Title: Gravitoelectromagnetism

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Abstract: Gravitomagnetism is a subtle concept. Adding Lorentz invariance to Newtonian gravity leads to magnetism, but Einsteinian gravitomagnetism differs from Maxwell\'s electromagnetism. The differences lead to confusion when Lense-Thirring precession is wrongly ascribed to gyroscopes, and when authors disagree about whether lunar laser ranging has measured gravitomagnetism. To clarify these issues, we analyze electric and magnetic effects in local Lorentz frames using the tetrad formalism.

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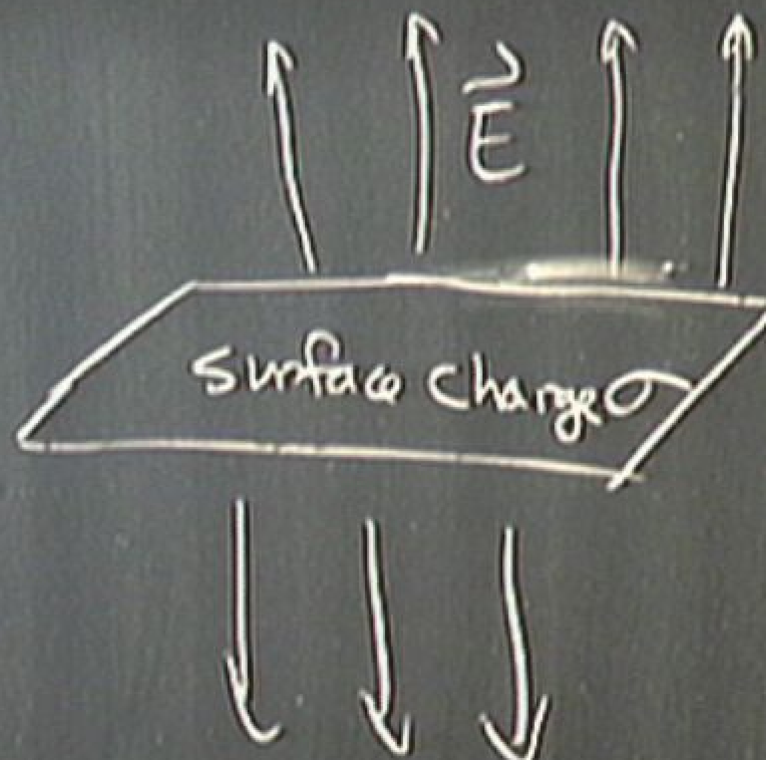

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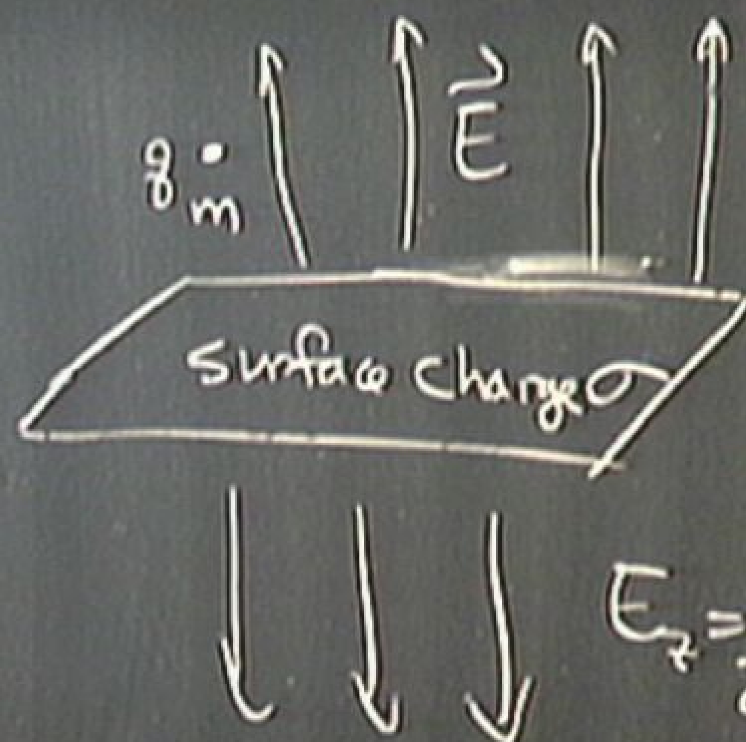

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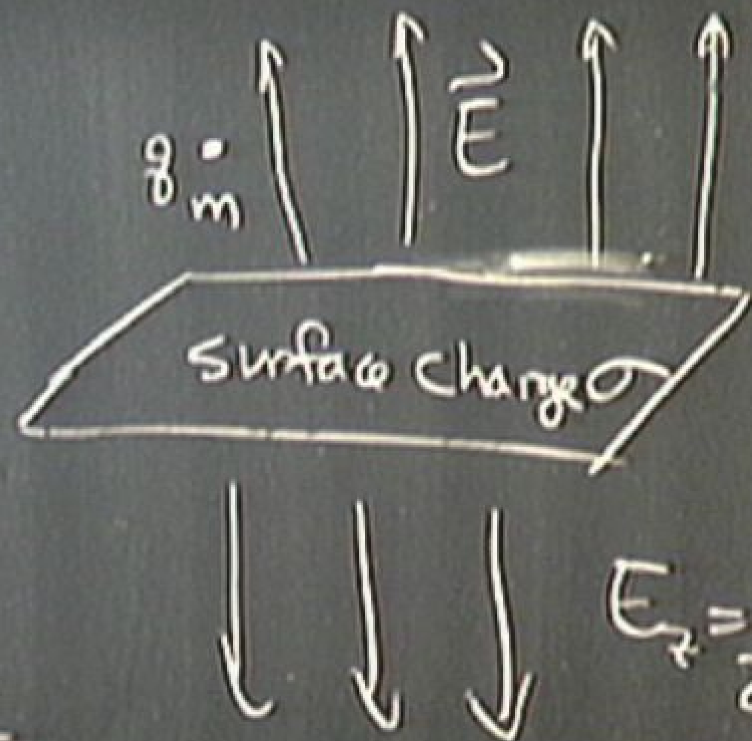
a



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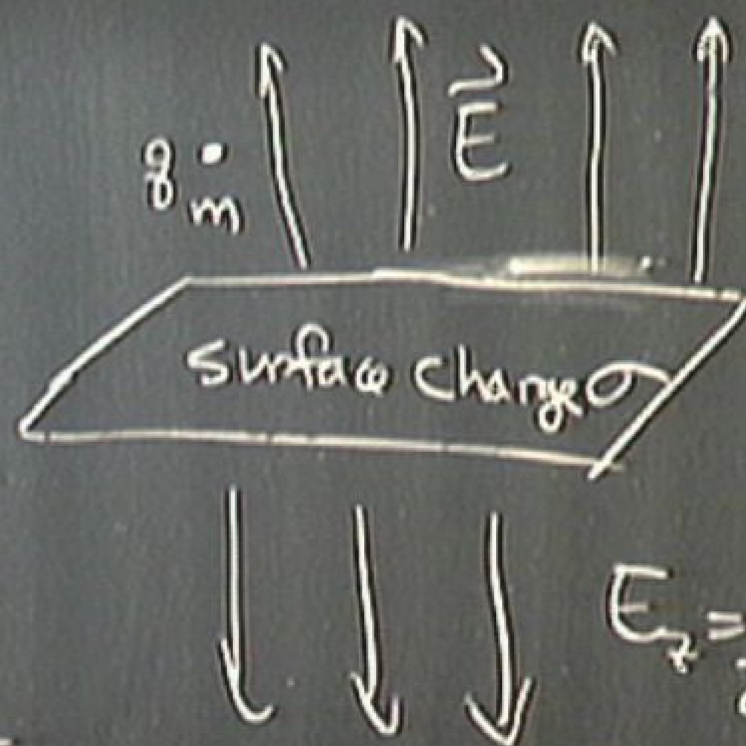
$$a = \frac{qE_z}{m}$$



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Lorentz boost $\vec{u} = u\,\vec{e}_z$

Along the worldline, $\gamma_u \equiv \frac{1}{\sqrt{1-u^2}}$

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Substituting into $\vec{v}' = \frac{d\vec{x}'}{dt'}$, $\vec{p}' = \frac{m\vec{v}'}{\sqrt{1-v'^2}}$

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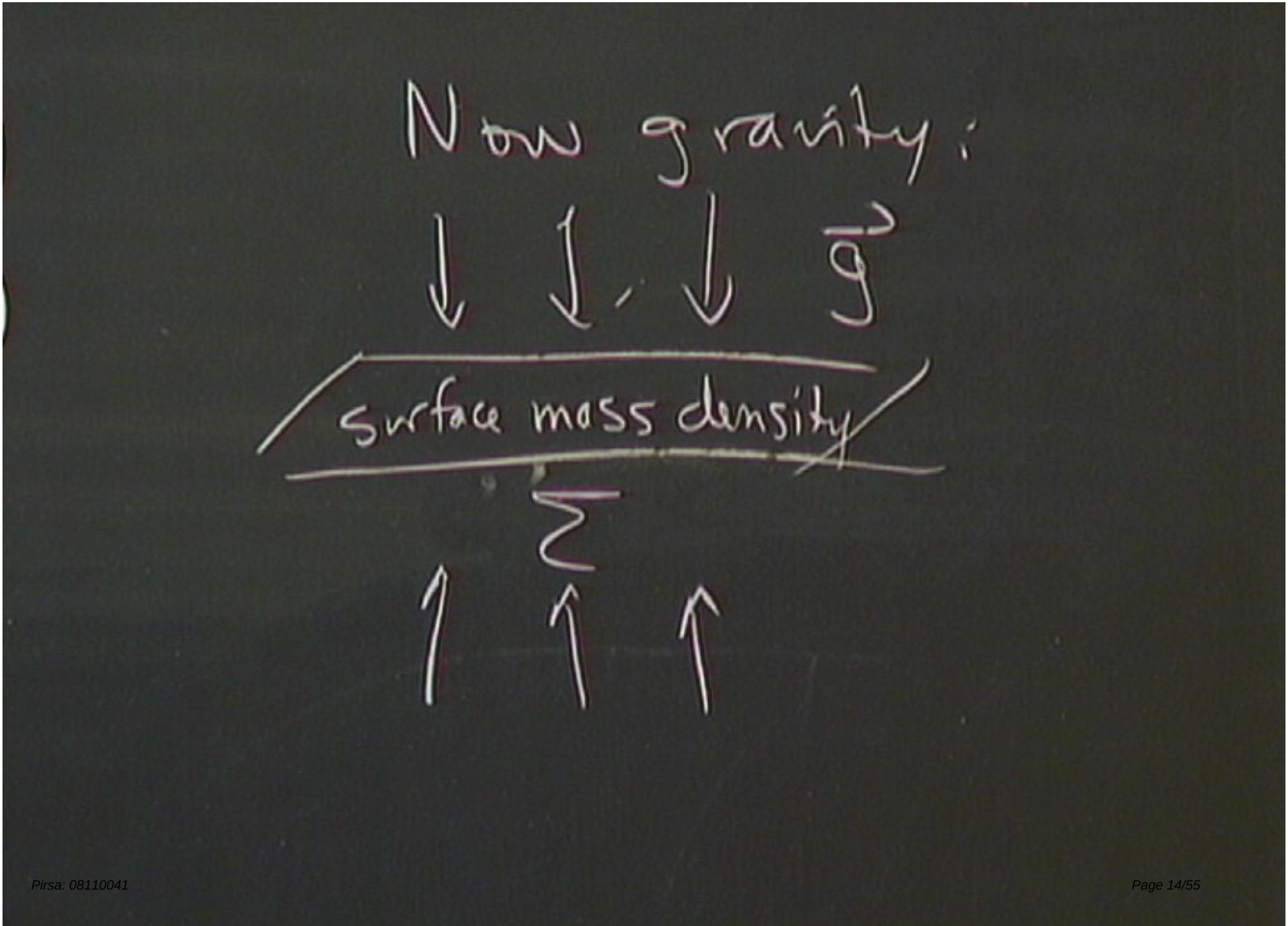
$$\frac{d\vec{p}'}{dt'} = q\left(\vec{E}' + \vec{v}' \times \vec{B}'\right)$$

$$\vec{E}' = \gamma_u \left(\vec{E} + \vec{u} \times \vec{B} \right)$$

$$\vec{B}' = \gamma_u \left(\vec{B} - \vec{u} \times \vec{E} \right)$$


Now gravity:
$\vec{g}$
surface mass density
Σ

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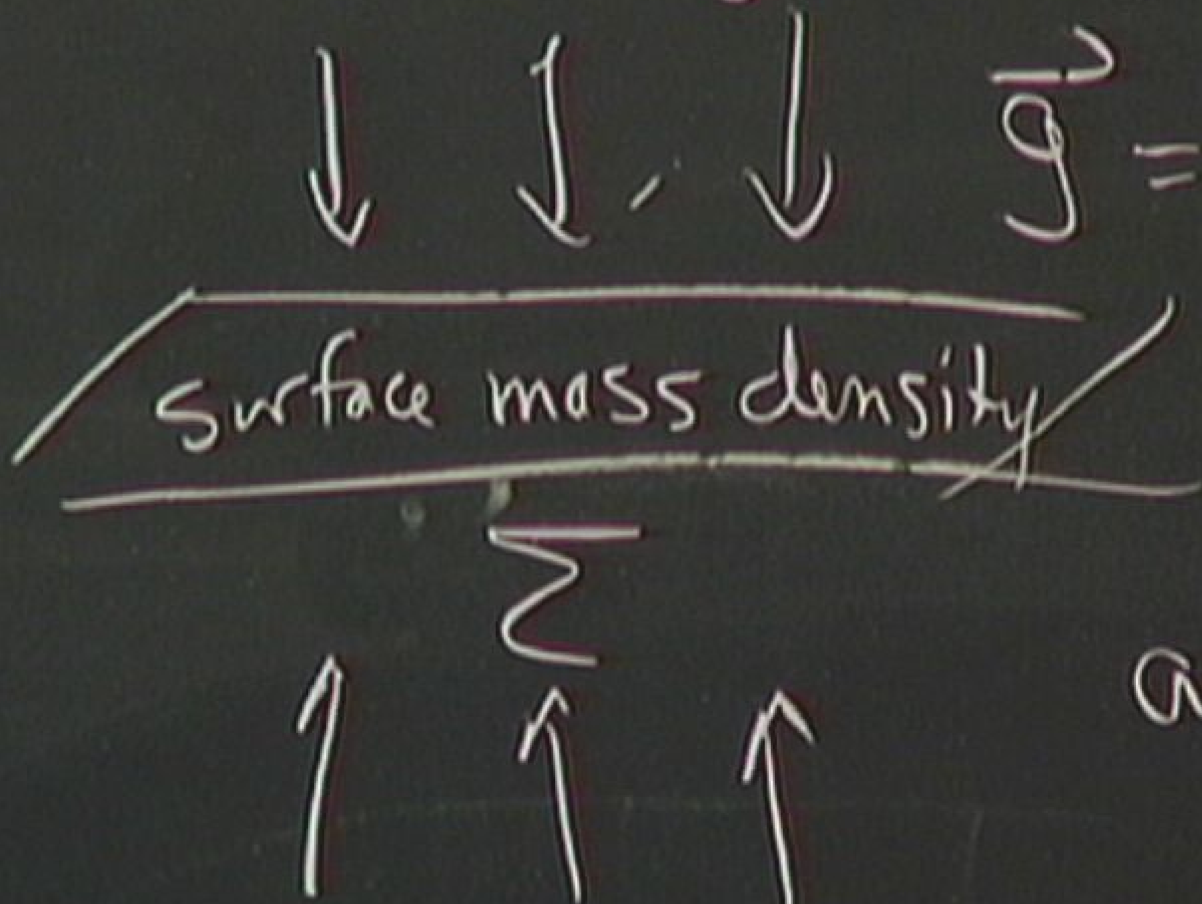

$$\vec{g} = a\,\vec{e}_z$$

$$a = \frac{G\Sigma}{2\pi}$$

$\vec{E} + \vec{u} \times \vec{B})$

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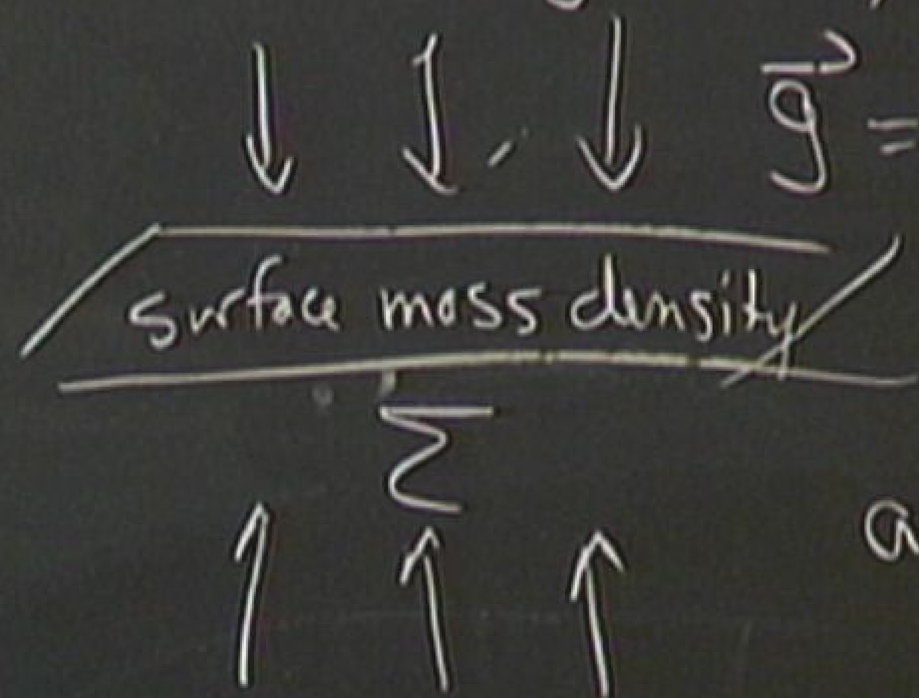

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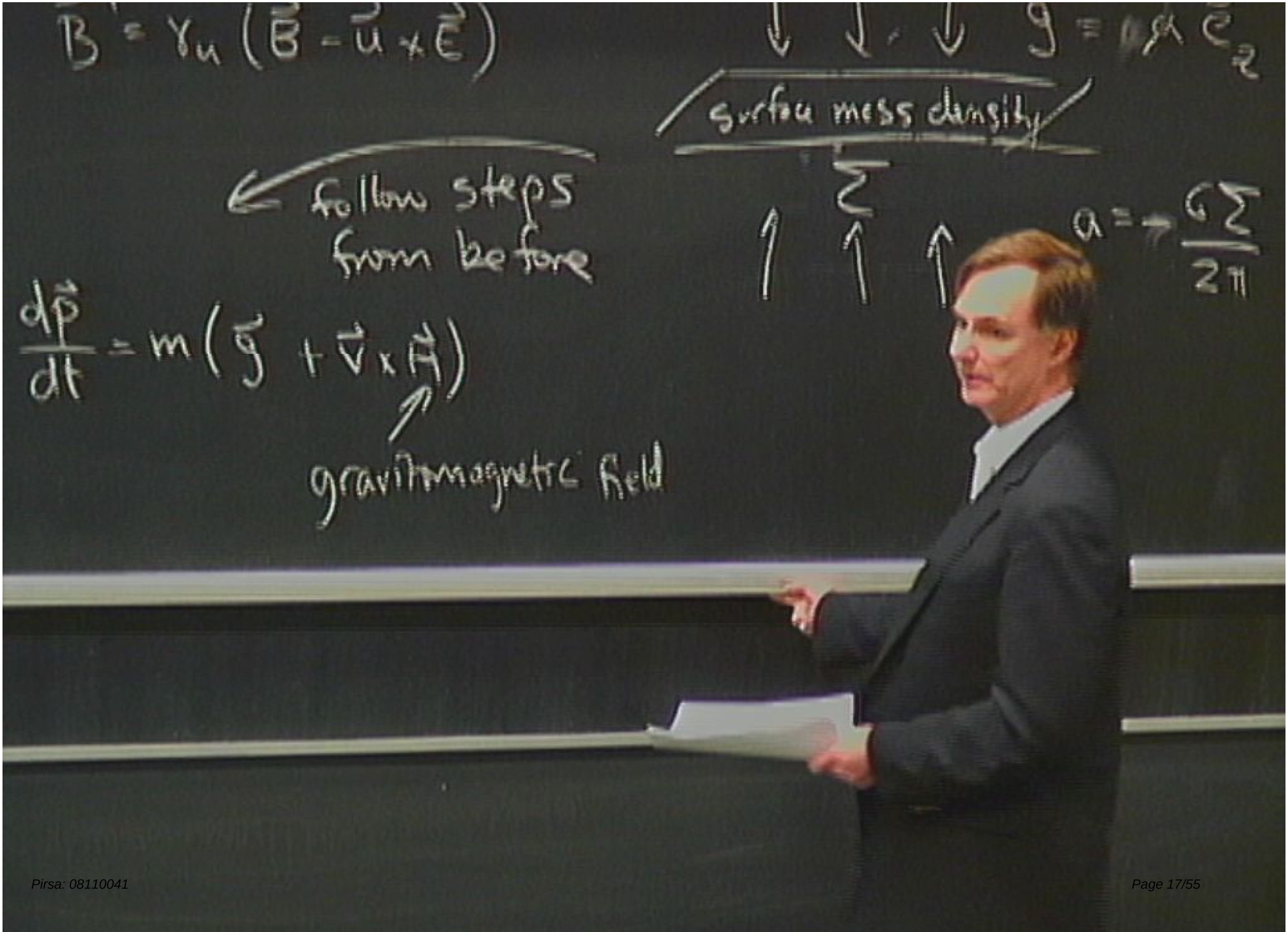
follow steps from before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$$\vec{E}' = \gamma_u (\vec{E} + \vec{u} \times \vec{B})$$

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Now gravity!

$$\vec{g} = a\,\hat{e}_z$$

surface mass density

$$\Sigma$$

$$a = \frac{G\Sigma}{2\pi}$$

follow steps from before

$$\frac{d\vec{p}}{dt} = m(\vec{g} + \vec{v} \times \vec{H})$$

gravitomagnetic field

$$\vec{g}' = \gamma_u (\vec{g} - \vec{u} \times \vec{H}),$$

$$\vec{H}' = \gamma_u (\vec{H} + \vec{u} \times \vec{g})$$

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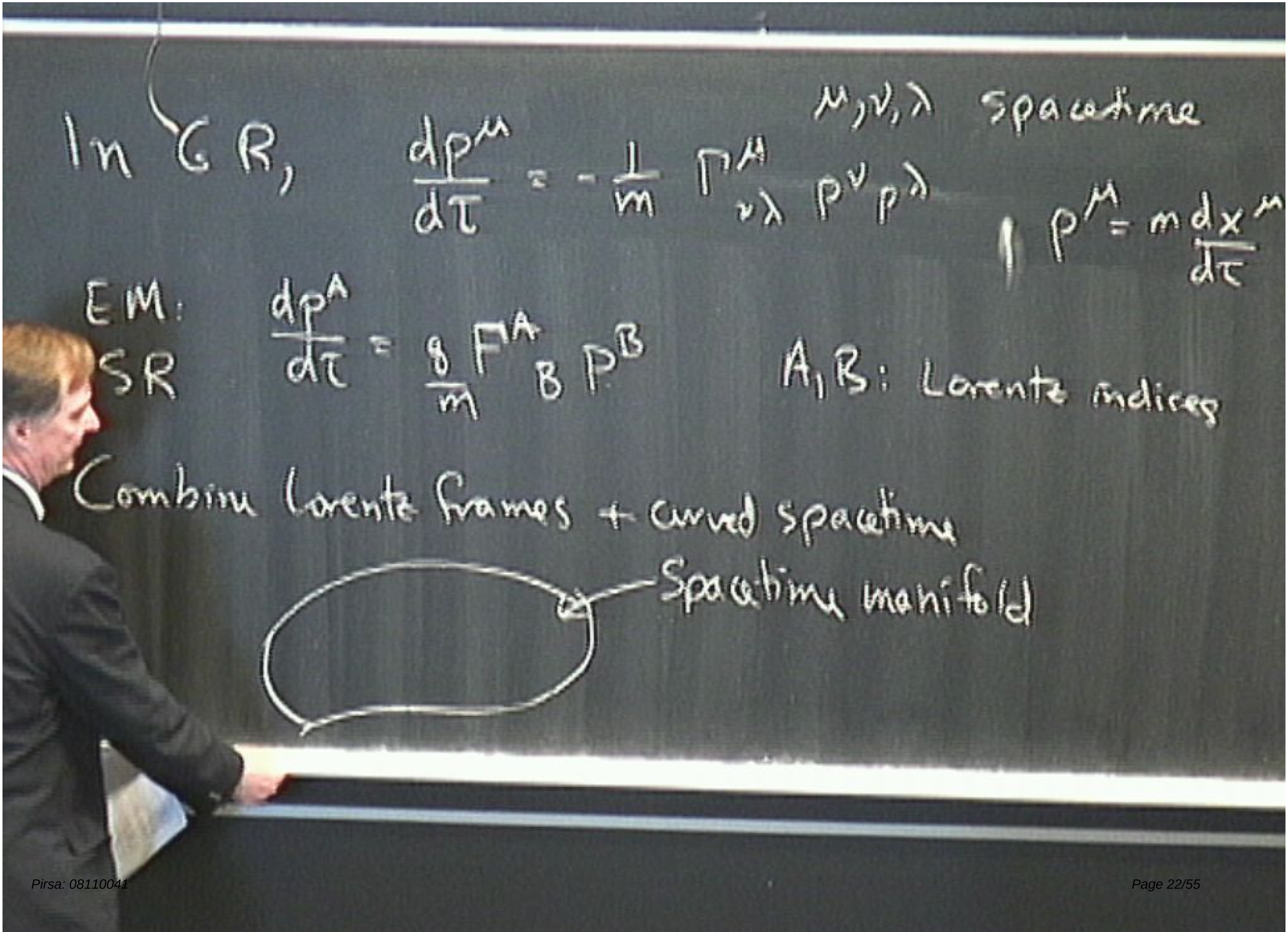
EM in SR: $\frac{dp^A}{d\tau} = \frac{q}{m} F^A{}_B p^B$ A, B: Lorentz indices



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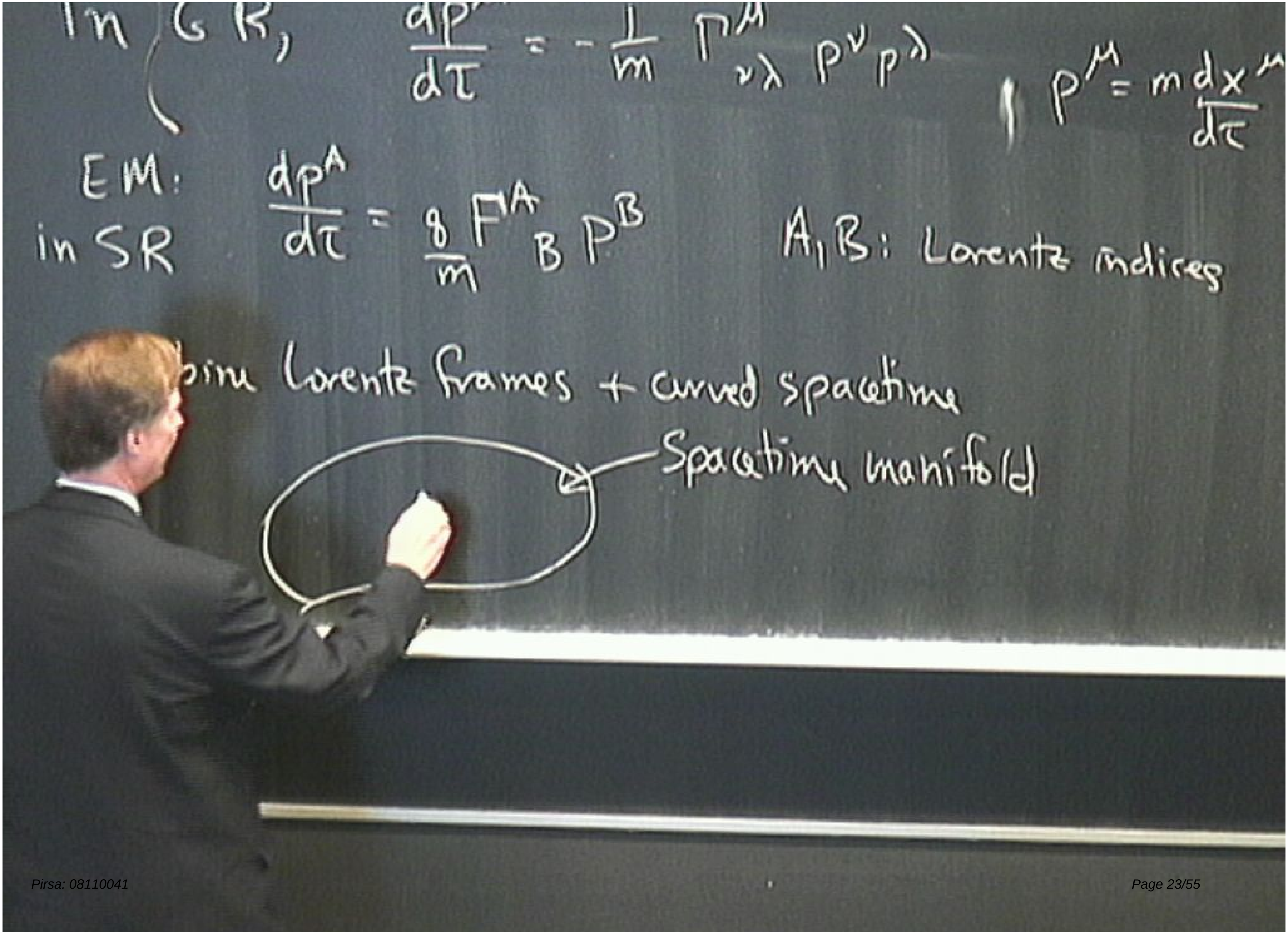
A, B: Lorentz indices
Combine Lorentz frames + curved spacetime
Spacetime manifold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Spacetime manifold

Orthonormal basis at x: $\{\vec{e}_A(x)\}$

Relating coordinate

Relating coordinate and orthonormal bases:

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Spin connection

Ricci Rotation

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Lorentz connection

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2 Kinds of transformations:

Coordinate $x^\mu \to x'^\mu(x)$

Lorentz: $p^A \to \Lambda^A{}_B(x) p^B$

Given $V^A(x)$

$$D_\mu V^A \equiv \frac{\partial V^A}{\partial x^\mu} + \omega_{\mu}{}^{A}{}_{B} V^B$$

Under L.T.,

$$V^A \rightarrow \Lambda^A{}_B V^B$$

$$D_\mu V^A \rightarrow \Lambda^A{}_B (D_\mu V^B)$$

Given $V^A(x)$

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Under L.T., $\quad D_\mu V^A \to \Lambda^A_B (D_\mu V^B)$ iff $\omega^A_{\mu B} \to \Lambda^A_C (\Lambda^{-1})^D_B \, \omega^C_{\mu D}$

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cf. $A_\mu \to A_\mu - \partial_\mu \Phi$ $\quad e^{ig\Phi}$

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$$P^A = E(1, v^I)$$

$$p^\mu = p^t(1, v^i)$$

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g^I

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$$\omega^I{}_{t0} = \omega^0{}_{tI} = -g^I, \quad \omega^I{}_{tJ} = \epsilon^I{}_{JK}\,\Omega^K$$

$$\omega^I{}_{j0} = \omega^0{}_{jI} = -M^I{}_j, \quad \omega^I{}_{jK} = \epsilon^I{}_{KL}\,N^L{}_j$$

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$\downarrow$

$\underline{g} = -\underline{\nabla}\Phi - \partial_t \underline{w}$, $\underline{\Omega} = -\frac{1}{2}\underline{\nabla}\times\underline{w}$

$\uparrow$

3-vector

$$M^{\hat{i}}{}_{\hat{j}} = -\frac{1}{2}\partial_t h^{i}{}_{j} - \epsilon^{i}{}_{jk}(\underline{\nabla}\times\underline{w})^k, \quad N^{\hat{i}}{}_{\hat{j}} = -\frac{1}{2}\epsilon^{ik\ell}\partial_k h_{\ell j}$$

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$$\omega^I{}_{t0} = \omega^0{}_{tI} = -g^I, \quad \omega^I{}_{tJ} = \epsilon^I{}_{JK}\,\Omega^K$$

Spin Precession

$$\omega^I{}_{j0} = \omega^0{}_{jI} = -M^I{}_j, \quad \omega^I{}_{jK} = \epsilon^I{}_{KL}\, N^L{}_j \qquad \frac{dS^A}{dt} = -\omega^A{}_{\mu B}\frac{p^\mu}{m} S^B$$

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$$= E\left(\underline{g} + 2\underline{\Omega} \times \underline{v}\right)$$

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Coriolis!

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$$= E(\underline{g} + 2\underline{\Omega} \times \underline{v})$$

$\uparrow$

Coriolis! $\rightarrow$ Frame Dragging!

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$$= E(\underline{g} + 2\underline{\Omega} \times \underline{v})$$

Coriolis! → Frame Dragging!

2 Equivalence Principles

Acceleration, Rotation