Title: Renormalization, an overview

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Abstract: We review how renormalization, born in quantum field theory has evolved into a rather universal tool to analyze the change of physical laws under scaling. Recent developments in non commutative geometry with hopefully potential applications to the quantization of gravity will be discussed.

Renormalization, an Overview

Vincent Rivasseau

LPT Orsay

Perimeter Institute, November 5th, 2008

First of all

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Cheers, Obama!

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Now let's renormalize the world together!

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- Perspectives and Conclusion (link NCQFT/Quantum Gravity)

Quantum Field Theory

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For me the soul of quantum field theory is renormalization.

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Noncommutative renormalization may allow a fresh look at these questions.

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Step towards understanding what is fundamental in QFT.

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In quantum gravity we expect still more beautiful and complicated variations to be woven...

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- Constructive field theory adresses the second problem. Species of Graphs → Species of Trees.

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This phenomenon always occur (either at the "infrared" or at the "ultraviolet" end of the renormalization group) in field theory on ordinary four dimensional space time (except possibly for extremely special models). This is somewhat frustrating.

Ordinary ϕ^4_4

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$d\mu_C$ Gaussian measure with propagator

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$$S_N(z_1, ..., z_N) = \int \phi(z_1)...\phi(z_N) d\nu(\phi).$$

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$$A_G(z_1,\cdots,z_N) = \int \prod_{v=1}^{n} d^d x_v \prod_{\ell} C(x_\ell, x'_\ell)$$

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The first two elements are quite universal. The third depends on the details of the model.

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At fixed scale attribution, some subgraphs play an essential role. They are the connected subgraphs whose internal lines all have higher scale index than all the external lines of the subgraph. Let's call them the "high" subgraphs. They form a single forest for the inclusion relation.

Locality principle

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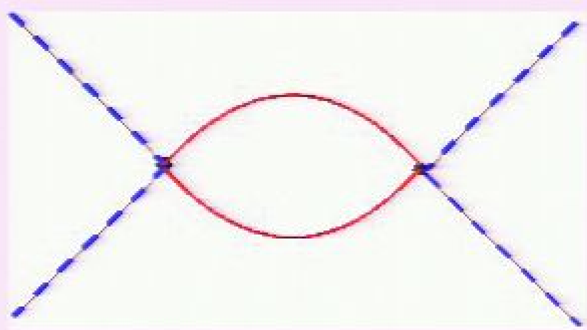

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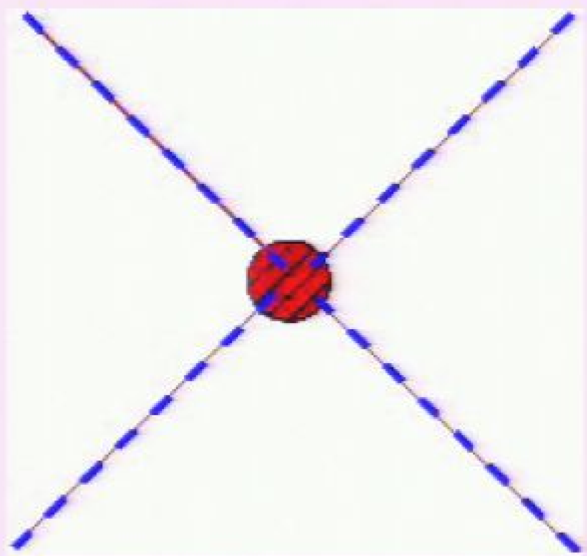

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In four dimension by the previous estimates of a single scale propagator C, power counting delivers a factor M^{2i} per line and M^{-4i} per vertex integration $\int d^4x$. There are $n-1$ "internal" integrations to perform to compare a high connected subgraph to a local vertex. For a connected ϕ^4 graph, the net factor is $2l(G) - 4(n(G)-1) = 4 - N(G)$ (because $4n = 2l + N$). When this factor is strictly negative, the sum is geometrically convergent, otherwise it diverges.

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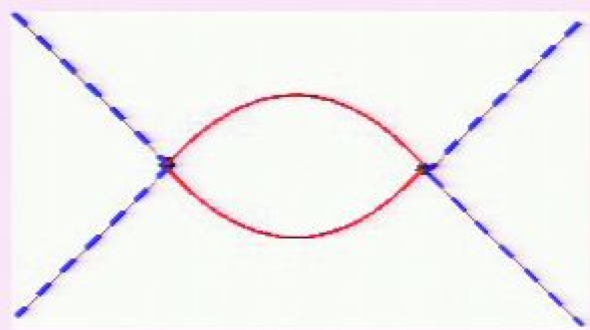

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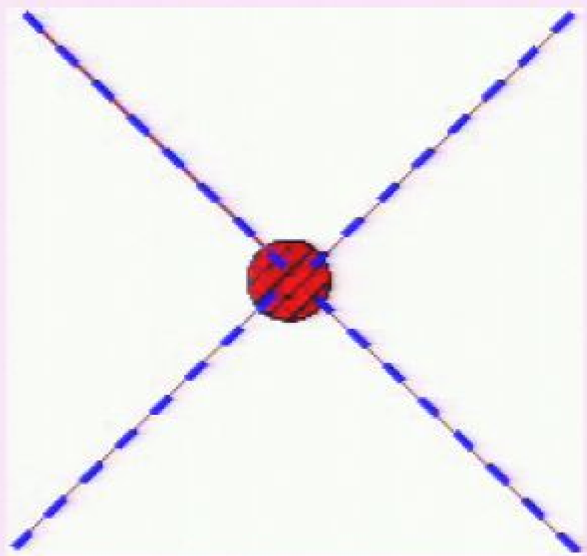

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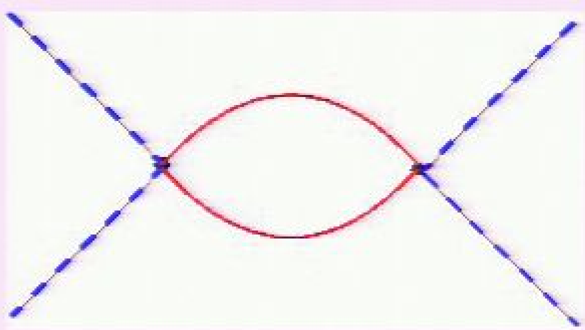

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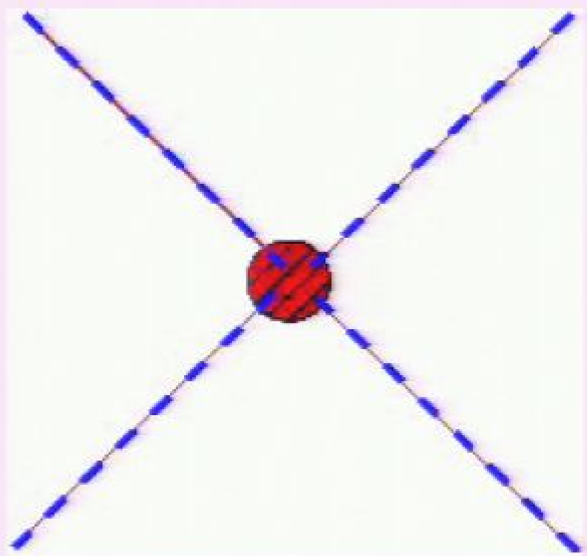

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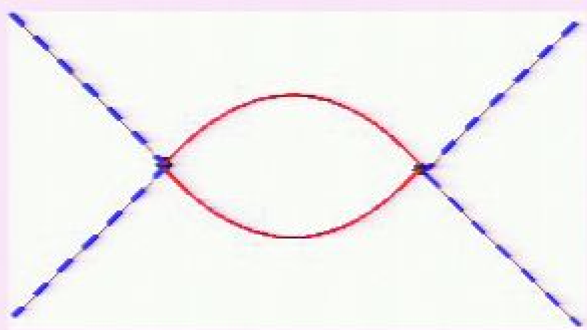

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For instance the previous graph diverges (logarithmically) because there are two line factors M^{2i} and a single internal integration M^{-4i}.

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Such models are called (perturbatively) renormalizable. But...

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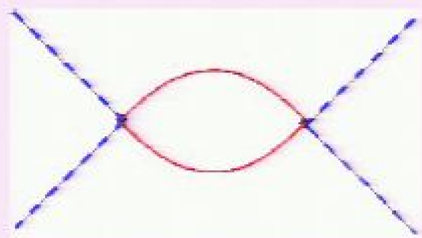

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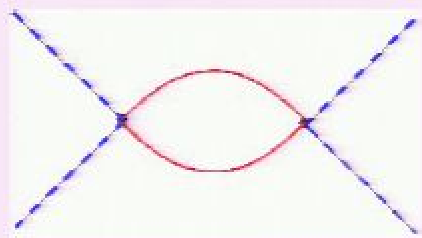

It gives the flow equation

$$-\lambda_{i-1} = -\lambda_i + \beta(-\lambda_i)^2, \quad \frac{d\lambda_i}{di} = +\beta(\lambda_i)^2,$$

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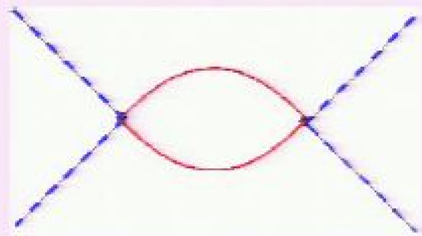

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This lead to the explanation by D. Gross and F. Wilczek of nuclear strong interactions through the QCD theory. However the counterpart of the lack of Landau ghost is infrared slavery, which prevents until now analytic understanding of the confinement.

Noncommutative geometry

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Noncommutative geometry, pioneered by A. Connes, is a large branch of mathematics that generalizes ordinary geometry. Ordinary smooth functions or observables form a commutative algebra under ordinary multiplication. For instance in classical mechanics observables are smooth functions on phase space. Quantum mechanics replaces this commutative algebra by a noncommutative algebra of operators. This is the first physical example of noncommutative geometry.

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Space-time itself could be of this type; at a certain yet unobservable scale new uncertainty relations could appear between length and width which would generalize Heisenberg's relations.

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In $d = 4$ every index m, n... is a pair $m = (m_1, m_2)$ ...

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- The relevant species is now the species of ribbon graphs.

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Examples of ribbon graphs

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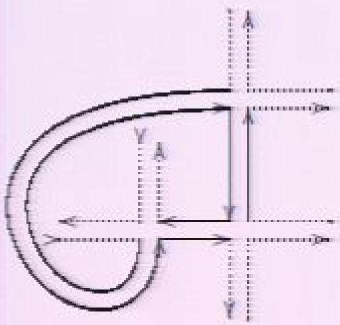

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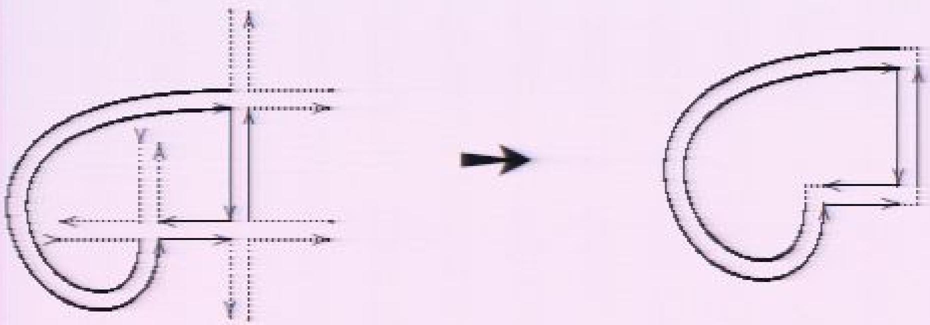

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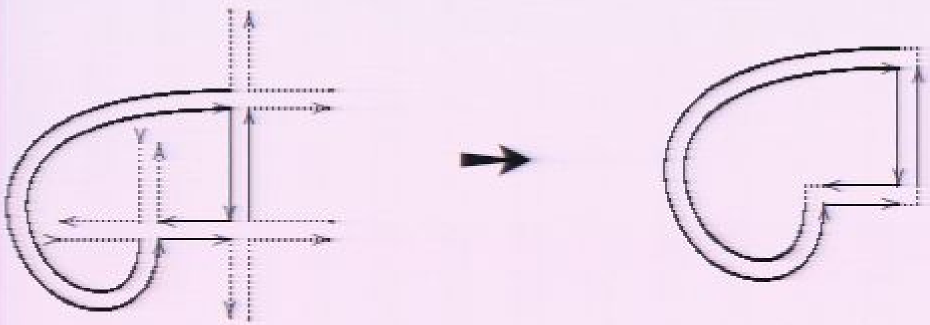

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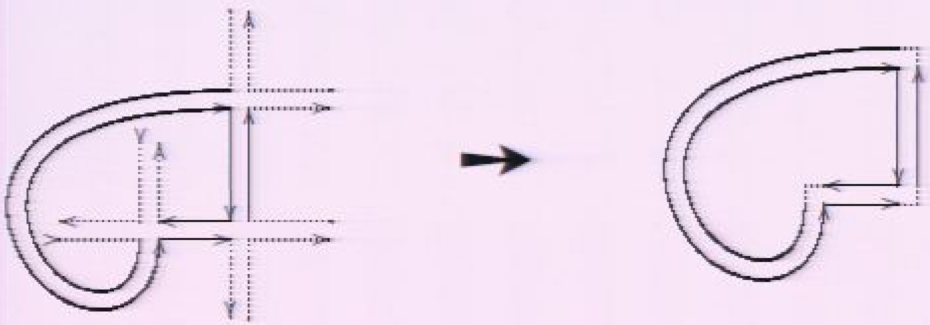

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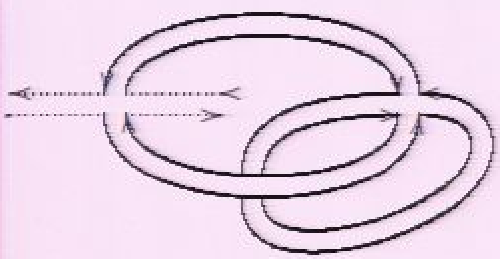

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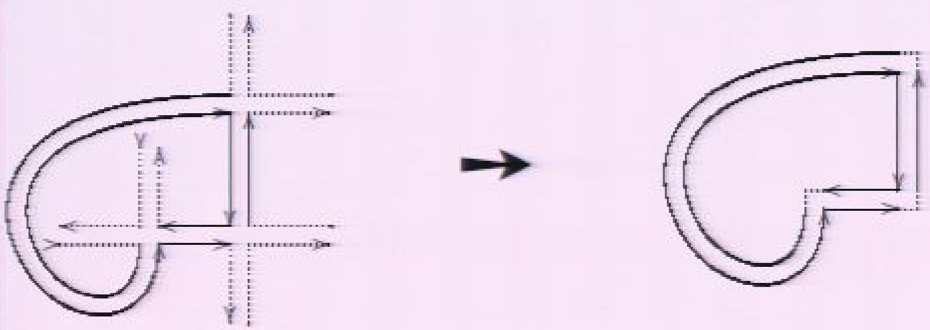

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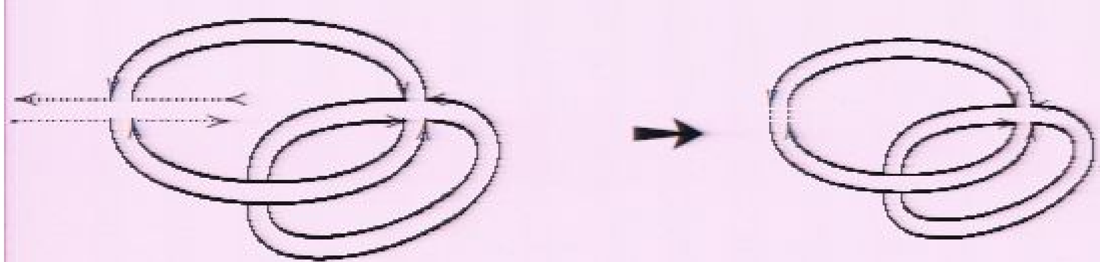

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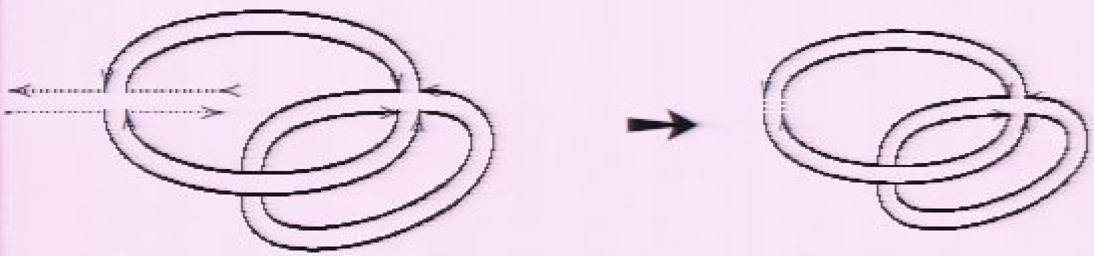

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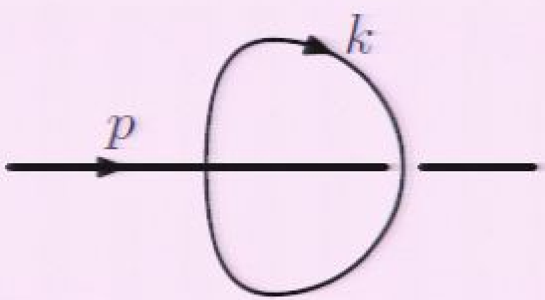

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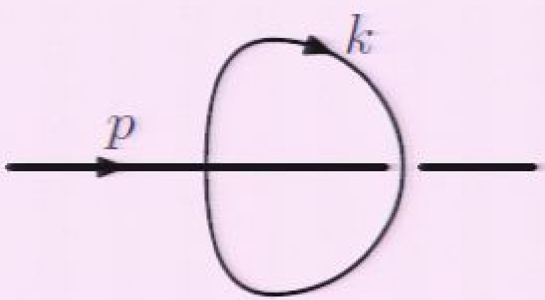

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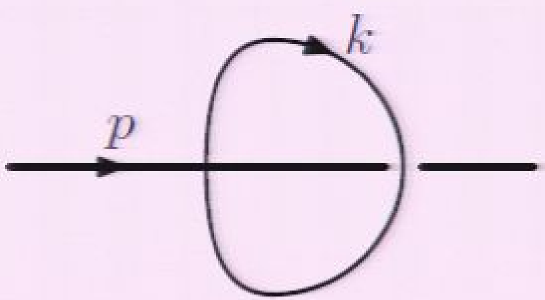

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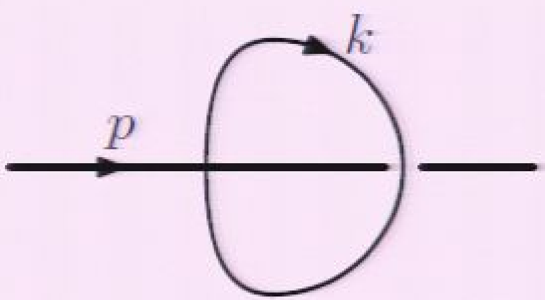

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- The theory cannot be renormalized.

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The Euclidean theory with an additional harmonic potential (we also say "vulcanized" for short:

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- If the covariance is $G(m,n;,k,l) = \delta_{mk}\delta_{nl} g(m,n)$ we say that the measure is independently distributed.
- If furthermore $g(m,n)$ is a constant, we say that the measure is independently identically distributed or iid.

Most specialists of random matrices only work on independent identically distributed, because this is the case where one can compute the large N limit ($\simeq$ law of large numbers for ordinary random variables) .

Among other aspects, the GW model is a theory of non independent, non identically distributed random matrices for which one can understand the large N limit.

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In the matrix base, the action is

$$S = (2\pi\theta)^2 \sum_{m,n,k,l} \left\{ \frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right\}$$

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$$G_{m,m+h;l+h,l} = \frac{\theta}{8\Omega}\int_0^1 d\alpha \, \frac{(1-\alpha)^{\frac{\mu_0^2\theta}{8\Omega}}}{(1+C\alpha)^2}\left(\frac{\sqrt{1-\alpha}}{1+C\alpha}\right)^{m+l+h}$$
$$\times \sum_{u=\max(0,-h)}^{\min(m,l)} \mathcal{A}(m,l,h,u)\ \left(\frac{C\alpha(1+\Omega)}{\sqrt{1-\alpha}(1-\Omega)}\right)^{m+l-2u}$$

with $\mathcal{A}(m,l,h,u) = \sqrt{\binom{m}{m-u}\binom{m+h}{m-u}\binom{l}{l-u}\binom{l+h}{l-u}}$ and $C(\Omega) = \frac{(1-\Omega)^2}{4\Omega}$.

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It is a Gaussian distribution for random matrices which is independent, non-identically distributed.

The propagator in direct space

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The propagator in x space has a parametric representation

$$G(x,y) = \frac{\theta}{4\Omega}\Big(\frac{\Omega}{\pi\theta}\Big)\int_0^\infty d\alpha\; e^{-\frac{\mu_0^2\theta}{4\Omega}\alpha} \frac{1}{(\sinh\alpha)^2}\exp\Big(-\frac{\Omega}{\theta\sinh\alpha}\|x-y\|^2 - \frac{\Omega}{\theta}\tanh\frac{\alpha}{2}(\|x\|^2+\|y\|^2)\Big)\,.$$

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which involves the Mehler kernel rather than the heat kernel.

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$$\int \prod_{i=1}^{4} d^4x^i \phi(x^i)\, \delta(x_1 - x_2 + x_3 - x_4) \; \exp\left(2\imath\theta^{-1}(x_1 \wedge x_2 + x_3 \wedge x_4)\right)$$

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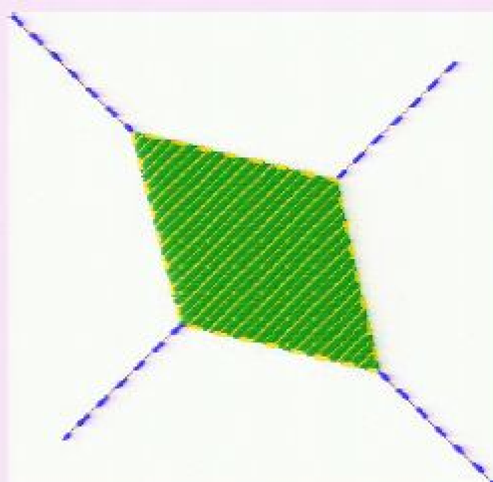

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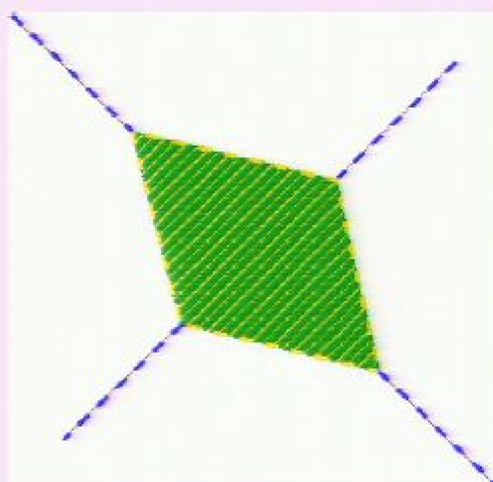

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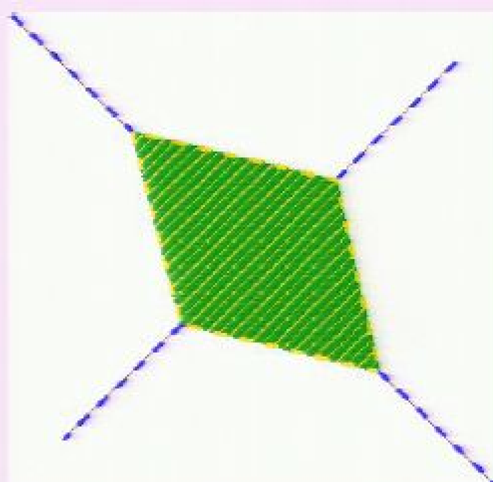

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As in the commutative case the first two elements are quite universal. The third depends on details of the model, such as dimension and interaction.

The new multiscale analysis

It relies again on a slicing of the propagator according to a geometric sequence:

$$G^i(x,y) = \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha \cdots \leqslant K M^{2i} e^{-c_1 M^{2i}\|x-y\|^2 - c_2 M^{-2i}(\|x\|^2+\|y\|^2)}$$

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The corresponding new renormalization group corresponds to a new mixture of the previous ultraviolet or short-distance and infrared or long-distance notions. There exists only a half direction for this RG.

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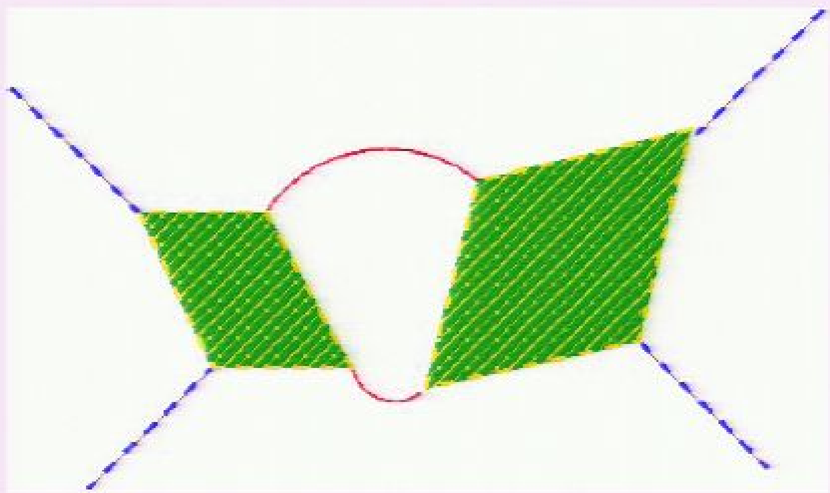

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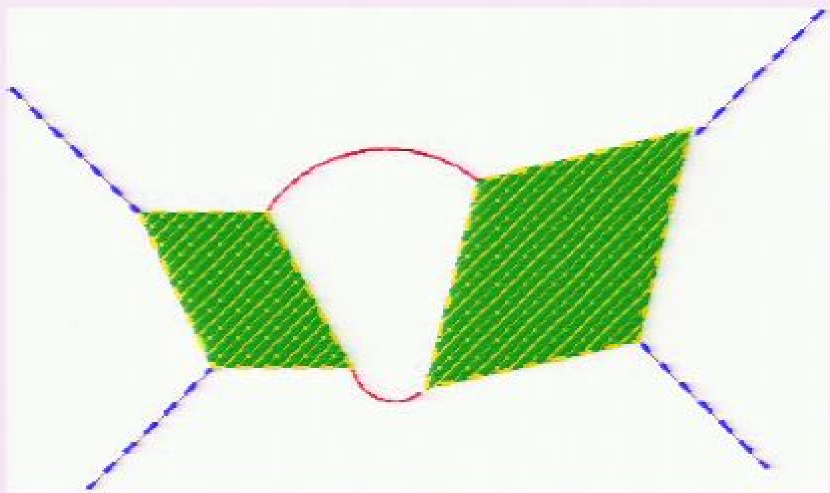

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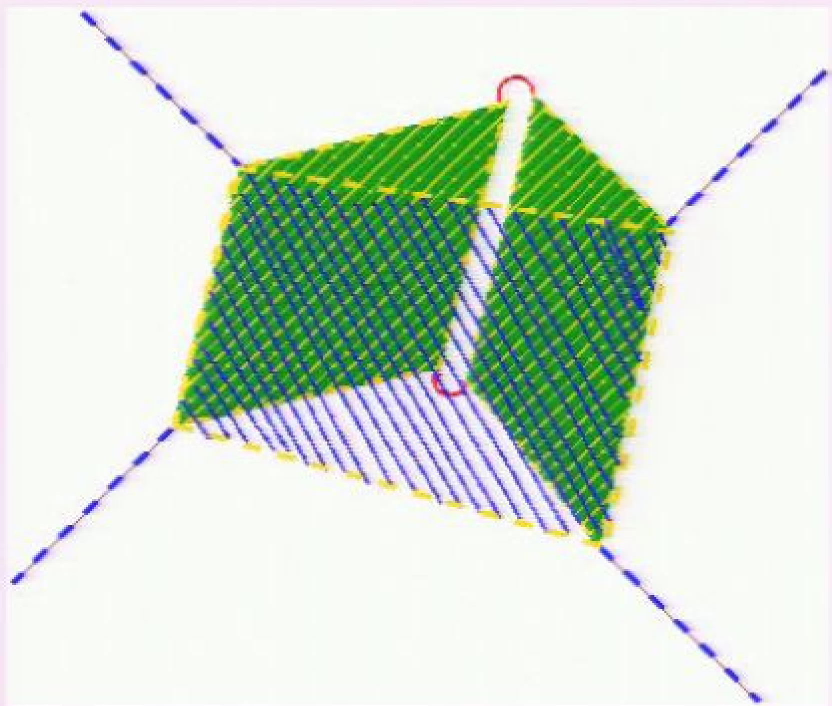

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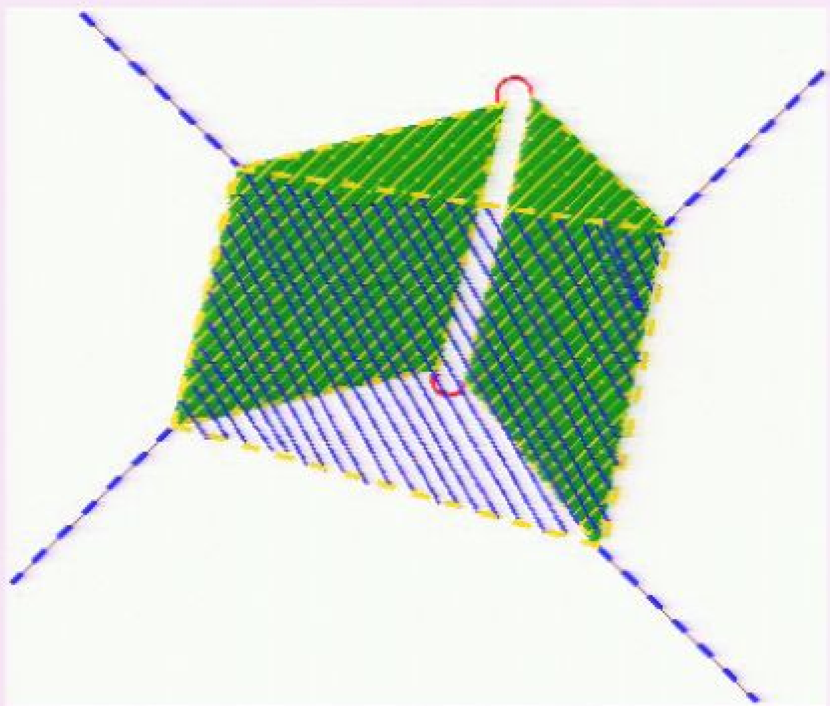

This principle applies only to planar graphs with a single external face.

The new Power Counting

It is best seen in the matrix base at $\Omega = 1$. Up to constants, at scale i one has to pays $M^{di/2}$ per internal ribbon face, and one earns M^{-i} per internal line.

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$$\omega = \frac{d}{2}(F - EF) - L = \left(2 - \frac{E}{2}\right) - 4g - 2(EF - 1) \text{ if } d = 4.$$

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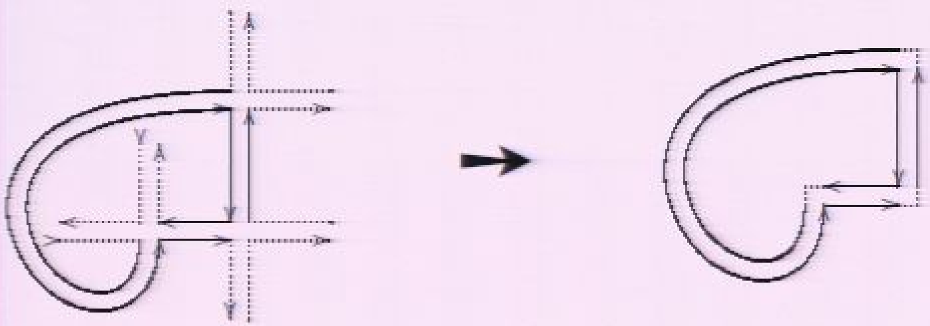

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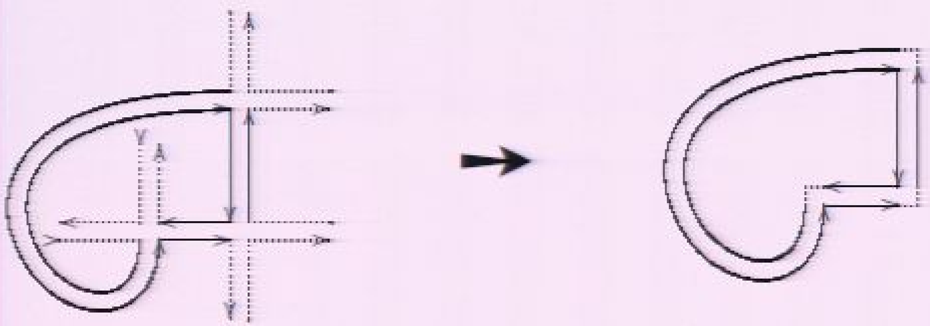

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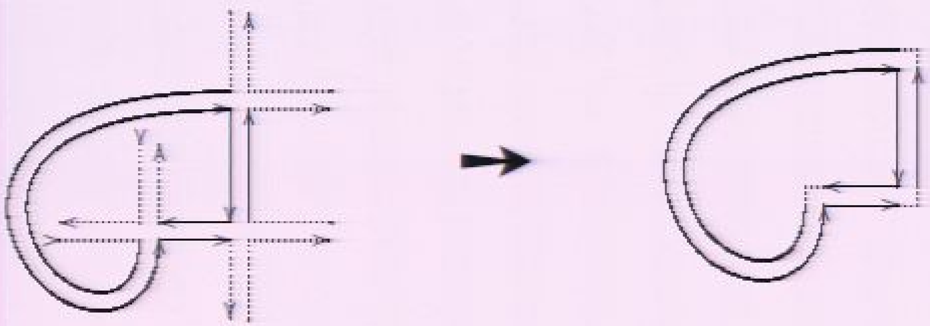

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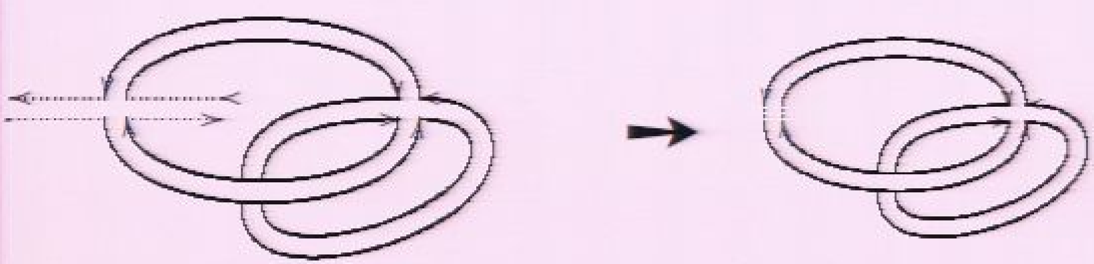

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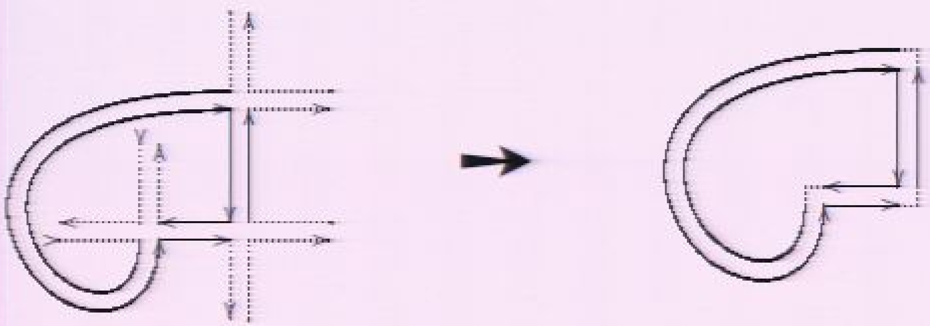

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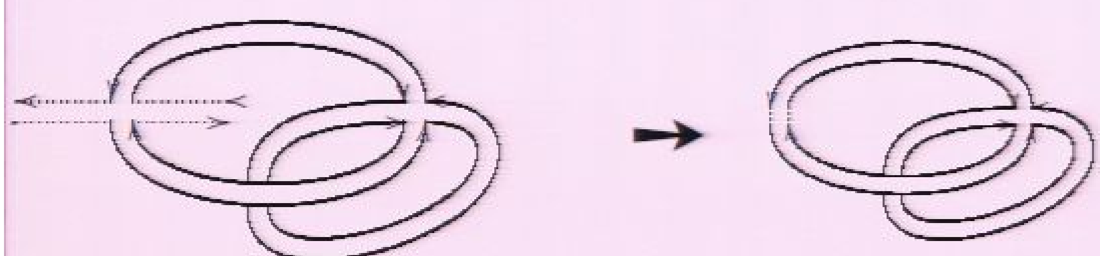

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The Moyality principle tells us that when the gap grows between internal and external lines in the sense of the new renormalization group slicing, these terms look like Moyal products. The corresponding counterterms are therefore of the form of the initial theory!

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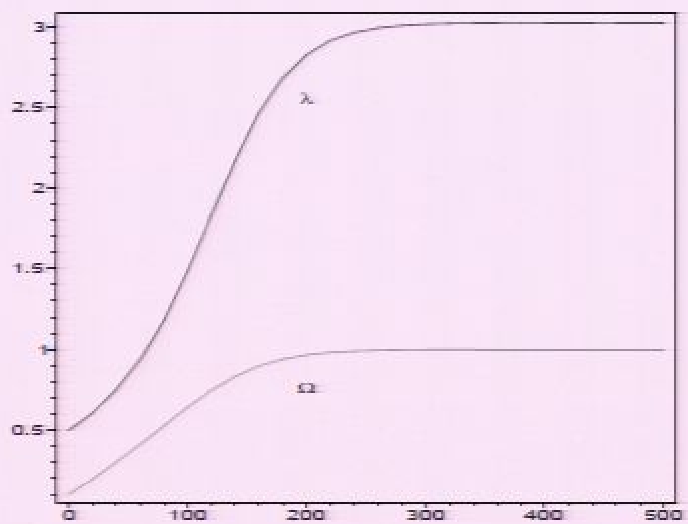
- The proof relies on techniques adapted from the 2D Thirring model.

The flows of λ and Ω

We must now follow two main parameters under renormalization group flow, namely λ and Ω. At first order one finds

$$\frac{d\lambda_i}{di} \simeq a(1-\Omega_i)\lambda_i^2 \ , \quad \frac{d\Omega_i}{di} \simeq b(1-\Omega_i)\lambda_i \ ,$$

whose solution is:



What happened

At $\Omega = 1$ (selfdual point) the field strength renormalization compensates the coupling constant renormalization so that $\lambda\phi^4$ remains invariant.

HU as a positive sum

The incidence matrix of a ribbon graph contains roughly speaking twice as many variables as for an ordinary graph.

$$HU_{G,\bar{v}}(t) = \sum_{I,J} \Omega^{k_{IJ}-2g} (Pf_{IJ})^2 \prod_{\ell \in I} t_\ell \prod_{\ell' \in J} t_{\ell'}$$

$$k_{IJ} = |I| + |J| - L - F + 1$$

where Pf_{IJ} is the Pfaffian of a certain antisymmetric matrix with integer entries where lines and columns corresponding to two sets I and J have been deleted.

The Forest Formula or "constructive swiss knife"

Let F be a smooth function of $n(n-1)/2$ line variables x_ℓ, $\ell = (i,j)$, $1 \le i < j \le n$. The BKAR forest formula states

$$F(1,...,1) = \sum_{\mathcal{F}} \Big\{ \prod_{\ell \in \mathcal{F}} [\int_0^1 dw_\ell] \Big\} \Big\{ \prod_{\ell \in \mathcal{F}} \frac{\partial}{\partial x_\ell} F \Big\} [x^{\mathcal{F}}(\{w\})], \text{ where}$$

- the sum over $\mathcal{F}$ is over all forests over n vertices,
- the "weakening parameter" $x_\ell^{\mathcal{F}}(\{w\})$ is 0 if $\ell = (i,j)$ with i and j in different connected components with respect to $\mathcal{F}$; otherwise it is the infimum of the $w_{\ell'}$ for ℓ' running over the unique path from i to j in $\mathcal{F}$.
- Furthermore the real symmetric matrix $x_{i,j}^{\mathcal{F}}(\{w\})$ (completed by 1 on the diagonal $i = j$) is positive.

Some recent developments, II

- Computation of beta functions and flows for new NC models (J. Ben Geloun, R. Gurau, A. Tanasa, V. R, arXiv:0805.2538, arXiv:0805.4362, arXiv:0806.3886)

Perspectives for future work

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- NCQFT lies "in between" quantum gravity and ordinary QFT
- 4D Moyal space is really 2+2. 4D NCQFT remains at the complexity level of 2D gravity. 3D and 4D gravity remain the big challenge ahead.
- I feel the discovery of the noncommutative RG associated to the GW model, which is nonlocal and mixes ordinary scales, is an encouraging step towards finding the renormalization group of quantum gravity.

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