Title: Quantum boolean functions

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Abstract: In recent years, the analysis of boolean functions has arisen as an important theme in theoretical computer science. In this talk I will discuss an extension of the concept of a boolean function to quantum computation. It turns out that many important classical results in the theory of boolean functions have natural quantum analogues. These include property testing of boolean functions; the Goldreich-Levin algorithm for approximately learning boolean functions; and a theorem of Friedgut, Kalai and Naor on the Fourier spectra of boolean functions. The quantum generalisation of this theorem uses a quantum extension of the hypercontractive inequality of Bonami, Gross and Beckner. This talk is based on joint work with Tobias Osborne.

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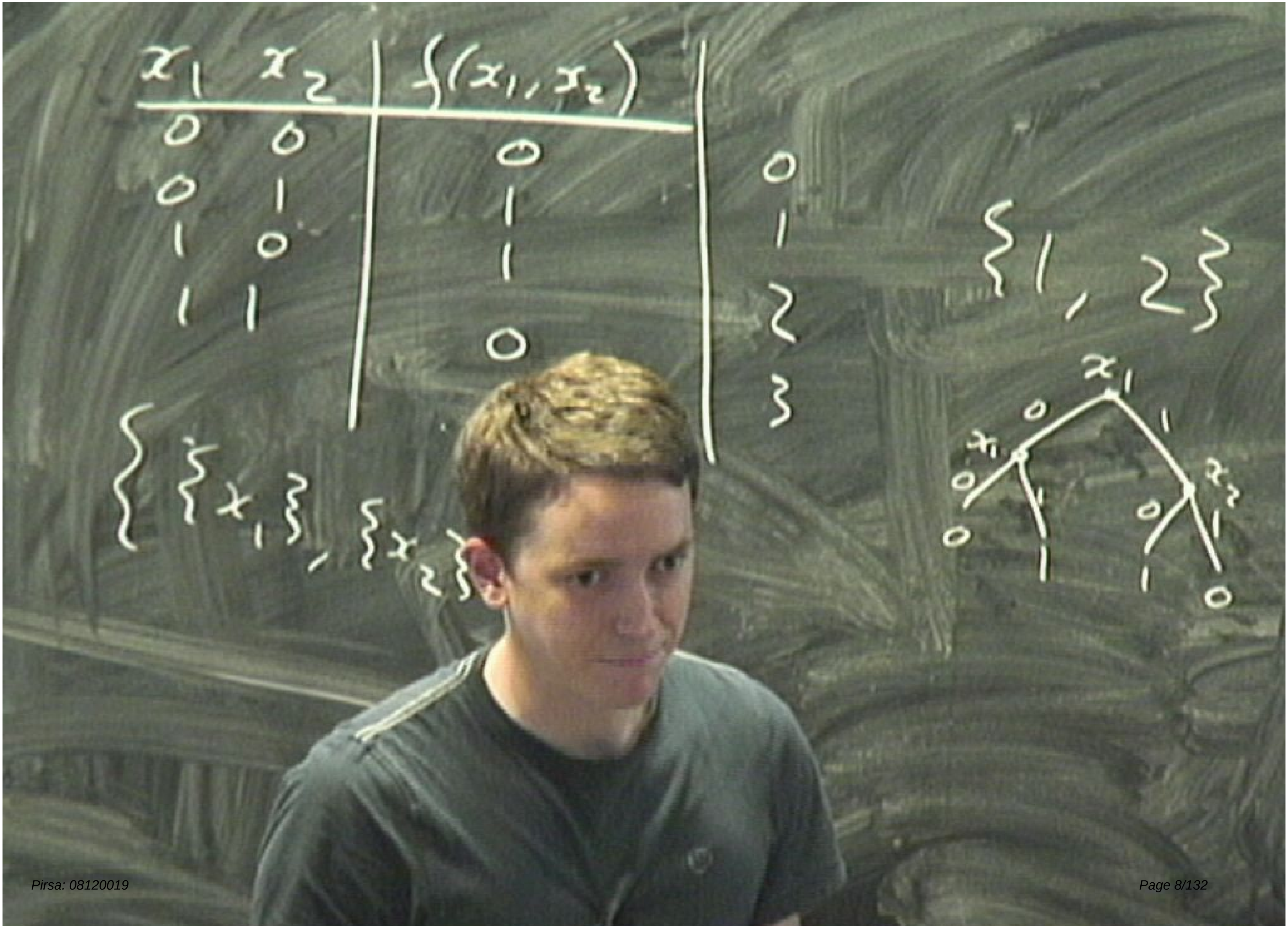

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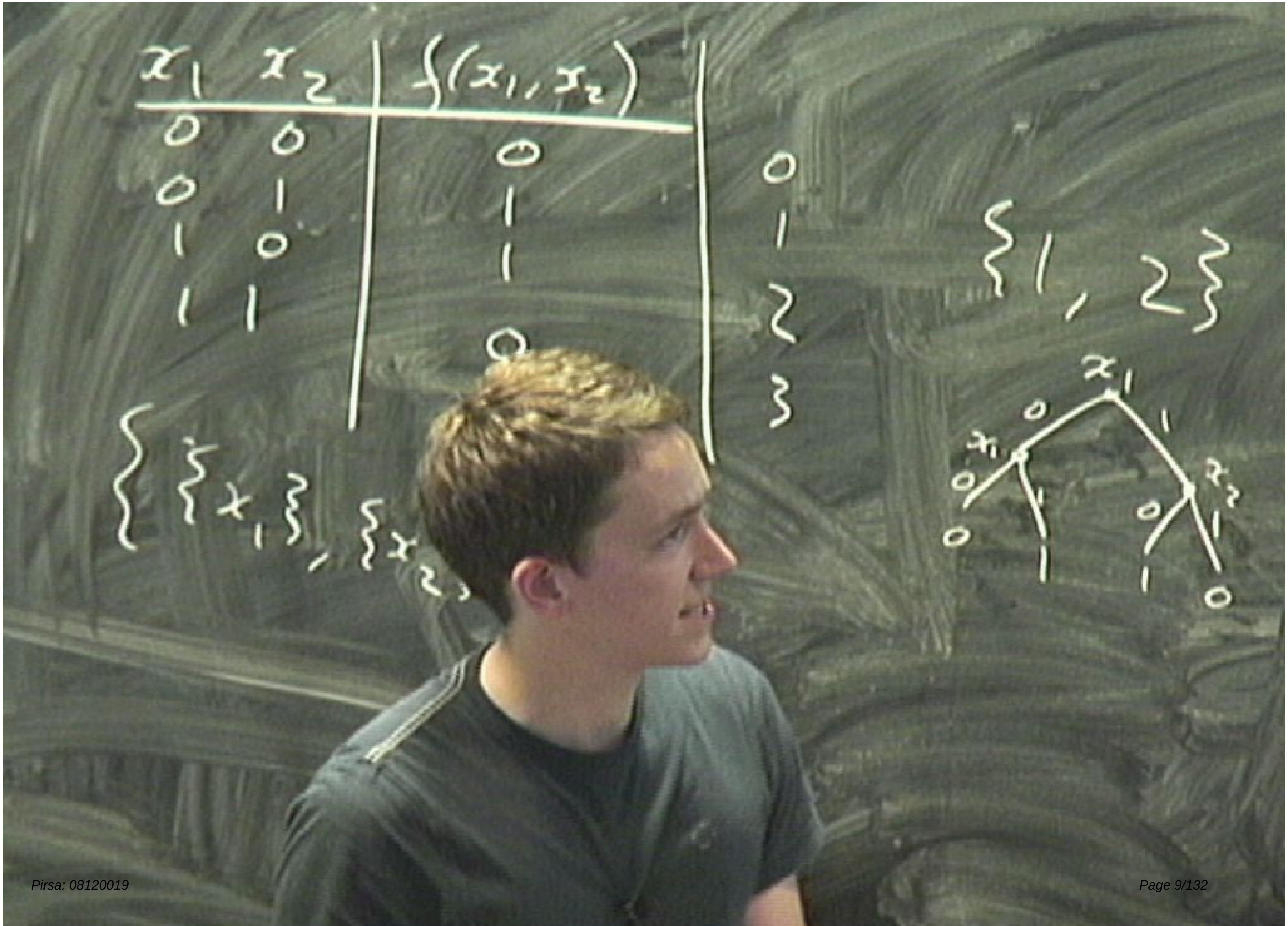
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Ryan O'Donnell:

"By analysis of boolean functions, roughly speaking we mean deriving information about boolean functions by looking at their 'Fourier expansion'."

(See http://www.cs.cmu.edu/~odonnell/boolean-analysis/ for an entire course on the subject.)

Fourier analysis of boolean functions

For an n-bit boolean function, we need to do Fourier analysis over the group $\mathbb{Z}_2^n$. This involves expanding functions

$$f : \{0,1\}^n \to \mathbb{R}$$

in terms of the characters of $\mathbb{Z}_2^n$. These characters are the parity functions

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for some $\{\hat{f}_S\}$ – the Fourier coefficients of f. How do we find them? By carrying out the Fourier transform over $\mathbb{Z}_2^n$ – i.e. a (renormalised) Hadamard transform!

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Think of f and $\hat{f}$ as 2^n-dimensional vectors; then

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So what can we do with Fourier analysis?

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- Linearity ($f(x+y) = f(x) + f(y)$ for all x, y)
- Dictatorship ($f(x) = x_i$ for some i)

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These results have been useful in social choice theory and hardness of approximation.

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Important extension: the Goldreich-Levin algorithm, which outputs a list of the "large" Fourier coefficients of f "efficiently".

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The remainder of this talk:

- Basic consequences of this definition (why it's the *right* definition)
- Generalisations of classical results to QBFs (why it's an *interesting* definition)

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Yes: Given any classical boolean function $f : \{0,1\}^n \to \{0,1\}$, there are two natural ways of implementing f on a quantum computer:

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- The *phase oracle* $|x\rangle \mapsto (-1)^{f(x)}|x\rangle$.

...and both of these give QBFs!

Other examples of QBFs

A projector P onto any subspace gives rise to a QBF: take $f = \mathbb{I} - 2P$. Thus:

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$$f = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Norms and inner products

Some definitions we'll need later:

- The (normalised) Schatten p-norm: for any d-dimensional operator f, $\|f\|_p \equiv \left(\frac{1}{d}\sum_{j=1}^{d}\sigma_j^p\right)^{\frac{1}{p}}$, where $\{\sigma_j\}$ are the singular values of f.

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- Note Hölder's inequality: for $1/p + 1/q = 1$, $|\langle f, g\rangle| \leqslant \|f\|_p \|g\|_q$.

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We write a tensor product of Paulis (a stabiliser operator) as $\chi_{\mathbf{s}} \equiv \sigma^{s_1} \otimes \sigma^{s_2} \otimes \cdots \otimes \sigma^{s_n}$, where $s_j \in \{0, 1, 2, 3\}$.

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We use the notation σ_i^j for the dictator which acts as σ^j at the i'th position, and trivially elsewhere.

"Fourier analysis" for QBFs (2)

The $\{\chi_{\mathbf{s}}\}$ operators form an orthonormal basis for the space of operators on n qubits, implying

- any n qubit Hermitian operator f has an expansion

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- Thus, if f is quantum boolean, $\sum_{\mathbf{s}} \hat{f}_{\mathbf{s}}^2 = 1$.

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- Quantum property testers that determine with a small number of uses of an unknown QBF whether it is close to having some property.
- Quantum analogues of computational learning results: an algorithm that outputs the large Fourier coefficients of an unknown QBF, accessed as an oracle.
- A quantum analogue of the FKN theorem regarding Fourier expansion of QBFs.

Generalising classical results to QBFs

Now we have our quantum analogue of a Fourier expansion, we can try to generalise classical results that depend on Fourier analysis. We find:

- Quantum property testers that determine with a small number of uses of an unknown QBF whether it is close to having some property.
- Quantum analogues of computational learning results: an algorithm that outputs the large Fourier coefficients of an unknown QBF, accessed as an oracle.
- A quantum analogue of the FKN theorem regarding Fourier expansion of QBFs.

In order to get this last result, we prove a quantum hypercontractive inequality which may be of independent interest.

Quantum property testing

We want to solve problems of the following kind.

Quantum property testing

Given access to a QBF f that is promised to either have some property, or to be "far" from having some property, determine which is the case, using a small number of uses of f.

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We first need to define a notion of closeness for QBFs.

Closeness

Let f and g be two QBFs. Then we say that f and g are ϵ-close if $\langle f, g \rangle \geqslant 1 - 2\epsilon$ (equivalently, $\|f - g\|_2^2 \leqslant 4\epsilon$).

Note that the use of the 2-norm gives an average-case, rather than worst-case, notion of closeness.

Quantum property testing

Consider the following representative example:

Stabiliser testing

Given oracle access to an unknown operator f on n qubits, determine whether f is a stabiliser operator $\chi_{\mathbf{s}}$ for some $\mathbf{s}$.

This problem is a generalisation of classical linearity testing.

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We give a test (the quantum stabiliser test) that has the following property.

Proposition

Suppose that a QBF f passes the quantum stabiliser test with probability $1 - \epsilon$. Then f is ϵ-close to a stabiliser operator $\chi_{\mathbf{s}}$.

The test uses 2 queries (best known classical test uses 3).

Quantum stabiliser testing

Algorithm (sketch):

1. Apply f to the halves of n maximally entangled states $|\Phi\rangle^{\otimes n}$ resulting in a quantum state $|f\rangle = f \otimes \mathbb{I}|\Phi\rangle^{\otimes n}$.

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5. Accept if all measurements say "yes".

Quantum stabiliser testing: proof of correctness

We can calculate the probability of saying "yes" using Fourier analysis. It turns out that for the stabiliser test

$$\Pr[\text{test accepts}] = \sum_{\mathbf{s}} \hat{f}_{\mathbf{s}}^4.$$

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Another obvious property we might want to test: locality.

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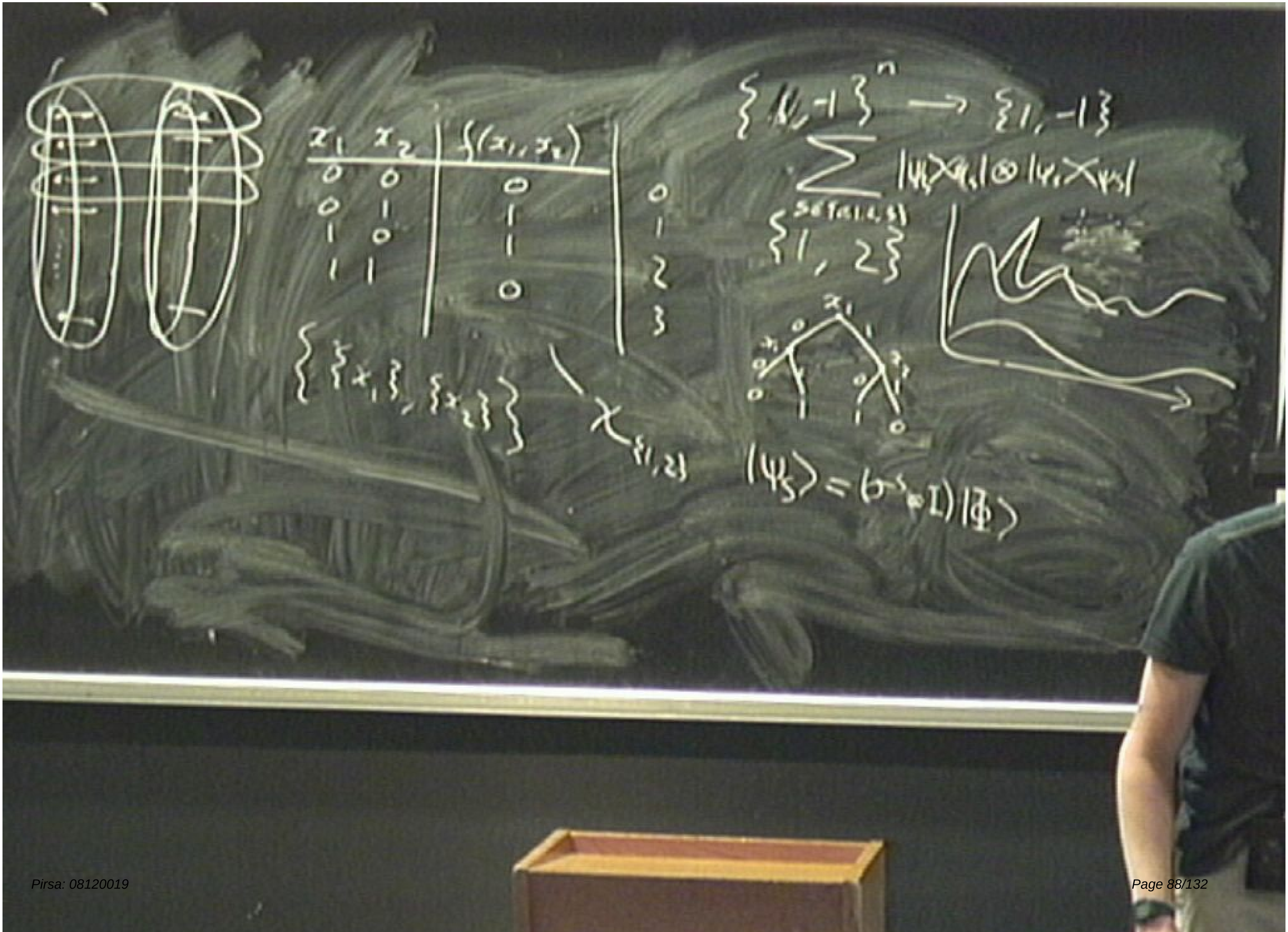
x1 x2 | f(x1, x2)
0 0 | 0 | 0
0 1 | 1 | 1
1 0 | 1 | 2
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{{x1}, {x2}}
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Hypercontractivity and noise

An essential component in many results in classical analysis of boolean functions is the hypercontractive inequality of Bonami, Gross and Beckner[1].

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For example, the inequality allows us to prove:

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This inequality is most easily defined in terms of a noise operator which performs local smoothing.

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For a bit-string $x \in \{0,1\}^n$, define the distribution $y \sim_\epsilon x$:

- $y_i = x_i$ with probability $1/2 + \epsilon/2$
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Equivalently, T_ϵ may be defined by its action on Fourier coefficients, as

$$T_\epsilon f = \sum_{S \subseteq [n]} \epsilon^{|S|} \hat{f}_S \chi_S.$$

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Bonami-Gross-Beckner inequality

Let f be a function $f : \{0,1\}^n \to \mathbb{R}$ and assume that $1 \leqslant p \leqslant q \leqslant \infty$. Then, provided that

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we have

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we have

$$\|T_\epsilon f\|_q \leqslant \|f\|_p.$$

Intuition behind this inequality:

- For $p \leqslant q$, it always holds that $\|f\|_p \leqslant \|f\|_q$.
- This inequality says that, if we smooth f enough, then the inequality holds in the other direction too.

A quantum noise operator

We can immediately find a quantum version of the Fourier-theoretic definition of the noise operator.

Noise superoperator

The noise superoperator with rate $-1/3 \leqslant \epsilon \leqslant 1$, written T_ϵ, is defined as

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Turns out that this has an equivalent definition in terms of the qubit depolarising channel!

Noise superoperator (2)

$T_\epsilon f = \mathcal{D}_\epsilon^{\otimes n} f$, where $\mathcal{D}_\epsilon$ is the qubit depolarising channel with noise rate ϵ, i.e. $\mathcal{D}_\epsilon(f) = \frac{(1-\epsilon)}{2} \operatorname{tr}(f)\mathbb{I} + \epsilon f$.

(This connection is well-known, see e.g. [Kempe et al '08].)

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- The proof is by induction on n. The case $n = 1$ follows immediately from the classical proof.

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- For $n > 1$, expand f as $f = \mathbb{I} \otimes a + \sigma^1 \otimes b + \sigma^2 \otimes c + \sigma^3 \otimes d$, and write it as a block matrix.
- Using a non-commutative Hanner's inequality for block matrices[2], can bound $\|T_\epsilon f\|_q$ in terms of the norm of a 2×2 matrix whose entries are the norms of the blocks of $T_\epsilon f$.
- Bound the norms of these blocks using the inductive hypothesis.
- The hypercontractive inequality for the base case $n = 1$ then gives an upper bound for this 2×2 matrix norm.

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Corollaries

There are some interesting corollaries of this result. We only mention one, about the degree of operators.

By analogy with the classical notion of degree, we define

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for n-qubit operators f. Then:

Different norms of low-degree operators are close

Let f be a Hermitian operator on n qubits with degree at most d. Then, for any $q \geqslant 2$, $\|f\|_q \leqslant (q-1)^{d/2} \|f\|_2$.

A quantum FKN theorem

Once the hypercontractive inequality is established, the proof of the classical Friedgut-Kalai-Naor theorem goes through fairly straightforwardly (with one or two caveats).

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- This result is the first stab at understanding the structure of the Fourier expansion of QBFs.
- Applications? “Quantum voting”?

Example: a 1D spin chain

Consider a Hamiltonian which can be written

$$H = \sum_{j=1}^{n-1} h_j$$

with h_j Hermitian, $\|h_j\|_\infty = O(1)$, and $\mathrm{supp}(h_j) \subset \{j, j+1\}$ for $j \leqslant n-1$.

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The end

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