Title: Explaining Regularities: The Need for Singular Behaviour

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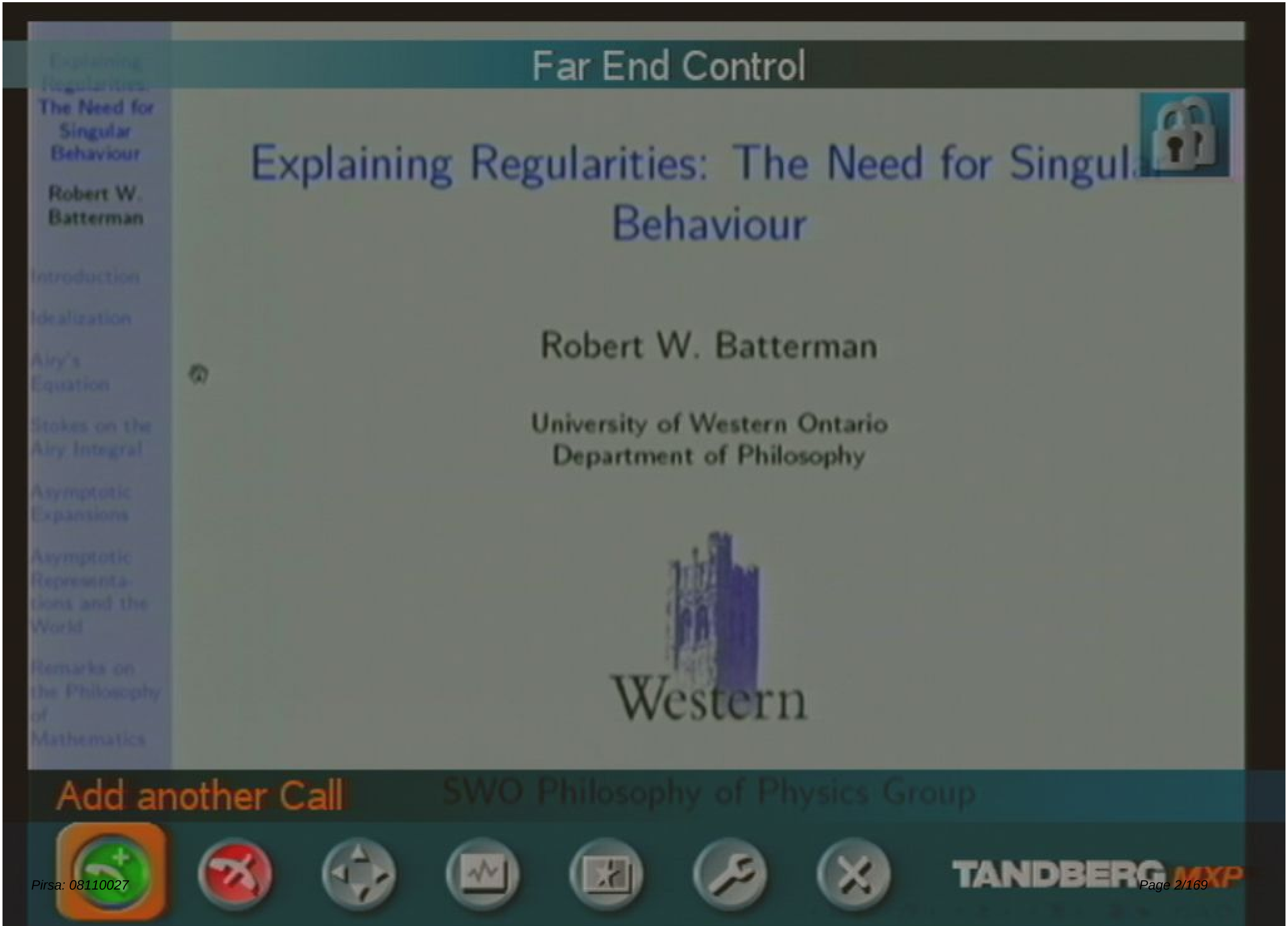
Abstract:


Far End Control
Explaining Regularities: The Need for Singular Behaviour
Robert W. Batterman
Introduction
Idealization
Airy's Equation
Stokes on the Airy Integral
Asymptotic Expansions
Asymptotic Representations and the World
Remarks on the Philosophy of Mathematics
Explaining Regularities: The Need for Singular Behaviour
Robert W. Batterman
University of Western Ontario
Department of Philosophy
Western
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SWO Philosophy of Physics Group
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- Aim is to discuss reasons for taking *mathematically* characterized idealizations very seriously in explaining *physical phenomena*.
- Suggest that the physical world, itself, requires or demands a certain type of mathematical representation and that, as a result, one should not be surprised that mathematical *asymptotic* explanations play an essential, ineliminable role in providing physical understanding.

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- The discussion relates to issues concerning the applicability of mathematics in the natural sciences (Wigner's Problem).

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Outline

Slogan:

> *"Singularity is almost invariably a clue." —Sherlock Holmes in The Boscome Valley Mystery*

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Structure

- Discuss two approaches to idealization in physics.

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  - Traditional (Galilean)
  - Nontraditonal
- Discuss the role of limits and asymptotics on the nontraditional approach.

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The Need for Singular Behaviour

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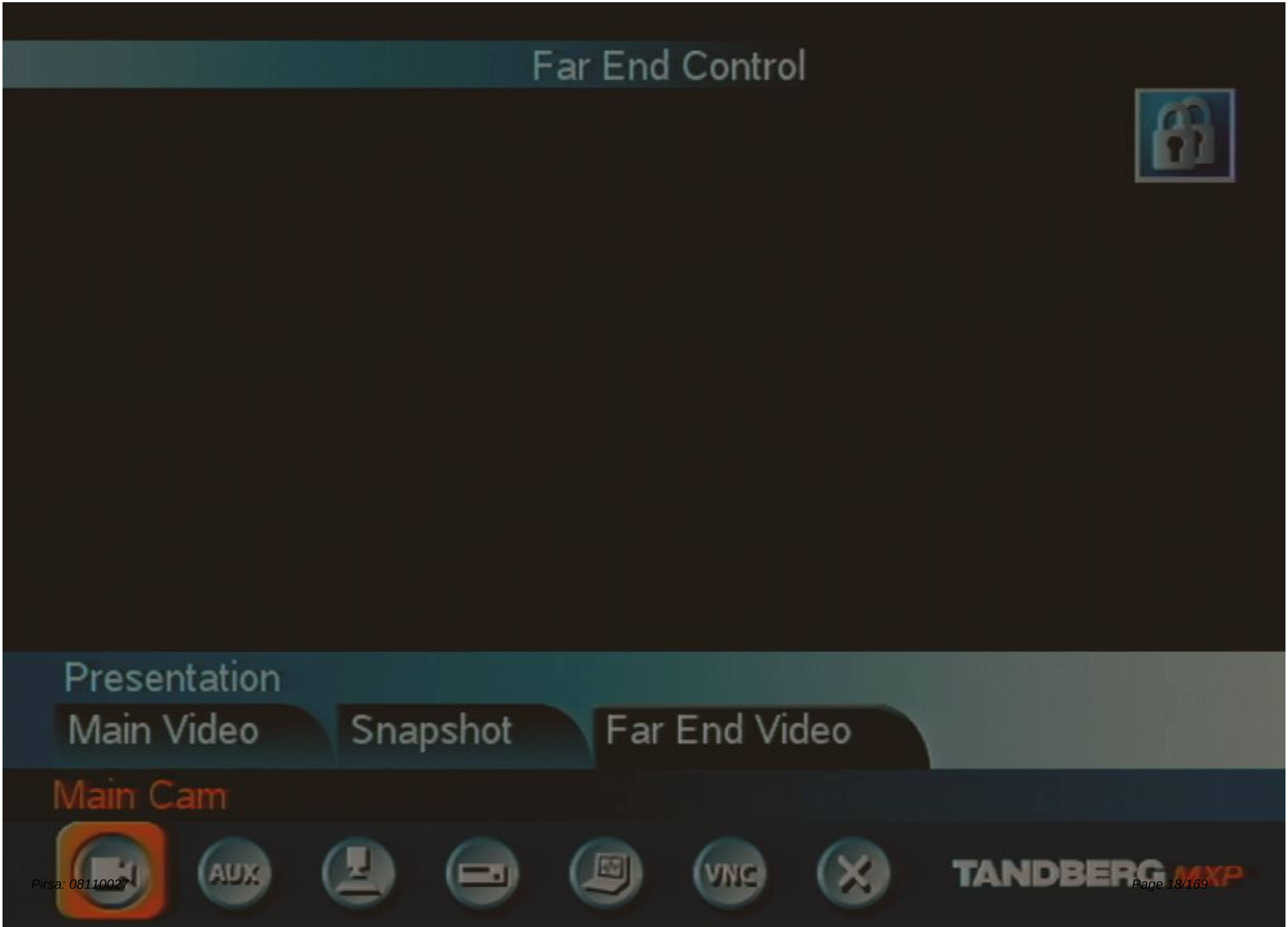
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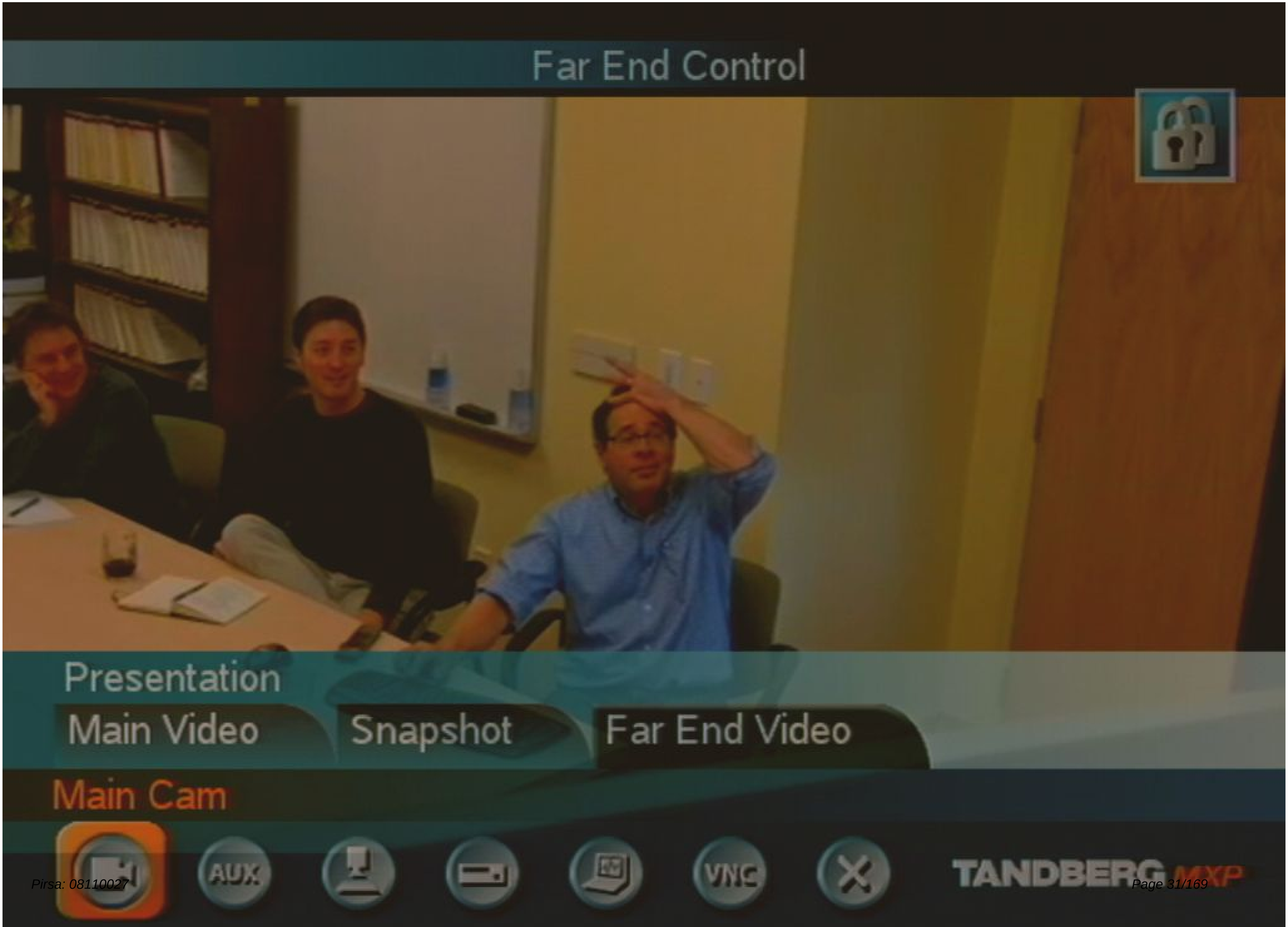
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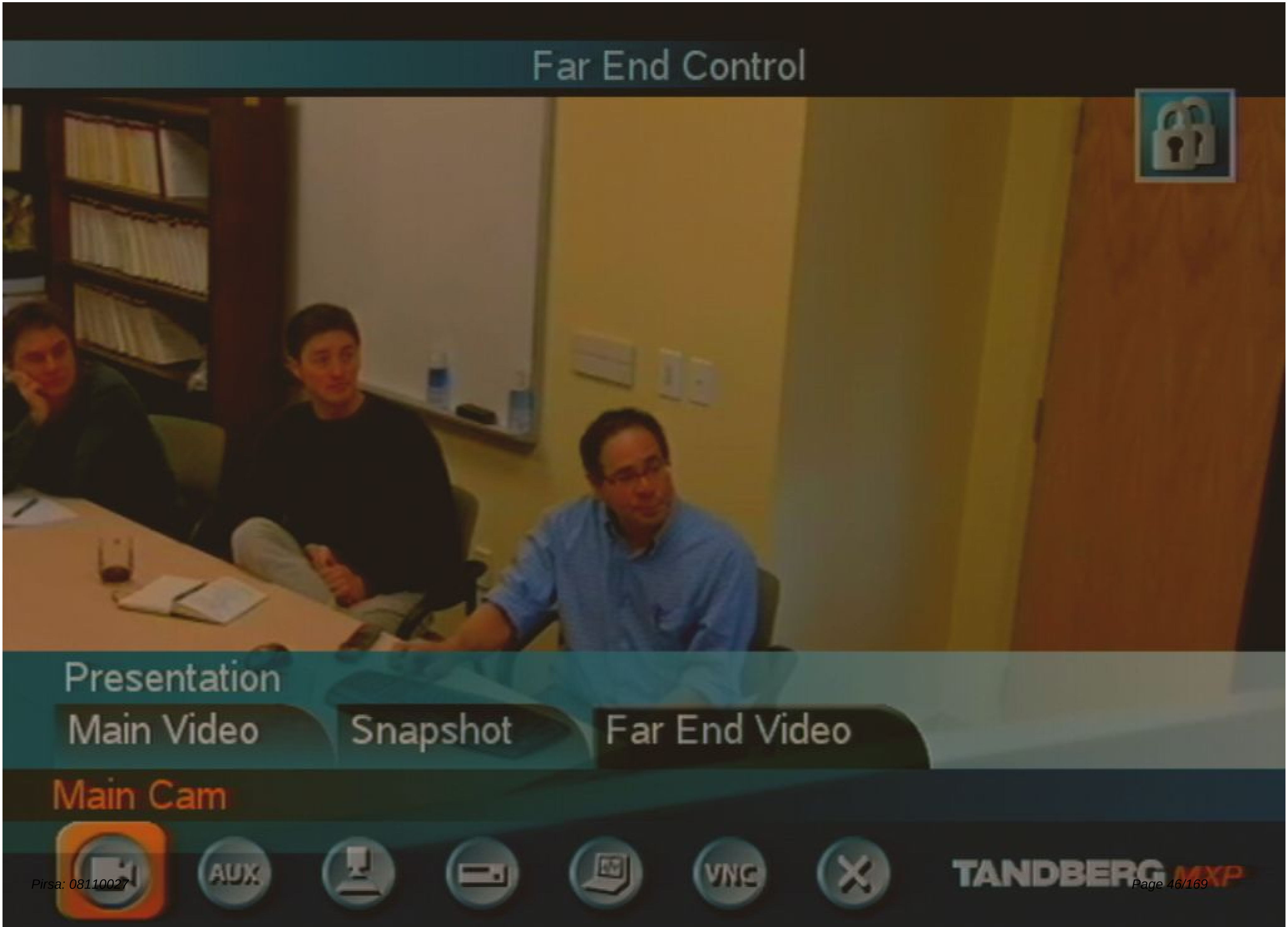
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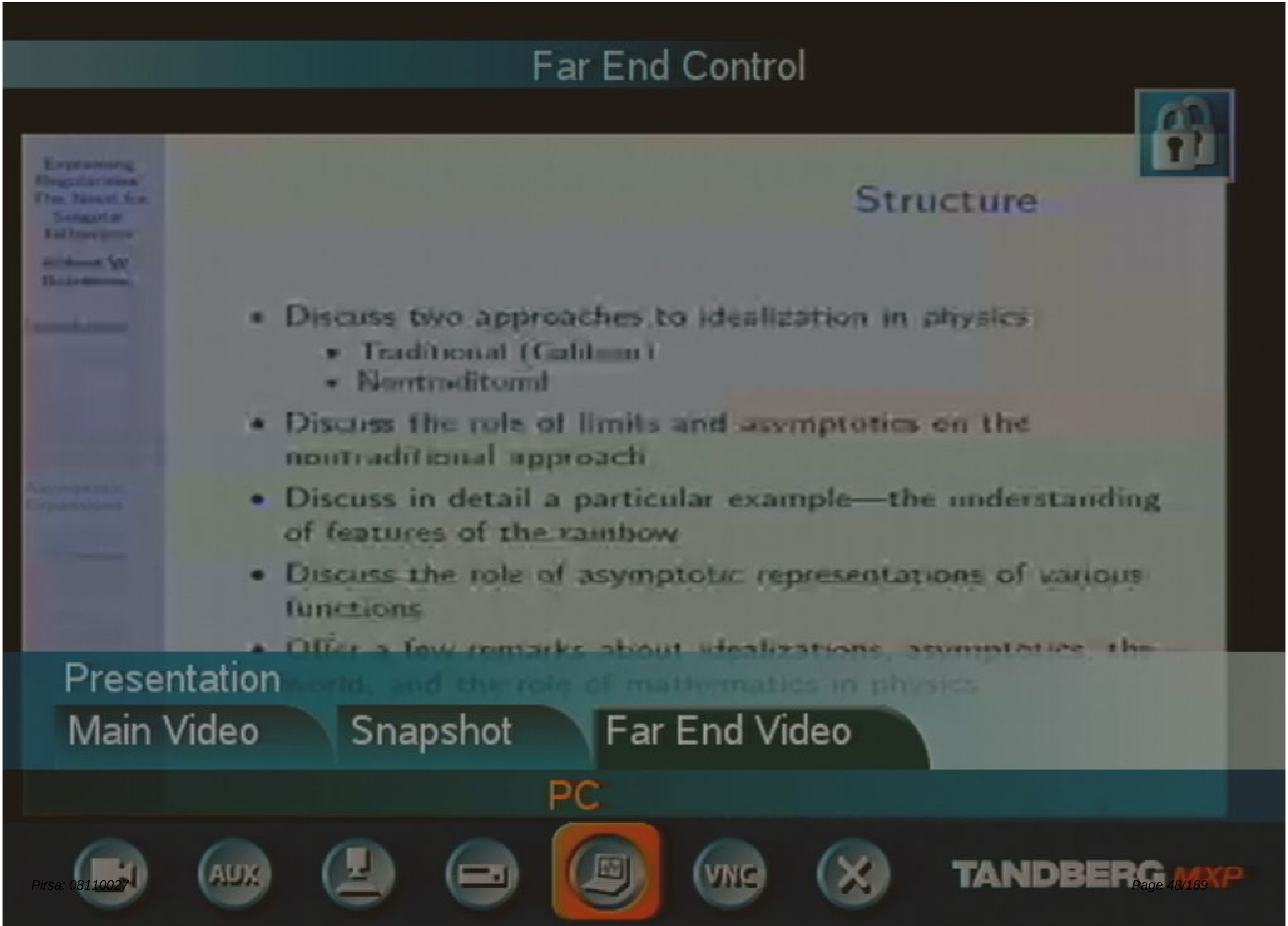
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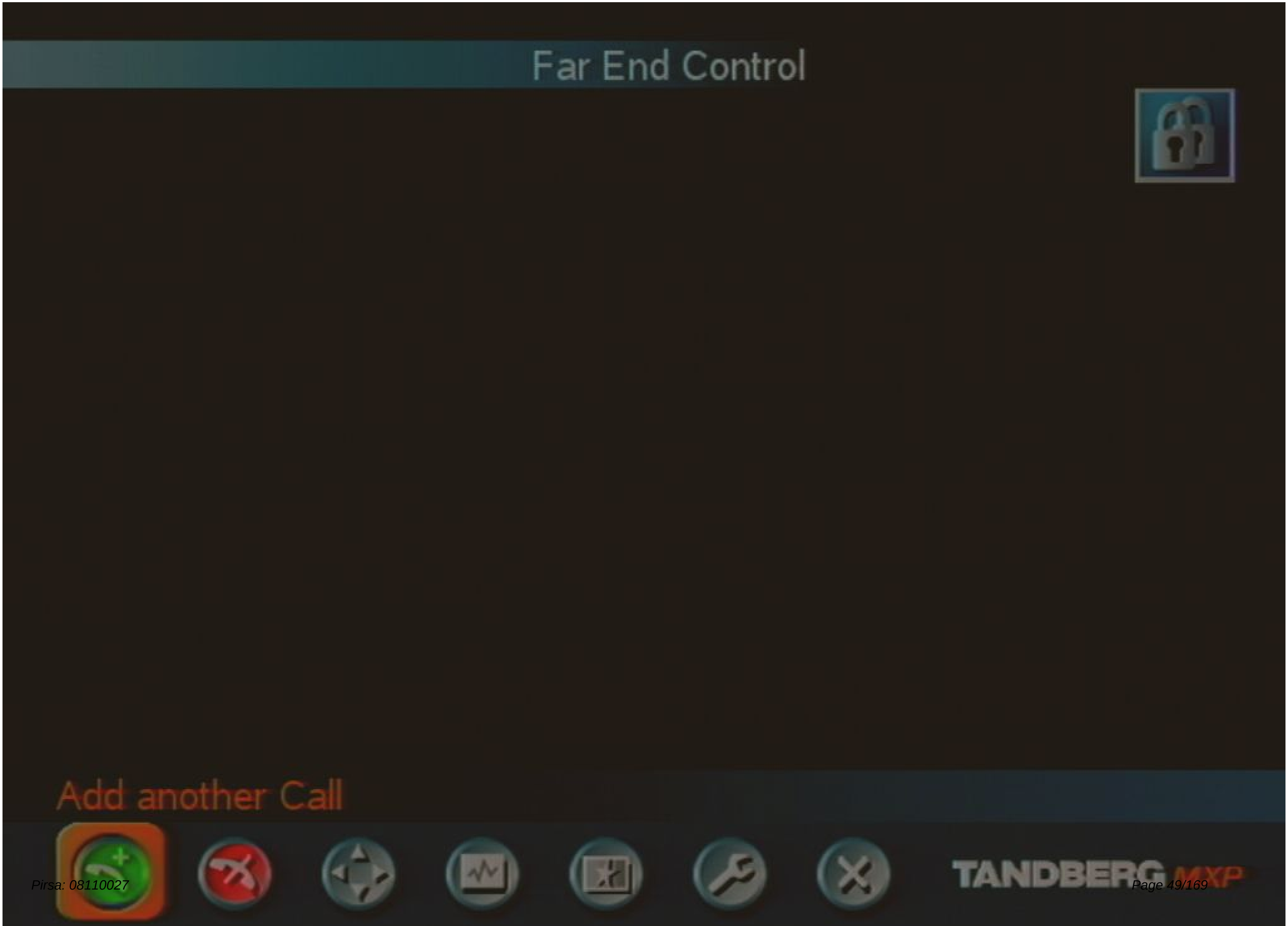
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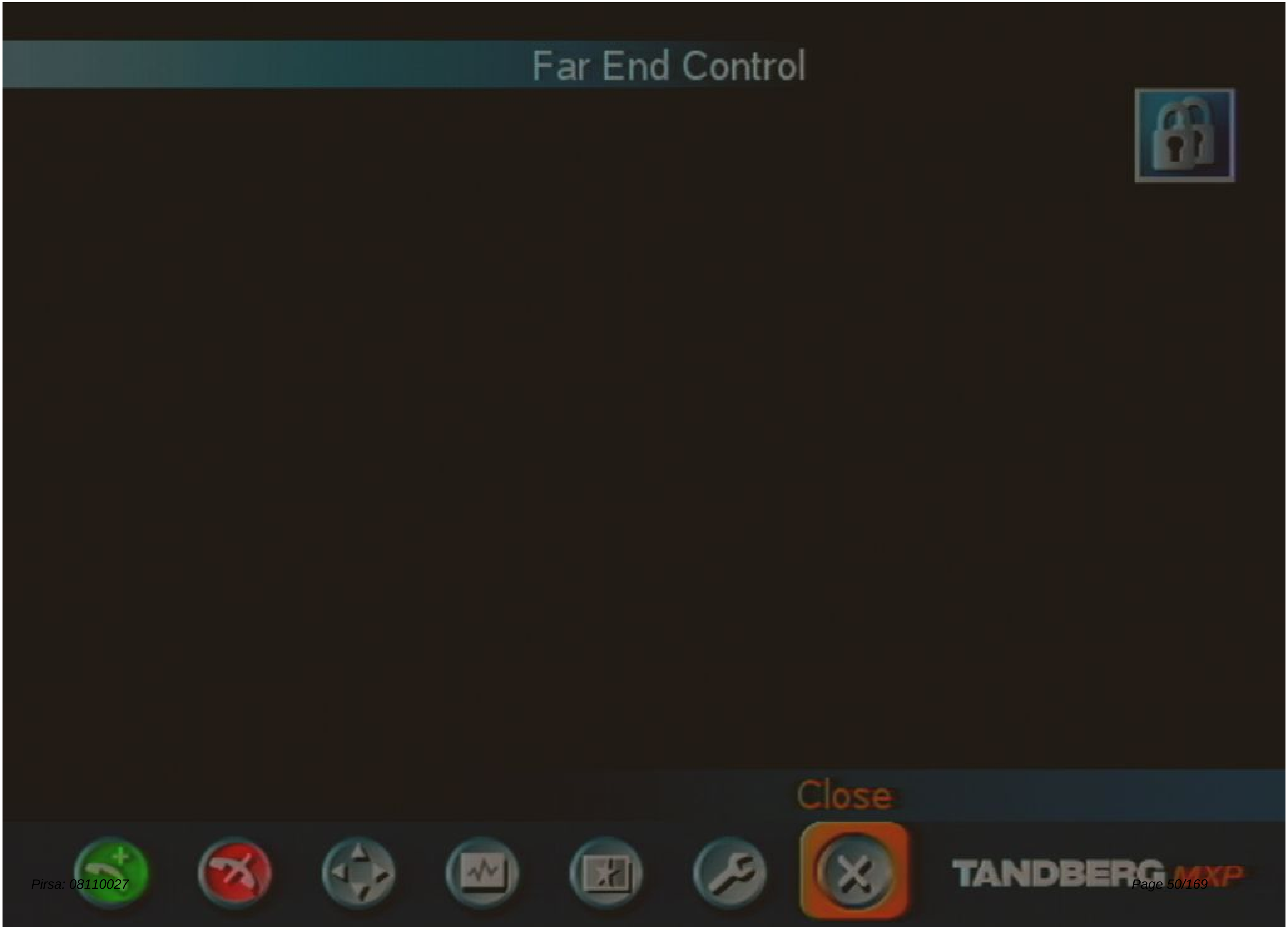
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- Discuss the role of limits and asymptotics on the nontraditional approach.
- Discuss in detail a particular example—the understanding of features of the rainbow.
- Discuss the role of asymptotic representations of various functions.
- Offer a few remarks about idealizations, asymptotics, the world, and the role of mathematics in physics.

Traditional Viewpoint

Almost all applications of mathematics to physical problems involve idealized models of some kind or other.

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- The aim is to try to effect a kind of convergence between model and reality. Ultimately, the goal is a complete (or true) description of the phenomenon of interest.
- On this view, a model is better the more details of the real phenomenon it is actually able to represent mathematically. In effect, *the idealizations are introduced only to be removed through further work on the details.*

Idealization

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- Prandtl (1948): "When the complete mathematical problem looks hopeless, it is recommended to enquire what happens when one essential parameter of the problem reaches the limit zero."
- This is simplification by limiting idealizations. Not *solely* a pragmatic exercise to yield exactly solvable equations.

Minimal Models

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  - Holds that adding details to "improve" the model is self-defeating—such improvements are illusory.

Modeling Shocks

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- Dense region is a shock. How can it be modeled?
    - Computationally track the molecules using the equations of molecular dynamics vs. ...

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Modeling Shocks

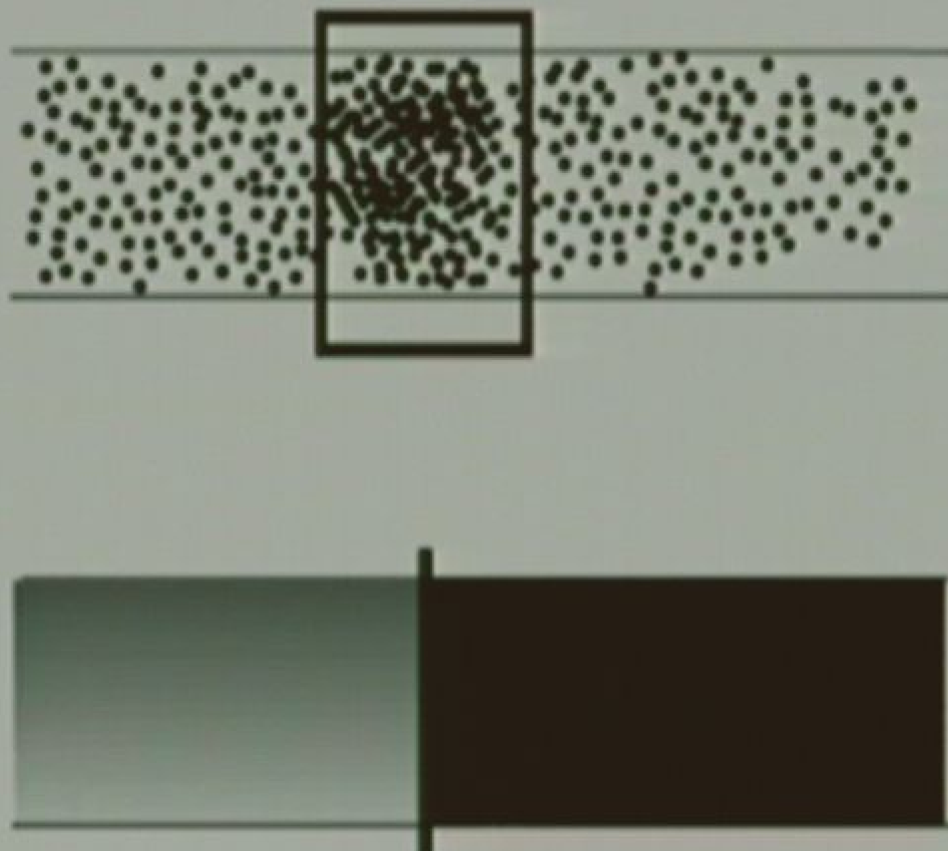
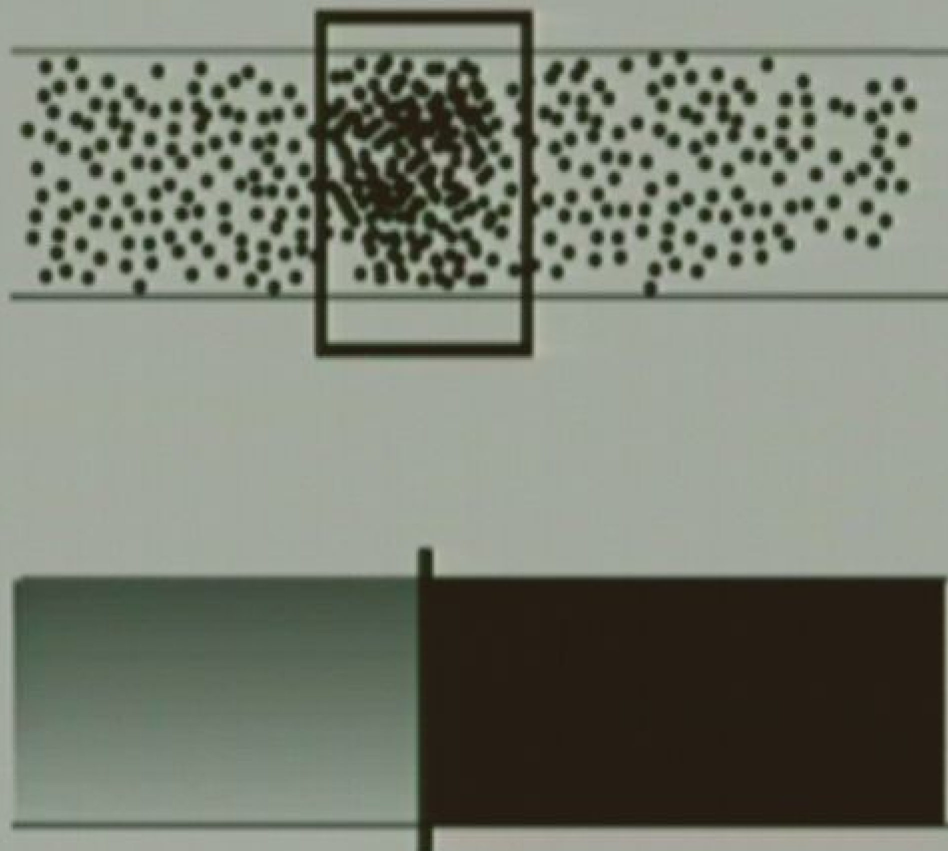


Figure: Modeling Shocks

Modeling Shocks



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Continuum Idealization:

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- On either side of the boundary the fluid's behavior is governed by the relevant partial differential equations of fluid dynamics.
- Behavior at the boundary is not law governed—no differential equation holds; instead we have algebraic "jump conditions"—singular behavior across the boundary.
- Standard view: Such boundaries are relatively unimportant to the physics—they are not law-governed. Covering law accounts of explanation relegate boundary conditions (and initial conditions) to secondary status in explanation.

Modeling Shocks

Continuum Idealization:

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Rays and the Rainbow Caustic

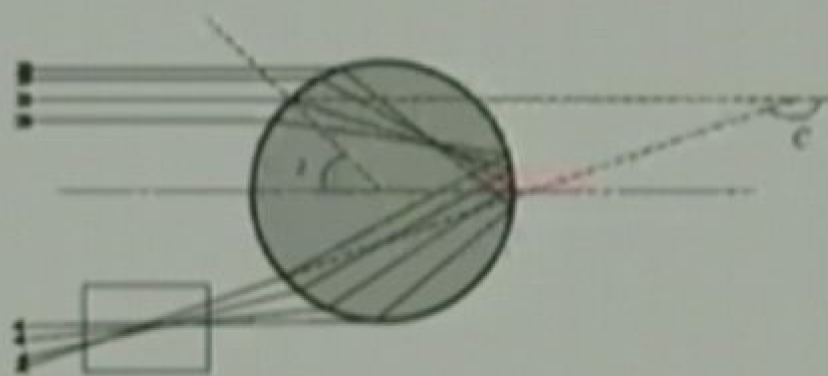

Figure: Raindrop and Light Rays

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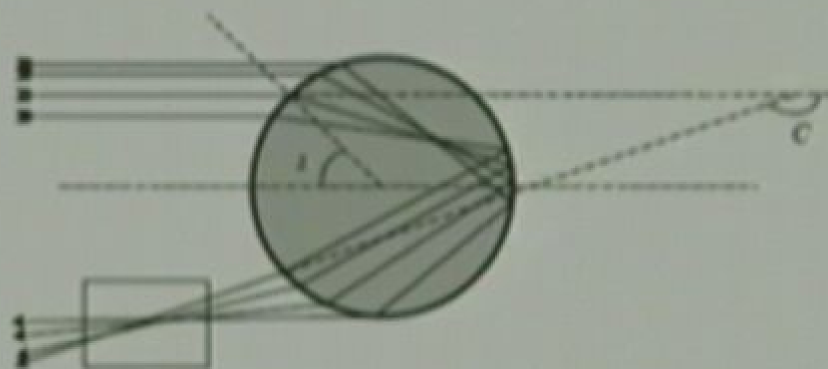

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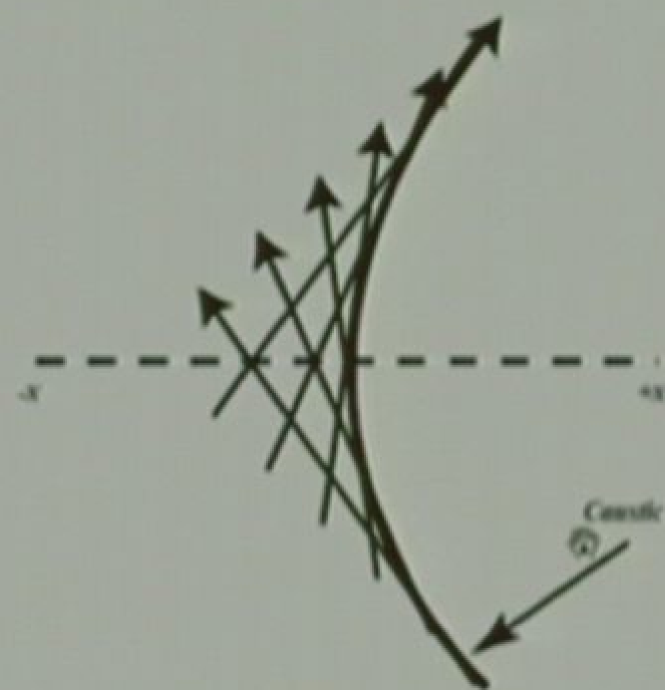

Figure: Fold Caustic

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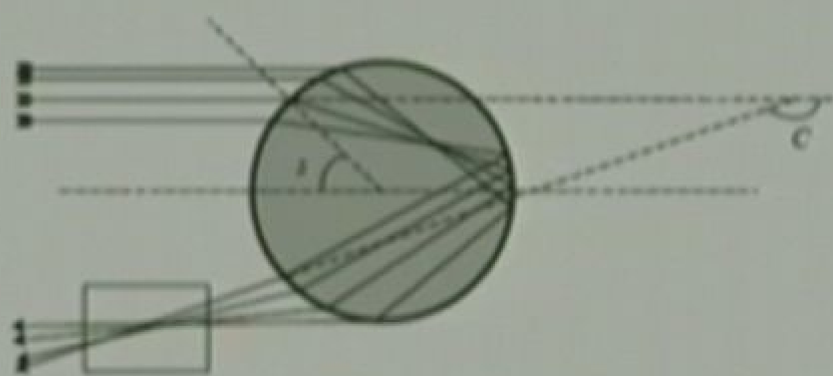

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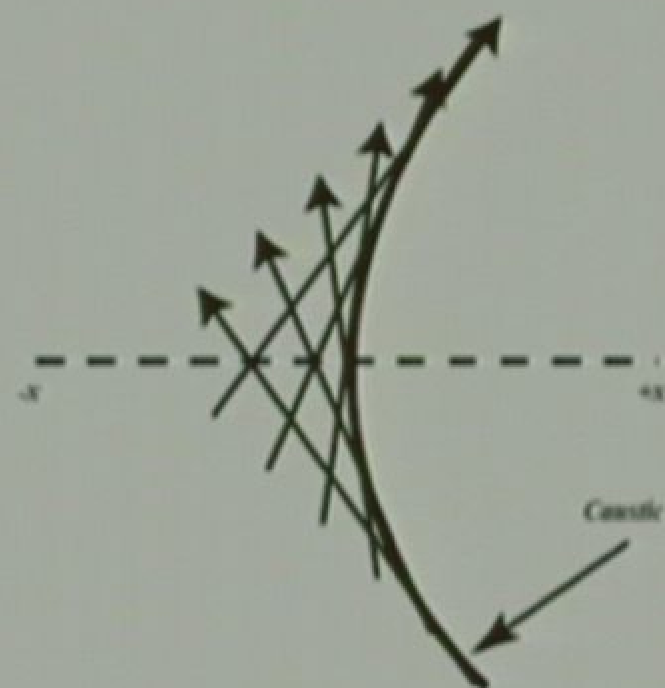

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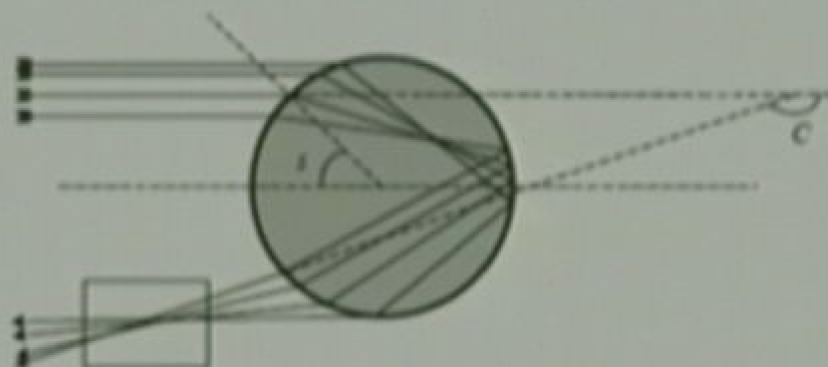

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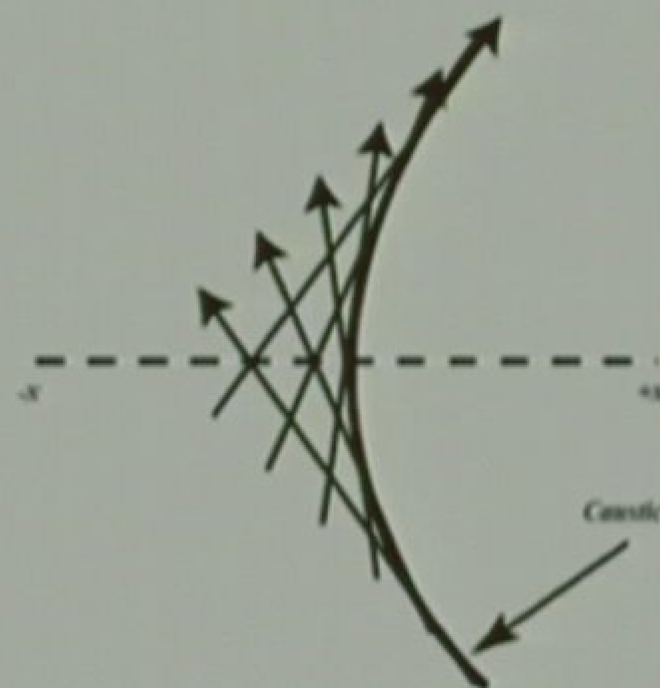

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Ray Theory or Geometrical Optics

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- This represents a kind of discontinuity or singularity in the theory. It is the $\lambda \to 0$ limit of the wave theory and is analogous to the continuum limit yielding a singular shock.
- The ray theory is incapable of describing the so-called supernumerary bows—bows resulting from wave interference—that are sometimes seen as faint arcs of alternating pink and green on the lit side of the main rainbow arc.

Ray Theory or Geometrical Optics

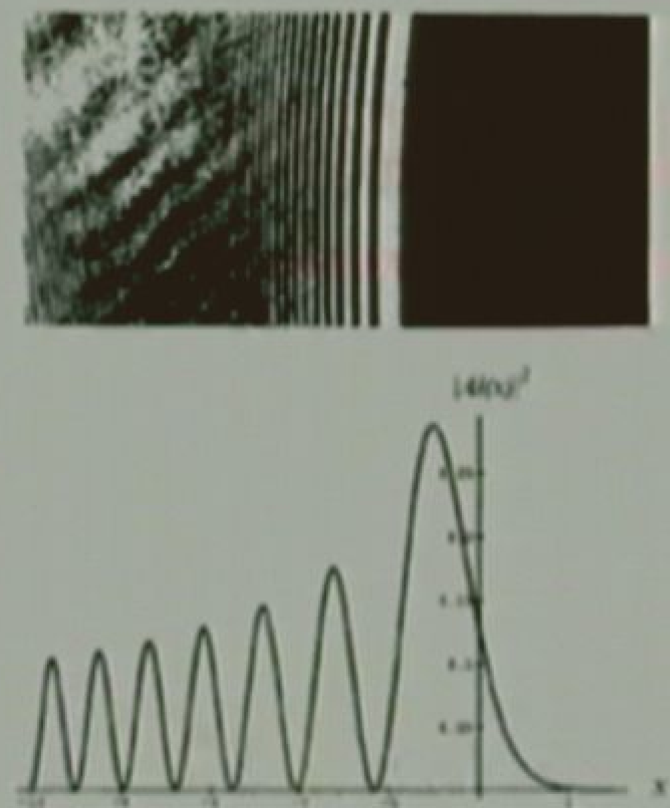

Figure: Supernumerary Bows

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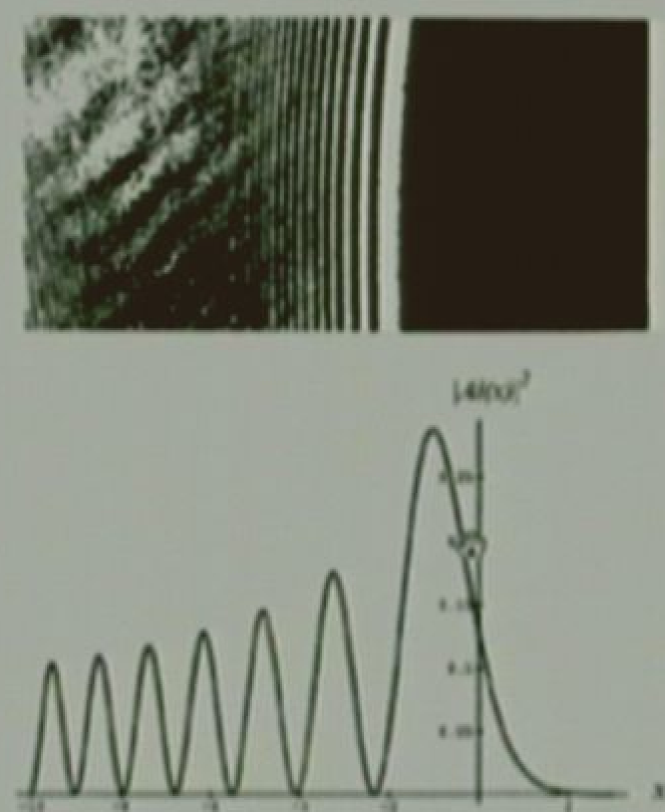

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Ray Theory or Geometrical Optics

So why do we need the ray theory at all?

Ray Theory or Geometrical Optics

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- Because the stability (under perturbation) of the caustics (of the ray-theoretic focal structures) plays an essential role in our understanding of the wave phenomena.

Airy Equation

In 1838 George Biddell Airy derived a definite integral from the undulatory or wave theory of light from which one can determine the variation in intensity of light near a caustic of geometrical or ray optics. The term "caustic" literally means "burning surface" and in nature they are extremely bright lines and surfaces caused by the natural focusing of light. They are, in a very obvious sense "dominant" features of light phenomena.

Airy Equation

Airy recognized that the singularity at the caustic was an artifact of the ray theory, and by properly incorporating effects of diffraction using the wave theory, he was able to derive the definite integral of equation (1).

$$Ai(x) \equiv \frac{1}{\pi}\int_0^\infty cos\left(\frac{t^3}{3}+xt\right)dt. \tag{1}$$

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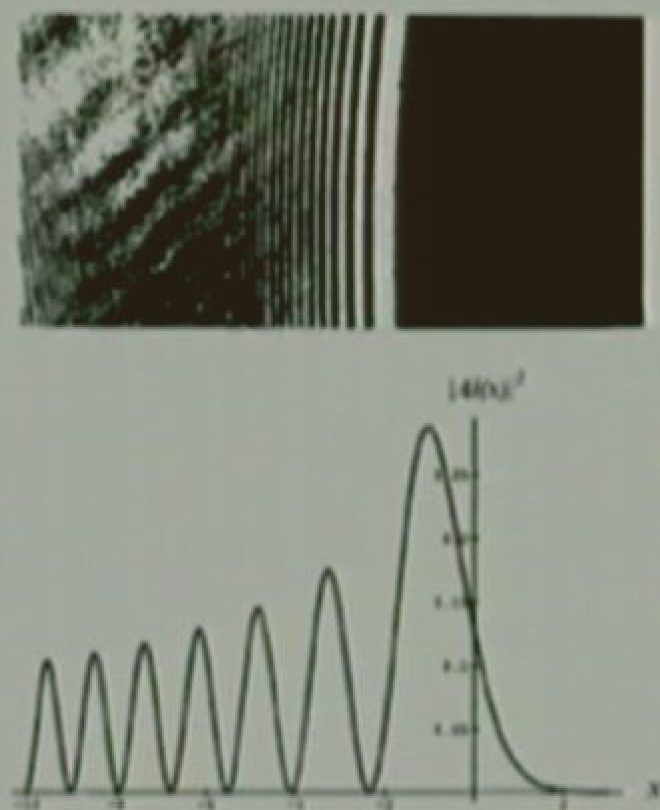

Figure: Airy Function and Supernumerary Bows

Airy's Convergent Expansion

Using a convergent series representation of this function, Airy was able to compute values for $Ai(x)$ for x between -5.6 to +5.6, where x is the distance along a normal to the caustic, negative values on the lit side, positive values in the shadow. Airy's series converges for all values of x and, as a result of this convergence, we may take it to be an exact representation of the Airy function. Unfortunately, this range of values allows one to locate only the first two dark bands on the lit side of the caustic. Because of the slow convergence of the series Airy was unable to extend his calculations beyond this limited range.

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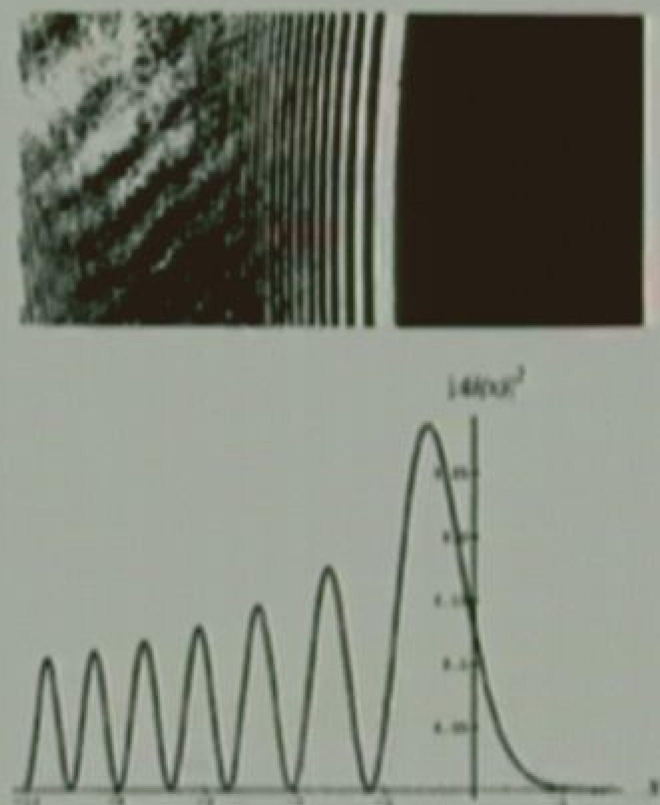

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Divergent Asymptotic Expansions

In 1850, in the first of several discussions having their genesis in Airy's caustic paper, Stokes noted that despite the fact that Airy's series converges, "...when [x] is at all large the calculation becomes exceeding laborious." [Stokes(1966b), p. 329] He was motivated, therefore, to express the Airy integral as a solution of a differential equation in a form that would exhibit "what terms are of most importance when x is large ...." [Stokes(1966b), p. 331] Stokes was aimed for physical understanding—a desire to display in the most perspicuous fashion *those structures or features that dominate the phenomenon of interest*, namely, the rainbow.

Divergent Asymptotic Expansions

- "After many trials I at last succeeded in putting Mr Airy's integral under a form from which its numerical value can be calculated with extreme facility when [x] [the distance from the geometrical caustic] is large, whether positive or negative, or even moderately large." [Stokes(1966b), p.330]

Divergent Asymptotic Expansions

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- "Moreover the form of the expression points out, without any numerical calculation the law of the progress of the function when [x] is large." [Stokes(1966b), pp. 330]

Divergent Asymptotic Expansions

Had Stokes had large supercomputers at his disposal, there still would be compelling reasons for engaging in this asymptotic investigation—namely, that such investigations can often highlight important mathematical structures that are hidden in (or obscured by) the exact, convergent expansion.

Divergent Asymptotic Expansions

Stokes derived his asymptotic (divergent) expansion by finding the differential equation for which the Airy integral (1) was a solution and by using what is now known as the WKB method. He examined the general case where the argument can be complex and arrived at the following differential equation:

$$\frac{d^2u}{dz^2} - zu = 0. \tag{2}$$

Divergent Asymptotic Expansions

Stokes reasoned as follows (and, at base, this is the type of asymptotic reasoning behind the WKB method). Let us focus on large values of $|z|$ which reflects our interest in being able to describe the locations and intensities of the bows relatively far from the caustic. If we increase $|z|$ by a small increment δz, the proportionate increase of $|z|$ will be small. That is, for large $|z|$ we can effectively regard z as a constant. If we make this assumption, then equation (2) has the approximate solution:

$$u(z) \equiv Ai(z) \approx Ae^{-\frac{2}{3}z^{\frac{3}{2}}} + Be^{\frac{2}{3}z^{\frac{3}{2}}}, \qquad (3)$$

where A and B are arbitrary constants.

Divergent Asymptotic Expansions

"The approximate integral [(3)] points out the existence of circular functions . . . in the true integral." In fact, the approximate 'solution' (3) shows that when z is real and positive, the behavior of the solution to equation (2) can be exponential in character and when z is real and negative, its behavior will be oscillatory or trigonometric in character.

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Explaining Regularities: The Need for Singular Behaviour

Robert W. Batterman

Introduction

Idealization

Airy's Equation

Stokes on the Airy Integral

Asymptotic Expansions

Asymptotic Representations and the World

Remarks on the Philosophy of Mathematics

Airy vs. Stokes

- Airy's convergent series:

$$
\begin{aligned}
Ai(z) &= A\left\{1 + \frac{9z+3}{2\cdot 3} + \frac{9^2 z^6}{2\cdot 3\cdot 5\cdot 6} + \frac{9^3 z^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \cdots\right. \\
&\quad + B\left\{z + \frac{9z^4}{3\cdot 4} + \frac{9^2 z^7}{3\cdot 4\cdot 6\cdot 7} + \frac{9^3 z^{10}}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} + \cdots\right.
\end{aligned}
$$

- Stokes' divergent series:

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\begin{aligned}
Ai(z) &\equiv Cz^{-\frac{1}{4}} e^{-2z^{\frac{3}{2}}}\left\{1 - \frac{1\cdot 5}{1\cdot 144 z^{\frac{3}{2}}} + \frac{1\cdot 5\cdot 7\cdot 11}{1\cdot 2\cdot 144^2 z^3} - \cdots\right. \\
&\quad + Dz^{-\frac{1}{4}} e^{2z^{\frac{3}{2}}}\left\{1 + \frac{1\cdot 5}{1\cdot 144 z^{\frac{3}{2}}} + \frac{1\cdot 5\cdot 7\cdot 11}{1\cdot 2\cdot 144^2 z^3} + \cdots\right\}
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Airy vs. Stokes

- Stokes' "solution" and, in particular, its leading terms, explicitly exhibits the oscillatory and exponential character of the "true integral." This provides a crucial component of our understanding of the physical phenomenon of interest. Our understanding of the patterns present in the rainbow is provided by the relatively *transparent* mathematical representation of these dominant characteristics.
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Asymptotic Representations

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- However, because the late terms of such series typically diverge, they have historically been taken to be inherently vague and without any coherent meaning.
- Contemporary understanding of asymptotic expansions rejects this skeptical assessment of the meaningfulness of asymptotic expansions, and, in fact, one can see, already in Stokes' own work, the seeds of this modern point of view.

Asymptotic Representations

- Stokes noted that "[a] semi-convergent [divergent] series (considered numerically, and apart from its analytical form) defines a function only subject to a certain amount of vagueness ...." [Stokes(1966a), p. 285]
- Stokes' parenthetical remark here seems to say that while there is numerical vagueness in asymptotic expansions, there may be exact formal information captured in the full asymptotic expansion.

Far End Control

R. B. Dingle (1973)

"[A]symptotic expansions are normal, immediately comprehensible, functions of their variables *in so far as functional form is concerned*. . . . A complete asymptotic expansion of a function $f(x)$ may therefore be defined as an expansion containing an asymptotic series which formally exactly obeys—throughout a certain phase sector—all those relations satisfied by $f(x)$ which do not involve any numerical value of x other than on the infinite circle $|x| \to \infty$: for instance,

1. Functional form as $|x| \to \infty$, i.e. boundary conditions on f and its derivatives at infinity.
2. Differential, difference and integral equations.
3. Relations involving other parameters incorporated, such as recurrence relations between orders." [Dingle(1973), pp. 19–20]

Far End Control

R. B. Dingle

- In maintaining that a complete asymptotic expansion for a given function $f(z)$ is not inherently vague or meaningless because of the divergence of the asymptotic series in the expansion, Dingle commits himself to providing *an interpretation of the late terms of the series.*

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- To put this another way, Dingle is committed to interpreting the infinities arise when one takes the singularities in the theory (the places of breakdown) seriously.
- This interpretation is based on a theorem of Darboux from 1878.

Far End Control

Asymptotic Interpretation

- Darboux's Theorem entails that the late terms in a convergent Taylor expansion depend only on the behavior of the function in the immediate neighborhood of the singularity closest to the point of expansion.

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- Dingle: This extends to asymptotic (that is, divergent) series as well.
- Almost without exception, the late terms of divergent series will have the same form.

Asymptotic Interpretation

The late terms ($n \gg 1$)

> *of any asymptotic power series will transpire to be expressible in a standard limiting form $(n + \text{constant})!/(\text{variable})^n$, the accuracy of this limiting representation increasing with n .... This conclusion ... is critically important in two ways: first, because it provides a valuable lead on how asymptotic power series and expansions containing them might best be defined; and second, because is shows that substantially a single theory of interpretation will apply equally to late terms of all such asymptotic series. [Dingle(1973), p. 4 with a slight change in notation]*

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- The singularities very often are associated with the essential or dominant physical features—features that are in some sense generic and not specific to any one single instance: the shock boundaries, the stable focal surfaces of light.
- Thus, even though there may not be *real* physical singularities, the mathematical singularities reflect essential features of the situation.

Far End Control

Asymptotic Representations and Idealizations

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- Most applications of mathematics to the description and explanation of physical phenomena do involve simplifications of a sort.
- Most applications involve idealizations in which certain parameters are treated asymptotically.
- Such asymptotic limiting idealizations are *essential* for obtaining genuine scientific explanation and understanding.

Asymptotic Representations and Idealizations

- Many (if not most) investigations of physical phenomena focus on patterns of physical behavior—those aspects of a system that replay themselves in varying circumstances. This reflects a kind of stability under various changes of the phenomenon of interest. (Details surrounding distinct instances don't matter.)

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- Asymptotic limits are most effective for examining what goes on at places where the 'laws' break down—that is, at places of singularities in the governing equations of the phenomena.
- These 'physical' singularities and their 'effects'—how they dominate the observed phenomena—are themselves best investigated through asymptotic representations of the solutions to the relevant governing equations. Stokes and the rainbow.

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  - The ray-theoretic singularities need to be dealt with by the wave theory—accomplished by Airy in deriving his integral equation.
  - However, the ray-theoretic singularity dominates the physical phenomenon.
- The *mathematical* singularities in the asymptotic equations constrain the structure of the solutions to the governing equations.

Asymptotic Representations and Idealizations

Alfred Seeger (2000) in the introduction to *Special Functions: A Unified Theory Based on Singularities*:
"In mapping a complex physical situation onto manageable mathematics, location and character of the singularities reflect the essentials of the situation, whereas the parameters not directly associated with the singularities usually carry incidental information, e.g. on the physical properties of the specific material under consideration. Recognizing this led to a new appreciation of the importance of asymptotic expansions and of the Stokes' phenomenon. On the mathematical side, it is the singularities of the differential equations resulting from the mapping that determine the character of the solutions."
[Slavyanov and Lay(2000), p. vii]

Far End Control

Conclusions

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- Such idealizations focus on the "essentials of the situation."
- In turn, our mathematical attention is focused on certain singularities in the relevant mathematical equations which require asymptotic investigations.
- Interplay between physics and mathematics mediated by singularities.

Wigner's Problem

The "unreasonable effectiveness of mathematics in the natural sciences."

- The "appropriateness of the language of mathematics for the formulation of the laws of physics" is a "miracle"—"a wonderful gift which we neither understand nor deserve." [Wigner(1967), p. 237]

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- The "appropriateness of the language of mathematics for the formulation of the laws of physics" is a "miracle"—"a wonderful gift which we neither understand nor deserve." [Wigner(1967), p. 237]
- Suggestion: The study of the asymptotics of various functions and the reasons for their usefulness in mathematical physics may very well help to dispel the appearance of the miraculous (and for some, the divine).

The Applicability of Mathematics

- Mark Steiner in, *The Applicability of Mathematics as a Philosophical Problem*, argues that contemporary mathematical physicists employ various (analogical) strategies in forming and discovering new theories about unobservable aspects of the world. He argues that these analogical strategies are fully anthropocentric—that is, that they depend for their success, upon humans having a special place in the world.
- He aims to maintain the miraculous nature of the applicability of mathematics to the world by demonstrating repeatedly that physicists employ analogies in discovery that are tied primarily to the formalism of existing theories and that cannot in any way be taken to be physically motivated.

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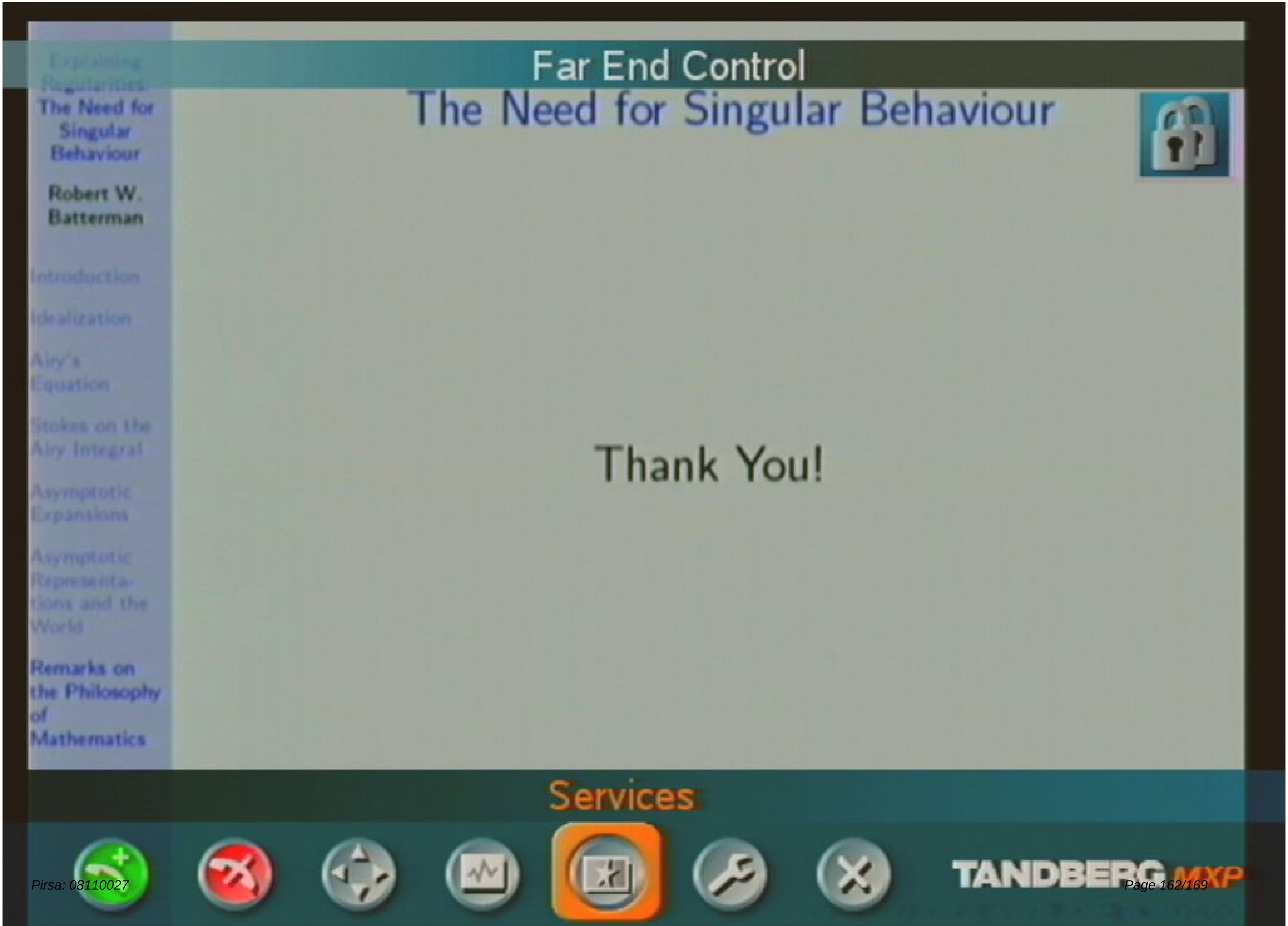
  - The very nature of many phenomena that are investigated—their stability under changes in detail, their association with singularities or places of breakdown of various theories—constrains the nature of those investigations.
  - The dominant and real features of phenomena require that we employ limiting idealizations in forming the mathematical equations with which we may represent the phenomena.

The Applicability of Mathematics

I hope that the discussion here leads us to question the anthropocentric role of the mathematician's appreciation for beauty (or formal analogy) as an important criterion for what arguably should be paradigm examples of mathematics' applicability to the world; namely, the extraordinary effectiveness of mathematical asymptotics in describing and explaining the physical world.

The Need for Singular Behaviour

Thank You!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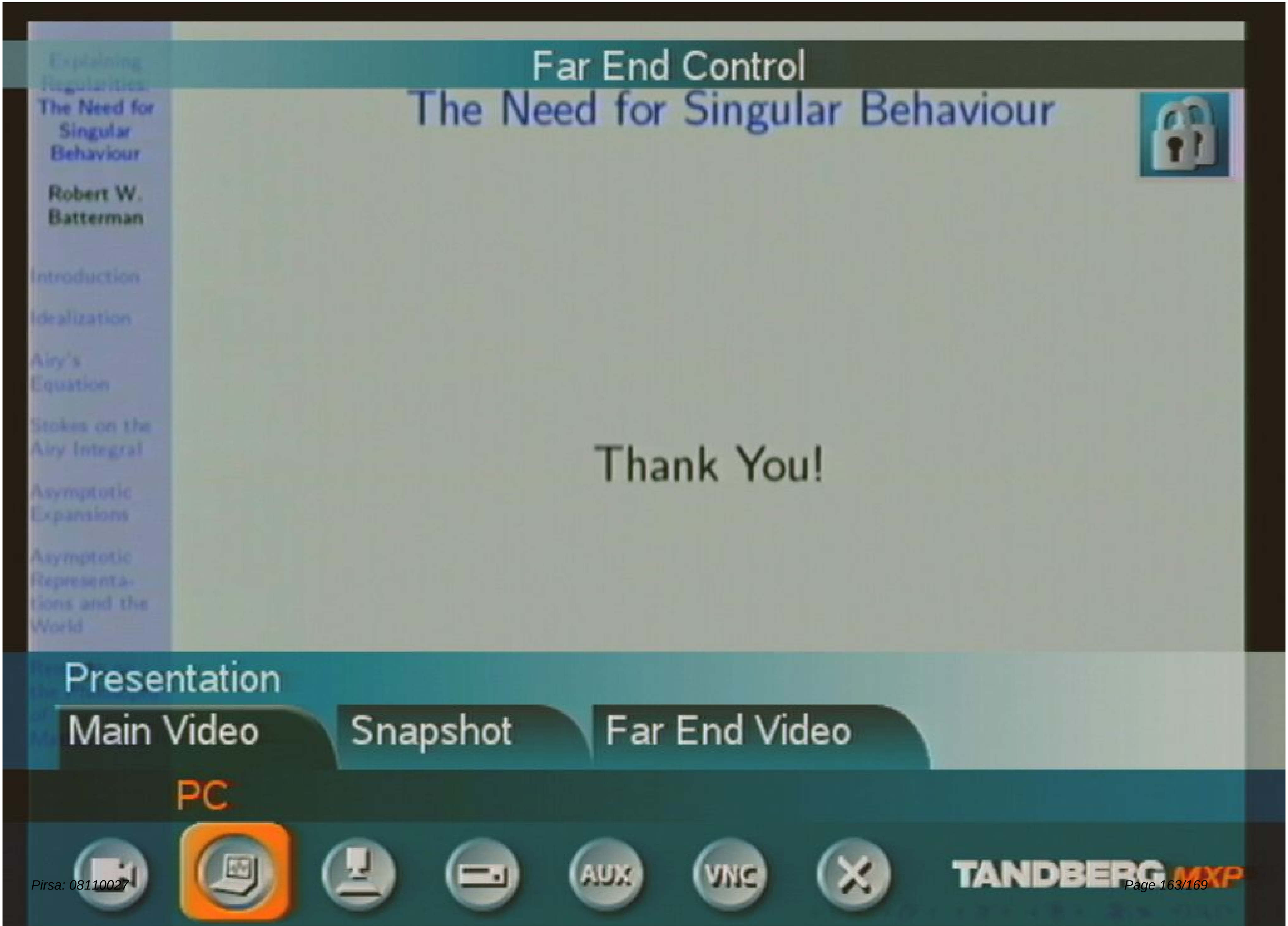
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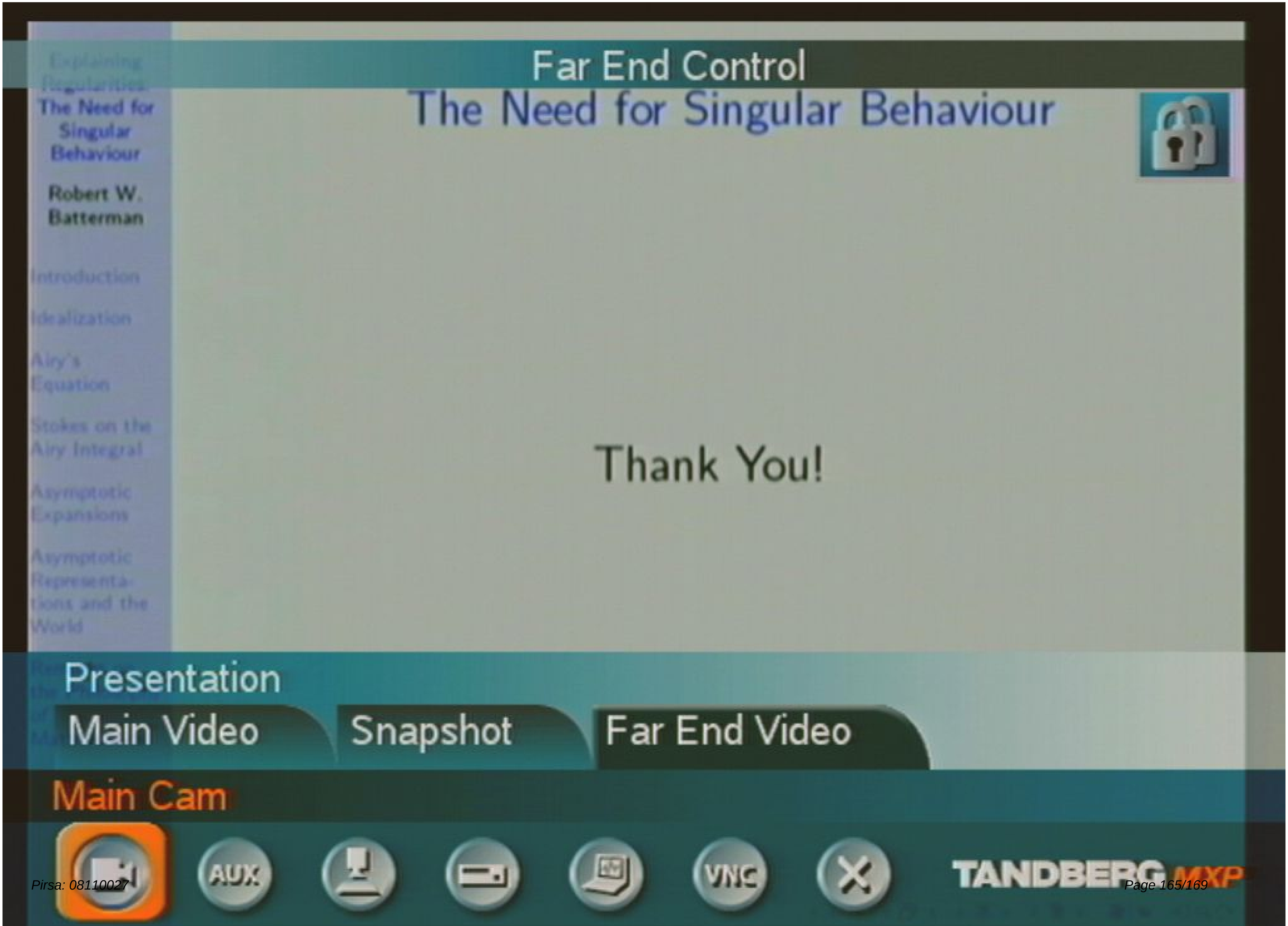
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