\, dp_T} = N_0(1 + 2v_2(p_T)\cos(2\phi))$$

Dual Fluctuations

$$\sim e^{i(qz-\omega t)}$$

$$\omega, q \ll T$$

Hydrodynamic fluctuations in gauge theory

Equations of motion admit two normal modes

$$\omega = -i\left(\frac{\eta}{\epsilon+P}\right)q^2$$ **Shear Mode**

$$\omega = v_s q - \frac{iq^2}{2(\epsilon+P)}\left(\frac{4\eta}{3}+\zeta\right)$$ **Sound Mode**

Long wavelength, long time gravitational perturbations in dual theory

Background Metric

$$ds^2 = g_{00}(r)dt^2 + g_{xx}(r)d\vec{x}^2 + g_{rr}(r)dr^2$$

Assumptions

$$r \, \epsilon \, (r_0, \infty)$$

$$g_{00}(r) < 0 \; ; \; g_{xx}(r) > 0 \; ; \; g_{rr}(r) > 0$$

Near Horizon Behavior

$$\begin{aligned} g_{00}(r \to r_0) &= -\gamma_0 (r - r_0) \\ g_{rr}(r \to r_0) &= \frac{\gamma_r}{(r - r_0)} \\ g_{xx}(r \to r_0) &= g_{xx}(r_0) \end{aligned}$$

$$T = \frac{1}{4\pi}\sqrt{\frac{\gamma_0}{\gamma_r}}$$

Metric Perturbations

- Classify perturbations under *O(2)* rotations in the *x*-*y* plane $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$
- Choose radial gauge where $h_{\mu r} = 0$

$h_{yz}, h_{0y} \neq 0$	**Shear Mode**
$h_{00}, h_{ii}, h_{0z} \neq 0$ with $h_{xx} = h_{yy}$	**Sound Mode**

The Prescription

- Solve Einstein equations $G^{(1)}_{\mu\nu} = -8\pi G T^{(1)}_{\mu\nu}$ to 1st order in $h_{\mu\nu}$ and for $w, q << T$

- Assume $h_{\mu\nu} = H_{\mu\nu}(r) e^{i(qz - wt)}$

- Look for solutions with a dispersion relation of the appropriate form

- Compare with hydrodynamic dispersion relations to get transport coefficients

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Einstein Equations (Shear Mode)

$$\frac{1}{\sqrt{-g}}\partial_r\left[\sqrt{-g}g^{rr}g^{00}g_{xx}(A')\right] - qg^{00}(qA+wB) = 16\pi G g_{xx}(T^{0y}_{(1)} + T^{00}_{(0)}A)$$

$$\frac{1}{\sqrt{-g}}\partial_r\left[\sqrt{-g}g^{rr}(B')\right] - wg^{00}(qA+wB) = 16\pi G g_{xx}(T^{yz}_{(1)} + T^{xx}_{(0)}B)$$

$$-i\left(qg^{xx}B' - wg^{00}A'\right) = 16\pi G g_{rr}T^{yr}_{(1)}$$

$$A = g_{00}H^{0y}$$
$$B = g_{xx}H^{yz}$$

Background Fields

- Simple example: multiple scalar fields

$$16\pi G_5 T^{(0)}_{\mu\nu} = \sum_{k=1}^{n} (\partial_\mu \phi_k(r) \partial_\nu \phi_k(r)) - g_{\mu\nu} \mathcal{L}_{\phi k}$$

$$\mathcal{L}_{\phi k} = \sum_{k=1}^{n} \frac{1}{2} \partial_\lambda \phi_k(r) \partial^\lambda \phi_k(r) + U(\phi_1, \phi_2 ... \phi_n)$$

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- Shear mode:

$$16\pi G_5 T^{(1)}_{0y} = -h_{0y} \mathcal{L}_{\phi k} = 16\pi G_5 g^{00} h_{0y} T^{(0)}_{00}$$

...similarly for the other components

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Shear Mode Solution (Hydrodynamic)

$$w = -iDq^2 - i\tau D^2 q^4 + \ldots$$

$$A^{(0)}(r) = a_1 \int_\infty^r dr' \frac{g_{rr}(r')g_{00}(r')}{\sqrt{-g(r')}g_{xx}(r')}$$

$$B^{(1)}(r) = -iDa_1 \int_\infty^r dr' \frac{g_{rr}(r')}{\sqrt{-g(r')}}$$

$$A^{(2)}(r) = \int_\infty^r dr' \frac{g_{rr}(r')g_{00}(r')}{\sqrt{-g(r')}g_{xx}(r')} \left\{ a_3 + \int_\infty^{r'} dr'' \sqrt{-g(r'')} g^{00}(r'') A^{(0)}(r'') \right\}$$

$$B^{(3)}(r) = -iD \int_\infty^r dr' \frac{g_{rr}(r')}{\sqrt{-g(r')}} \left\{ a_1 \tau D + a_3 - \int_\infty^{r'} dr'' \sqrt{-g(r'')} g^{00}(r'') A^{(0)}(r'') \right\}$$

Dirichlet BCs on A,B are assumed as $r \to \infty$

Boundary Condition

- Examine Einstein equations near the horizon
- Fix r at the stretched horizon r_h so that

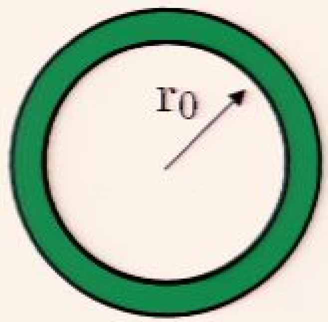
$$r_0 e^{-T^2/q^2} \ll (r_h - r_0) \ll r_0 \frac{q^2}{T^2}$$

- Assume only incoming waves at horizon
- Then, the Einstein equations give

$$(r - r_0)\frac{dB}{dr} = -i\sqrt{\frac{\gamma_r}{\gamma_0}}\,(qA + \omega B) \quad \text{as} \quad r \to r_h$$

The Membrane Paradigm

- Physical Interpretation of Stretched Horizon: a membrane imbued with hydrodynamic properties



- Alternative derivation of transport coefficients:
  - Define stress energy tensor on stretched horizon
  - Show it has the appropriate form (Saremi, 2007)
- Hydrodynamics of the stretched horizon agrees with AdS/CFT results in dual gauge theory.

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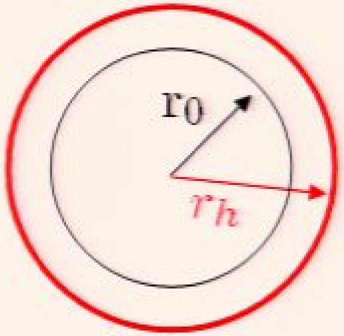

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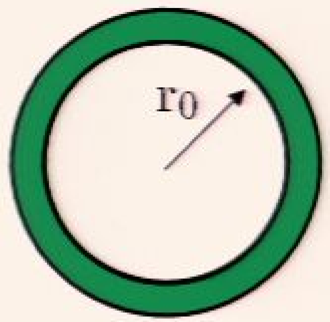

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Transport Coefficient Solutions

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- Lowest order (Shear Viscosity)

$$D \equiv \frac{\eta}{Ts} = D(r_0)$$

In Agreement with KSS (2003)

- Next Order ("Shear Relaxation Time")

$$\tau = \frac{\sqrt{-g(r_0)}}{\sqrt{-g_{00}(r_0) g_{rr}(r_0)}} \int_{r_0}^\infty dr \frac{g_{rr}(r)}{\sqrt{-g(r)}} \left[1 - \left(\frac{D(r)}{D(r_0)}\right)^2\right]$$

Applications

$$H_n = \int_0^1 \frac{1-x^n}{1-x}\,dx$$

- SAdS_{p+2} $$ds^2_{p+2} = \frac{r^2}{L^2}\left[-\left(1-\left(\frac{r_0}{r}\right)^{p+1}\right)dt^2 + d\vec{x}_p^2\right] + \frac{L^2}{r^2\left[1-\left(\frac{r_0}{r}\right)^{p+1}\right]}dr^2$$

$$(4\pi T)\tau = H_{\frac{2}{p+1}}$$

- D*p* Brane (p < 7)

$$ds_E^2 = \left(\frac{r}{L}\right)^{\frac{(7-p)^2}{8}}\left[-\left(1-\left(\frac{r_0}{r}\right)^{7-p}\right)dt^2 + d\vec{x}_p^2\right] + \left(\frac{L}{r}\right)^{\frac{(7-p)(p+1)}{8}}\left[\frac{dr^2}{\left(1-\left(\frac{r_0}{r}\right)^{7-p}\right)} + L^2 d\Omega_{8-p}^2\right]$$

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- All known results reproduced, new ones added. ($\tau \to \infty$ for D*p* brane with p>4?)

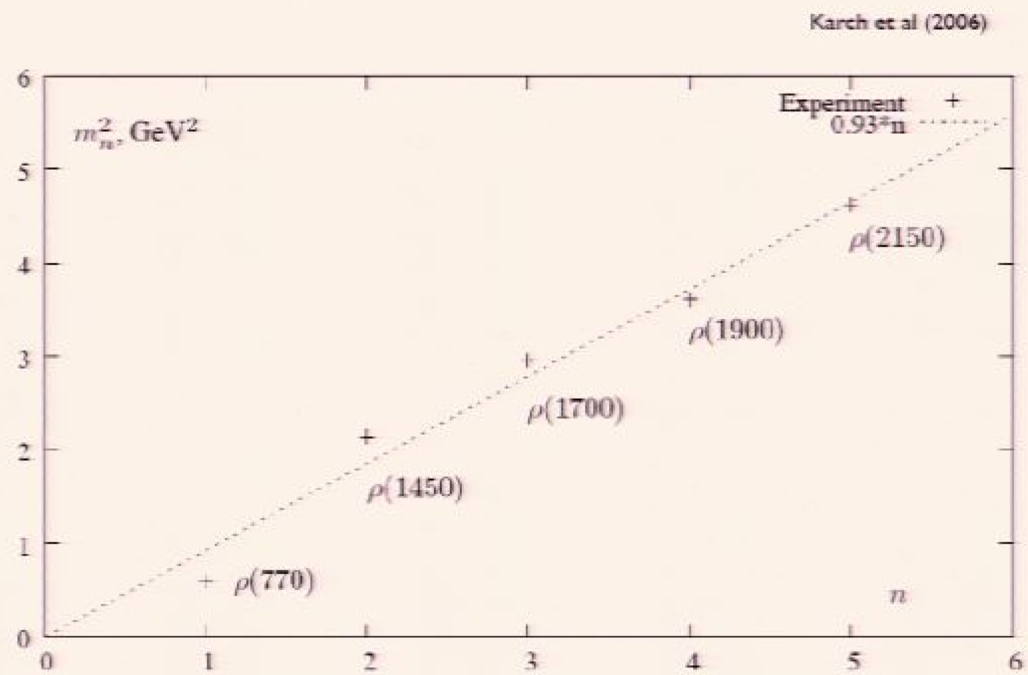
Natsuume, arXiv:0807.1392

$$V_s^2 = \frac{S - p}{q - p}$$

Applications (continued)

- "Soft Wall" introduced to get linear Regge trajectories for mesons $m_n^2 \sim n$
- Soft Wall Model (1) Karch, Katz, Son, Stephanov (2006)

$$g_{\mu\nu} = SAdS_5$$
$$\phi(r) = \frac{cL^4}{r^2}$$
$$\frac{\eta}{s} = 4\pi^{-1}$$
$$(2\pi T)\tau = 1 - \ln(2)$$



- Soft Wall Model (2)
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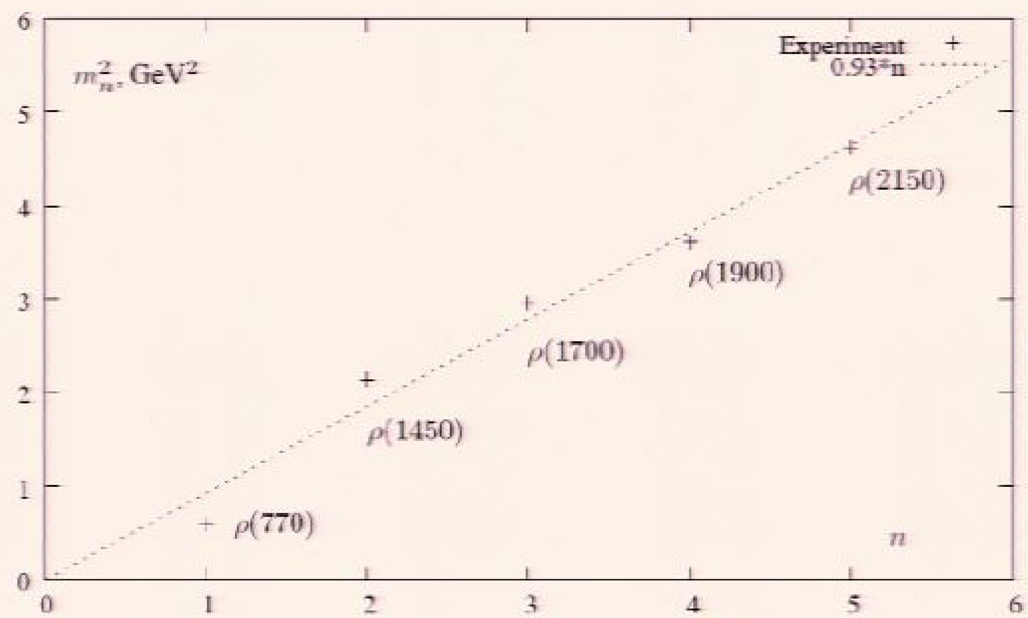

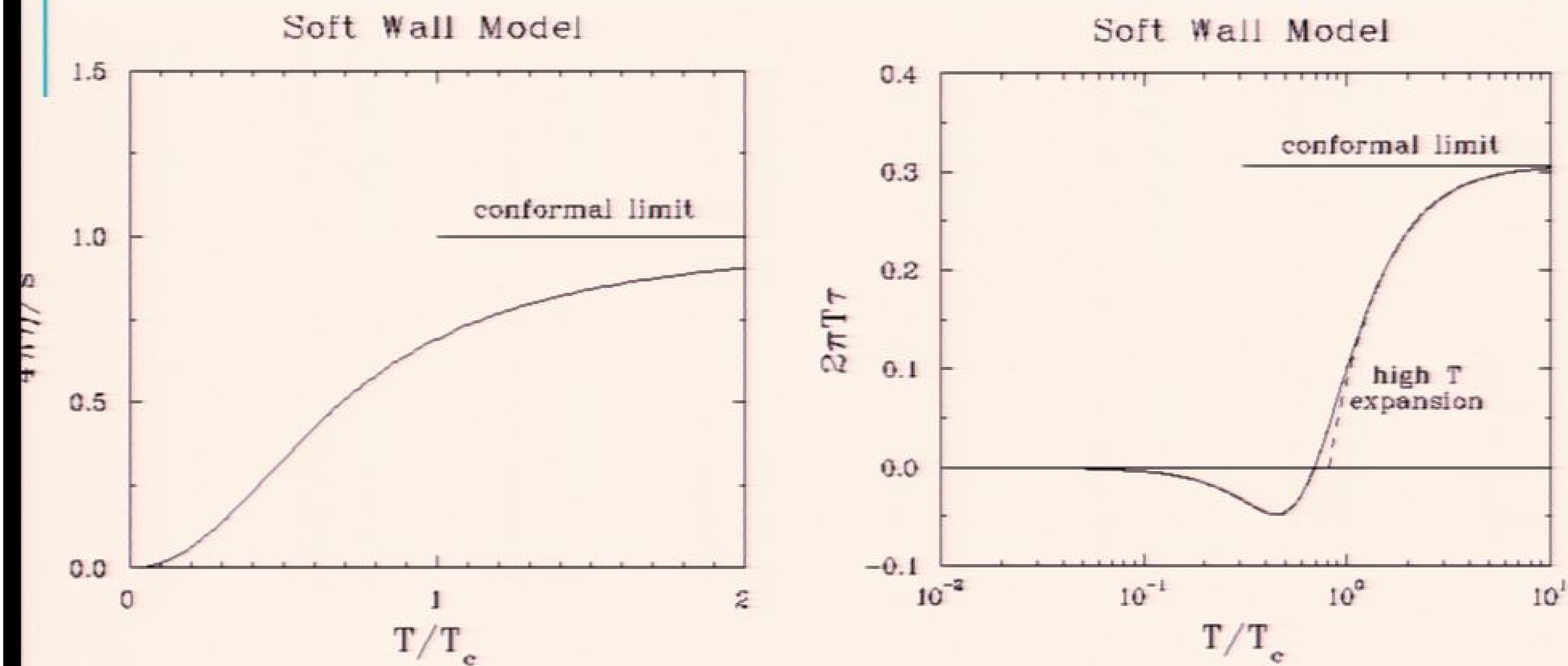

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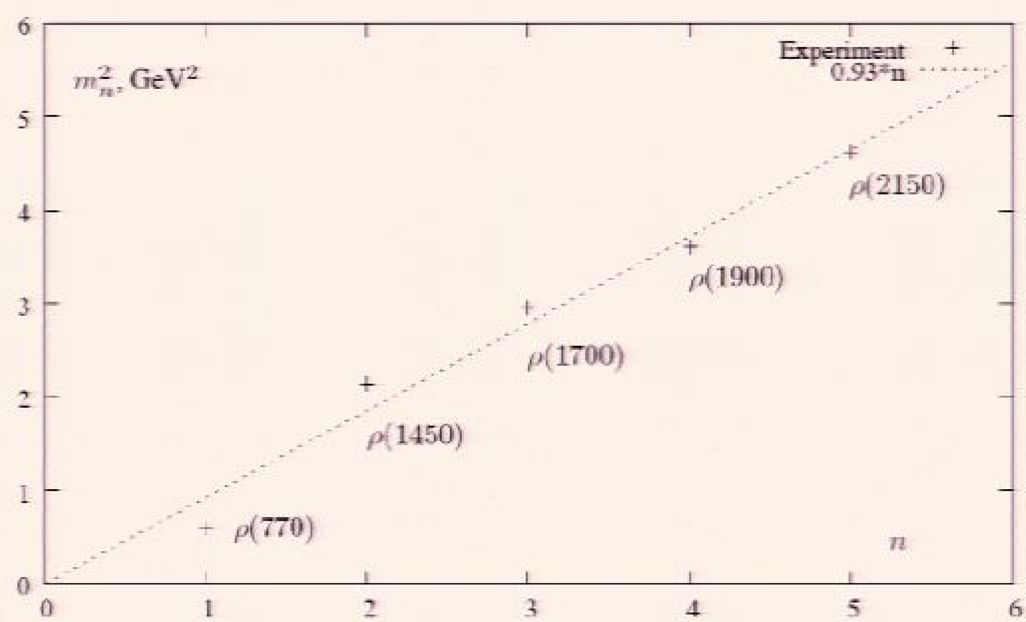

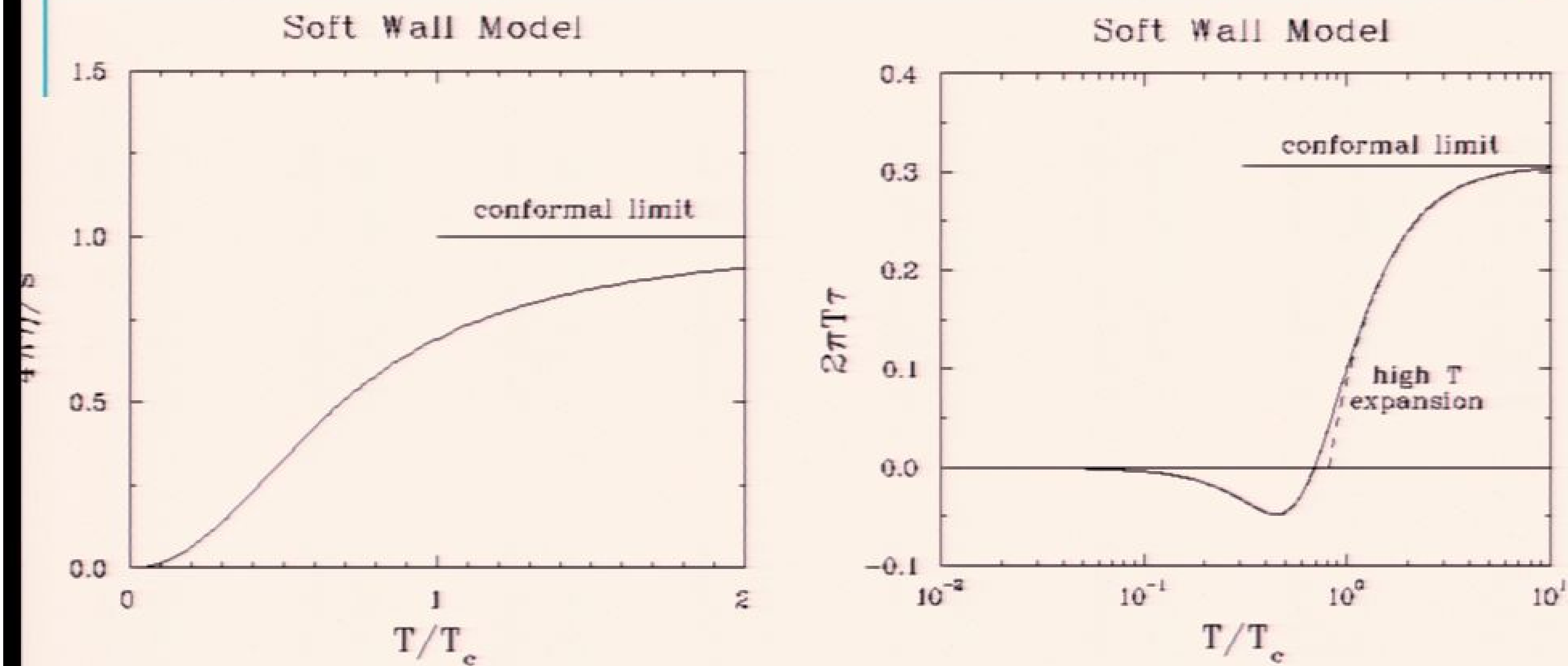

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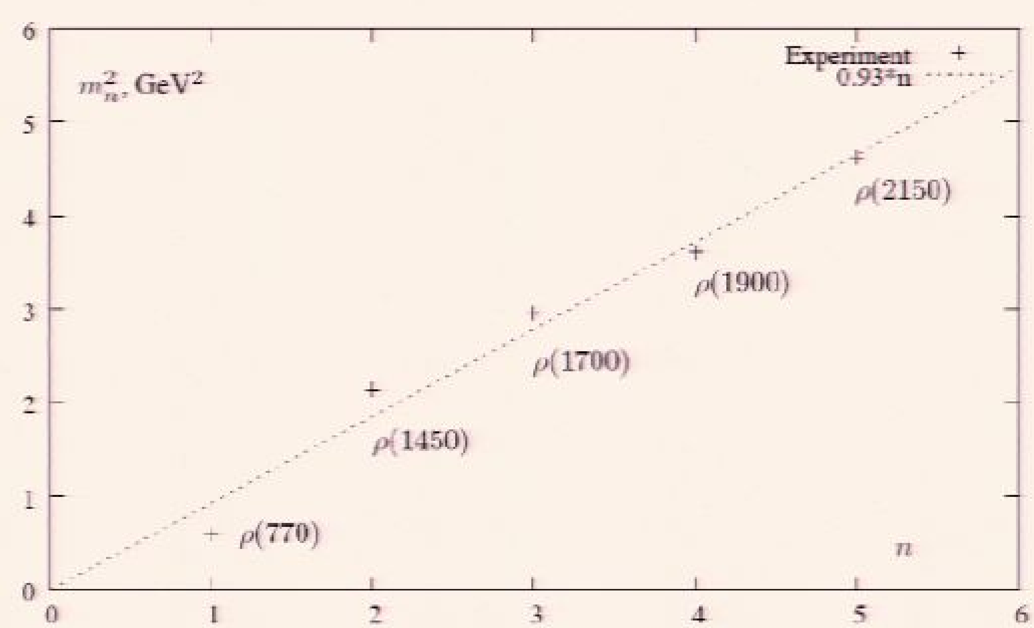

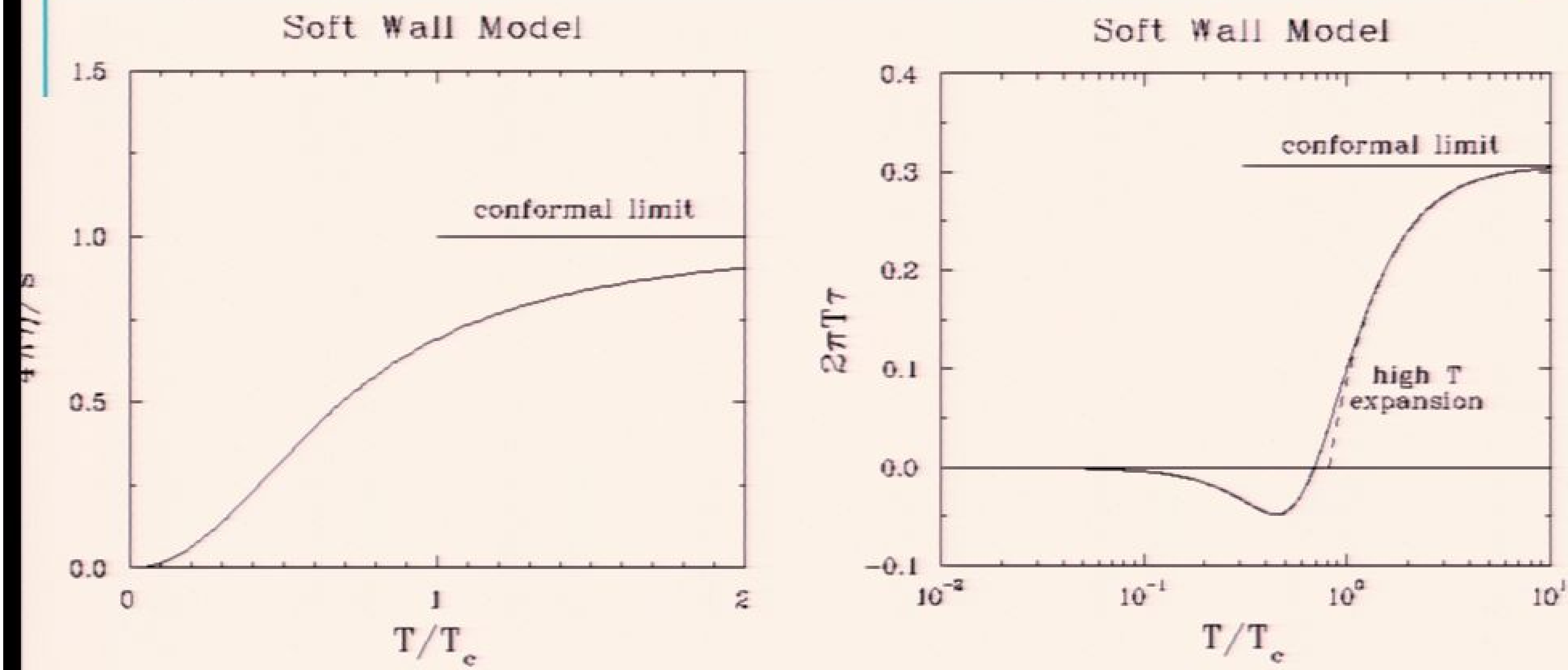

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$$D \equiv \frac{\eta}{Ts} = D(r_0)$$

In Agreement with KSS (2003)

- Next Order ("Shear Relaxation Time")

$$\tau = \frac{\sqrt{-g(r_0)}}{\sqrt{-g_{00}(r_0)g_{rr}(r_0)}} \int_{r_0}^\infty dr \frac{g_{rr}(r)}{\sqrt{-g(r)}} \left[1 - \left(\frac{D(r)}{D(r_0)}\right)^2\right]$$

Einstein Equations (Shear Mode)

$$\frac{1}{\sqrt{-g}}\partial_r\left[\sqrt{-g}g^{rr}g^{00}g_{xx}(A')\right] - qg^{00}(qA+wB) = 16\pi G g_{xx}(T^{0y}_{(1)} + T^{00}_{(0)}A)$$

$$\frac{1}{\sqrt{-g}}\partial_r\left[\sqrt{-g}g^{rr}(B')\right] - wg^{00}(qA+wB) = 16\pi G g_{xx}(T^{yz}_{(1)} + T^{xx}_{(0)}B)$$

$$-i\left(qg^{xx}B' - wg^{00}A'\right) = 16\pi G g_{rr}T^{yr}_{(1)}$$

$$A = g_{00}H^{0y}$$
$$B = g_{xx}H^{yz}$$

Background Fields

- Simple example: multiple scalar fields

$$
\begin{aligned}
16\pi G_5 T^{(0)}_{\mu\nu} &= \sum_{k=1}^{n} (\partial_\mu \phi_k(r) \partial_\nu \phi_k(r)) - g_{\mu\nu} \mathcal{L}_{\phi k} \\
\mathcal{L}_{\phi k} &= \sum_{k=1}^{n} \frac{1}{2} \partial_\lambda \phi_k(r) \partial^\lambda \phi_k(r) + U(\phi_1, \phi_2 ... \phi_n) \\
16\pi G_5 T^{(1)}_{\mu\nu} &= \sum_{k=1}^{n} (\partial_\mu \phi_k(r) \partial_\nu (\delta\phi_k) + \partial_\mu (\delta\phi_k) \partial_\nu \phi_k(r)) - g_{\mu\nu} (\delta \mathcal{L}_{\phi k}) - h_{\mu\nu} \mathcal{L}_{\phi k}
\end{aligned}
$$

- Shear mode:

$$
16\pi G_5 T^{(1)}_{0y} = -h_{0y} \mathcal{L}_{\phi k} = 16\pi G_5 g^{00} h_{0y} T^{(0)}_{00}
$$

...similarly for the other components

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0

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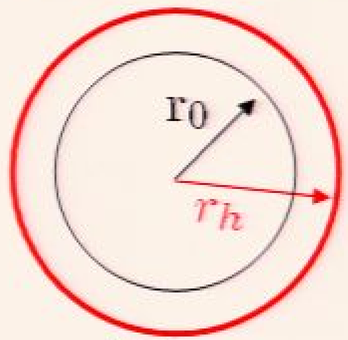
The Membrane Paradigm

- Physical Interpretation of Stretched Horizon: a membrane imbued with hydrodynamic properties



- Alternative derivation of transport coefficients:
  - Define stress energy tensor on stretched horizon
  - Show it has the appropriate form (Saremi, 2007)
- Hydrodynamics of the stretched horizon agrees with AdS/CFT results in dual gauge theory.

Applications (continued)

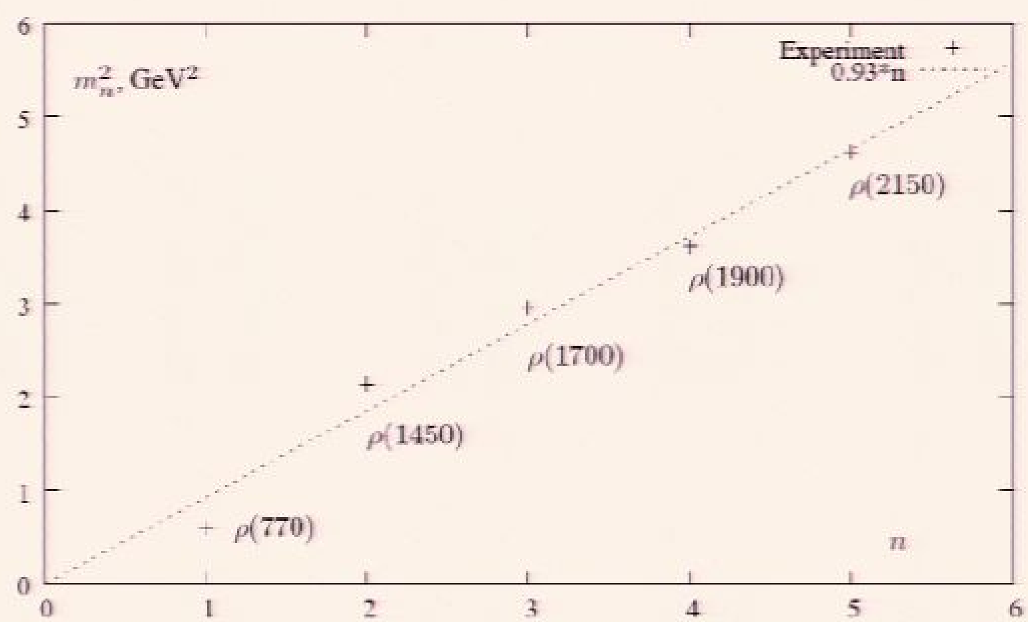
- "Soft Wall" introduced to get linear Regge trajectories for mesons $m_n^2 \sim n$
- Soft Wall Model (1) Karch, Katz, Son, Stephanov (2006)

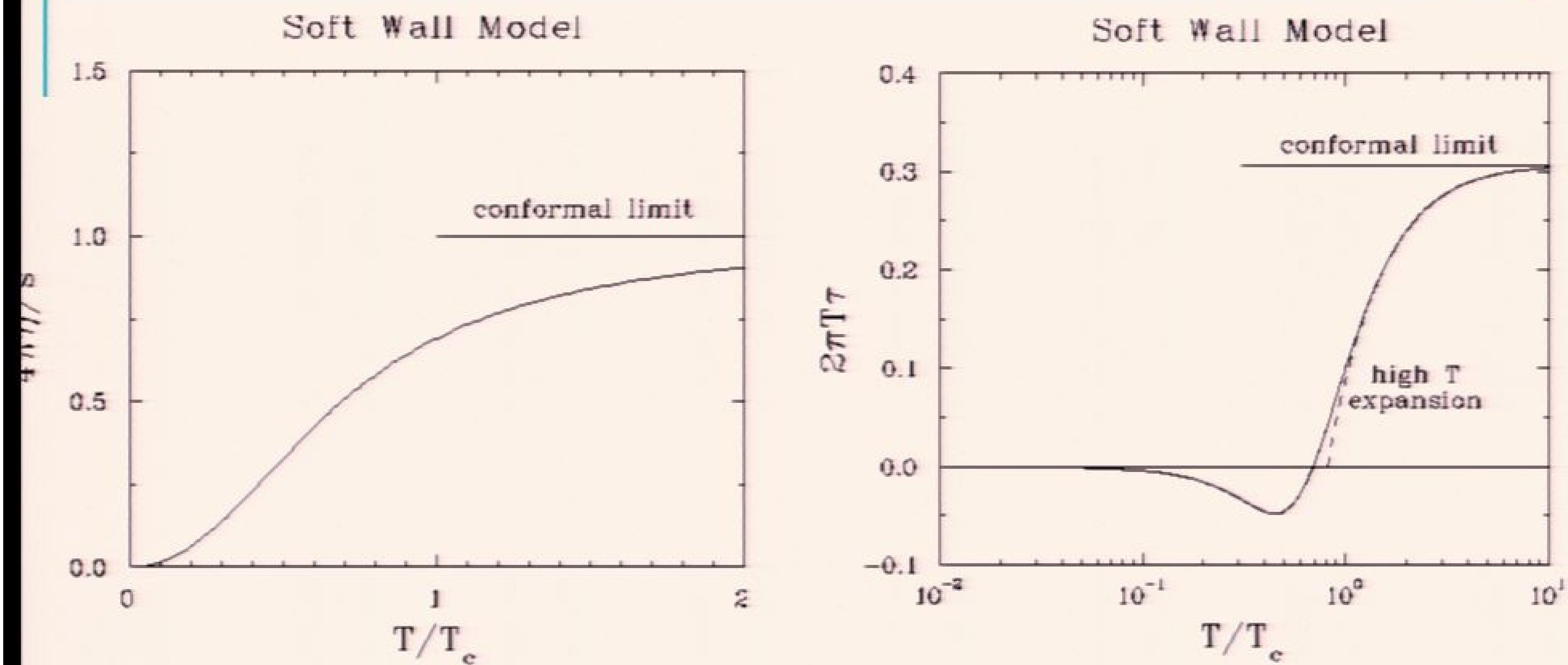
$$g_{\mu\nu} = SAdS_5$$
$$\phi(r) = \frac{cL^4}{r^2}$$
$$\frac{\eta}{s} = 4\pi^{-1}$$
$$(2\pi T)\tau = 1 - \ln(2)$$

- Soft Wall Model (2) Andreev, Zakharov (2006)

$$g_{\mu\nu} = e^{-2\phi(r)}(SAdS_5)$$
$$\text{Dilaton} = \text{Constant}$$

Karch et al (2006)





- $\frac{\eta}{s} \neq 4\pi^{-1} \rightarrow$ such a metric is not a solution to the background equations we assume
- Negative τ is not unphysical

Transport Coefficient Solutions

$$w = -iDq^2 - i\tau D^2 q^4 + \ldots$$

$$D(r) \equiv \frac{\sqrt{-g(r)}}{\sqrt{-g_{00}(r)g_{rr}(r)}} \int_r^\infty dr' \frac{-g_{00}(r')g_{rr}(r')}{\sqrt{-g(r')g_{xx}(r')}}$$

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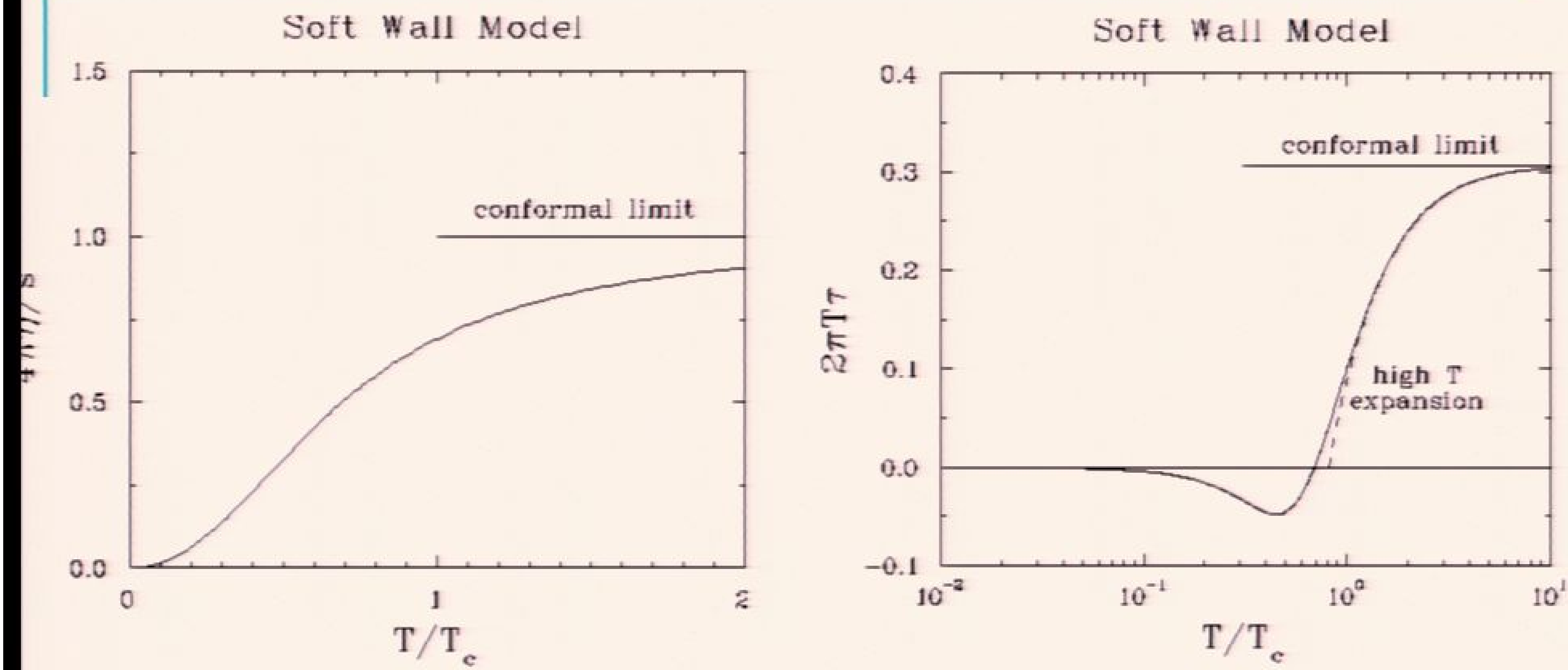

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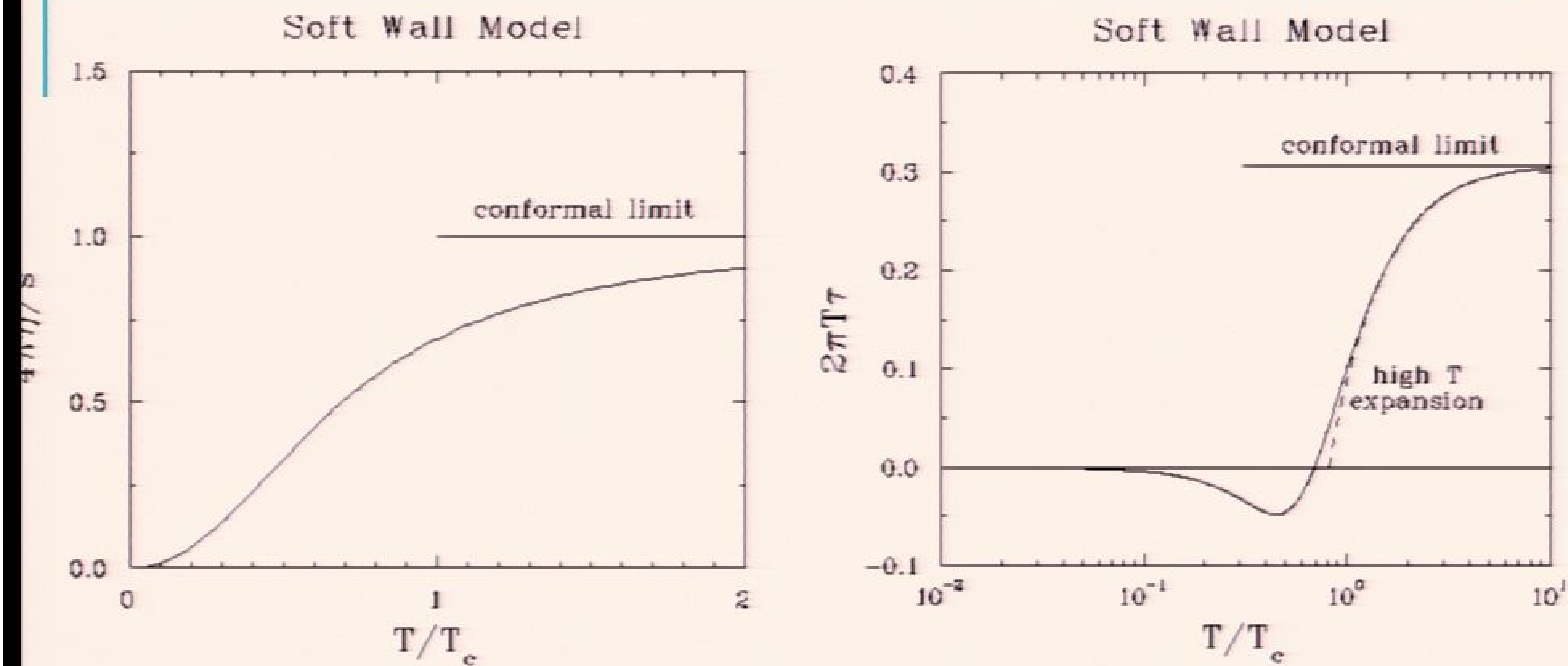
Sound Mode

- No analogous formulas exist for speed of sound, or bulk viscosity

$\omega = -i\left(\frac{\eta}{\epsilon+P}\right)q^2$	**Shear Mode**
$\omega = v_s q - \frac{iq^2}{2(\epsilon+P)}\left(\frac{4\eta}{3}+\zeta\right)$	**Sound Mode**

- Special cases
  - D*p-Brane* (Mas and Tarrio, 2007)
  - 5D metric supported by single scalar (Gubser, Pufu, Rocha, 2008)



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Sound Mode

- p+2 dimensions

$$ds^2 = g_{00}(r)dt^2 + g_{xx}(r)dx_i dx^i + g_{rr}(r)dr^2$$
$$i = 1, 2 \ldots p$$

- Arbitrary number of scalars

$$16\pi G_{p+2} T^{(0)}_{\mu\nu} = \sum_{k=1}^{n} \left(\partial_\mu \phi_k(r) \partial_\nu \phi_k(r)\right) - g_{\mu\nu} \mathcal{L}_{\phi k}$$

- Scalars must obey constraint

$$\sum_{k=1}^{n} \phi'_k(r)^2 = 2 g_{rr} \left(R^0_0(r) - R^r_r(r)\right)$$

Sound Mode Difficulties

- Equations are lengthy
  - More perturbations
  - 7 Einstein equations (4 independent)
- Matter fields couple
  - New perturbations $\delta\phi_k$
  - Additional equations for each field

$$\begin{aligned} A(r) &= g_{00}H^{00} \\ B(r) &= g_{xx}\frac{1}{p-1}\sum_{i=1}^{p-1} H^{ii} \\ C(r) &= g_{xx}H^{zz} \\ D(r) &= g_{xx}H^{0z} \end{aligned}$$

$$\Box^{(0)}(\delta\phi) + \Box^{(1)}\phi = \delta\left(\frac{dU(\phi)}{d\phi}\right)$$

Gauge Invariance

$$\begin{aligned} h_{\mu\nu} &\rightarrow h_{\mu\nu} - \nabla^{(0)}_{\mu}\xi_{\nu} - \nabla^{(0)}_{\nu}\xi_{\mu} \\ (\delta\phi) &\rightarrow (\delta\phi) - \xi_{\mu}(\partial^{\mu}\phi) \end{aligned}$$

- Gauge invariant variables (Kovtun, Starinets, 2005)

$$Z_0(r) = g_{00}(r)g^{xx}(r)\left(q^2A(r) + 2qwD(r)\right) + w^2C(r) - \left(q^2\frac{g'_{00}(r)}{g'_{xx}(r)} + w^2\right)B(r)$$

$$Z_{\phi i}(r) = \delta\phi_i(r) - \frac{\phi_i(r)g_{xx}(r)}{g'_{xx}(r)}B(r)$$

- Boundary Conditions determine *w(q)*
  - Incoming wave at horizon
  - Dirichlet at $r \rightarrow \infty$

Equations (Single Scalar)

- Two coupled 2nd order equations result
- *α,f* are known functions of the metric

$$\frac{g_{rr}}{\sqrt{-g}}\alpha^2 f^4 \partial_r \left[\frac{\sqrt{-g}g^{rr}}{\alpha^2 f^4} Z_0'\right] + Z_0 \left(\mathcal{D}_L[f^2]\mathcal{D}_L[f^2\alpha] - g_{rr}\left(w^2 g^{00} + q^2 g^{xx}\right)\right) \quad +$$

$$2Z_\phi \phi' f^2 \left(\alpha\partial_r \left[\frac{1}{\alpha}\left(\frac{w^2}{f^2} - q^2\right)\right] + \frac{q^2\mathcal{D}_L[f^2]}{p\mathcal{D}_L[g_{xx}]}\mathcal{D}_L\left[\sqrt{-g}g^{rr}\phi'\right]\right) \quad = \quad 0$$

$$\frac{g_{rr}}{\sqrt{-g}}\partial_r\left[\sqrt{-g}g^{rr}Z_\phi'\right] + \frac{2}{\alpha}\partial_r\left[\frac{\phi'}{\mathcal{D}_L[g_{xx}]}\right]\left\{\frac{1}{f}\partial_r\left[\frac{Z_0}{f}\right] - \left(w^2 g^{00} + q^2 g^{xx}\right)\phi' g_{xx} Z_\phi\right\}$$

$$-Z_\phi\left\{g_{rr}\left(q^2 g^{xx} + w^2 g^{00}\right) + \frac{\left(\mathcal{D}_L[g_{xx}]\right)^2}{f^2\phi'}\partial_r\left[\frac{f^2\phi'}{\left(\mathcal{D}_L[g_{xx}]\right)^2}\mathcal{D}_L\left[\sqrt{-g}g^{rr}\phi'\right]\right]\right\} \quad = \quad 0$$

- Correctly reduce to the form given by Mas, Tarrio in the limit of the D*p*-brane

$$\mathcal{D}_L[F(r)] \equiv \frac{F'(r)}{F(r)}$$

A Solvable Special Case

- Assume

$$\phi(r) = C \log\left[g_{xx}(r)\right]$$

- What sort of metric does this produce?

$$g_{00}(r) = g_{xx}(r)\left(\frac{a_0}{a_2 - p} - a_1 g_{xx}(r)^{a2-p}\right)$$

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Applications

- SAdS$_{p+2}$

$$a_0 = \frac{p+1}{2}, \quad a_2 = \frac{p-1}{2} \longrightarrow v_s = \frac{1}{\sqrt{p}}, \quad \frac{\zeta}{\eta} = 0$$

- Dp Brane

$$a_0 = \frac{p(7-p)}{9-p}, \quad a_2 = \frac{2p}{9-p} \longrightarrow v_s = \sqrt{\frac{5-p}{9-p}}, \quad \frac{\zeta}{\eta} = \frac{2(3-p)^2}{9-p}$$

- Bulk viscosity conjecture $\frac{\zeta}{\eta} \geq 2\left(\frac{1}{p} - v_s^2\right)$

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Discussion – Black Hole Thermodynamics

- Black hole thermodynamics gives a simple expression for v_s, which only depends on the metric.

$$v_s^2 = \frac{d\log(T)}{d\log(s)} = \frac{d\log(T)}{d\log\left[gxx(r_0)^{p/2}\right]}$$

- Does a simple analytic solution exist for the gauge invariant equations?
- Why do the scalar field profiles appear important? (Unnecessary in the BH thermo approach).

Summary

- Membrane Paradigm results agree with AdS/CFT calculations in many cases

(Iqbal, Liu, 2008)

- Results for τ have extended shear mode analysis to the next hydrodynamic order
- Interpretation of these results requires a consistent theory of ~~2^{nd}~~ 3^{rd} order hydrodynamics

(Baier, Romatschke, Son, Starinets, Stephanov, 2008)

- General sound mode equations derived for a black brane type metric supported by scalar fields
- Special case solution was presented which is a generalization of previous work

Works in Progress

- Extend sound mode solution to next hydrodynamic order
  - Are there compact formulas for τ_{Π}?
  - What can we learn about 3rd order hydrodynamics?
- Bulk viscosity should be calculated in phenomenological models
  - Batell, Gherghetta (2008)
  - Gursoy, Kiritsis, Mazzanti, Nitti (2008)

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No Signal
VGA-1

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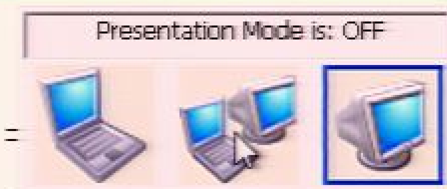

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