Title: Doing physics with non-diagonalizable Hamiltonians and the solution to the ghost problem in fourth-order derivative theories

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Abstract: It has long been thought that theories based on equations of motion possessing derivatives of order higher than second are not unitary. Specifically, they are thought to possess unphysical ghost states with negative norm. However, it turns out that the appropriate Hilbert space for such theories had not been correctly constructed, and when the theory is formulated properly [Bender and Mannheim, PRL 100, 110402 (2008). (arXiv:0706.0207 [hep-th]] there are no ghost states at all and time evolution is fully unitary. Unitarity can be established for theories based on both second and fourth order derivatives, and for theories based on fourth order derivatives alone. In this latter case the Hamiltonian is a non-diagonalizable, Jordan-block operator which possesses fewer eigenstates than eigenvalues. Despite the lack of completeness of the energy eigenstates, a consistent, unitary quantum mechanics for the theory can still be formulated [Bender and Mannheim, PRD 78, 025022 (2008). (arXiv:0807.2607 [hep-th]).] The implications of these results for the construction of a consistent theory of quantum gravity in four spacetime dimensions will be briefly discussed.

DOING PHYSICS WITH NON-DIAGONALIZABLE HAMILTONIANS AND THE SOLUTION TO THE GHOST PROBLEM IN FOURTH-ORDER DERIVATIVE THEORIES

Philip D. Mannheim

Presentation at Perimeter Institute, November 2008

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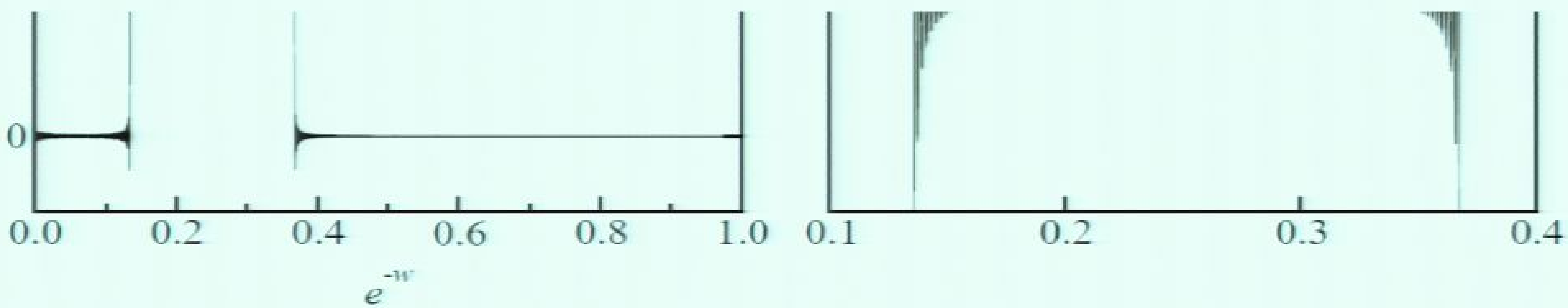
EVEN IF NON-NORMALIZABLE, STATES CAN STILL BE COMPLETE

Completeness of non-normalizable modes

Philip D. Mannheim and Ionel Simbotin

J. Phys. A 39, 13783 (2006). (hep-th/0607090)

$$\int_{-\infty}^{\infty} dw e^{-2A(w)} f_m(w) f_{m'}(w) = \delta_{m,m'} \quad (1)$$



GHOST PROBLEM AND UNITARITY

1. C. M. Bender and P. D. Mannheim, *No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model*, June 2007 (0706.0207 [hep-th]). Phys. Rev. Lett. **100**, 110402 (2008).

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PT QUANTUM MECHANICS

5. C. M. Bender, *Making Sense of Non-Hermitian Hamiltonians*, March 2007 (hep-th/070309). Rep. Prog. Phys. **70**, 947 (2007).

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6. P. D. Mannheim and A. Davidson, *Fourth order theories without ghosts*, January 2000 (hep-th/0001115).

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CONFORMAL GRAVITY AND THE COSMOLOGICAL CONSTANT PROBLEM

9. P. D. Mannheim, *Alternatives to dark matter and dark energy*, May 2005 (astro-ph/0505266). Progress in Particle and Nuclear Physics **56**, 340 (2006).

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THE QUANTUM GRAVITY UNITARITY PROBLEM

In four spacetime dimensions invariance under

$$g_{\mu\nu}(x) \to e^{2\alpha(x)} g_{\mu\nu}(x) \tag{7}$$

leads to a unique gravitational action

$$I_W = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x(-g)^{1/2} \left[R^{\mu\nu} R_{\mu\nu} - \frac{1}{3}(R^\alpha{}_\alpha)^2 \right] \tag{8}$$

where α_g is dimensionless and

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^\alpha{}_\alpha [g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}] - \frac{1}{2} [g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}] \tag{9}$$

is the conformal Weyl tensor. The associated gravitational equations of motion are the **fourth-order derivative**:

$$4\alpha_g [2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu} \tag{10}$$

Conformal gravity is thus a renormalizable theory of gravity since α_g is dimensionless. Moreover, conformal gravity controls the cosmological constant. Specifically, in a Robertson-Walker cosmology we have $C^{\mu\lambda\nu\kappa} = 0$, to yield

$$T^{\mu\nu} = 0, \tag{11}$$

so unlike the double-well Higgs potential, conformal gravity knows where the zero of energy is. However, since the field equations are fourth-order derivative equations, the theory is thought to have negative norm ghost states and not be unitary.

To illustrate the issues involved, consider the typical second- plus fourth-order derivative theory:

$$I = \frac{1}{2}\int d^4x \left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi - M^2\partial_\mu\phi\partial^\mu\phi\right] \tag{12}$$

$$(\partial_t^2 - \nabla^2)(\partial_t^2 - \nabla^2 + M^2)\phi(\bar{x}, t) = 0 \tag{13}$$

$$D^{(4)}(k^2) = \frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2}\left(\frac{1}{k^2 - M^2} - \frac{1}{k^2}\right) \tag{14}$$

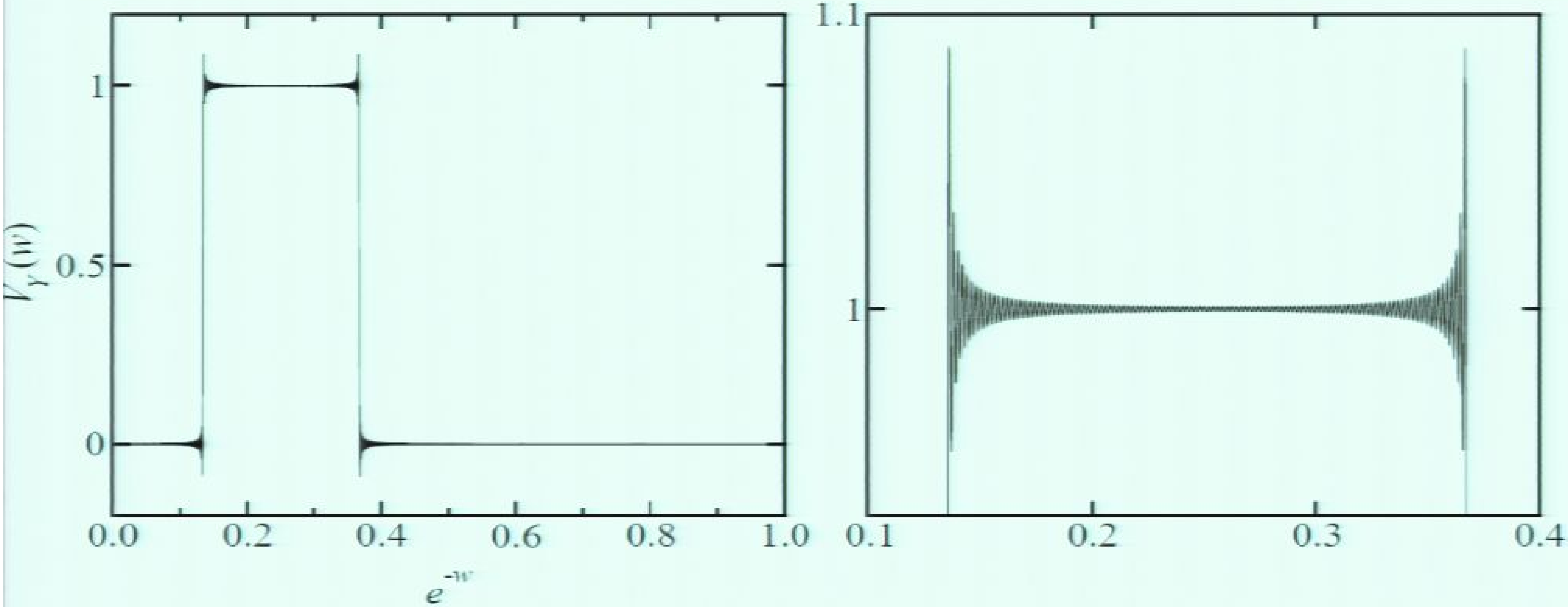
Does the relative minus sign in propagator mean ghost states with negative norm and loss of unitarity, since anticipate that one can write the propagator as

$$D(\bar{x}, \bar{x}', E) = \sum \frac{\psi_n(\bar{x})\psi_n^*(\bar{x}')}{E - E_n} - \sum \frac{\psi_m(\bar{x})\psi_m^*(\bar{x}')}{E - E_m}. \tag{15}$$

and the completeness relation as

$$\sum |n\rangle\langle n| - \sum |m\rangle\langle m| = 1. \tag{16}$$

Reconstruction of the square step $V_Y(|w|) = 1$, $1 < |w| < 2$, $V_Y = 0$ otherwise via sum $V_Y(|w|) = \Sigma a_i Y_2(y_i e^{-|w|})$ on DIVERGENT NON-NORMALIZABLE modes with basis states which obey $Y_1(y_i) = 0$.



CONCLUSION: All that matters is LINEAR relation: $\psi(w) = \Sigma_m a_m f_m(w)$.
No need to require BILINEAR relation $\Sigma_m f_m(w') f_m(w) = e^{2A(w)}\delta(w - w')$.

IMPLICATION: $H|\psi\rangle = E|\psi\rangle$ is linear. No reference to $\langle\psi|\psi\rangle$. Thus left eigenvector which obeys $\langle L|H = \langle L|E$ need not be conjugate of right eigenvector which obeys $H|R\rangle = E|R\rangle$.

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THINGS WE TAKE FOR GRANTED IN QUANTUM MECHANICS.....

(1) The Hamiltonian must be Hermitian

(2) The momentum operator must be Hermitian

(3) In the $[x, p] = i\hbar$ commutator the momentum operator can always be represented by $p = -i\hbar\frac{\partial}{\partial x}$.

(4) States such as energy eigenstates must form a complete set

(5) To be complete states must be normalizable

(6) The scalar product must be given as $\langle m|n\rangle = \delta_{m,n}$

(7) The completeness relation must be given by $\Sigma\,|n\rangle\langle n| = 1$

(8) Theories in which $\langle n|n\rangle$ is negative are unphysical and cannot be formulated in Hilbert space

(9) The Hamiltonian must be diagonalizable

.....AIN'T NECESSARILY SO

AND FOR THEORIES BASED ON FOURTH-ORDER DERIVATIVES....
ALL THESE THINGS ARE NECESSARILY NOT SO,......

AND CAN ENABLE FOURTH-ORDER DERIVATIVE CONFORMAL GRAVITY TO BE A CONSISTENT THEORY OF QUANTUM GRAVITY IN FOUR SPACETIME DIMENSIONS

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$$\phi(\bar{x},t) \sim z(t)e^{i\bar{k}\cdot\bar{x}}, \;\; \omega_1 = (\bar{k}^2 + M^2)^{1/2}, \;\; \omega_2 = |\bar{k}| \tag{17}$$

$$\frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2 z}{dt^2} + \omega_1^2\omega_2^2 z = 0 \tag{18}$$

$$I_{\mathrm{PU}} = \frac{\gamma}{2}\int dt[\ddot{z}^2 - (\omega_1^2 + \omega_2^2)\dot{z}^2 + \omega_1^2\omega_2^2 z^2] \tag{19}$$

$$H_{\mathrm{PU}} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2}\left(\omega_1^2 + \omega_2^2\right)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 z^2, \qquad x = \dot{z} \tag{20}$$

$$[x, p_x] = i, \qquad [z, p_z] = i \tag{21}$$

$$\begin{aligned} z &= a_1 + a_1^\dagger + a_2 + a_2^\dagger, \\ p_z &= i\gamma\omega_1\omega_2^2(a_1 - a_1^\dagger) + i\gamma\omega_1^2\omega_2(a_2 - a_2^\dagger), \\ x &= -i\omega_1(a_1 - a_1^\dagger) - i\omega_2(a_2 - a_2^\dagger), \\ p_x &= -\gamma\omega_1^2(a_1 + a_1^\dagger) - \gamma\omega_2^2(a_2 + a_2^\dagger) \end{aligned} \tag{22}$$

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$$[a_1, a_1^\dagger] = \frac{1}{2\gamma\omega_1\left(\omega_1^2 - \omega_2^2\right)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\gamma\omega_2\left(\omega_1^2 - \omega_2^2\right)} \tag{24}$$

$$\begin{aligned} & a_1|\Omega\rangle = a_2|\Omega\rangle = 0, \qquad H_{\mathrm{PU}}|\Omega\rangle = \frac{1}{2}(\omega_1 + \omega_2)|\Omega\rangle, \\ & \langle\Omega|a_2 a_2^\dagger|\Omega\rangle < 0, \end{aligned} \tag{25}$$

Negative norm state problem looks insurmountable, but.....

QUANTUM MECHANICS IS A GLOBAL THEORY. NEED TO SUPPLY GLOBAL INFORMATION. NEED TO LOOK AT WAVE FUNCTIONS. FIND THAT $H_{\rm PU}$, z, p_z ARE NOT HERMITIAN

$$[x,p_x]=i,\quad p_x=-i\frac{\partial}{\partial x},\quad [z,p_z]=i,\quad p_z=-i\frac{\partial}{\partial z} \tag{26}$$

$$\psi_0(z,x)=\exp\left[\frac{\gamma}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2+i\gamma\omega_1\omega_2 zx-\frac{\gamma}{2}(\omega_1+\omega_2)x^2\right] \tag{27}$$

The states of negative norm are also states of INFINITE norm since $\int dxdz\psi_0^*(z,x)\psi_0(z,x)$ **is divergent, and when acting on such states, one CANNOT set** $p_z=-i\partial/\partial z$

$$\left[e^{i\theta}z,-\frac{i}{e^{i\theta}}\frac{\partial}{\partial z}\right]\psi(e^{i\theta}z)=i\psi(e^{i\theta}z),\quad z\rightarrow -iz,\qquad p_z\rightarrow\frac{\partial}{\partial z} \tag{28}$$

p_z **and** z **not Hermitian – they are anti-Hermitian.**

$$y=e^{\pi p_z z/2}ze^{-\pi p_z z/2}=-iz,\ q=e^{\pi p_z z/2}p_z e^{-\pi p_z z/2}=ip_z \tag{29}$$

$$H=\frac{p^2}{2\gamma}-iqx+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2+\frac{\gamma}{2}\omega_1^2\omega_2^2y^2\neq H^{\dagger},\qquad p=p_x \tag{30}$$

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Negative norm state problem looks insurmountable, but.....

To illustrate the issues involved, consider the typical second- plus fourth-order derivative theory:

$$I = \frac{1}{2}\int d^4x \left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi - M^2\partial_\mu\phi\partial^\mu\phi\right] \tag{12}$$

$$(\partial_t^2 - \nabla^2)(\partial_t^2 - \nabla^2 + M^2)\phi(\bar{x}, t) = 0 \tag{13}$$

$$D^{(4)}(k^2) = \frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2}\left(\frac{1}{k^2 - M^2} - \frac{1}{k^2}\right) \tag{14}$$

Does the relative minus sign in propagator mean ghost states with negative norm and loss of unitarity, since anticipate that one can write the propagator as

$$D(\bar{x}, \bar{x}', E) = \sum \frac{\psi_n(\bar{x})\psi_n^*(\bar{x}')}{E - E_n} - \sum \frac{\psi_m(\bar{x})\psi_m^*(\bar{x}')}{E - E_m}. \tag{15}$$

and the completeness relation as

$$\sum |n\rangle\langle n| - \sum |m\rangle\langle m| = 1. \tag{16}$$

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$$\left[e^{i\theta}z, -\frac{i}{e^{i\theta}}\frac{\partial}{\partial z}\right]\psi(e^{i\theta}z) = i\psi(e^{i\theta}z), \quad z \rightarrow -iz, \qquad p_z \rightarrow \frac{\partial}{\partial z} \tag{28}$$

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PAIS-UHLENBECK OSCILLATOR

$$\phi(\bar{x},t) \sim z(t)e^{i\bar{k}\cdot\bar{x}}, \;\; \omega_1 = (\bar{k}^2 + M^2)^{1/2}, \;\; \omega_2 = |\bar{k}| \tag{17}$$

$$\frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2 z}{dt^2} + \omega_1^2\omega_2^2 z = 0 \tag{18}$$

$$I_{\rm PU} = \frac{\gamma}{2}\int dt[\ddot{z}^2 - (\omega_1^2 + \omega_2^2)\dot{z}^2 + \omega_1^2\omega_2^2 z^2] \tag{19}$$

$$H_{\rm PU} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2}\left(\omega_1^2 + \omega_2^2\right)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 z^2, \qquad x = \dot{z} \tag{20}$$

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$$\begin{aligned} z &= a_1 + a_1^\dagger + a_2 + a_2^\dagger, \\ p_z &= i\gamma\omega_1\omega_2^2(a_1 - a_1^\dagger) + i\gamma\omega_1^2\omega_2(a_2 - a_2^\dagger), \\ x &= -i\omega_1(a_1 - a_1^\dagger) - i\omega_2(a_2 - a_2^\dagger), \\ p_x &= -\gamma\omega_1^2(a_1 + a_1^\dagger) - \gamma\omega_2^2(a_2 + a_2^\dagger) \end{aligned} \tag{22}$$

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LEFT-EIGENVECTOR IS NOT DIRAC CONJUGATE OF RIGHT EIGENVECTOR

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THE QUANTUM GRAVITY UNITARITY PROBLEM

In four spacetime dimensions invariance under

$$g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x) \tag{7}$$

leads to a unique gravitational action

$$I_W = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x(-g)^{1/2} \left[R^{\mu\nu} R_{\mu\nu} - \frac{1}{3}(R^{\alpha}{}_{\alpha})^2 \right] \tag{8}$$

where α_g is dimensionless and

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} + \frac{1}{6} R^{\alpha}{}_{\alpha} [g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu}] - \frac{1}{2} [g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu}] \tag{9}$$

is the conformal Weyl tensor. The associated gravitational equations of motion are the **fourth-order derivative**:

$$4\alpha_g [2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu} \tag{10}$$

Conformal gravity is thus a renormalizable theory of gravity since α_g is dimensionless. Moreover, conformal gravity controls the cosmological constant. Specifically, in a Robertson-Walker cosmology we have $C^{\mu\lambda\nu\kappa} = 0$, to yield

$$T^{\mu\nu} = 0, \tag{11}$$

so unlike the double-well Higgs potential, conformal gravity knows where the zero of energy is. However, since the field equations are fourth-order derivative equations, the theory is thought to have negative norm ghost states and not be unitary.

GHOST PROBLEM AND UNITARITY

1. C. M. Bender and P. D. Mannheim, *No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model*, June 2007 (0706.0207 [hep-th]). Phys. Rev. Lett. **100**, 110402 (2008).

2. P. D. Mannheim, *Conformal Gravity Challenges String Theory*, Pascos-07, July 2007 (0707.2283 [hep-th]).

3. C. M. Bender and P. D. Mannheim, *Giving up the ghost*, October 2007 (0807.2607 [hep-th]). Jour. Phys. A **41**, 304018 (2008)

4. C. M. Bender and P. D. Mannheim, *Exactly solvable PT-symmetric Hamiltonian having no Hermitian counterpart*, April 2008 (0804.4190 [hep-th]). Phys. Rev. D **78**, 025022 (2008).

PT QUANTUM MECHANICS

5. C. M. Bender, *Making Sense of Non-Hermitian Hamiltonians*, March 2007 (hep-th/070309). Rep. Prog. Phys. **70**, 947 (2007).

PAIS-UHLENBECK FOURTH ORDER OSCILLATOR

6. P. D. Mannheim and A. Davidson, *Fourth order theories without ghosts*, January 2000 (hep-th/0001115).

7. P. D. Mannheim and A. Davidson, *Dirac quantization of the Pais-Uhlenbeck fourth order oscillator*, August 2004 (hep-th/0408104). Phys. Rev. A **71**, 042110 (2005).

8. P. D. Mannheim, *Solution to the ghost problem in fourth order derivative theories*, August 2006 (hep-th/0608154). Found. Phys. **37**, 532 (2007).

CONFORMAL GRAVITY AND THE COSMOLOGICAL CONSTANT PROBLEM

9. P. D. Mannheim, *Alternatives to dark matter and dark energy*, May 2005 (astro-ph/0505266). Progress in Particle and Nuclear Physics **56**, 340 (2006).

10. P. D. Mannheim, *Dynamical symmetry breaking and the cosmological constant problem*, September 2008 (0809.1200 [hep-th]). Proceedings of ICHEP08.

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$$C^2 = 1, \quad [C, PT] = 0, \quad [C, H] = 0, \quad C = e^Q P \tag{32}$$

$$Q = \alpha[pq + \gamma^2 \omega_1^2 \omega_2^2 xy], \quad \alpha = \frac{1}{\gamma\omega_1\omega_2} \log\left(\frac{\omega_1 + \omega_2}{\omega_1 - \omega_2}\right) \tag{33}$$

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2 \tag{34}$$

Hamiltonian can be diagonalized by a similarity transformation which is non-unitary since Q is Hermitian rather than anti-Hermitian. Original Hamiltonian H is thus a Hermitian Hamiltonian as written in a skew basis. The eigenstates of H and $\tilde{H}$ are not unitarily equivalent, and thus ....

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QUANTUM MECHANICS IS A GLOBAL THEORY. NEED TO SUPPLY GLOBAL INFORMATION. NEED TO LOOK AT WAVE FUNCTIONS. FIND THAT $H_{\rm PU}$, z, p_z ARE NOT HERMITIAN

$$[x,p_x]=i,\quad p_x=-i\frac{\partial}{\partial x},\quad [z,p_z]=i,\quad p_z=-i\frac{\partial}{\partial z} \tag{26}$$

$$\psi_0(z,x)=\exp\left[\frac{\gamma}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2+i\gamma\omega_1\omega_2 zx-\frac{\gamma}{2}(\omega_1+\omega_2)x^2\right] \tag{27}$$

The states of negative norm are also states of INFINITE norm since $\int dxdz\psi_0^*(z,x)\psi_0(z,x)$ **is divergent, and when acting on such states, one CANNOT set** $p_z=-i\partial/\partial z$

$$\left[e^{i\theta}z,-\frac{i}{e^{i\theta}}\frac{\partial}{\partial z}\right]\psi(e^{i\theta}z)=i\psi(e^{i\theta}z),\quad z\rightarrow -iz,\qquad p_z\rightarrow\frac{\partial}{\partial z} \tag{28}$$

p_z **and** z **not Hermitian – they are anti-Hermitian.**

$$y=e^{\pi p_z z/2}ze^{-\pi p_z z/2}=-iz,\ q=e^{\pi p_z z/2}p_z e^{-\pi p_z z/2}=ip_z \tag{29}$$

$$H=\frac{p^2}{2\gamma}-iqx+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2+\frac{\gamma}{2}\omega_1^2\omega_2^2y^2\neq H^{\dagger},\qquad p=p_x \tag{30}$$

$$\text{Hermitian}\quad [x,p]=i,\quad \text{Hermitian}\quad [y,q]=i,\quad \text{non}-\text{Hermitian}\ \ H \tag{31}$$

To illustrate the issues involved, consider the typical second- plus fourth-order derivative theory:

$$I = \frac{1}{2}\int d^4x \left[\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi - M^2\partial_\mu\phi\partial^\mu\phi\right] \tag{12}$$

$$(\partial_t^2 - \nabla^2)(\partial_t^2 - \nabla^2 + M^2)\phi(\bar{x}, t) = 0 \tag{13}$$

$$D^{(4)}(k^2) = \frac{1}{k^2(k^2 - M^2)} = \frac{1}{M^2}\left(\frac{1}{k^2 - M^2} - \frac{1}{k^2}\right) \tag{14}$$

Does the relative minus sign in propagator mean ghost states with negative norm and loss of unitarity, since anticipate that one can write the propagator as

$$D(\bar{x}, \bar{x}', E) = \sum \frac{\psi_n(\bar{x})\psi_n^*(\bar{x}')}{E - E_n} - \sum \frac{\psi_m(\bar{x})\psi_m^*(\bar{x}')}{E - E_m}. \tag{15}$$

and the completeness relation as

$$\sum |n\rangle\langle n| - \sum |m\rangle\langle m| = 1. \tag{16}$$

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$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \qquad -i\frac{d}{dt}\langle\alpha_S(t)| = \langle\alpha_S(t)|H^\dagger \tag{40}$$

$$|\alpha_S(t)\rangle = e^{-iHt}|\alpha_S(0)\rangle, \qquad \langle\alpha_S(t)| = \langle\alpha_S(0)|e^{iH^\dagger t} \tag{41}$$

$$\langle\alpha_S(t)|\alpha_S(t)\rangle = \langle\alpha_S(0)|e^{iH^\dagger t}e^{-iHt}|\alpha_S(0)\rangle \neq \langle\alpha_S(0)|\alpha_S(0)\rangle \qquad \textbf{NOT UNITARY} \tag{42}$$

$$\langle\alpha_S(t)|A_S|\alpha_S(t)\rangle = \langle\alpha_S(0)|e^{iH^\dagger t}A_S e^{-iHt}|\alpha_S(0)\rangle \tag{43}$$

$$A_H(t) = e^{iH^\dagger t}A_S e^{-iHt} \tag{44}$$

$$i\frac{d}{dt}A_H(t) = A_H(t)H - H^\dagger A_H(t), \qquad i\frac{d}{dt}A_H(t) = A_H(t)H - HA_H(t) \tag{45}$$

$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \qquad -i\frac{d}{dt}\langle\hat{\alpha}_S(t)| = \langle\hat{\alpha}_S(t)|H \tag{46}$$

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LEFT-EIGENVECTOR IS NOT DIRAC CONJUGATE OF RIGHT EIGENVECTOR

$$\langle\hat{\alpha}_S(t)|\alpha_S(t)\rangle = \langle\hat{\alpha}_S(0)|e^{iHt}e^{-iHt}|\alpha_S(0)\rangle = \langle\hat{\alpha}_S(0)|\alpha_S(0)\rangle \qquad \textbf{UNITARY} \tag{48}$$

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In equal frequency limit the diagonalizing operator Q becomes singular and partial fraction decomposition of propagator becomes undefined.

$$\psi_0(x,y,t) = \exp\left[-\frac{\gamma}{2}(\omega_1+\omega_2)(x^2+\omega_1\omega_2 y^2) - \gamma\omega_1\omega_2 yx\right]\exp(-iE_0 t),$$
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TWO one-particle states have collapsed into ONE state.

THE MISSING ENERGY EIGENSTATES....

$$H_{1\mathrm{P}}(\epsilon) = \frac{1}{2\omega}\begin{pmatrix} 4\omega^2+\epsilon^2 & 4\omega^2-\epsilon^2 \\ \epsilon^2 & 4\omega^2-\epsilon^2 \end{pmatrix}, \quad \omega_1 \equiv \omega+\epsilon, \;\; \omega_2 \equiv \omega-\epsilon, \quad (54)$$

$$|2\omega+\epsilon\rangle \equiv \begin{pmatrix} 2\omega+\epsilon \\ \epsilon \end{pmatrix}, \qquad |2\omega-\epsilon\rangle \equiv \begin{pmatrix} 2\omega-\epsilon \\ -\epsilon \end{pmatrix} \quad (55)$$

$$S^{-1}\left(\frac{1}{2\omega}\right)\begin{pmatrix} 4\omega^2+\epsilon^2 & 4\omega^2-\epsilon^2 \\ \epsilon^2 & 4\omega^2-\epsilon^2 \end{pmatrix} S = \begin{pmatrix} 2\omega+\epsilon & 0 \\ 0 & 2\omega-\epsilon \end{pmatrix} \quad (56)$$

$$S = \frac{1}{2\epsilon\omega^{1/2}(2\omega+\epsilon)^{1/2}}\begin{pmatrix} 2\omega+\epsilon & -(4\omega^2-\epsilon^2)\epsilon \\ \epsilon & (2\omega+\epsilon)\epsilon^2 \end{pmatrix} \quad (57)$$

$$H_{1P}(\epsilon=0) = 2\omega\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (58)$$

Non-diagonalizable, Jordan-block matrix with TWO eigenvalues ($\lambda_1 = 1,\ \lambda_2 = 1$ since $Tr = 1,\ Det = 1$), but only ONE eigenvector.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c+d \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \qquad d = 0 \quad (59)$$

... BECAME NONSTATIONARY

$$\hat{\psi}_{1a}(x,y,t) = \lim_{\epsilon \longrightarrow 0} \frac{\psi_2(x,y,t) - \psi_1(x,y,t)}{2\epsilon}$$
$$= [(x + \omega y)it + y]\, \hat{\psi}_0(x,y,t)e^{-i\omega t} \qquad (60)$$

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Stationary plus non-stationary together are complete since just the right number of independent polynomial functions of x and y.

$$i\frac{\partial}{\partial t}\int dx\, dy\, \hat{\psi}^c_B(x,y,t)\hat{\psi}_A(x,y,t) = -\int dx\, dy\, x\frac{\partial}{\partial y}\left[\hat{\psi}^c_B(x,y,t)\hat{\psi}_A(x,y,t)\right] \qquad (62)$$

$$i\frac{\partial}{\partial t}\int dx\, dy\, \hat{\psi}^c_B(x,y,t)\hat{\psi}_A(x,y,t) = 0 \qquad (63)$$

Norm preserved in time so time evolution is unitary.

Conformal Supergravity in Twistor-String Theory

N. Berkovits and E. Witten

June 2004 (arXiv:hep-th/040605). JHEP 0408 (2004) 009

"The net effect is that the translation generator D acts as

$$\begin{pmatrix} P & * \\ 0 & P \end{pmatrix}$$

where P would represent ordinary translations and the off-diagonal $*$ arises from $[D, \partial_I] \neq 0$.

This matrix is not diagonalizable. This clashes with our usual experience. We are accustomed to the idea that the translation generators are Hermitian operators and so can be diagonalized. However, conformal supergravity is not a unitary theory, and one symptom of this is that the translation generators are undiagonalizable."

========

Bender and Mannheim: Not so fast.

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$$i\frac{\partial}{\partial t}\int dx\,dy\,\hat{\psi}^c_B(x,y,t)\hat{\psi}_A(x,y,t) = -\int dx\,dy\,x\frac{\partial}{\partial y}\left[\hat{\psi}^c_B(x,y,t)\hat{\psi}_A(x,y,t)\right] \quad (62)$$

$$i\frac{\partial}{\partial t}\int dx\,dy\,\hat{\psi}^c_B(x,y,t)\hat{\psi}_A(x,y,t) = 0 \quad (63)$$

Norm preserved in time so time evolution is unitary.

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$$\begin{pmatrix} P & * \\ 0 & P \end{pmatrix}$$

where P would represent ordinary translations and the off-diagonal $*$ arises from $[D, \partial_I] \neq 0$.

This matrix is not diagonalizable. This clashes with our usual experience. We are accustomed to the idea that the translation generators are Hermitian operators and so can be diagonalized. However, conformal supergravity is not a unitary theory, and one symptom of this is that the translation generators are undiagonalizable."

========

Bender and Mannheim: Not so fast.

... BECAME NONSTATIONARY

$$\hat{\psi}_{1a}(x,y,t) = \lim_{\epsilon \longrightarrow 0} \frac{\psi_2(x,y,t) - \psi_1(x,y,t)}{2\epsilon}$$
$$= [(x + \omega y)it + y]\,\hat{\psi}_0(x,y,t)e^{-i\omega t} \tag{60}$$

$$i\frac{\partial}{\partial t}\hat{\psi}(x,y,t) = \left(-\frac{1}{2\gamma}\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial y} + \gamma\omega^2 x^2 + \frac{\gamma}{2}\omega^4 y^2\right)\hat{\psi}(x,y,t) \tag{61}$$

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The Hamiltonian is not Hermitian – but it is PT symmetric, and thus still has real eigenvalues. Bender and collaborators showed that $H = p^2 + ix^3$ has a completely real energy spectrum.

$$C^2 = 1, \quad [C, PT] = 0, \quad [C, H] = 0, \quad C = e^Q P \tag{32}$$

$$Q = \alpha[pq + \gamma^2\omega_1^2\omega_2^2 xy], \quad \alpha = \frac{1}{\gamma\omega_1\omega_2}\log\left(\frac{\omega_1 + \omega_2}{\omega_1 - \omega_2}\right) \tag{33}$$

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2 \tag{34}$$

Hamiltonian can be diagonalized by a similarity transformation which is non-unitary since Q is Hermitian rather than anti-Hermitian. Original Hamiltonian H is thus a Hermitian Hamiltonian as written in a skew basis. The eigenstates of H and $\tilde{H}$ are not unitarily equivalent, and thus ....

QUANTUM MECHANICS IS A GLOBAL THEORY. NEED TO SUPPLY GLOBAL INFORMATION. NEED TO LOOK AT WAVE FUNCTIONS. FIND THAT $H_{\rm PU}$, z, p_z ARE NOT HERMITIAN

$$[x, p_x] = i, \quad p_x = -i\frac{\partial}{\partial x}, \quad [z, p_z] = i, \quad p_z = -i\frac{\partial}{\partial z} \tag{26}$$

$$\psi_0(z, x) = \exp\left[\frac{\gamma}{2}(\omega_1+\omega_2)\omega_1\omega_2 z^2 + i\gamma\omega_1\omega_2 zx - \frac{\gamma}{2}(\omega_1+\omega_2)x^2\right] \tag{27}$$

The states of negative norm are also states of INFINITE norm since $\int dxdz\psi_0^*(z,x)\psi_0(z,x)$ **is divergent, and when acting on such states, one CANNOT set** $p_z = -i\partial/\partial z$

$$\left[e^{i\theta}z, -\frac{i}{e^{i\theta}}\frac{\partial}{\partial z}\right]\psi(e^{i\theta}z) = i\psi(e^{i\theta}z), \quad z \rightarrow -iz, \qquad p_z \rightarrow \frac{\partial}{\partial z} \tag{28}$$

p_z **and** z **not Hermitian – they are anti-Hermitian.**

$$y = e^{\pi p_z z/2} z e^{-\pi p_z z/2} = -iz, \; q = e^{\pi p_z z/2} p_z e^{-\pi p_z z/2} = ip_z \tag{29}$$

$$H = \frac{p^2}{2\gamma} - iqx + \frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2 \neq H^\dagger, \qquad p = p_x \tag{30}$$

$$\text{Hermitian} \quad [x, p] = i, \quad \text{Hermitian} \quad [y, q] = i, \quad \text{non}-\text{Hermitian} \;\; H \tag{31}$$

PAIS-UHLENBECK OSCILLATOR

$$\phi(\bar{x},t) \sim z(t)e^{i\bar{k}\cdot\bar{x}}, \quad \omega_1 = (\bar{k}^2 + M^2)^{1/2}, \quad \omega_2 = |\bar{k}| \tag{17}$$

$$\frac{d^4 z}{dt^4} + (\omega_1^2 + \omega_2^2)\frac{d^2 z}{dt^2} + \omega_1^2\omega_2^2 z = 0 \tag{18}$$

$$I_{\mathrm{PU}} = \frac{\gamma}{2}\int dt[\ddot{z}^2 - (\omega_1^2 + \omega_2^2)\dot{z}^2 + \omega_1^2\omega_2^2 z^2] \tag{19}$$

$$H_{\mathrm{PU}} = \frac{p_x^2}{2\gamma} + p_z x + \frac{\gamma}{2}\left(\omega_1^2 + \omega_2^2\right)x^2 - \frac{\gamma}{2}\omega_1^2\omega_2^2 z^2, \qquad x = \dot{z} \tag{20}$$

$$[x, p_x] = i, \qquad [z, p_z] = i \tag{21}$$

$$\begin{aligned} z &= a_1 + a_1^\dagger + a_2 + a_2^\dagger, \\ p_z &= i\gamma\omega_1\omega_2^2(a_1 - a_1^\dagger) + i\gamma\omega_1^2\omega_2(a_2 - a_2^\dagger), \\ x &= -i\omega_1(a_1 - a_1^\dagger) - i\omega_2(a_2 - a_2^\dagger), \\ p_x &= -\gamma\omega_1^2(a_1 + a_1^\dagger) - \gamma\omega_2^2(a_2 + a_2^\dagger) \end{aligned} \tag{22}$$

$$H_{\rm PU} = 2\gamma(\omega_1^2 - \omega_2^2)(\omega_1^2 a_1^\dagger a_1 - \omega_2^2 a_2^\dagger a_2) + (\omega_1 + \omega_2)/2 \tag{23}$$

$$[a_1, a_1^\dagger] = \frac{1}{2\gamma\omega_1\left(\omega_1^2 - \omega_2^2\right)}, \quad [a_2, a_2^\dagger] = -\frac{1}{2\gamma\omega_2\left(\omega_1^2 - \omega_2^2\right)} \tag{24}$$

$$a_1|\Omega\rangle = a_2|\Omega\rangle = 0, \qquad H_{\rm PU}|\Omega\rangle = \frac{1}{2}(\omega_1 + \omega_2)|\Omega\rangle,$$
$$\langle\Omega|a_2 a_2^\dagger|\Omega\rangle < 0, \tag{25}$$

Negative norm state problem looks insurmountable, but.....

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....THE NORM IS NOT THE DIRAC NORM

$$\tilde{H} = e^{-Q/2} H e^{Q/2} = \frac{p^2}{2\gamma} + \frac{q^2}{2\gamma\omega_1^2} + \frac{\gamma}{2}\omega_1^2 x^2 + \frac{\gamma}{2}\omega_1^2\omega_2^2 y^2 \qquad (35)$$

$$\tilde{H}|\tilde{n}\rangle = E_n|\tilde{n}\rangle, \quad H|n\rangle = E_n|n\rangle, \quad |n\rangle = e^{Q/2}|\tilde{n}\rangle \qquad (36)$$

$$\langle\tilde{n}|\tilde{H} = E_n\langle\tilde{n}|, \quad \langle n| \equiv \langle\tilde{n}|e^{Q/2}, \quad \langle n|e^{-Q}H = \langle n|e^{-Q}E_n \qquad (37)$$

The energy eigenbra $\langle n|e^{-Q} = \langle n|PC$ is not the Dirac conjugate of the energy eigenket $|n\rangle$, since $\langle n|H^\dagger = \langle n|E_n$ is not an eigenvalue equation for H.

$$\langle\tilde{n}|\tilde{m}\rangle = \delta_{m,n}, \quad \Sigma|\tilde{n}\rangle\langle\tilde{n}| = \mathbf{1}, \quad \tilde{H} = \Sigma|\tilde{n}\rangle E_n\langle\tilde{n}| \qquad (38)$$

$$\langle n|e^{-Q}|m\rangle = \delta_{m,n}, \quad \Sigma|n\rangle\langle n|e^{-Q} = \mathbf{1}, \quad H = \Sigma|n\rangle E_n\langle n|e^{-Q} \qquad (39)$$

The e^{-Q} norm is positive and so theory is unitary. Since $C^2 = 1$, its eigenvalues are ± 1, with the relative plus and minus signs in the fourth-order propagator being due to the fact that the two poles have opposite signed eigenvalues of C.

1 NON-HERMITICITY AND UNITARITY

$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \qquad -i\frac{d}{dt}\langle\alpha_S(t)| = \langle\alpha_S(t)|H^\dagger \tag{40}$$

$$|\alpha_S(t)\rangle = e^{-iHt}|\alpha_S(0)\rangle, \qquad \langle\alpha_S(t)| = \langle\alpha_S(0)|e^{iH^\dagger t} \tag{41}$$

$$\langle\alpha_S(t)|\alpha_S(t)\rangle = \langle\alpha_S(0)|e^{iH^\dagger t}e^{-iHt}|\alpha_S(0)\rangle \neq \langle\alpha_S(0)|\alpha_S(0)\rangle \qquad \textbf{NOT UNITARY} \tag{42}$$

$$\langle\alpha_S(t)|A_S|\alpha_S(t)\rangle = \langle\alpha_S(0)|e^{iH^\dagger t}A_S e^{-iHt}|\alpha_S(0)\rangle \tag{43}$$

$$A_H(t) = e^{iH^\dagger t}A_S e^{-iHt} \tag{44}$$

$$i\frac{d}{dt}A_H(t) = A_H(t)H - H^\dagger A_H(t), \qquad i\frac{d}{dt}A_H(t) = A_H(t)H - HA_H(t) \tag{45}$$

$$i\frac{d}{dt}|\alpha_S(t)\rangle = H|\alpha_S(t)\rangle, \qquad -i\frac{d}{dt}\langle\hat{\alpha}_S(t)| = \langle\hat{\alpha}_S(t)|H \tag{46}$$

$$|\alpha_S(t)\rangle = e^{-iHt}|\alpha_S(0)\rangle, \qquad \langle\hat{\alpha}_S(t)| = \langle\hat{\alpha}_S(0)|e^{iHt} \tag{47}$$

LEFT-EIGENVECTOR IS NOT DIRAC CONJUGATE OF RIGHT EIGENVECTOR

$$\langle\hat{\alpha}_S(t)|\alpha_S(t)\rangle = \langle\hat{\alpha}_S(0)|e^{iHt}e^{-iHt}|\alpha_S(0)\rangle = \langle\hat{\alpha}_S(0)|\alpha_S(0)\rangle \qquad \textbf{UNITARY} \tag{48}$$

$$\langle\hat{\alpha}_S(t)| = \langle\alpha_S(t)|e^{-Q}, \quad H^\dagger = e^{-Q}He^{Q} \tag{49}$$

NON-DIAGONALIZABILITY AND UNITARITY
THE SINGULAR EQUAL-FREQUENCY LIMIT

In equal frequency limit the diagonalizing operator Q becomes singular and partial fraction decomposition of propagator becomes undefined.

$$\psi_0(x,y,t) = \exp\left[-\frac{\gamma}{2}(\omega_1+\omega_2)(x^2+\omega_1\omega_2 y^2) - \gamma\omega_1\omega_2 yx\right]\exp(-iE_0 t),$$
$$E_0 = (\omega_1+\omega_2)/2 \quad (50)$$

$$\psi_1(x,y,t) = (x+\omega_2 y)\psi_0(x,y,t)e^{-i\omega_1 t}, \qquad E_1 = E_0+\omega_1$$
$$\psi_2(x,y,t) = (x+\omega_1 y)\psi_0(x,y,t)e^{-i\omega_2 t}, \qquad E_2 = E_0+\omega_2 \quad (51)$$

$$\hat{\psi}_0(x,y,t) = \exp\left[-\gamma\omega^3 y^2 - \gamma\omega^2 yx - \gamma\omega x^2 - i\omega t\right], \quad \hat{E}_0 = \omega \quad (52)$$

$$\hat{\psi}_1(x,y,t) = (x+\omega y)\hat{\psi}_0(x,y,t)e^{-i\omega t}, \qquad \hat{E}_1 = \hat{E}_0+\omega \quad (53)$$

TWO one-particle states have collapsed into ONE state.

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where P would represent ordinary translations and the off-diagonal $*$ arises from $[D, \partial_I] \neq 0$.

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=======

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WHAT IS SO SPECIAL ABOUT FOURTH-ORDER THEORIES TO CAUSE ALL THIS

Consider the Lehmann Representation for a scalar field. Assume translation invariance and a bounded, real energy eigenspectrum with states of 4-momentum k_μ^n with $k_0^n > 0$. We can set $\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}$. Provisionally set $\Sigma\,|n\rangle\langle n| = 1$. Then can show

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_n |\langle\Omega|\phi(0)|k_n^\mu\rangle|^2 e^{-ik^n\cdot(x-y)} \tag{64}$$

Now introduce the spectral function

$$\rho(q^2) = (2\pi)^3 \sum_n \delta^4(k_\mu^n - q_\mu)|\langle\Omega|\phi(0)|k_n^\mu\rangle|^2\theta(q_0) \tag{65}$$

and set

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty dm^2\rho(m^2)\int \frac{d^4q}{(2\pi)^3}\theta(q_0)\delta(q^2 - m^2)e^{-iq\cdot(x-y)} \tag{66}$$

Thus finally obtain the Lehmann representation

$$\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle = \int_0^\infty dm^2\rho(m^2)\Delta_2^{\text{free}}(x-y;m^2) \tag{67}$$

This relation holds for any interacting two-point function no matter what its equation of motion, with it always being the **FREE SECOND-ORDER** Feynman propagator which appears in the integral because the mass shell condition $p^2 = m^2$ is always second-order. The Lehmann representation thus holds in fourth-order theories also. However, for large k^2 the second-order Feynman propagator behaves as $1/k^2$, whereas for fourth-order theories the propagator behaves as $1/k^4$. Hence (Smolin) we have a contradiction if $\rho(m^2)$ is positive definite.

Solution is that spectral function cannot be positive definite, and we cannot set $\Sigma|n\rangle\langle n|=1$. Rather we must set $\Sigma|n\rangle\langle n|e^{-Q}=1$ (and not $\Sigma|n\rangle\langle n|-\Sigma|m\rangle\langle m|=1$), and distinguish between left and right momentum eigenvectors. Thus must use $|R\rangle$ and $\langle L|=\langle R|e^{-Q}$. With this choice, the spectral function is replaced by

$$\rho(q^2)=(2\pi)^3\sum_n \delta^4(k_\mu^n-q_\mu)\langle\Omega|e^{-Q}\phi(0)|k_n^\mu\rangle\langle k_n^\mu|e^{-Q}\phi(0)|\Omega\rangle\theta(q_0) \tag{68}$$

Now there is no positivity requirement and we can cancel the $1/k^2$ behavior without having to give up unitarity. Since we still impose the reality of the momentum eigenvalues, the Hamiltonian of the theory must be PT invariant rather than Hermitian. From the Lehmann representation we thus conclude that the Hamiltonian of theories such as the fourth-order Pais-Uhlenbeck oscillator cannot be Hermitian and must instead be PT invariant, just as we had found directly.

THE REMARKABLE MORAL OF THE STORY

Consider **ANY** higher derivative theory in which the equation of motion is of the form $f(D)\phi=0$ where $D=\partial_\mu\partial^\mu$ and $f(D)=\Sigma a_n D^n$. Also require that all momentum eigenvalues be real. In such a theory, at large k^2 the propagator will behave as $1/k^{2n}$. Hence there will be a contradiction with the Lehmann representation if we require the standard $\Sigma|n\rangle\langle n|=1$. Rather, such theories must be PT theories rather than standard Hermitian ones. Hence once we depart from second-order equations we are forced to PT-invariant, non-Hermitian Hamiltonians. PT-invariance is thus the general rule, and it is only a historical accident (second-order theories were encountered first) that it was not discovered earlier.

DYNAMICAL SYMMETRY BREAKING AND THE COSMOLOGICAL CONSTANT PROBLEM

The zero-point energy and cosmological constant problems are two separate problems. With dynamical symmetry breaking they solve each other.

The zero-point energy problem already exists in a free field theory, and as such is separate from any cosmological constant term that might be induced by spontaneous symmetry breaking. The cosmological constant is associated with the minimum of the symmetry breaking potential while the zero-point energy is associated with the fluctuations about it. Moreover, the zero-point term and the cosmological constant term even transform differently under a general coordinate transformation, the former possessing a fluid velocity and being maximally 3-symmetric, with the latter possessing no fluid velocity and being maximally 4-symmetric. The zero-point fluctuation term is associated with a perfect matter fluid in which both ρ_m and p_m are positive,– so that $T_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m g_{\mu\nu}$. While the cosmological constant term is associated with a perfect fluid in which $p = -\rho$, – so that $T_{\mu\nu} = -\Lambda g_{\mu\nu}$.

When the symmetry is broken dynamically by fermion condensates, it is the fermionic zero-point fluctuations which cause the change in the vacuum in the first place, to thus actually produce the cosmological constant. In this case the zero-point and cosmological constant terms are not independent, and are related in a way which allows each one to cancel the other, so that both the zero-point and cosmological constant problems solve each other.

For a free quantum fermion field, in a mode of the form

$$\psi(x) = \sum_{\pm s} \int \frac{d^4p}{(2\pi)^{3/2}} \left(\frac{m}{E_p}\right)^{1/2} \left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x}\right] \qquad (69)$$

the vacuum expectation value of $T_{\mu\nu} \sim \bar{\psi}\gamma_\mu\partial_\nu\psi$ is due to the non-vanishing of

$$\langle\Omega|d(p,s)d^\dagger(p,s)|\Omega\rangle = \langle\Omega|(d(p,s)d^\dagger(p,s) + d^\dagger(p,s)d(p,s))|\Omega\rangle, \qquad (70)$$

to yield the zero-point contribution

$$\langle\Omega|T^{\mu\nu}|\Omega\rangle \sim -2\int \frac{d^3p}{(2\pi)^3 E_p} p^\mu p^\nu. \qquad (71)$$

When averaged over all directions of $\bar{p}$ it takes the form of a perfect matter fluid $T_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m g_{\mu\nu}$ with both a zero-point energy and a zero-point pressure. Specifically, for a mode with $p^\mu = (E_p, 0, 0, p)$, the quantity $p^\mu p^\nu / E_p$ evaluates to the non-diagonal

$$\begin{pmatrix} E_p & 0 & 0 & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & p^2/E_p \end{pmatrix}$$

Similarly, for a mode with $p^\mu = (E_p, 0, 0, -p)$, we get

$$\begin{pmatrix} E_p & 0 & 0 & -p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & 0 & p^2/E_p \end{pmatrix}$$

On incoherently adding them we thus get the now diagonal

$$\begin{pmatrix} 2E_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p^2/E_p \end{pmatrix}$$

FOUR-FERMI MEAN-FIELD THEORY IN TWO DIMENSIONS

In $D = 2$ the four-Fermi theory is defined via

$$L = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - (g/2)[\bar{\psi}\psi]^2, \qquad i\gamma^{\mu}\partial_{\mu}\psi - g[\bar{\psi}\psi]\psi = 0. \tag{74}$$

$$T^{\mu\nu} = i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - g^{\mu\nu}\frac{g}{2}[\bar{\psi}\psi]^2, \qquad T^{\mu}{}_{\mu} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - g[\bar{\psi}\psi]^2 = 0, \tag{75}$$

with the energy-momentum tensor being traceless. In the mean-field approximation one looks for self-consistent, translation invariant, states $|S\rangle$ in which

$$\langle S|\bar{\psi}\psi|S\rangle = m/g, \qquad \langle S|[\bar{\psi}\psi - m/g]^2|S\rangle = 0, \qquad i\gamma^{\mu}\partial_{\mu}\psi - m\psi = 0, \tag{76}$$

$$\langle S|T^{\mu\nu}|S\rangle = \langle S|i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi|S\rangle - g^{\mu\nu}\frac{m^2}{2g}, \qquad \langle S|T^{\mu}{}_{\mu}|S\rangle = m\langle S|\bar{\psi}\psi|S\rangle - \frac{m^2}{g} = 0, \tag{77}$$

with the mean-field approximation preserving tracelessness.

In a plane wave solution

$$\psi(x) = \sum_{\pm s}\int d^2p/(2\pi)^{1/2}(m/E_p)^{1/2}[b(p,s)u(p,s)e^{-ip\cdot x} + d^{\dagger}(p,s)v(p,s)e^{ip\cdot x}] \tag{78}$$

we find that

$$\langle\Omega|i\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi|\Omega\rangle = -2\int d^2p p^{\mu}p^{\nu}/(2\pi E_p) \tag{79}$$

due to zero-point fluctuations since

$$\langle\Omega|bb^{\dagger}|\Omega\rangle = \langle\Omega|(bb^{\dagger} + b^{\dagger}b)|\Omega\rangle. \tag{80}$$

On incoherently adding the contributions of modes with $p^\mu = (E_p, \pm p, 0, 0)$, $p^\mu = (E_p, 0, \pm p, 0)$, we finally obtain

$$\begin{pmatrix} 6E_p & 0 & 0 & 0 \\ 0 & 2p^2/E_p & 0 & 0 \\ 0 & 0 & 2p^2/E_p & 0 \\ 0 & 0 & 0 & 2p^2/E_p \end{pmatrix}$$

to yield a perfect fluid form with $\rho_m \sim 6E_p$, $p_m \sim 2p^2/E_p$. That this is **NOT** a perfect fluid with $p_m = -\rho_m$ is because the averaging is only 3-dimensional, i.e. for modes which obey the wave equation only the 3-momentum is averaged over. Moreover, the velocity of the fluid U^μ is a timelike vector, and is allowed simply because a Minkowski metric distinguishes between timelike and spacelike.

In flat space, for $T^{\mu\nu} \sim k^\mu k^\nu / E_k$ one can also construct the total energy and momentum as $P^\mu = \int d^3x T^{0\mu}$, to find that

$$P^0 \sim E_k, \qquad P^i = 0. \tag{72}$$

with there being no zero-point total momentum. However one can also evaluate

$$\int d^3x T^{ij} \sim \int d^3x \frac{k^i k^j}{E_k} \sim \delta^{ij} \tag{73}$$

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$$\psi(x) = \sum_{\pm s}\int d^2p/(2\pi)^{1/2}(m/E_p)^{1/2}[b(p,s)u(p,s)e^{-ip\cdot x} + d^{\dagger}(p,s)v(p,s)e^{ip\cdot x}] \tag{78}$$

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On incoherently adding the contributions of modes with $p^\mu = (E_p, \pm p, 0, 0)$, $p^\mu = (E_p, 0, \pm p, 0)$, we finally obtain

$$\begin{pmatrix} 6E_p & 0 & 0 & 0 \\ 0 & 2p^2/E_p & 0 & 0 \\ 0 & 0 & 2p^2/E_p & 0 \\ 0 & 0 & 0 & 2p^2/E_p \end{pmatrix}$$

to yield a perfect fluid form with $\rho_m \sim 6E_p$, $p_m \sim 2p^2/E_p$. That this is **NOT** a perfect fluid with $p_m = -\rho_m$ is because the averaging is only 3-dimensional, i.e. for modes which obey the wave equation only the 3-momentum is averaged over. Moreover, the velocity of the fluid U^μ is a timelike vector, and is allowed simply because a Minkowski metric distinguishes between timelike and spacelike.

In flat space, for $T^{\mu\nu} \sim k^\mu k^\nu / E_k$ one can also construct the total energy and momentum as $P^\mu = \int d^3x T^{0\mu}$, to find that

$$P^0 \sim E_k, \qquad P^i = 0, \tag{72}$$

with there being no zero-point total momentum. However one can also evaluate

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For a free quantum fermion field, in a mode of the form

$$\psi(x) = \sum_{\pm s} \int \frac{d^4p}{(2\pi)^{3/2}} \left(\frac{m}{E_p}\right)^{1/2} [b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{ip\cdot x}] \tag{69}$$

the vacuum expectation value of $T_{\mu\nu} \sim \bar{\psi}\gamma_\mu\partial_\nu\psi$ is due to the non-vanishing of

$$\langle\Omega|d(p,s)d^\dagger(p,s)|\Omega\rangle = \langle\Omega|(d(p,s)d^\dagger(p,s) + d^\dagger(p,s)d(p,s))|\Omega\rangle, \tag{70}$$

to yield the zero-point contribution

$$\langle\Omega|T^{\mu\nu}|\Omega\rangle \sim -2\int \frac{d^3p}{(2\pi)^3 E_p} p^\mu p^\nu. \tag{71}$$

When averaged over all directions of $\bar{p}$ it takes the form of a perfect matter fluid $T_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu + p_m g_{\mu\nu}$ with both a zero-point energy and a zero-point pressure. Specifically, for a mode with $p^\mu = (E_p, 0, 0, p)$, the quantity $p^\mu p^\nu/E_p$ evaluates to the non-diagonal

$$\begin{pmatrix} E_p & 0 & 0 & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & p^2/E_p \end{pmatrix}$$

Similarly, for a mode with $p^\mu = (E_p, 0, 0, -p)$, we get

$$\begin{pmatrix} E_p & 0 & 0 & -p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -p & 0 & 0 & p^2/E_p \end{pmatrix}$$

On incoherently adding them we thus get the now diagonal

$$\begin{pmatrix} 2E_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2p^2/E_p \end{pmatrix}$$

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CANCELLATION OF ZERO-POINT AND Λ IN TRACE OF $T_{\mu\nu}$

Incoherently adding together modes with $p^\mu = (E_p, p)$ and $(E_p, -p)$ gives

$$\Sigma \frac{p^\mu p^\nu}{E_p} = \begin{pmatrix} E_p & p \\ p & p^2/E_p \end{pmatrix} + \begin{pmatrix} E_p & -p \\ -p & p^2/E_p \end{pmatrix} = \begin{pmatrix} 2E_p & 0 \\ 0 & 2p^2/E_p \end{pmatrix} = (\rho_m + p_m)U^\mu U^\nu + p_m g^{\mu\nu} \quad (81)$$

On setting $m^2/g = g[\langle S|\bar{\psi}\psi|S\rangle]^2 = \Lambda$, we can thus set

$$\langle S|T^{\mu\nu}|S\rangle = (\rho_m + p_m)U^\mu U^\nu + p_m g^{\mu\nu} - g^{\mu\nu}\Lambda, \qquad \langle S|T^\mu{}_\mu|S\rangle = p_m - \rho_m - 2\Lambda = 0. \quad (82)$$

Thus $2\Lambda = p_m - \rho_m$, neither bigger nor smaller. The reason for this is that all of Λ, ρ_m and p_m are determined in one the same state $|S\rangle$. For the case of a fundamental scalar field $\phi(x)$, $\Lambda = \lambda\phi^4$ is determined by the vacuum (location of the minimum of the Higgs double-well potential), while ρ_m and p_m are determined by whichever matter field frequency modes are occupied. Hence one cannot relate Λ to ρ_m and p_m in standard cosmology with a fundamental Higgs field, to thus give rise to the cosmological constant problem. If however scalar field is a c-number Ginzburg-Landau condensate $\langle S|\bar{\psi}\psi|S\rangle$ then can relate relate Λ, ρ_m and p_m, and even have them cancel each other.

Comparison test: If Higgs is a c-number condensate then will **NOT** be produced in an accelerator such as the LHC, while if Higgs is fundamental then will be produced in an accelerator.

CANCELLATION OF ZERO-POINT AND Λ IN $T_{\mu\nu}$ ITSELF

In four spacetime dimensions invariance under

$$g_{\mu\nu}(x) \rightarrow e^{2\alpha(x)} g_{\mu\nu}(x), \tag{83}$$

leads to a unique gravitational action

$$I_W = -\alpha_g \int d^4x(-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha_g \int d^4x(-g)^{1/2} \left[R^{\mu\nu} R_{\mu\nu} - \frac{1}{3}(R^{\alpha}{}_{\alpha})^2 \right] \tag{84}$$

where α_g is dimensionless and $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor. The associated gravitational equations of motion are the **fourth-order derivative**:

$$4\alpha_g [2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}] = T^{\mu\nu} \tag{85}$$

and thus in cosmology where $C_{\lambda\mu\nu\kappa} = 0$, we obtain

$$T^{\mu\nu} = 0 \tag{86}$$

Thus again get cancellation of zero-point and Λ terms in $T_{\mu\nu}$ itself, but now need to include zero-point fluctuations in gravitational field as well. While $\langle S|[2C^{\mu\lambda\nu\kappa}{}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa} R_{\lambda\kappa}]|S\rangle$ vanishes in a classical cosmological background, quantum-mechanically there is a zero-point fluctuation contribution. Using the gravitational zero-point fluctuations to cancel the matter field zero-point fluctuations and Λ is only achievable in a renormalizable theory of gravity. Hence works in conformal gravity but not in standard gravity.

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