

Title: Gravitoelectromagnetism

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Abstract: Gravitomagnetism is a subtle concept. Adding Lorentz invariance to Newtonian gravity leads to magnetism, but Einsteinian gravitomagnetism differs from Maxwell's electromagnetism. The differences lead to confusion when Lense-Thirring precession is wrongly ascribed to gyroscopes, and when authors disagree about whether lunar laser ranging has measured gravitomagnetism. To clarify these issues, we analyze electric and magnetic effects in local Lorentz frames using the tetrad formalism.

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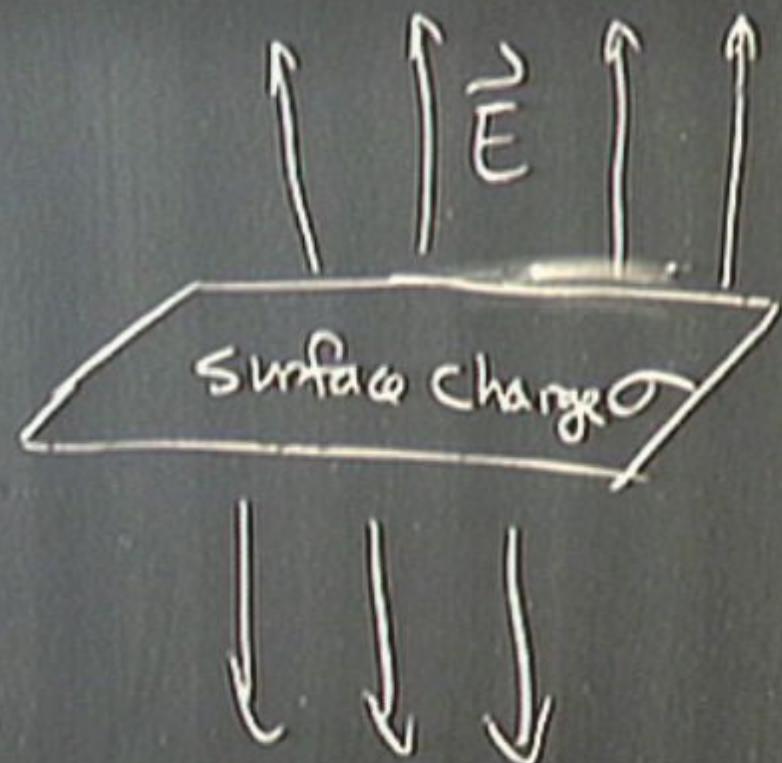
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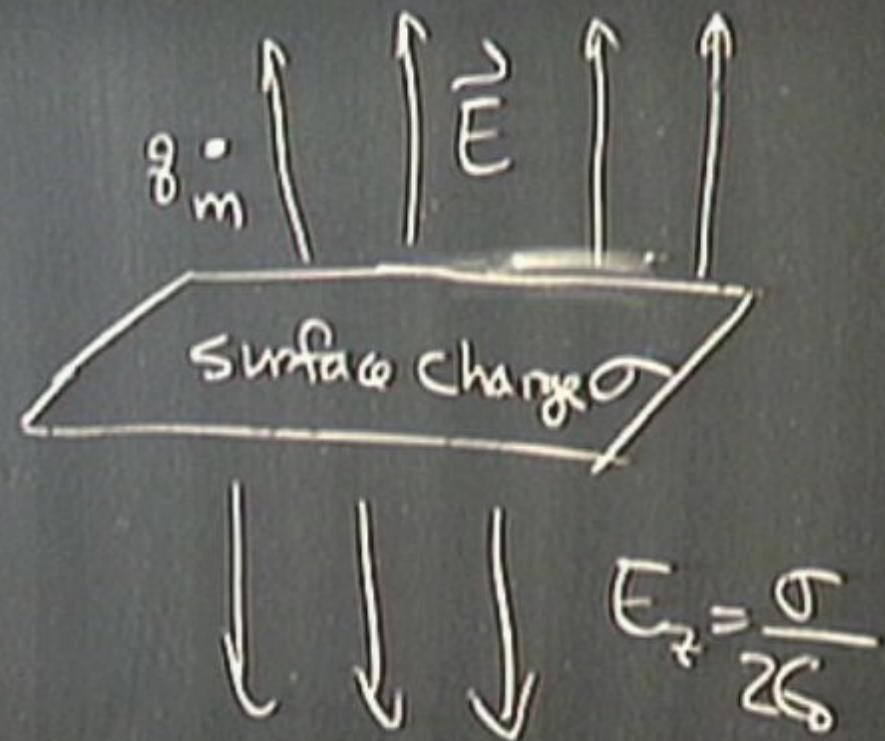
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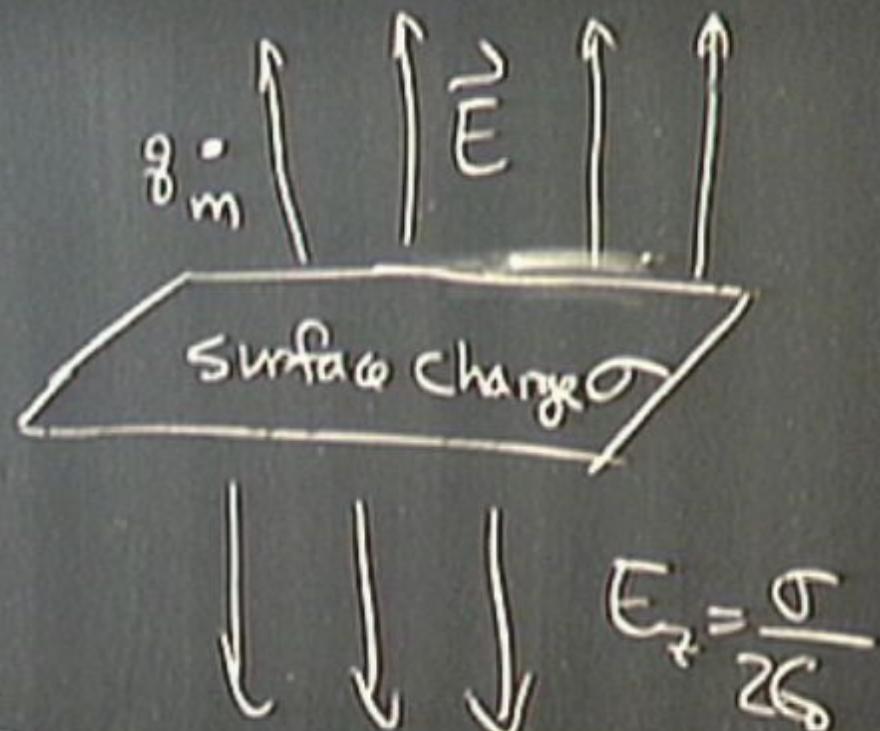
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$$X=0, Y=0, Z = Z_0 + \frac{1}{a} \ln \sqrt{1+a^2 v^2}$$



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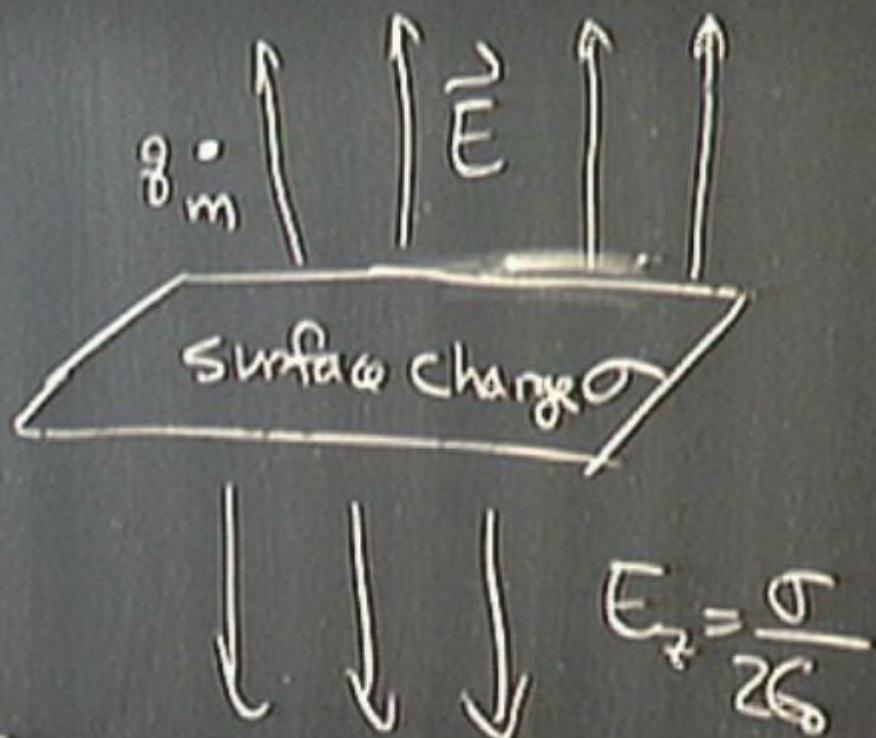
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$$a = \frac{1}{m} E_r$$



$$E_r = \frac{\sigma}{2\epsilon_0}$$

Lorentz boost  $\vec{u} = \gamma \vec{c}_z$

Along the worldline,  $\gamma_u = \frac{1}{\sqrt{1-u^2}}$

$$t' = \gamma_u t$$

Lorentz boost  $\vec{u} = \gamma \vec{c}_x$

Along the worldline,  $\gamma_u \equiv \frac{1}{\sqrt{1-u^2}}$

$$t' = \gamma_u t, \quad x' = -ut, \quad y' = 0, \quad z' = z$$

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Substituting into  $\vec{v}' = \frac{d\vec{x}'}{dt}, \quad \vec{p}' = \frac{m\vec{v}'}{\sqrt{1-v'^2}}$

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$$\frac{d\vec{p}'}{dt'} = q(\vec{E}' + \vec{v}' \times \vec{B}')$$

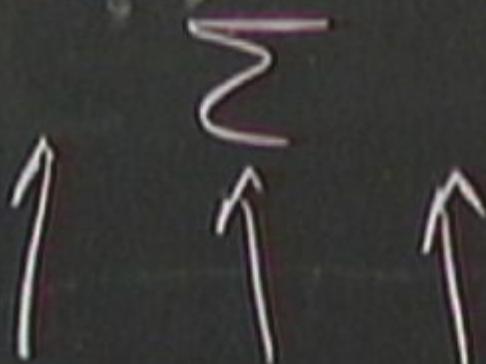
$$\vec{E}' = \gamma_u (\vec{E} + \vec{u} \times \vec{B})$$

$$\vec{B}' = \gamma_u (\vec{B} - \vec{u} \times \vec{E})$$

Now gravity:



Surface mass density



Non gravity:

$$\downarrow \quad \downarrow \quad \downarrow \quad \vec{g} = \alpha \hat{e}_z$$

Surface mass density

$$1 \quad \sum \quad \uparrow \quad \uparrow$$

$$a = -\frac{G \sum}{2\pi}$$

$$\vec{E} + \vec{u} \times \vec{B})$$

$$(\vec{B} - \vec{u} \times \vec{E})$$

follow steps  
from before

Now gravity:

$$\downarrow \downarrow \downarrow \vec{g} = \mu a \hat{e}_z$$

surface mass density

$$1 \sum \uparrow \uparrow$$

$$a = \frac{G \sum}{2\pi}$$

$$\vec{B}' = \gamma_n (\vec{B} - \vec{u} \times \vec{E})$$

follow steps  
from before

$$\frac{d\vec{p}}{dt} = m(\vec{g} + \vec{v} \times \vec{B})$$

gravitomagnetic field

$$g = \mu \kappa \vec{e}_z$$

surface mass density

$$a = -G \sum$$



$$\vec{E}' = \gamma_u (\vec{E} + \vec{u} \times \vec{B})$$

$$\vec{B}' = \gamma_u (\vec{B} - \vec{u} \times \vec{E})$$

follow steps  
from before

$$\frac{d\vec{p}}{dt} = m(\vec{g}' + \vec{\nabla} \times \vec{H})$$

gravitomagnetic field

now gravity:

$$\downarrow \downarrow \downarrow \downarrow \vec{g}' = \alpha \vec{e}_z$$

surface mass density

$$\sum \uparrow \uparrow \uparrow$$

$$\alpha = \frac{G \sum}{2\pi}$$

$$\vec{g}' = \gamma_u (\vec{g} - \vec{u} \times \vec{H}),$$

$$\vec{H}' = \gamma_u (\vec{H} + \vec{u} \times \vec{g})$$

In GR,  $\frac{dp^\mu}{d\tau} = -\frac{1}{m} \Gamma_{\nu\lambda}^\mu p^\nu p^\lambda$  ||  $p^\mu = m \frac{dx^\mu}{d\tau}$

$$\text{In GR, } \frac{dp^\mu}{d\tau} = -\frac{1}{m} \Gamma_{\nu\lambda}^\mu p^\nu p^\lambda \quad | \quad p^\mu = m \frac{dx^\mu}{d\tau}$$

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in SR

$$\frac{dp^A}{d\tau} = \frac{q}{m} F_A^B p^B$$

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*M, N, λ Spacetime*

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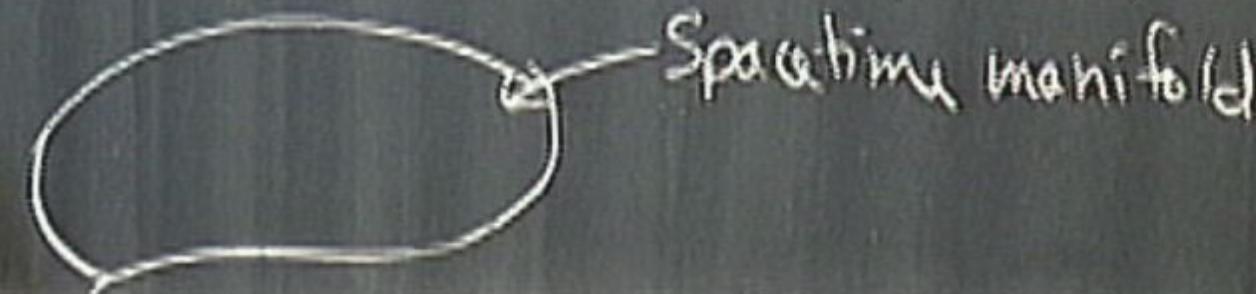
*A, B: Lorentz indices*

In GR,  $\frac{dp^\mu}{d\tau} = -\frac{1}{m} \Gamma_{\nu\lambda}^{\mu} p^\nu p^\lambda$   $\stackrel{\mu, \nu, \lambda}{\text{Spacetime}}$

$$|| p^\mu = m \frac{dx^\mu}{d\tau}$$

EM:  $\frac{dp^A}{d\tau} = \frac{q}{m} F_A^B p^B$   $A, B: \text{Lorentz indices}$   
 SR

Combining Lorentz frames + curved spacetime



$$\text{In GR, } \frac{dP^{\lambda}}{d\tau} = -\frac{1}{m} \Gamma_{\nu\lambda}^{\mu} P^{\nu} P^{\lambda} \quad | \quad P^{\mu} = m \frac{dx^{\mu}}{d\tau}$$

EM:  $\frac{dp^A}{d\tau} = \frac{q}{m} F_A^B P^B$       A, B: Lorentz indices  
 in SR

Spin Lorentz frames + curved spacetime

Spacetime manifold

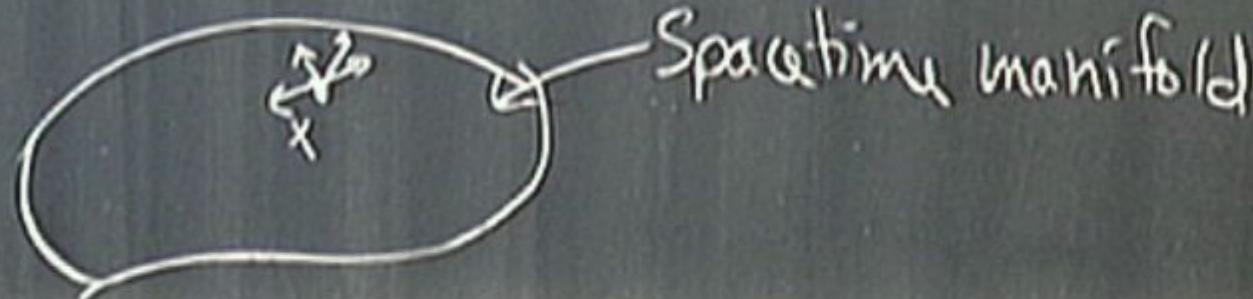


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In GR,  $\frac{dP^\mu}{d\tau} = -\frac{1}{m} \Gamma_{\nu\lambda}^\mu P^\nu P^\lambda$   <sup>$\mu, \nu, \lambda$</sup>  Spacetime  
 $P^\mu = m \frac{dx^\mu}{d\tau}$

EM:  
in SR  $\frac{dP^A}{d\tau} = \frac{q}{m} F_A^B P_B$   $A, B$ : Lorentz indices

Combining Lorentz frames + curved spacetime



Spacetime manifold

Orthonormal basis at  $x$ :  $\{\vec{e}_A(x)\}$

Relating coordinate

Relating coordinate and orthonormal bases:

$$\vec{e}_\mu(x) = \underbrace{e_\mu^A(x)}_{{\text{tetrad or vierbein}}} \vec{e}_A$$

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tetrad or vierbein

Connection:  
form

$$\partial_M \vec{e}_A - \underbrace{\omega^\beta_{\mu}(x)}_{\text{Spin connection}} \vec{e}_B$$

Spin connection  
Ricci Rotation

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Spin connection  
Ricci Rotation  
Lorentz connection

2 kinds of transformations:

Coordinate  $x^\mu \rightarrow x'^\mu(x)$

Lorentz:  $P^A \rightarrow \Lambda^A_B P^B$

Given  $V^A(\omega)$

$$D_\mu V^A = \frac{\partial V^A}{\partial x^\mu} + \omega_{\mu B}^A V^B$$

Under L.T.,  $D_\mu V^A \rightarrow P_B^A (D_\mu V^B)$

$$V^A \rightarrow P_B^A V^B$$

Given  $v^A(x)$

$$D_m v^A = \frac{\partial v^A}{\partial x^m} + \omega_{mB}^A v^B$$

Under L.T.,  $D_m v^A \rightarrow \Lambda^A_B (D_m v^B)$  iff  $\omega_{mB}^A \rightarrow \Lambda_c^A (\Lambda^{-1})_B^D \omega_{mD}^c$   
 $v^A \rightarrow \Lambda^A_B v^B$   $- (\Lambda^{-1})_B^C (\partial_m \Lambda_C^A)$

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Under L.T.,  $D_m V^A \rightarrow D_m V^A - (\Lambda')_B^A (D_m \Lambda^B)$  iff  $\omega_{mB}^A \rightarrow \Lambda_c^A (\Lambda^{-1})_B^D \omega_{mD}^C$

$$V^A \rightarrow \Lambda^A_B V^B$$

Consequence:  $\omega_{mB}^A$  is a tensor under global L.T. re. constant  $\Lambda(\beta)$

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Consequences:  $\omega_{mB}^A$  is a tensor under global L.T., ie constant  $\Lambda$   $\cancel{\text{if}}$   
" " " not " " local

Given  $V^A(x)$

$$D_\mu V^A = \frac{\partial V^A}{\partial x^\mu} + \omega_{\mu B}^A V^B$$

Under L.T.,  $D_\mu V^A \rightarrow D_\mu V^A - (\Lambda^{-1})_B^A D_\mu \Lambda^B$  iff  $\omega_{\mu B}^A \rightarrow \Lambda_c^A (\Lambda^{-1})_B^D \omega_{\mu D}^c - (\Lambda^{-1})_B^C (\partial_\mu \Lambda^A)_C$

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Given  $v^A(x)$

$$D_\mu v^A = \frac{\partial v^A}{\partial x^\mu} + \omega_{\mu B}^A v^B$$

Under L.T.,  $D_\mu v^A \rightarrow D_\mu^{(\Lambda)} v^B$  iff  $\omega_{\mu B}^A \rightarrow \Lambda_c^A (\Lambda^{-1})_B^D \omega_{\mu D}^c$   
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Consequence:  $\omega_{\mu B}^A$  is a tensor under global L.T., ie constant  $\Lambda(\not{x})$   
" " not " " local

Given  $v^A(x)$

$\omega_{\mu B}^A$  is a pseudotensor

$$D_\mu v^A = \frac{\partial v^A}{\partial x^\mu} + \omega_{\mu B}^A v^B \quad (\text{f. } A_\mu \rightarrow A_\mu - \partial_\mu \Phi)$$

eig.  
eig.

Under L.T.,  $D_\mu v^A \rightarrow D_\mu v^B$  iff  $\omega_{\mu B}^A \rightarrow \Lambda_c^A(\lambda) \Big|_B^D \omega_{\mu D}^c$   
 $v^A \rightarrow \Lambda_B^A v^B$   $- (\Lambda^c)_B^D (\partial_\mu \Lambda_c^A)$

Consequences:  $\omega_{\mu B}^A$  is a tensor under global L.T., ie constant  $\Lambda(\lambda)$   
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Given  $V^A(x)$

$\omega_{\mu B}^A$  is a pseudo tensor

$$D_\mu V^A = \frac{\partial V^A}{\partial x^\mu} + \omega_{\mu B}^A V^B \quad \text{cf. } A_\mu \rightarrow A_\mu - \partial_\mu \lambda$$

Under L.T.,  $D_\mu V^A \rightarrow D_\mu V^B$  iff  $\omega_{\mu B}^A \rightarrow \Lambda_c^A (\Lambda^{-1})_B^D \omega_{\mu D}^c$   
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Consequences:  $\omega_{\mu B}^A$  is a tensor under global L.T., i.e. constant  $\Lambda(x)$   
" " not " local

$$P^A = C(I, v^T)$$

$$P^M = P^L(I, v^i)$$

$$\left. \begin{array}{l} P^A = C(I, v^T) \\ P^M = P^L(I, v^i) \end{array} \right\} \quad \frac{dP^T}{dt} = E \left[ (g^T + M^T, w^j) + \epsilon_{TL}^T (\Omega^T + N^T, v^j) v^L \right]$$

$g^T$

$$\left. \begin{array}{l} P^A = E(I, v^T) \\ P^M = P^L(I, v^i) \end{array} \right\} \quad \frac{dP^I}{dt} = E \left[ (g^I + M^I, \dot{v}^i) + \epsilon_{KL}^I (\Omega^E + N^R_S v^i) v^L \right]$$

$$\omega^I_{t0} = \omega^0_{tI} = -g^I, \quad \omega^I_{+J} = \epsilon^I_{JK} \Omega^K$$

$$\omega^I_{j0} = \omega^0_{jI} = -M^I_{,j}, \quad \omega^I_{jK} = \epsilon^I_{KL} N^L_{,j}$$

$$\left. \begin{array}{l} P^A = E(l, v^I) \\ P^M = P^L(l, v^i) \end{array} \right\} \quad \frac{dP^I}{dt} = E \left[ (g^I + M^I, l^j) + \epsilon_{KL}^I (\Omega^E + N_s^K v^j) v^L \right]$$

$$\omega_{t0}^I = \omega_{tI}^0 = -g^I, \quad \omega_{+J}^I = \epsilon_{JK}^I \Omega^K$$

$$\omega_{j0}^I = \omega_{JI}^0 = -M_{,j}^I, \quad \omega_{jK}^I = \epsilon_{KL}^I N^L_{,j}$$

$$\left. \begin{aligned} P^A &= C(l, v^T) \\ P^M &= P^L(l, v^i) \end{aligned} \right\} \quad \frac{dP^I}{dt} = E \left[ (g^I + M^I_{\ j} \omega_j) + \epsilon^I_{KL} (\Omega^P N^R_{\ j} v^j) v^L \right]$$

$$\begin{aligned} \omega^I_{t0} &= \omega^0_{tI} = -g^I, \quad \omega^I_{+J} = \epsilon^I_{JK} \Omega^K \\ \omega^I_{j0} &= \omega^0_{jI} = -M^I_{\ j}, \quad \omega^I_{jP} = \epsilon^I_{KL} N^L_{\ j} \end{aligned}$$

| Weak-field limit.

Weak-field limit:

$$ds^2 = - (1 + 2\phi) dt^2 + 2 w_i dx^i dt + h_{ij} dx^i dx^j$$

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Choose Lorentz frames so  $\vec{e}_{A=0} = \vec{v}$  four-velocity  
of a coordinate-stationary observer

## Weak-field limit

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Choose Lorentz frames so  $\vec{e}_{A_0} = \vec{v}$  four-velocity  
 ↓  
 of a coordinate-stationary observer

$$\vec{w} = -\nabla\Phi - \partial_t \underline{w}, \underline{\Omega} = -\frac{1}{2}\nabla \times \underline{w}$$

↑  
 3-vector

$$M_j^I = -\frac{1}{2}\partial_t h_j^I - \epsilon_{jk}^I (\nabla \times \underline{w})^k, N_j^I = -\frac{1}{2}\epsilon_{IJK}^{KL} \partial_K h_{LJ}$$

Weak-field limit:

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3-vector  $\underline{g} = -\nabla\Phi - \partial_t \underline{w}$ ,  $\underline{\Omega} = \left(-\frac{1}{2}\right)\nabla \times \underline{w}$

$$M_j^I = -\frac{1}{2}\partial_t h_j^I - \epsilon_{jk}^I (\underline{g} \times \underline{w})^k, N_j^I = -\frac{1}{2}\epsilon_{jkl}^{Ikl} \partial_k h_l$$

$$\left. \begin{aligned} P^A &= E(I, v^I) \\ P^M &= P^L(I, v^i) \end{aligned} \right\} \quad \frac{dP^I}{dt} = \Theta [ (g^I + M^I_j \omega^j) + \epsilon_{BL}^I (\Omega^R N^R_j v^j) ]$$

Electric    magnetic

$$\omega^I = \omega^0 + \omega^I_{tI} - g^I, \quad \omega^I_{+J} = \epsilon^I_{JR} \Omega^R$$

Spin Precession

$$\omega^I_{j0} = \omega^0_{jI} - M^I_j, \quad \omega^I_{jR} = \epsilon^I_{kjL} N^L_j$$

$$\frac{dS^I}{dt} = -\omega^A_{MB} \frac{P^I}{m} S^B$$

If neglect  $\partial_t h_{ij}$ ,  $O(v^2)$

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$$\rightarrow \frac{dP}{dt} = E(g + \underline{v} \times H), \quad H = -2\Omega = \underline{r} \times \underline{w}$$

If neglect  $\partial_t h_{ij}$ ,  $O(v^2)$

$$\rightarrow \frac{d\mathbf{P}}{dt} = \mathbf{E}(\mathbf{g} + \mathbf{v} \times \mathbf{H}), \quad \mathbf{H} = -2\mathbf{\Omega} = \mathbf{P} \times \underline{\mathbf{w}}$$

$$= \mathbf{E}(\mathbf{g} + 2\mathbf{\Omega} \times \mathbf{v})$$

↑  
Coriolis!

If neglect  $\partial_t h_{ij}$ ,  $O(v^2)$

$$\rightarrow \frac{d\mathbf{P}}{dt} = \mathbf{E}(\mathbf{g} + \mathbf{v} \times \mathbf{H}), \quad \underline{\mathbf{H}} = -2 \underline{\mathbf{A}} = \underline{\mathbf{B}} \times \underline{\mathbf{v}}$$

$$= \mathbf{E}(\mathbf{g} + 2 \underline{\mathbf{A}} \times \mathbf{v})$$

$\uparrow$   
(Coriolis!  $\rightarrow$  Frame Dragging!)

If neglect  $\partial_t h_{ij}$ ,  $O(v^2)$

$$\rightarrow \frac{d\mathbf{F}}{dt} = E(g + \mathbf{v} \times \mathbf{H}), \quad \underline{\mathbf{H}} = -2\underline{\Omega} = \nabla \times \underline{\mathbf{w}}$$
$$= E(g + 2\underline{\Omega} \times \mathbf{v})$$

↑  
2 Equivalence Principles  
Acceleration, Rotation  
(Coriolis!) → Frame Dragging!