Title: The Toric Code, Perturbed

Date: Nov 12, 2008 04:00 PM

URL: http://pirsa.org/08110020

Abstract: By storing quantum information in the degenerate ground state of a Hamiltonian, it is hoped that it can be made quite robust against noise processes. We will examine this situation, with particular emphasis on the Toric code in 2D, and show how adversarial effects, either perturbations to the Hamiltonian or interactions with an environment, destroy the stored information extremely quickly.

The Toric Code, Perturbed

A. Kay arXiv: 0807.0287

F. Pastawski, AK, N. Schuch, I. Cirac (in preparation)

1. Introduction

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I. Introduction

Quantum Memories

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Quantum Memories

– better scheme

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Quantum Memories

— better scheme

— fault tolerance

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Quantum Memories

- better scheme
- fault tolerance $0.1\% \leq p_c \leq 5\%$

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Quantum Memories

- better scheme
- fault tolerance $0.1\% \leq p_c \leq 5\%$

middle road?

take 2 state, $|\psi_0\rangle, |\psi_1\rangle$

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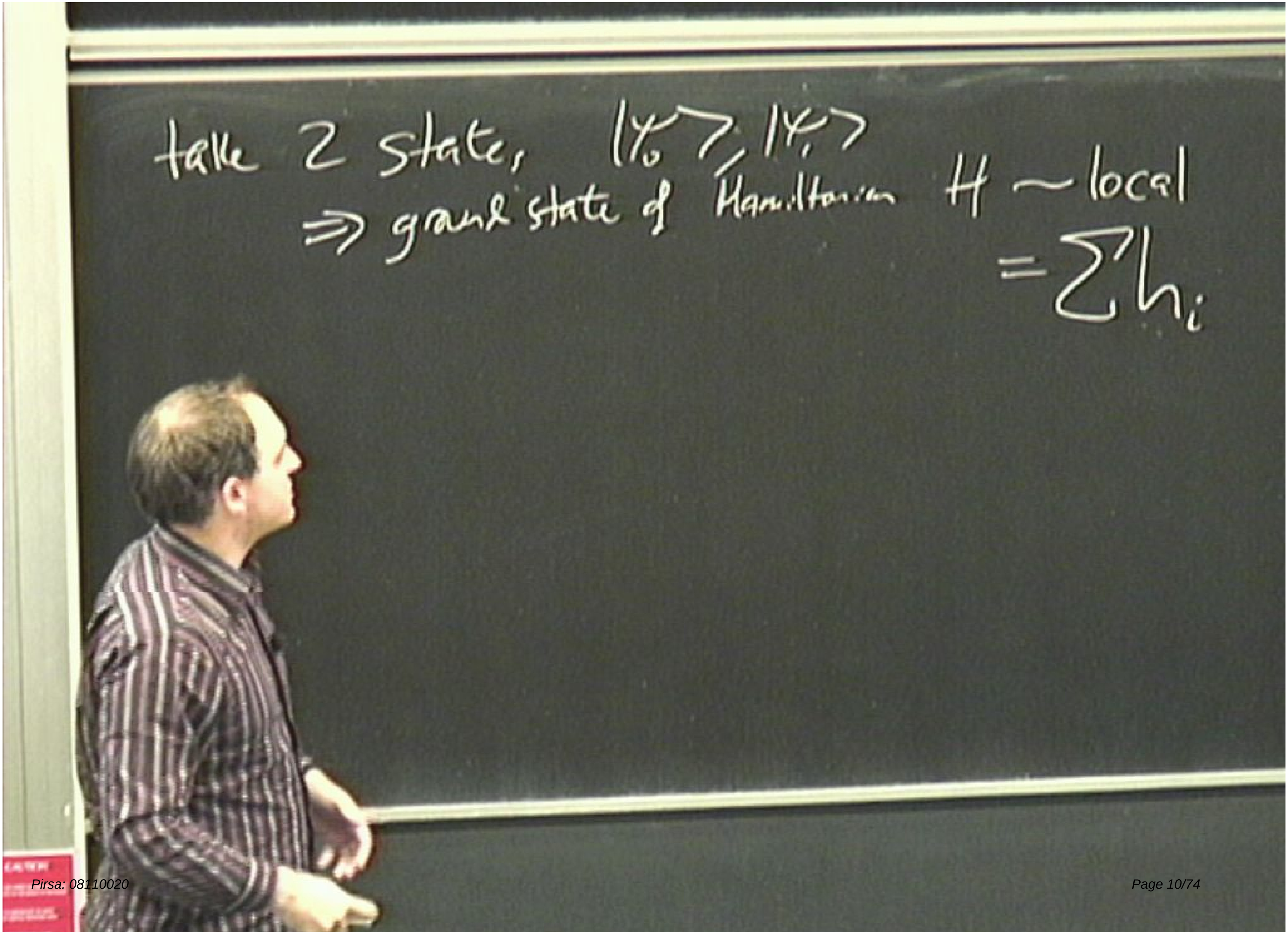
$\Rightarrow$ ground state of Hamiltonian H


take 2 state, $|\psi_0\rangle, |\psi_1\rangle$
$\Rightarrow$ ground state of Hamiltonian $H \sim$ local
$= \sum h_i$

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$\|h_i\| \leq 1$

take 2 state, $|\psi_0\rangle, |\psi_1\rangle$

$\Rightarrow$ ground state of Hamiltonian $H \sim$ local

$$= \sum h_i$$

$$\alpha|0\rangle + \beta|1\rangle \qquad \|h_i\| \leq 1$$

take 2 state, $|\psi_0\rangle, |\psi_1\rangle$

$\Rightarrow$ ground state of Hamiltonian $H \sim$ local

$$= \sum h_i$$

$$\|h_i\| \leq 1$$

$\alpha|0\rangle + \beta|1\rangle$

survival time

$\overline{|\psi_0\rangle}$ $\overline{|\psi_1\rangle}$

talk 2 state, $|\psi_0\rangle, |\psi_1\rangle$

$\Rightarrow$ grand state of Hamiltonian $H \sim$ local

$= \sum h_i$

$\|h_i\| \leq 1$

$\alpha|0\rangle + \beta|1\rangle$

survival time

ideal $t \sim \exp(N)$

$|\psi_0\rangle$ $|\psi_1\rangle$

take 2 state, $|\psi_0\rangle, |\psi_1\rangle$

$\Rightarrow$ ground state of Hamiltonian $H \sim$ local

$= \sum h_i$

$\| h_i \| \leq 1$

$\alpha|0\rangle + \beta|1\rangle$

survival time

ideal $t \sim \exp(N)$

"self correcting"

$|\psi_0\rangle \quad |\psi_1\rangle$

take 2 state, $|\Psi_0\rangle, |\Psi_1\rangle$

$\Rightarrow$ ground state of Hamiltonian $H \sim$ local

$= \sum h_i$

$\|h_i\| \leq 1$

$\alpha|0\rangle + \beta|1\rangle$

survival time

- ideal $t \sim \exp(N)$

"self correcting"

- potentially useful $t \sim \text{poly}(N)$

$|\Psi_0\rangle$ $|\Psi_1\rangle$

take 2 state, $|\psi_0\rangle, |\psi_1\rangle$

$\Rightarrow$ ground state of Hamiltonian $H \sim$ local

$= \sum h_i$

$\|h_i\| \leq 1$

$\alpha|0\rangle + \beta|1\rangle$

survival time

- ideal $t \sim \exp(N)$
  "self correcting"
- potentially useful $t \sim \mathrm{poly}(N)$
- useless $t \sim O(1)$

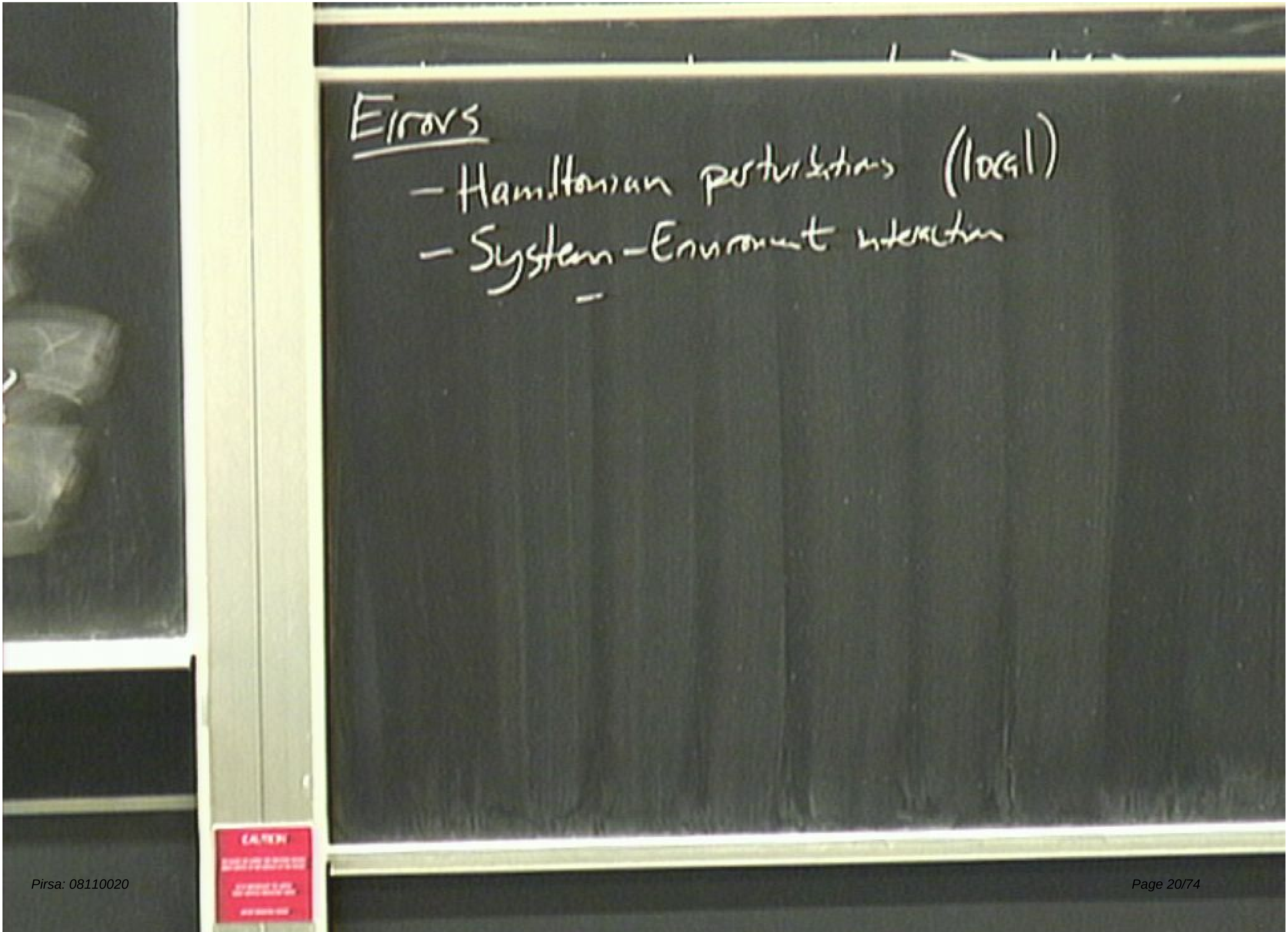
$|\psi_0\rangle$ $|\psi_1\rangle$

Errors

- Hamiltonian perturbations

Errors

– Hamiltonian perturbations (local)


Errors
- Hamiltonian perturbations (local)
- System-Environment interaction

Errors

- Hamiltonian perturbations (local)
- System-Environment interaction
  - Markovian

Errors

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  - Markovian
  - explicitly consider environment $\sim H'$ local $\sum h_i'$

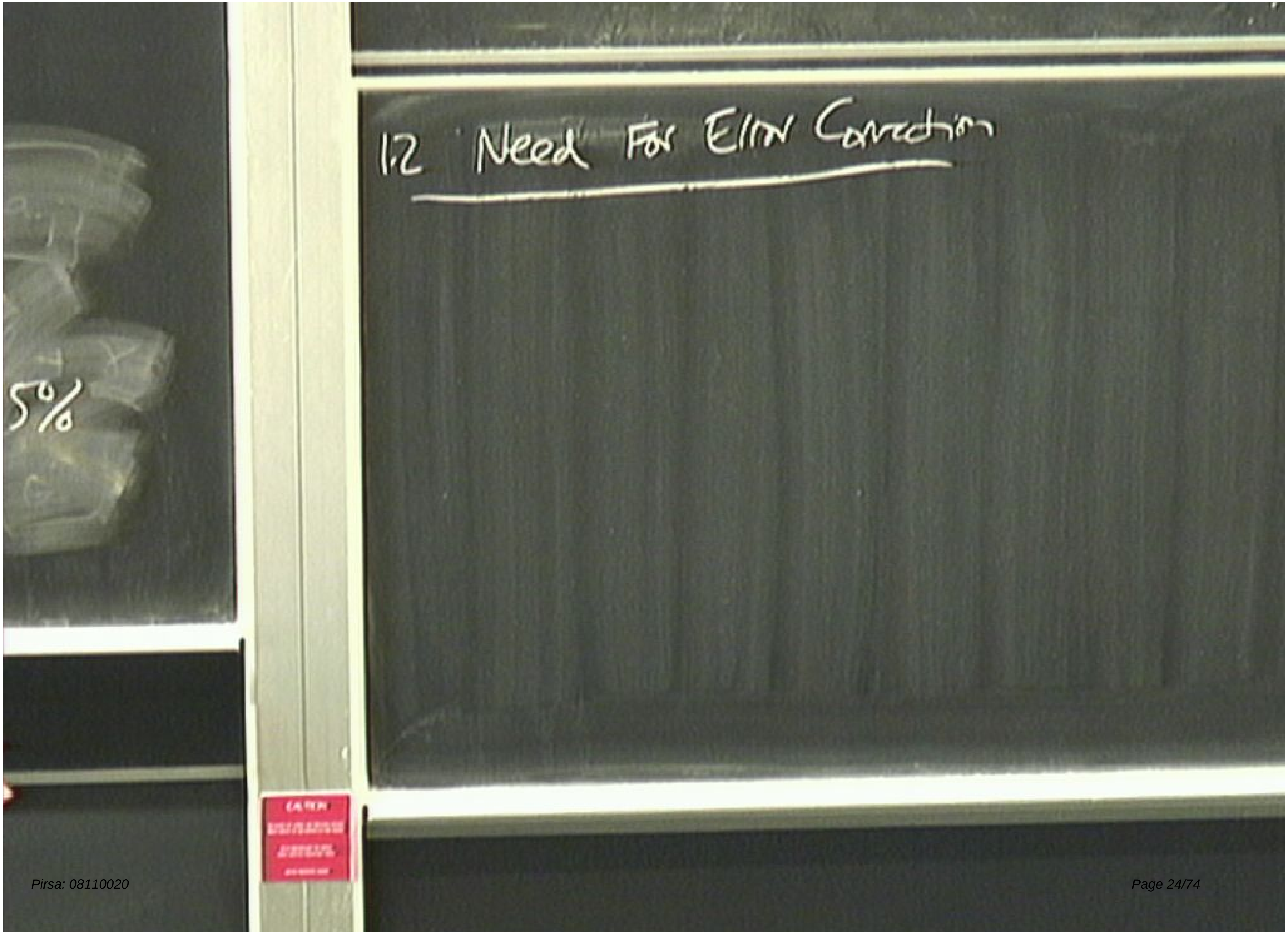
weak coupling to system $\|h_i\| < 1$

Errors

- Hamiltonian perturbations (local)
- System-Environment interaction
  - Markovian
  - explicitly consider environment — H' local $\sum h_i'$

weak coupling to system $\|h_i\| < 1$

Temperature analogue = $\dfrac{\text{total energy of } E}{\#\ \text{system spins}}$


12 Need For Error Correction
5%

1.2 Need For Error Correction

$|\psi\rangle$

1.2 Need For Error Correction

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle |\psi_i\rangle \qquad \langle \psi_i | \psi_j \rangle = \delta_{ij}$$

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$$\tilde{X} = \sum_{i=0}^{d-1} |i+1 \bmod d\rangle X_i \langle i|$$

1.2 Need For Error Correction

$$|\psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle |\psi_i\rangle \qquad \langle \psi_i | \psi_j \rangle = \delta_{ij}$$

$$\tilde{X} = \sum_{i=0}^{d-1} |i+1 \bmod d\rangle\langle i|$$

$$\langle \psi | \tilde{X} | \psi \rangle = 0$$

1.2 Need For Error Correction

$$|\psi\rangle = \sum_{i=0}^{d-1} \psi_i |i\rangle |\psi_i\rangle \qquad \langle\psi_i|\psi_j\rangle = \delta_{ij}$$

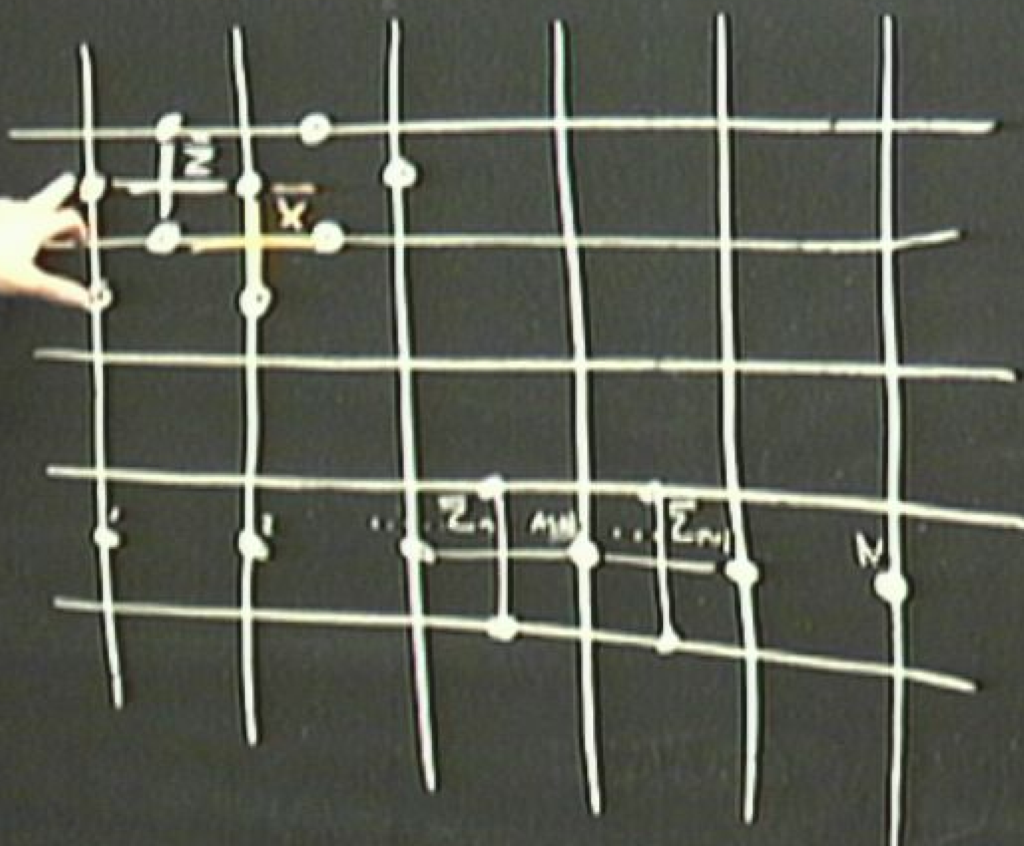
$$\tilde{X} = \sum_{i=0}^{d-1} |i+1 \bmod d\rangle\langle i| \qquad R\left(\tilde{X}|\psi\rangle\right) \longrightarrow |\psi\rangle$$

$$\langle\psi|\tilde{X}|\psi\rangle = 0$$

2. Hamiltonian Perturbations

2.1 2D Toric Code

$$H = -\Delta \sum \bar{X} + \bar{Z}$$



2. Hamiltonian Perturbations

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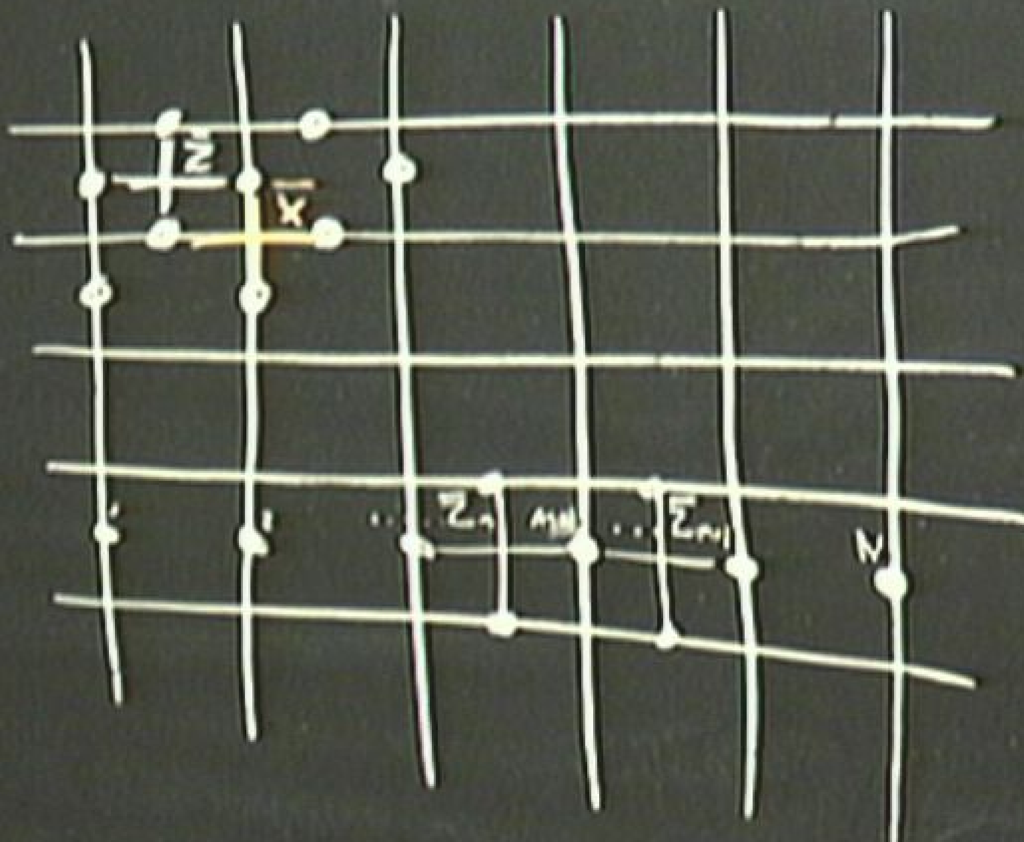

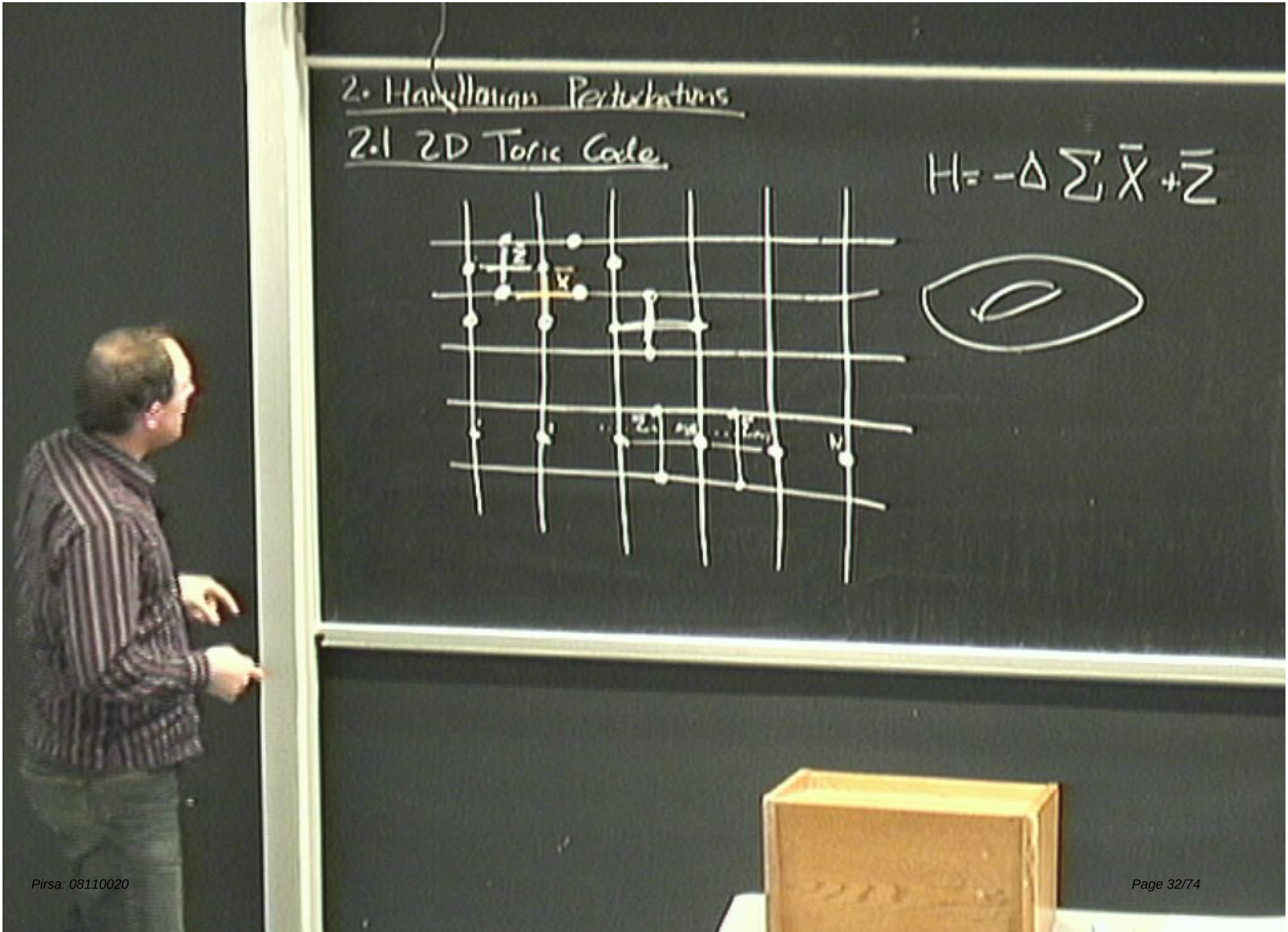
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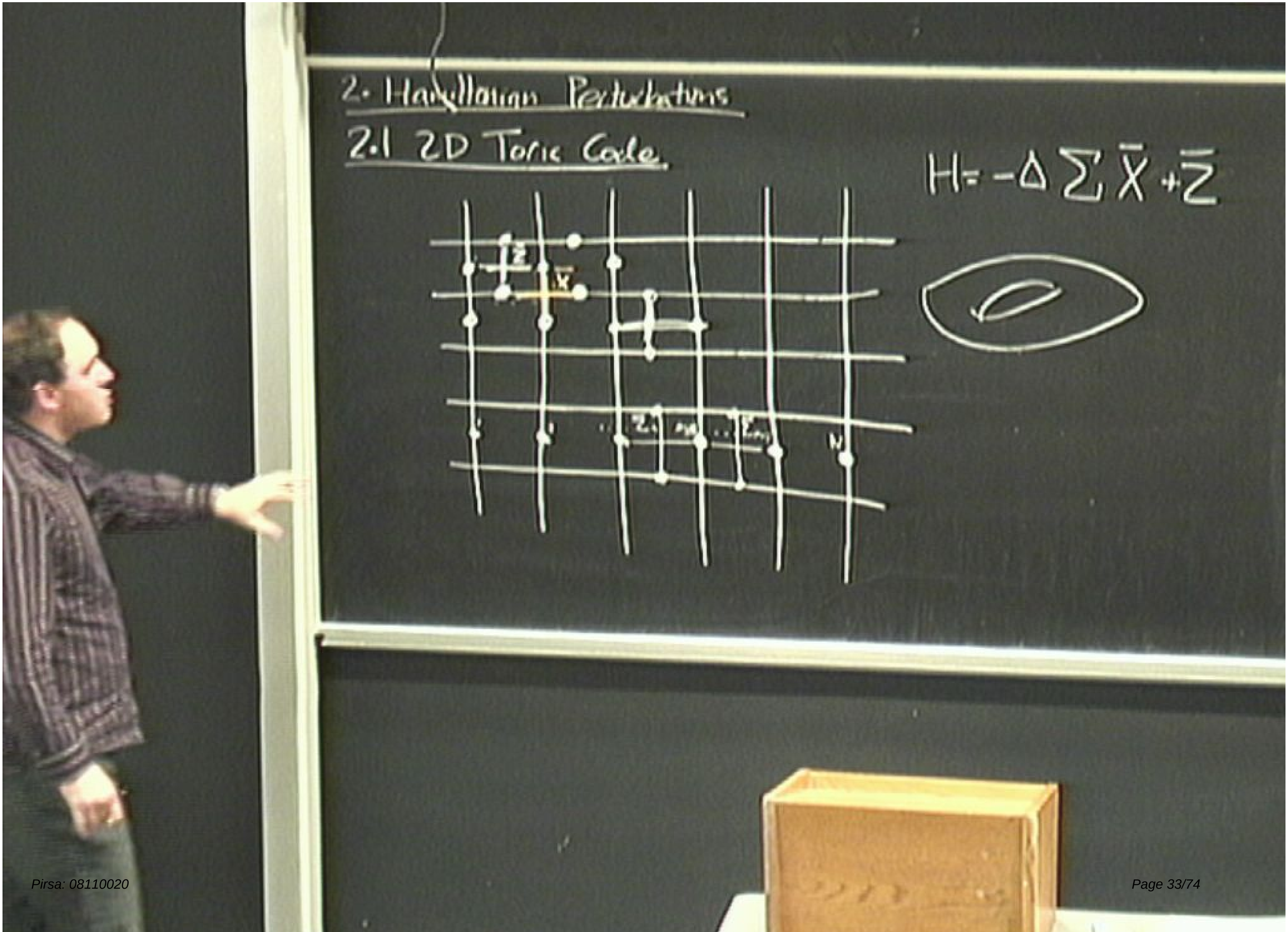
2.1 2D Toric Code
H = −Δ Σ X̄ + Z̄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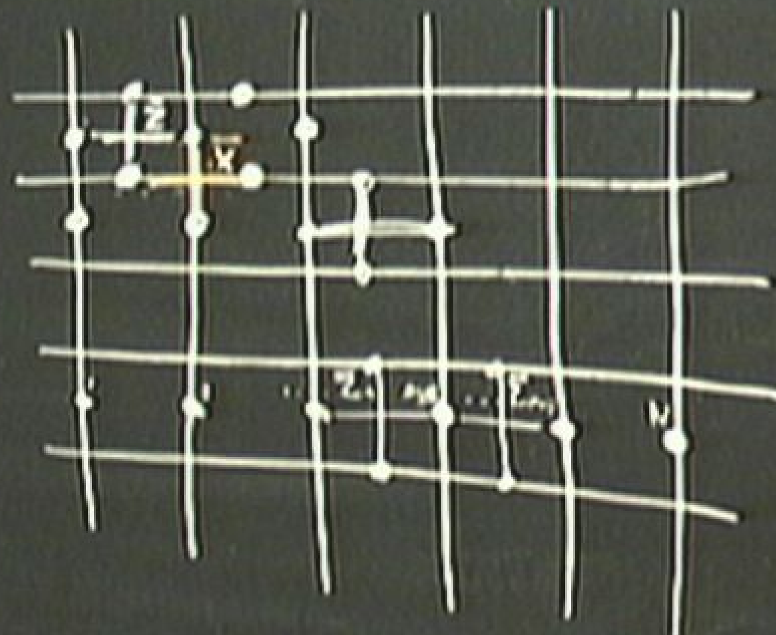

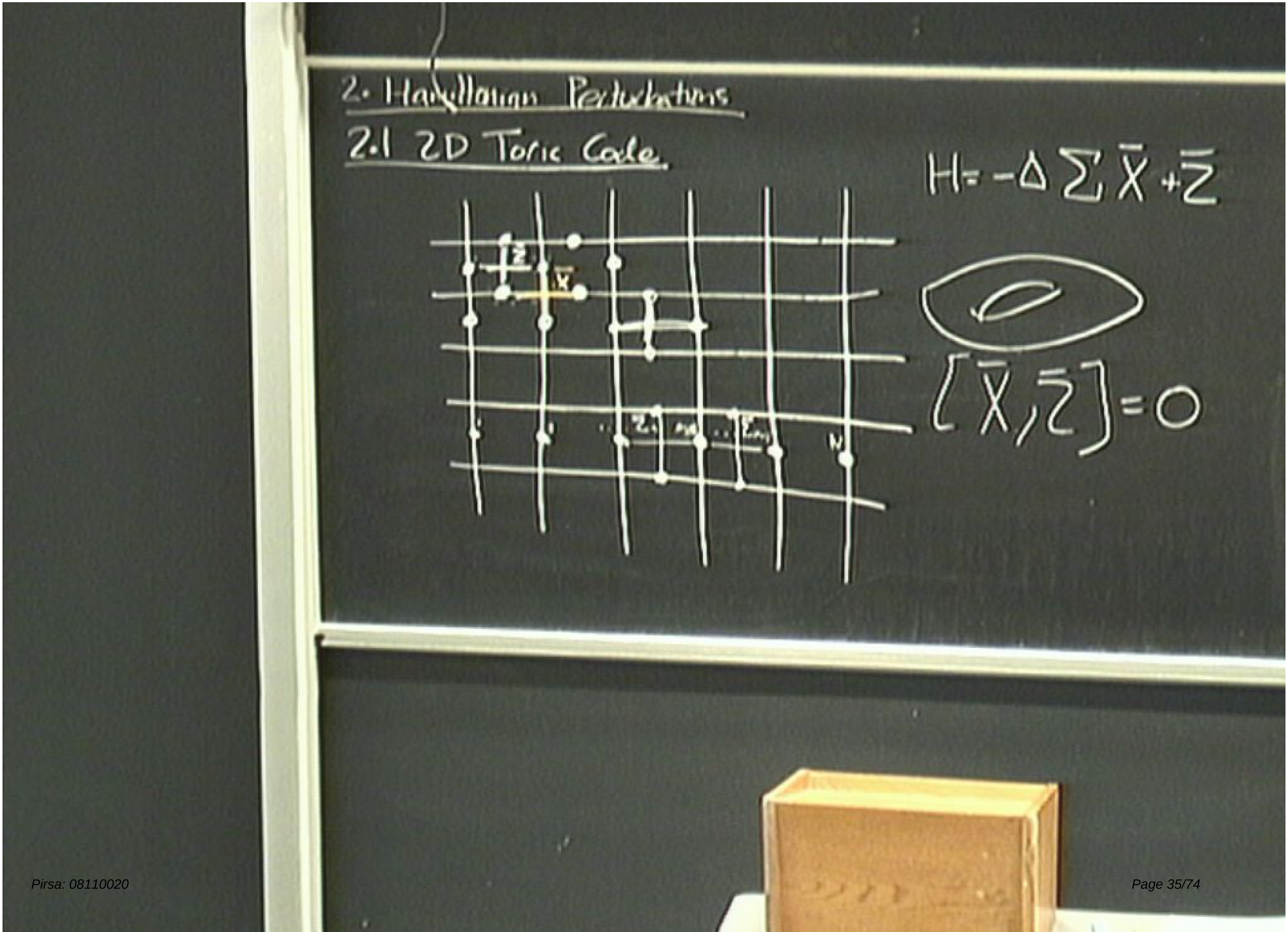
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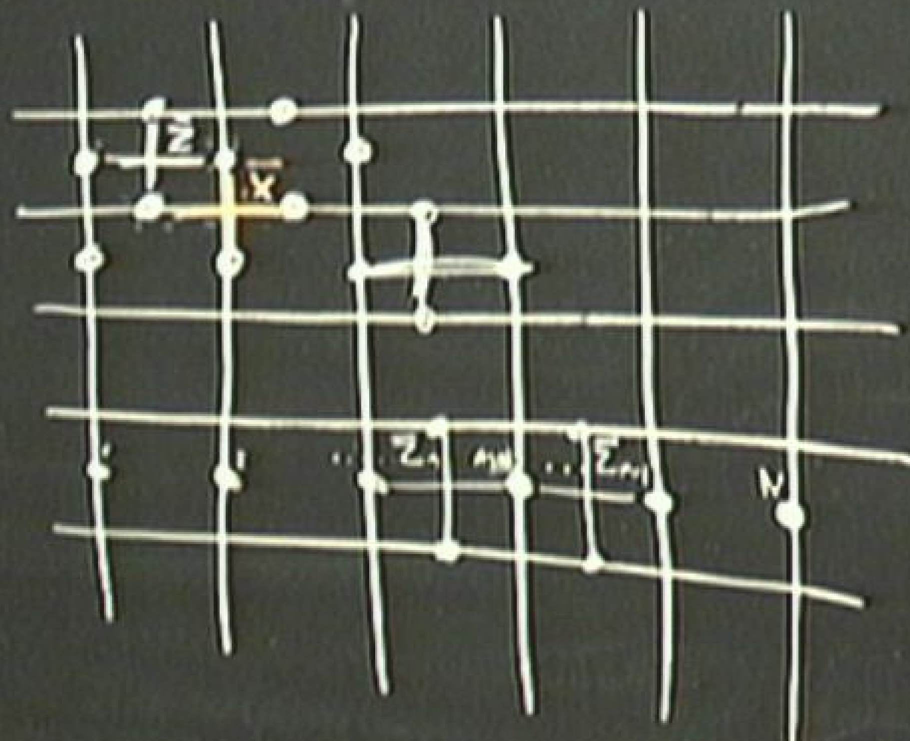
2. Hamiltonian Perturbations

2.1 2D Toric Code

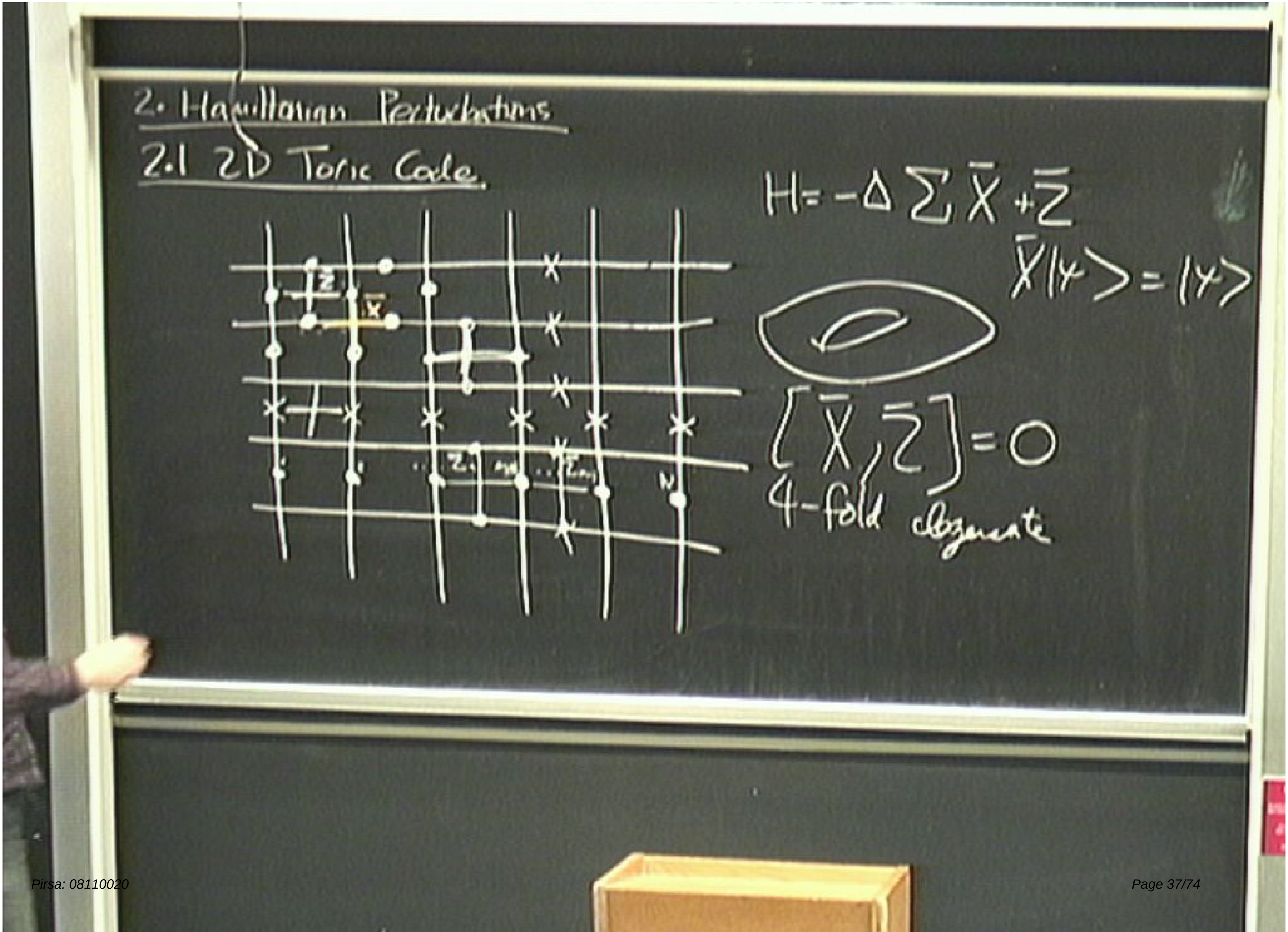


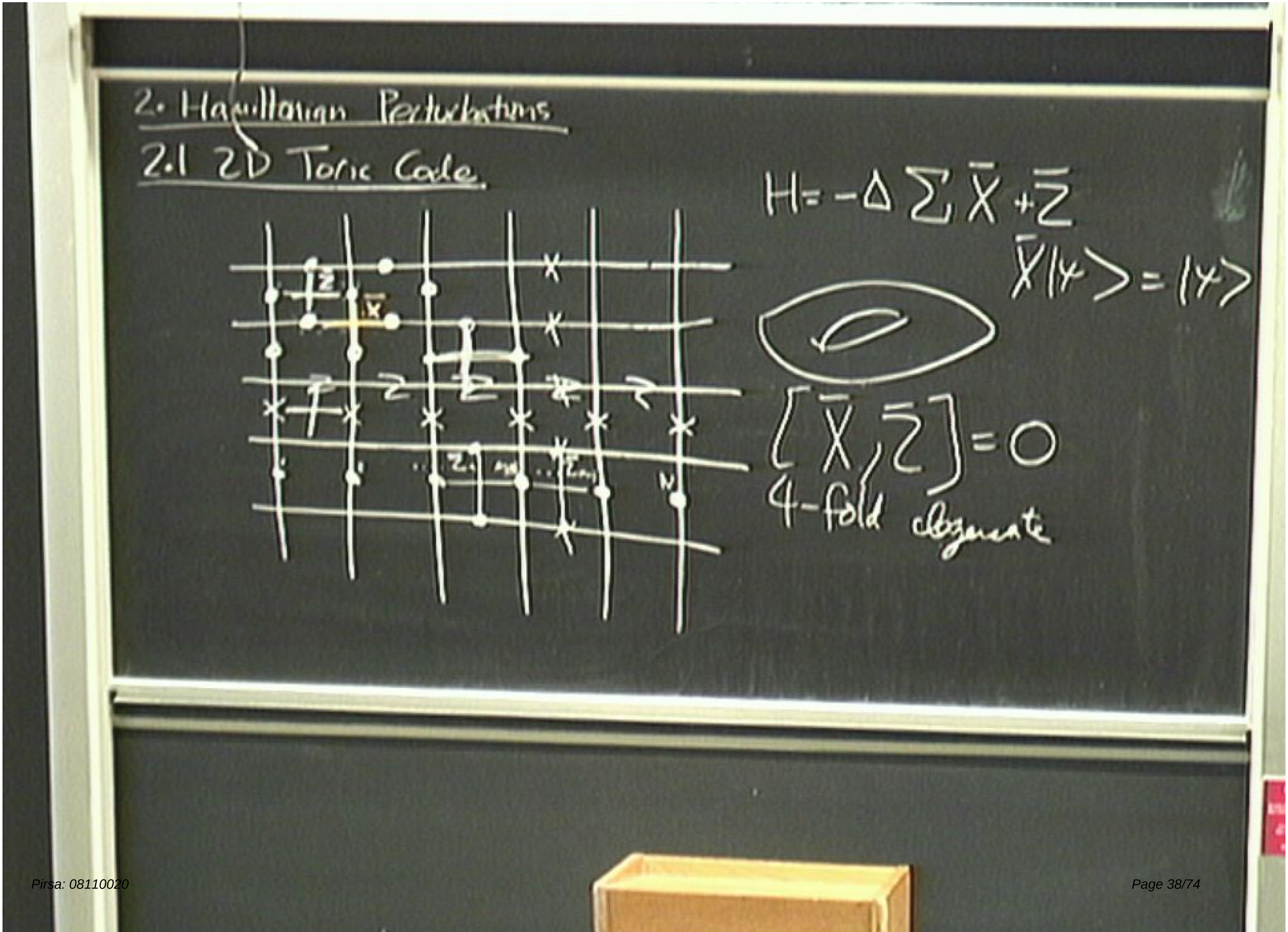
$$H = -\Delta \sum \bar{X} + \bar{Z}$$

$$\bar{X}|\psi\rangle = |\psi\rangle$$

$$[\bar{X}, \bar{Z}] = 0$$

4-fold degenerate


2. Hamiltonian Perturbations
2.1 2D Toric Code
H = −Δ Σ X̄ + Z̄
X̄|ψ> = |ψ>
[X̄, Z̄] = 0
4-fold degenerate
Z̄
X̄
N

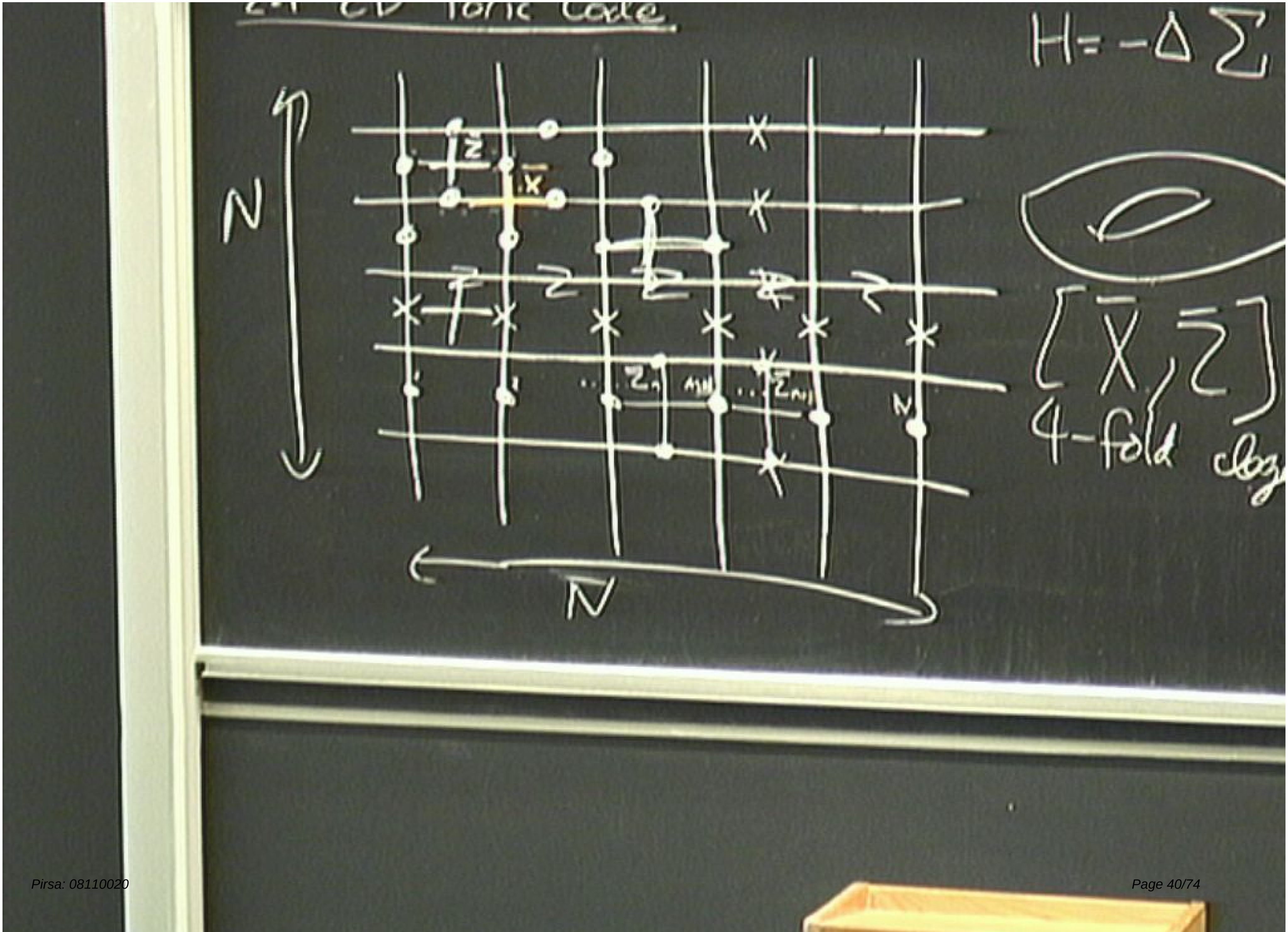


2. Hamiltonian Perturbations
2.1 2D Toric Code
$H = -\Delta \sum \bar{X} + \bar{Z}$
$\bar{X}|\psi\rangle = |\psi\rangle$
$[\bar{X}, \bar{Z}] = 0$
4-fold degenerate

$H + V$

$\delta \Sigma X$

$\langle \psi | X | \psi \rangle = 0$


H = -Δ Σ
N
[X̄, Z̄]
4-fold

Error Correction

$$H + V$$

$$\delta \Sigma X$$

perturbation theory $\Rightarrow \delta^N$

Error Correction

$\delta^N \updownarrow$ ——— $|1_L\rangle$

——— $|0_L\rangle$

$H + V$

$\delta \sum X$

$\frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$

perturbation theory $\Rightarrow \delta^N$ gap in g.s space

Error Correction

$H + V$

$\delta \sum X$

δ^N $|1_L\rangle$

$|0_L\rangle$ e^{Et}

$\frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$

perturbation theory $\Rightarrow$ δ^N gap in g.s. space

survival time δ^{-N}

Error Correction

$H + V$

$\delta \sum X$

δ^N — $|1_L\rangle$

$|0_L\rangle$ etc

$\frac{1}{\sqrt{2}}(|0_L\rangle + |1_L\rangle)$

perturbation theory $\Rightarrow \delta^N$ gap in g.s. space

survival time δ^{-N}

$\delta \ll 1$ $\quad \|V\| \ll 1$

Error Correction

$$H + V \begin{cases} \\ \delta \sum X \end{cases}$$

$\delta^N \updownarrow$ ——— $|1_L\rangle$

——— $|0_L\rangle$ etc

$\frac{1}{\sqrt{2}}(|0_L\rangle \pm |1_L\rangle)$

perturbation theory $\Rightarrow \delta^N$ gap in g.s space

survival time δ^{-N}

- $\delta \ll 1$ $\quad \|V\| \ll 1$
-

Error Correction

$|1_L\rangle$

$\delta^N \updownarrow$

$|0_L\rangle$ etc

$\frac{1}{\sqrt{2}}(|0_L\rangle \pm |1_L\rangle)$

$$H + V$$

$$V = \delta \sum X$$

perturbation theory $\Rightarrow \delta^N$ gap in g.s. space

survival time δ^{-cN}

- $\delta \ll 1 \qquad \|V\| \ll 1$
- grand state space changes

Error Correction

$$H + V \left(\delta \sum X \right.$$

$\delta^N \updownarrow$ $|1_L\rangle$

$|0_L\rangle$ etc

$\frac{1}{\sqrt{2}}(|0_L\rangle \pm |1_L\rangle)$

perturbation theory $\Rightarrow \delta^N$ gap in g.s. space

survival time δ^{-aN}

- $\delta \ll 1$ $\|V\| \ll 1$
- grand state space changes
-

Finite Corrections

$|1_L\rangle$

$\delta^N \updownarrow$

$|1_R\rangle$

$\frac{1}{\sqrt{2}}(|1_R\rangle \pm |1_L\rangle)$

$$H + V \qquad \delta\sum X$$

perturbation theory $\Rightarrow \delta^N$ gap in g.s. space

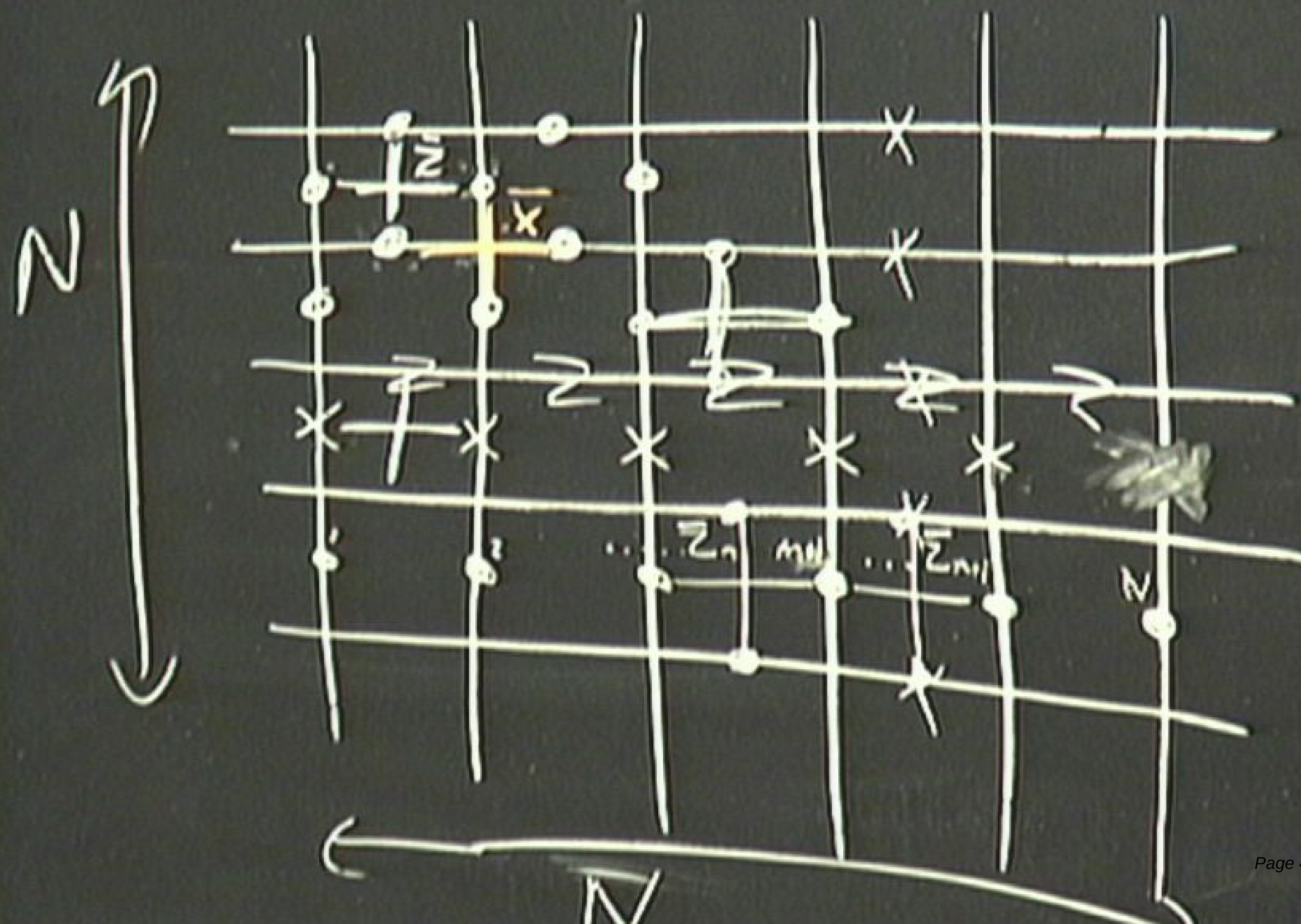
survival time $\delta^{-\alpha N}$

- $\delta \ll 1 \qquad \|V\| \ll 1$
- ground state space changes
- only tells you about state splitting

2. Hamiltonian Perturbations

2.1 2D Toric Code

$H=$



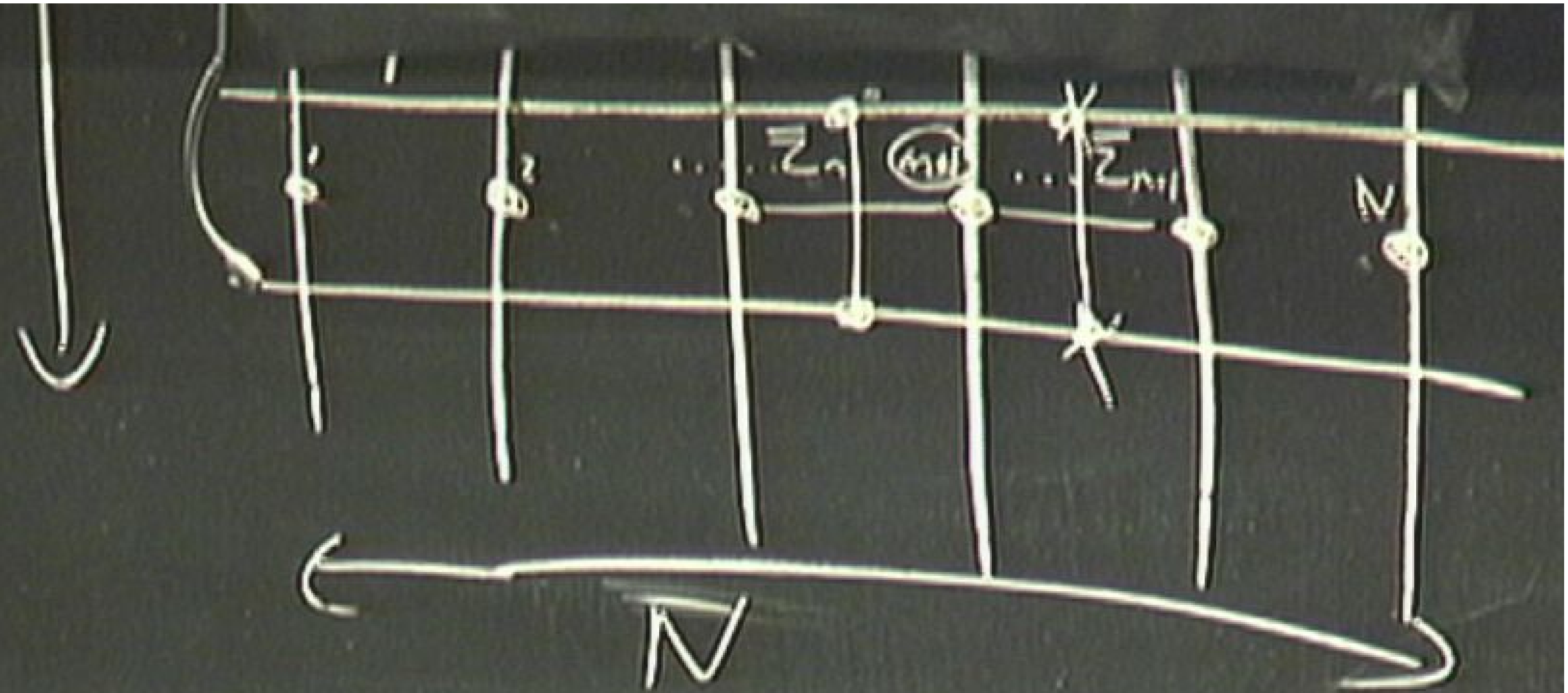
Nov 26 Catherine Kallin

Abby

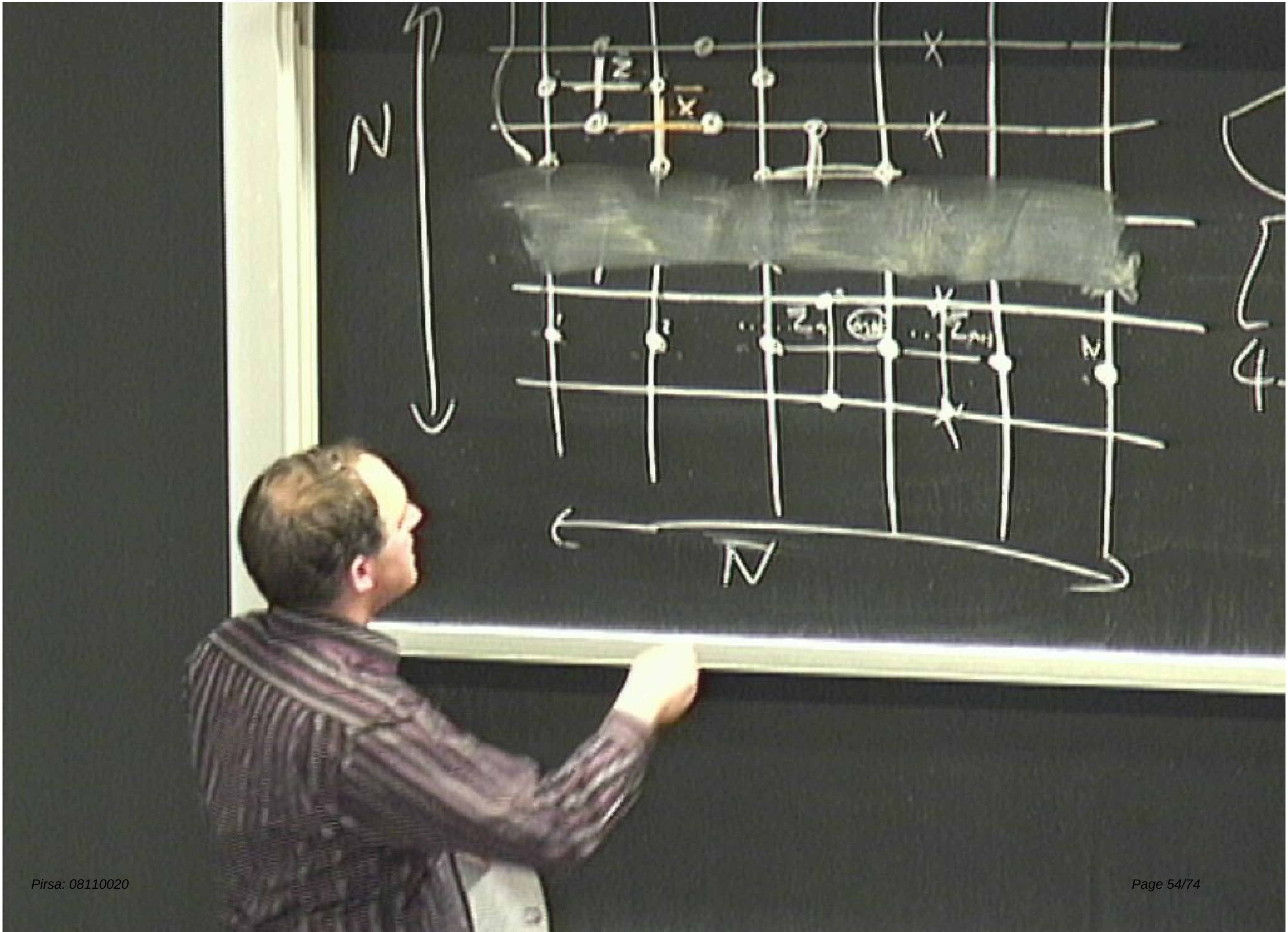
Dec 10 Keith Dienes

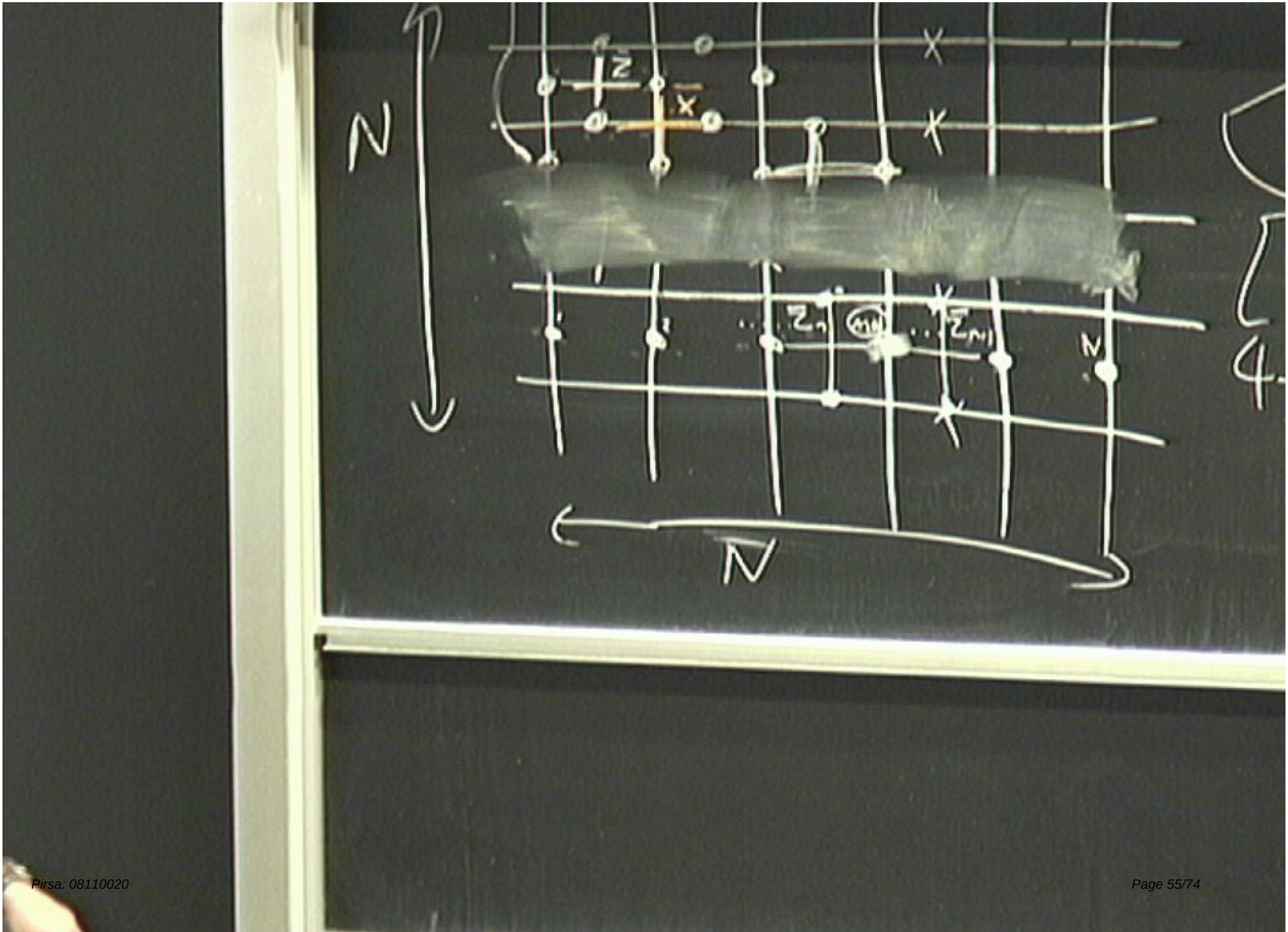
• if 1 local error occurs
what does PT do?

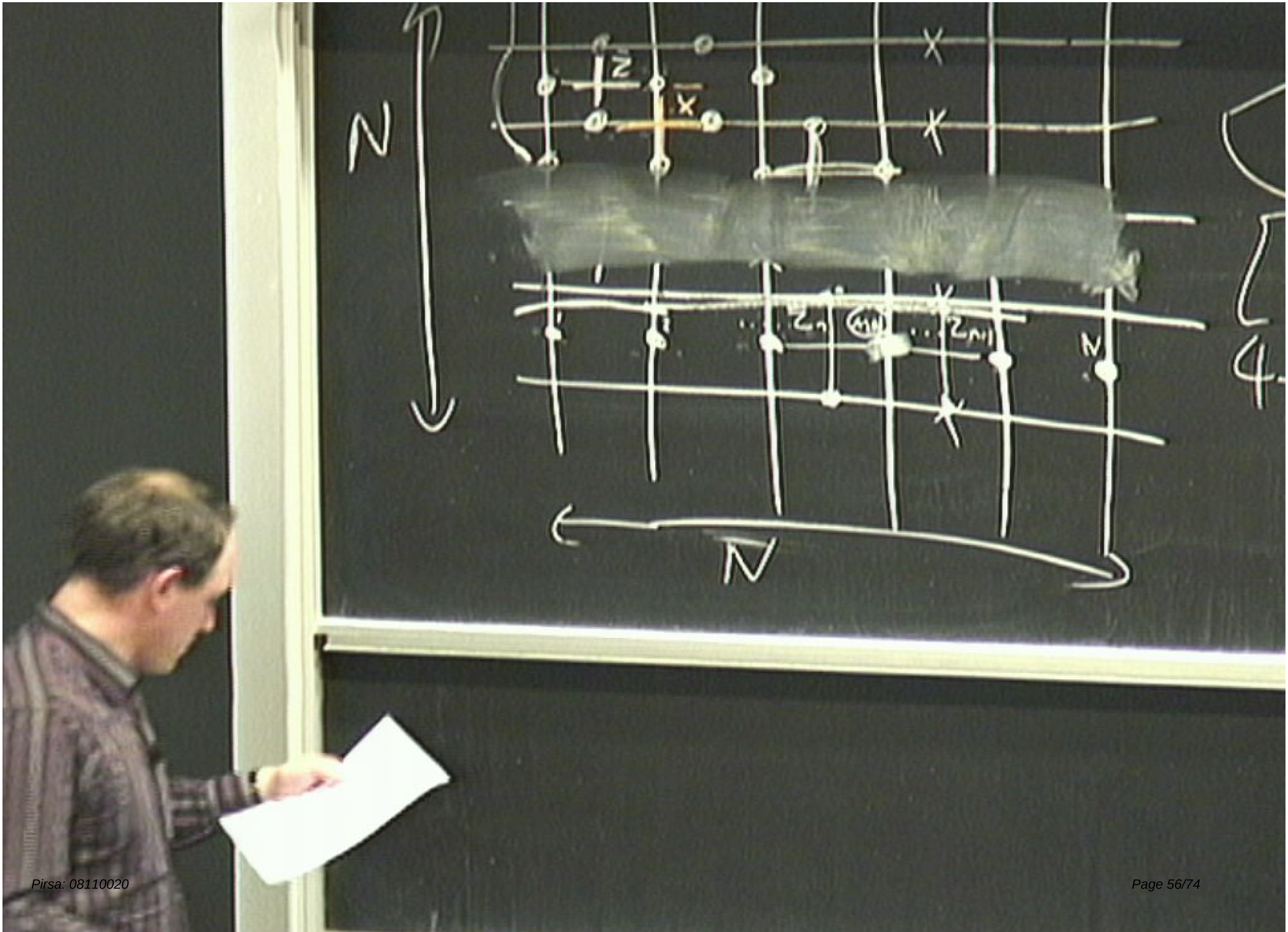
if 1 local error occurs, what does PT do?

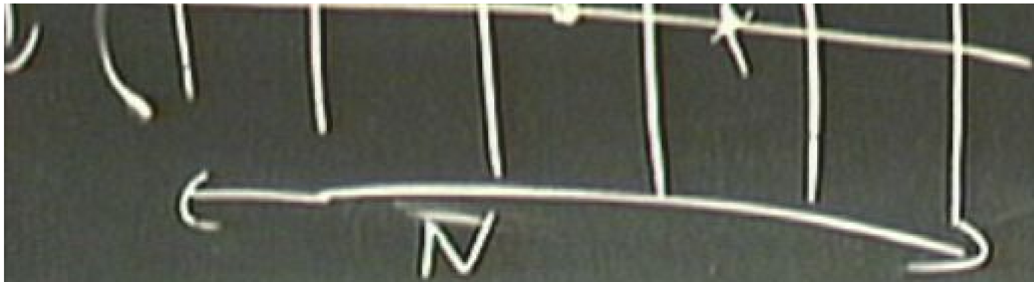


$$\bar{Z}_m \bar{Z}_{m+1}$$


N
Z
X
N
N


N
N

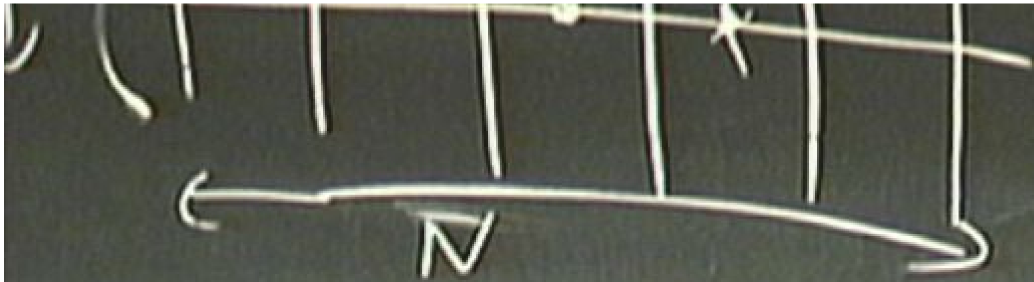




$$\bar{Z}_m Z_{m+1} |\psi\rangle = |\psi\rangle$$

$$|n\rangle = X_1 \cdots X_n |\psi\rangle$$

$$\bar{Z}_m \bar{Z}_{m+1} |n\rangle = \begin{cases} -|n\rangle & n = m, m+1 \\ |n\rangle & \text{otherwise} \end{cases}$$



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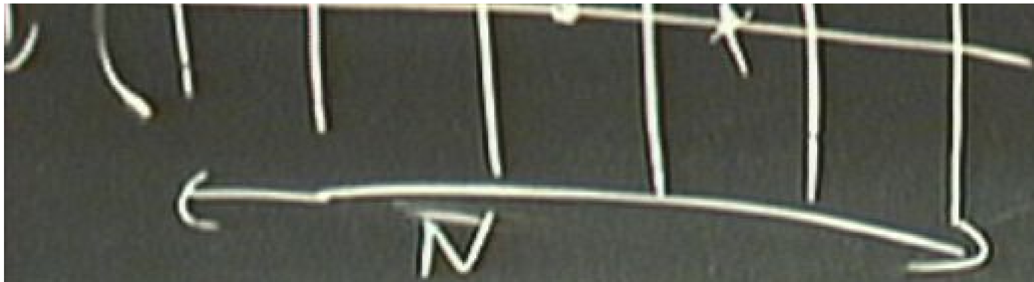
$$\bar{Z}_{m} \bar{Z}_{m+1} |\psi\rangle = |\psi\rangle$$

$$|n\rangle = X_1 \cdots X_n |\psi\rangle$$

$$J_m (X_{m+1} (\mathbb{1} - Z_m Z_{m+1})$$



$$\bar{Z}_m \bar{Z}_{m+1} |n\rangle = \begin{cases} -|n\rangle & n = m, m+1 \\ |n\rangle & \text{otherwise} \end{cases}$$

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$$|n\rangle = X_1 \cdots X_n |\psi\rangle$$

$$J_m X_{m+1} (\mathbb{1} - \bar{Z}_m \bar{Z}_{m+1}) |n\rangle = \begin{cases} J_m |m+1\rangle & n = m \\ J_m |m\rangle & n = m+1 \\ 0 & \text{otherwise} \end{cases}$$

$$V = \delta \sum J_m X_{m+1} \left(\mathbb{1} - Z_m Z_{m+1} \right)$$

$$V = \frac{\delta}{2} \sum_{m=1}^{N-2} J_m X_{m+1} \left(\mathbb{1} - Z_m Z_{m+1} \right)$$

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g.s with single local error $|1\rangle$

$$V = \frac{\delta}{2} \sum_{m=1}^{N-2} J_m X_{m+1} (\mathbb{1} - Z_m Z_{m+1})$$

g.s with single local error $|1\rangle$

$H + V$

$$V = \frac{\delta}{2} \sum_{m=1}^{N-2} J_m X_{m+1} (\mathbb{1} - Z_m Z_{m+1})$$

g.s with single local error

$|1\rangle \xrightarrow{t} |N-1\rangle$

$$H + V \Rightarrow \begin{pmatrix} 0 & J_1 & & & & \\ J_1 & 0 & J_2 & & & \\ & J_2 & 0 & & & \\ & & & 0 & & \\ & & & & \ddots & J_{N-2} \\ & & & & J_{N-2} & 0 \end{pmatrix}$$

$$V = \frac{\delta}{2} \sum_{m=1}^{N-2} J_m X_{m+1} (\mathbb{1} - Z_m Z_{m+1})$$

g.s with single local error $\quad |1\rangle \xrightarrow{t} |N-1\rangle$

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g.s with single local error

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$$J_m = \frac{2}{N-1} \sqrt{m(N-1-m)}$$

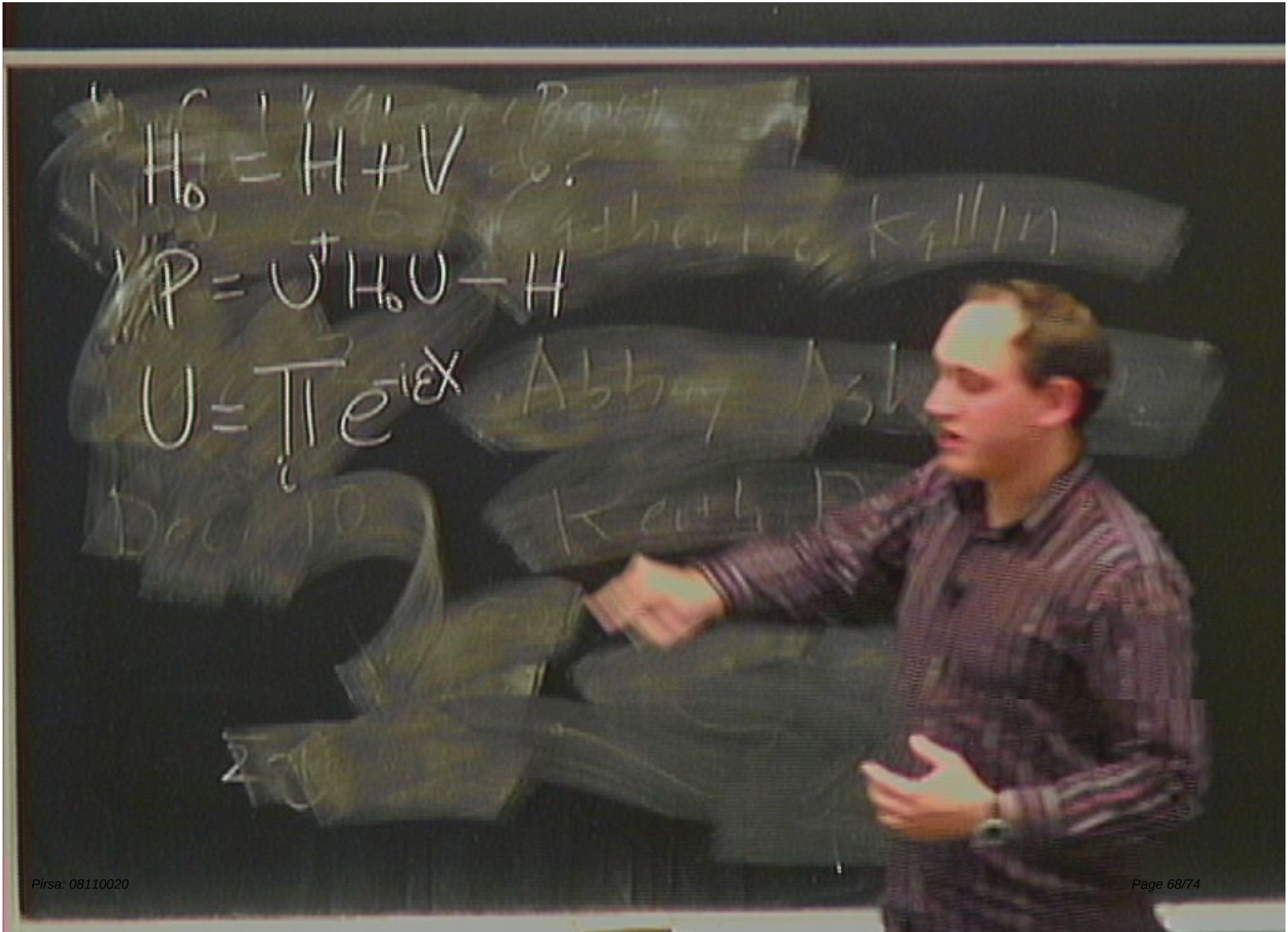
$$V = \frac{\delta}{2} \sum_{m=1}^{N-2} J_m X_{m+1} \left(\mathbb{1} - Z_m Z_{m+1} \right)$$

g.s with single local error $\quad |1\rangle \xrightarrow{t} |N-1\rangle$

$$H + V \Rightarrow \begin{pmatrix} 0 & J_1 & & & & \\ J_1 & 0 & J_2 & & & \\ & J_2 & 0 & & & \\ & & & 0 & & \\ & & & & \ddots & J_{N-2} \\ & & & & J_{N-2} & 0 \end{pmatrix}$$

$$J_m = \frac{2}{N-1} \sqrt{m(N-1-m)}$$

$$t = \frac{\pi (N-1)}{\delta}$$



$H_0 = H + V$
$V_P = U^\dagger H_0 U - H$
$U = \prod_i e^{-i\epsilon X}$

$$H_0 = H + V$$

$$P = U^{\dagger} H_0 U - H$$

$$U = \prod_i e^{-i\epsilon X}$$

$$H_0 = H + V_{\Sigma h_i}$$

$$P = U^\dagger H_0 U - H$$

$$U = \prod_i e^{-i\epsilon X} \qquad \epsilon \ll 1 \quad \sim 1/\sqrt{N}$$

$$H_0 = H + V$$

$\sum h.$

$$P = U^\dagger H_0 U - H$$

$\Rightarrow P$ is perturbation

$$U = \prod_i e^{-i\epsilon X} \qquad \epsilon \ll 1 \quad \sim 1/\sqrt{N}$$

$$e^{-i(H+P)t}|\psi\rangle$$

$$= U^\dagger e^{-iH_0 t} U|\psi\rangle$$

$$H_0 = H + V_{\Sigma h}$$

$$P = U^\dagger H_0 U - H$$

$\Rightarrow$ P is perturbation

$$U = \prod_i e^{-i\epsilon X}$$

$\epsilon \ll 1$

$\sim 1/\sqrt{N}$

$$e^{-i(H+P)t}|\psi\rangle$$

$$= U^\dagger e^{-iH_0 t} U |\psi\rangle$$

$\mathbb{R}(\quad)$

$$H_0 = H + V_{\text{Eh.}}$$

$$P = U^\dagger H_0 U - H$$

$$U = \prod_i e^{-i\varepsilon X} \qquad \varepsilon \ll 1 \quad \sim 1/\sqrt{N}$$

$\Rightarrow P$ is perturbation

$$e^{-i(H+P)t}|\psi\rangle$$

$$= U^\dagger e^{-iH_0 t} U |\psi\rangle$$

$\mathbb{P}(\quad)$

$\sin^2 \varepsilon$

$$H_0 = H + V_{Eh}$$

$$P = U^\dagger H_0 U - H$$

$$U = \prod_i e^{-i\varepsilon X}$$

$\Rightarrow P$ is perturbation

$\varepsilon \ll 1$

$\sim 1/\sqrt{N}$

$$e^{-i(H+P)t}|\psi\rangle$$

$$= U^\dagger e^{-iH_0 t} U |\psi\rangle$$

$\mathbb{P}(\quad)$

probability of logical error $\approx 50\%$ $\sin^2 \varepsilon$