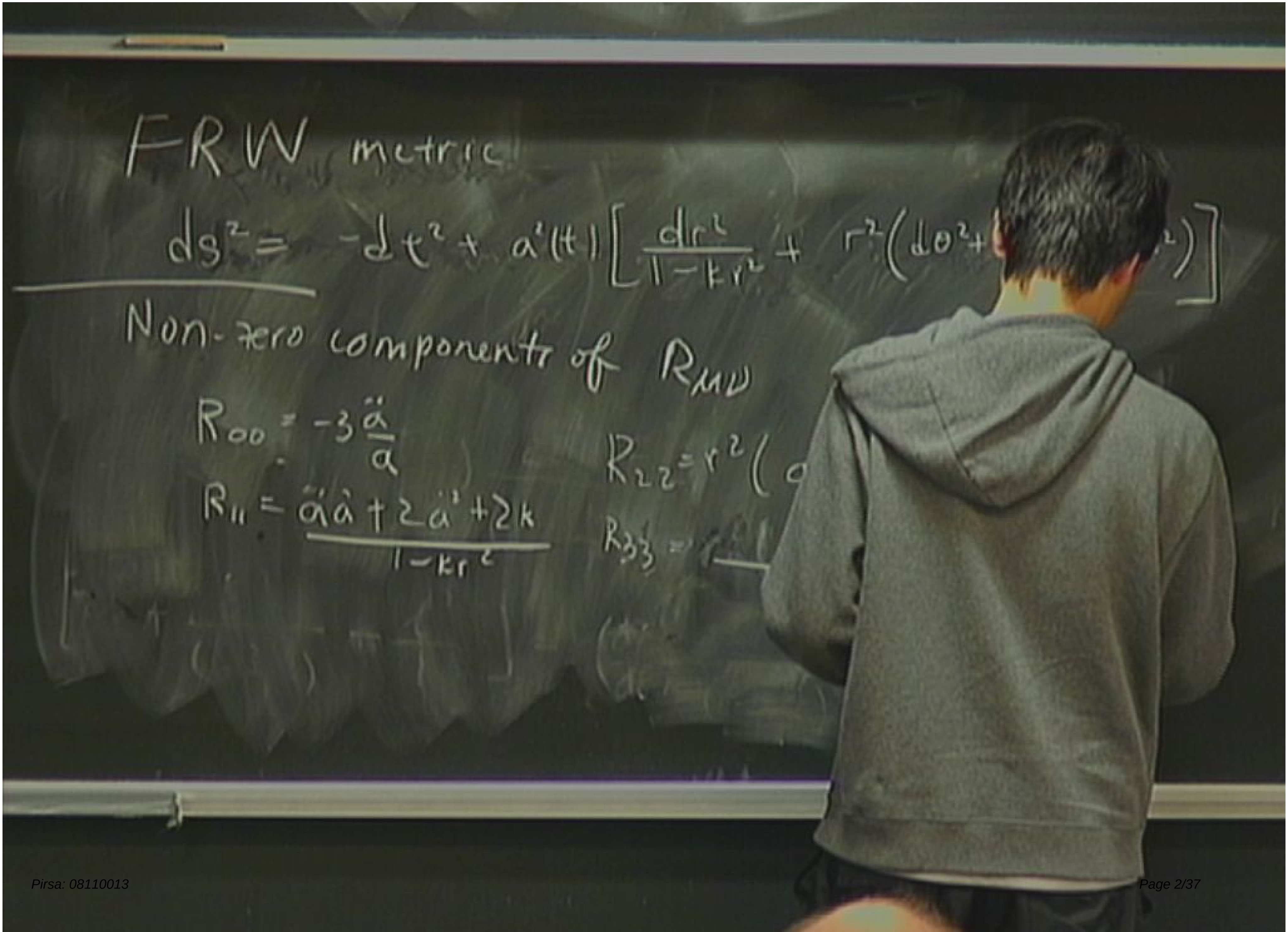
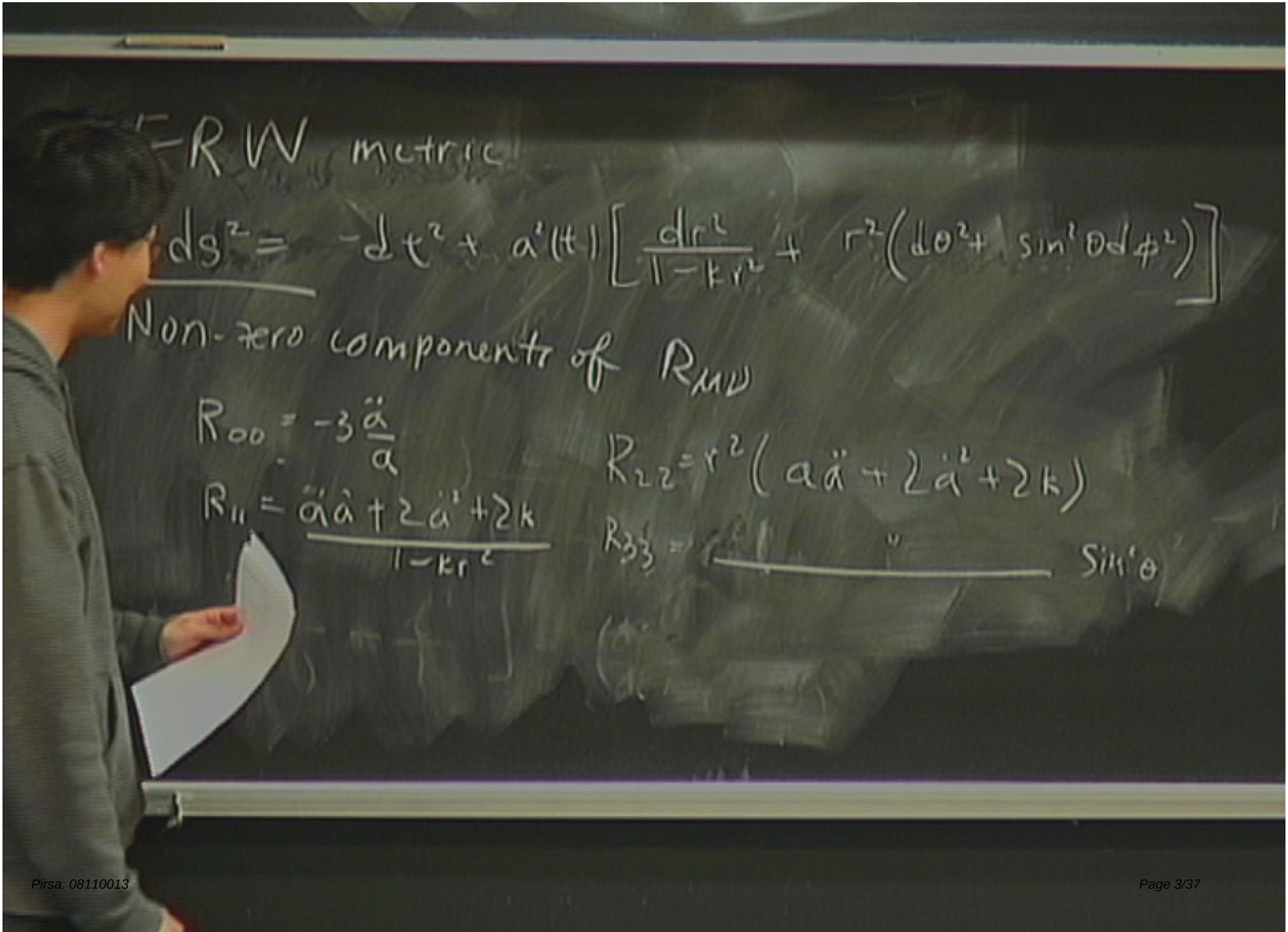
Title: Astrophysics and Cosmology through Problems - 11B

Date: Nov 13, 2008 12:30 PM

URL: http://pirsa.org/08110013

Abstract: This course is aimed at advanced undergraduate and beginning graduate students, and is inspired by a book by the same title, written by Padmanabhan. Each session consists of solving one or two pre-determined problems, which is done by a randomly picked student. While the problems introduce various subjects in Astrophysics and Cosmology, they do not serve as replacement for standard courses in these subjects, and are rather aimed at educating students with hands-on analytic/numerical skills to attack new problems.


FRW metric
$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2+\right.$
Non-zero components of $R_{\mu\nu}$
$R_{00} = -3\frac{\ddot{a}}{a}$
$R_{11} = \frac{\ddot{a}a + 2\dot{a}^2 + 2k}{1-kr^2}$
$R_{22} = r^2($
$R_{33} =$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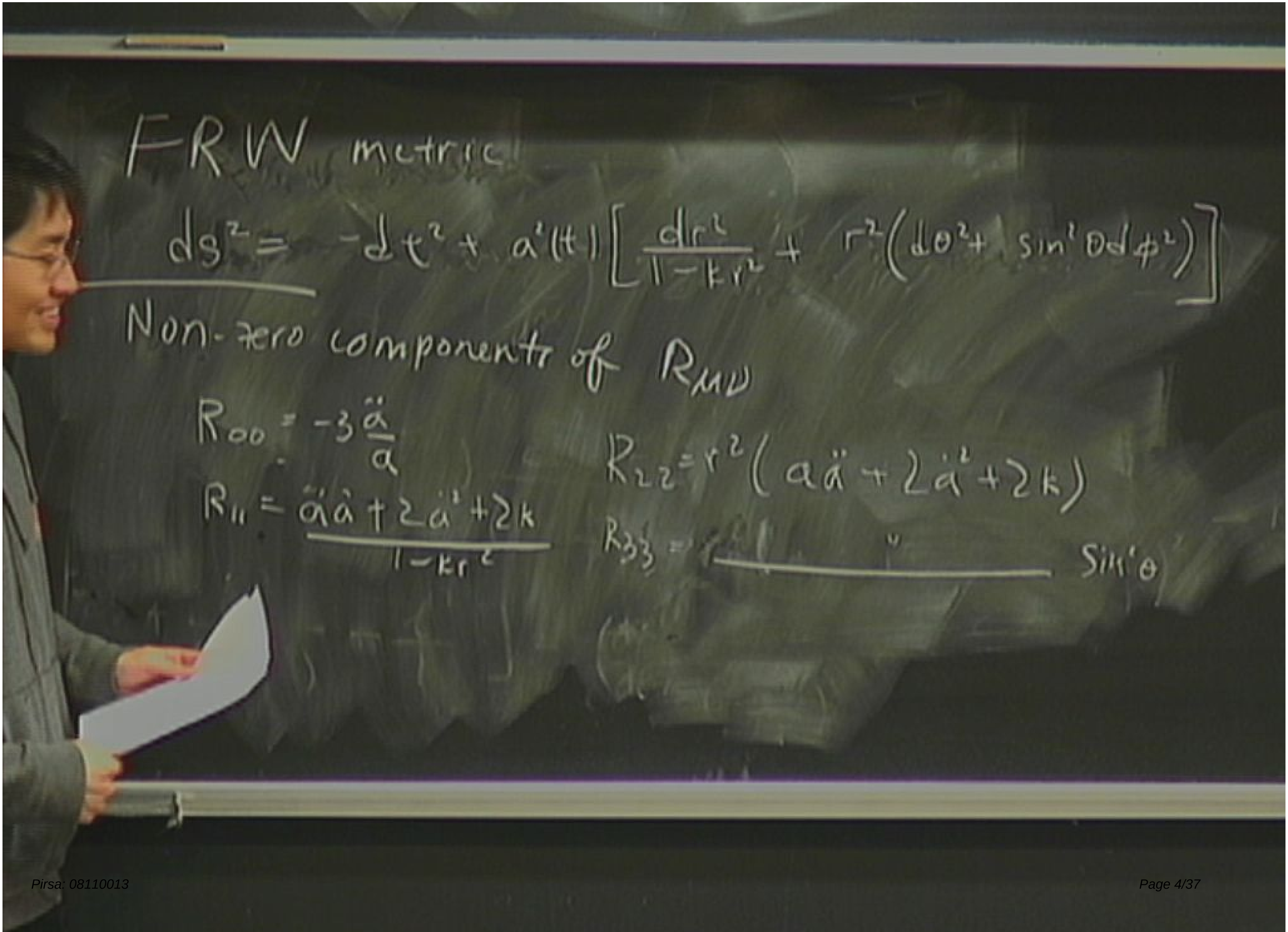
$R_{33} =$ " $\sin^2\theta$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Non-zero components of $R_{\mu\nu}$

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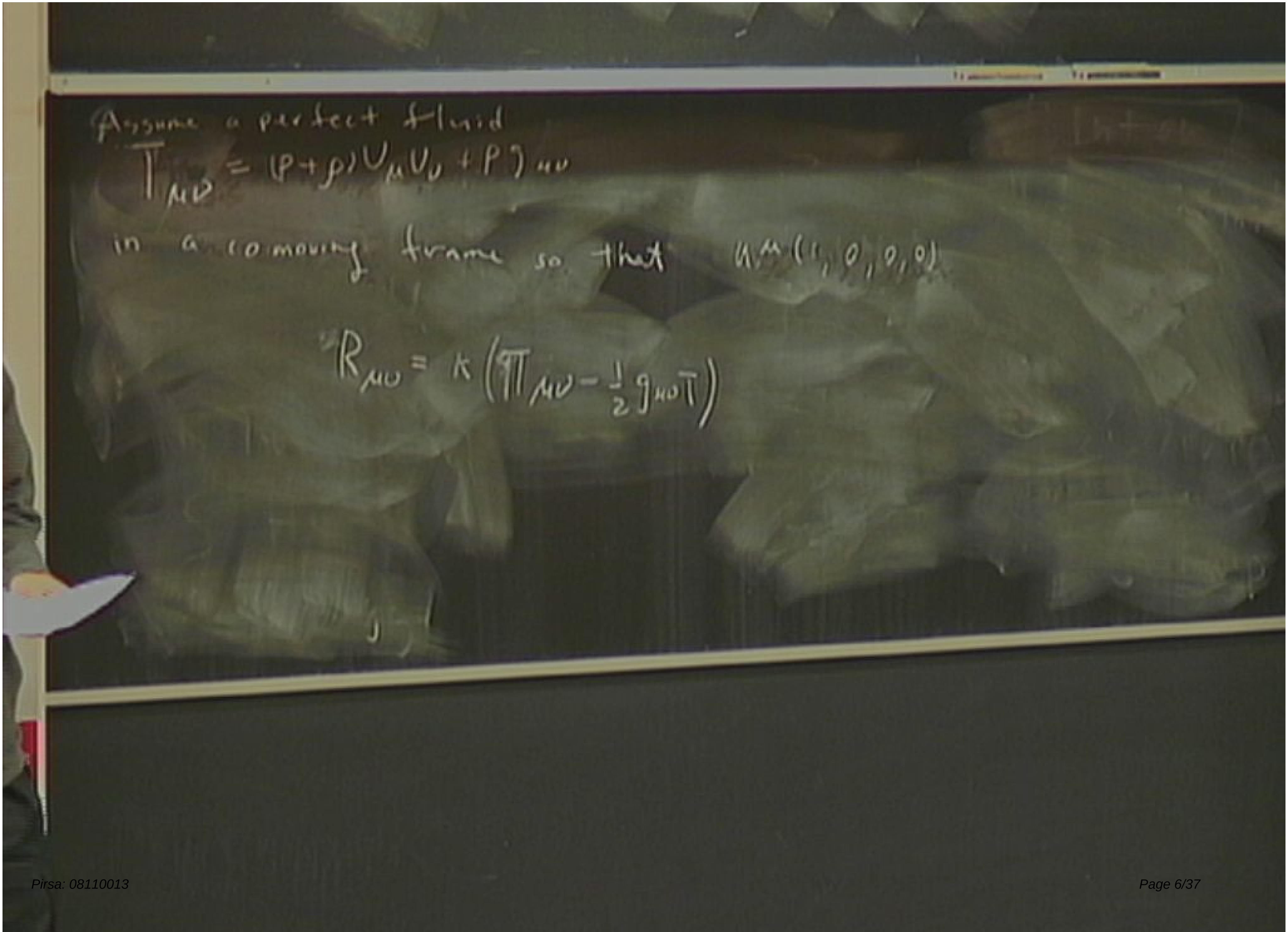
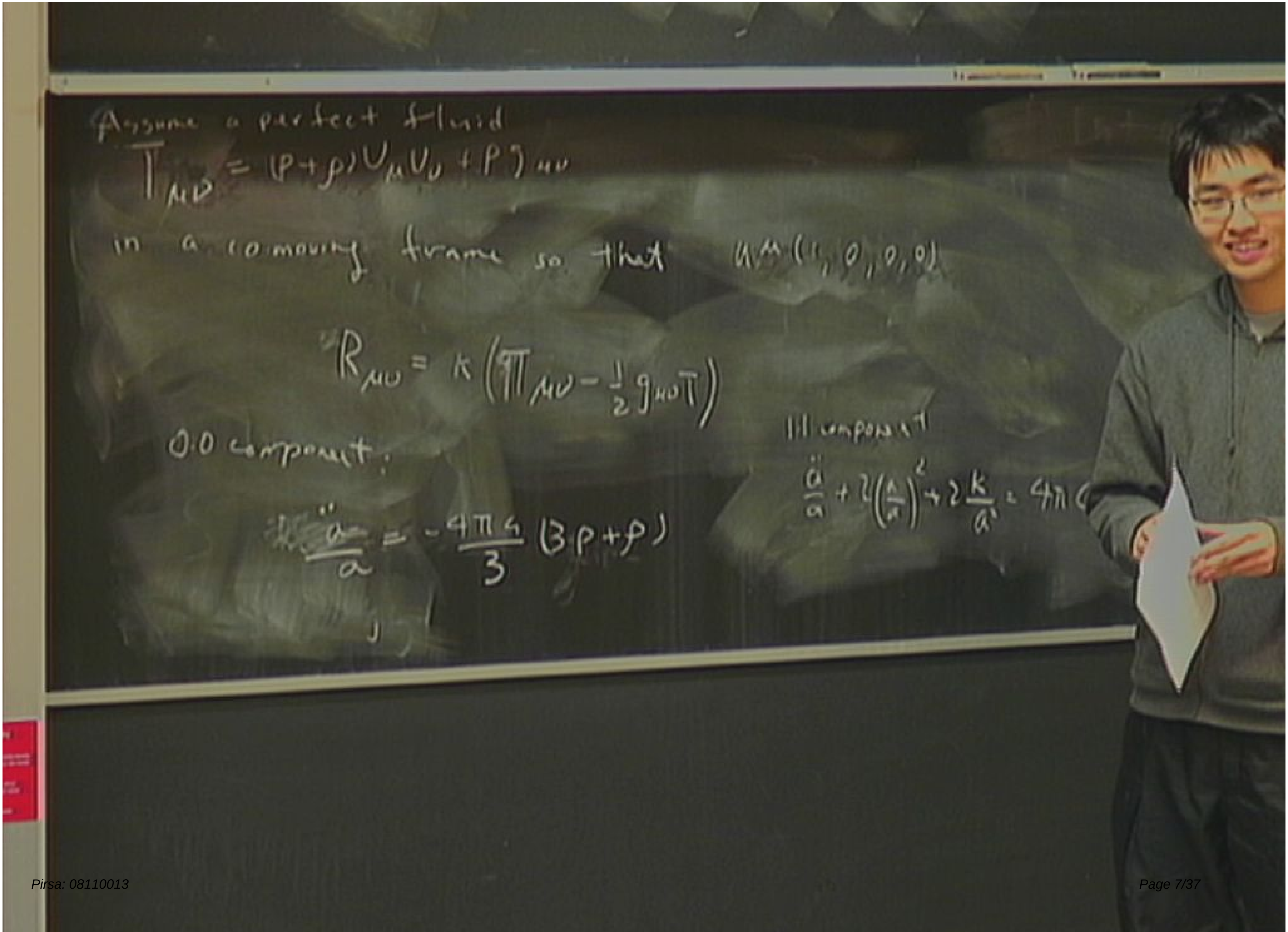

Assume a perfect fluid
T_μν = (ρ + p) U_μ U_ν + p g_μν
in a comoving frame so that u^μ (1, 0, 0, 0)
R_μν = k (8π T_μν − ½ g_μν T)



Assume a perfect fluid
$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p g_{\mu\nu}$
in a comoving frame so that $u^\mu(1,0,0,0)$
$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$
0.0 component:
$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p+\rho)$
11 component
$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G$

Assume a perfect fluid

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p\, g_{\mu\nu}$$

in a comoving frame so that $u^\mu(1,0,0,0)$

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

0.0 component:

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(3p+\rho)$$

1.1 component

$$\frac{\ddot a}{a} + 2\left(\frac{\dot a}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Assume a perfect fluid

$$T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p\, g_{\mu\nu}$$

in a comoving frame so that $u^\mu (1, 0, 0, 0)$

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

0.0 component:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p+\rho)$$

i i component

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)$$

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FRW metric $\frac{1}{r_c^2} \sim R_s \sim \frac{k}{a^2}$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

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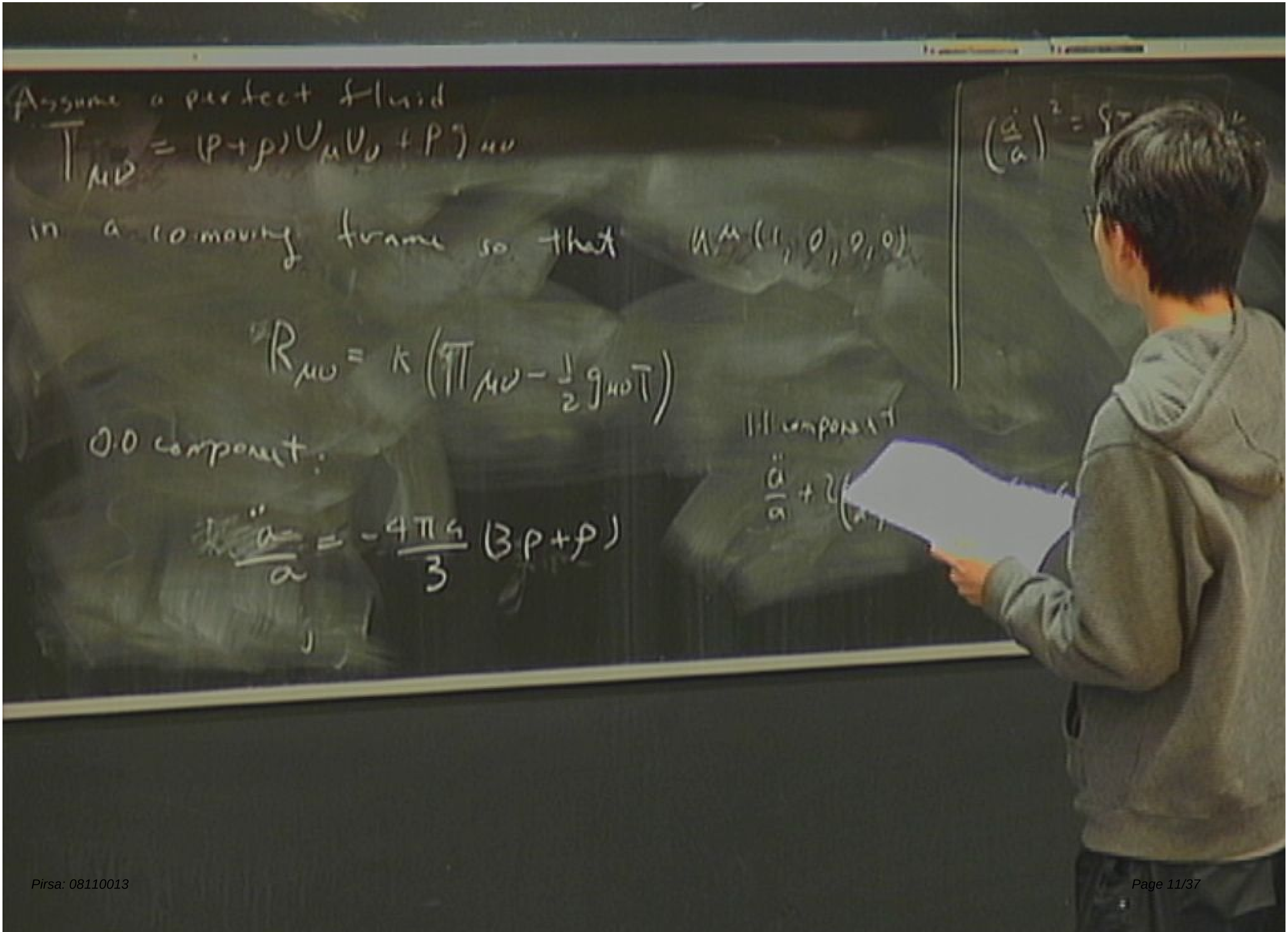
$$R_{33} = \ldots \sin^2\theta$$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in a comoving frame so that U^μ(1,0,0,0)
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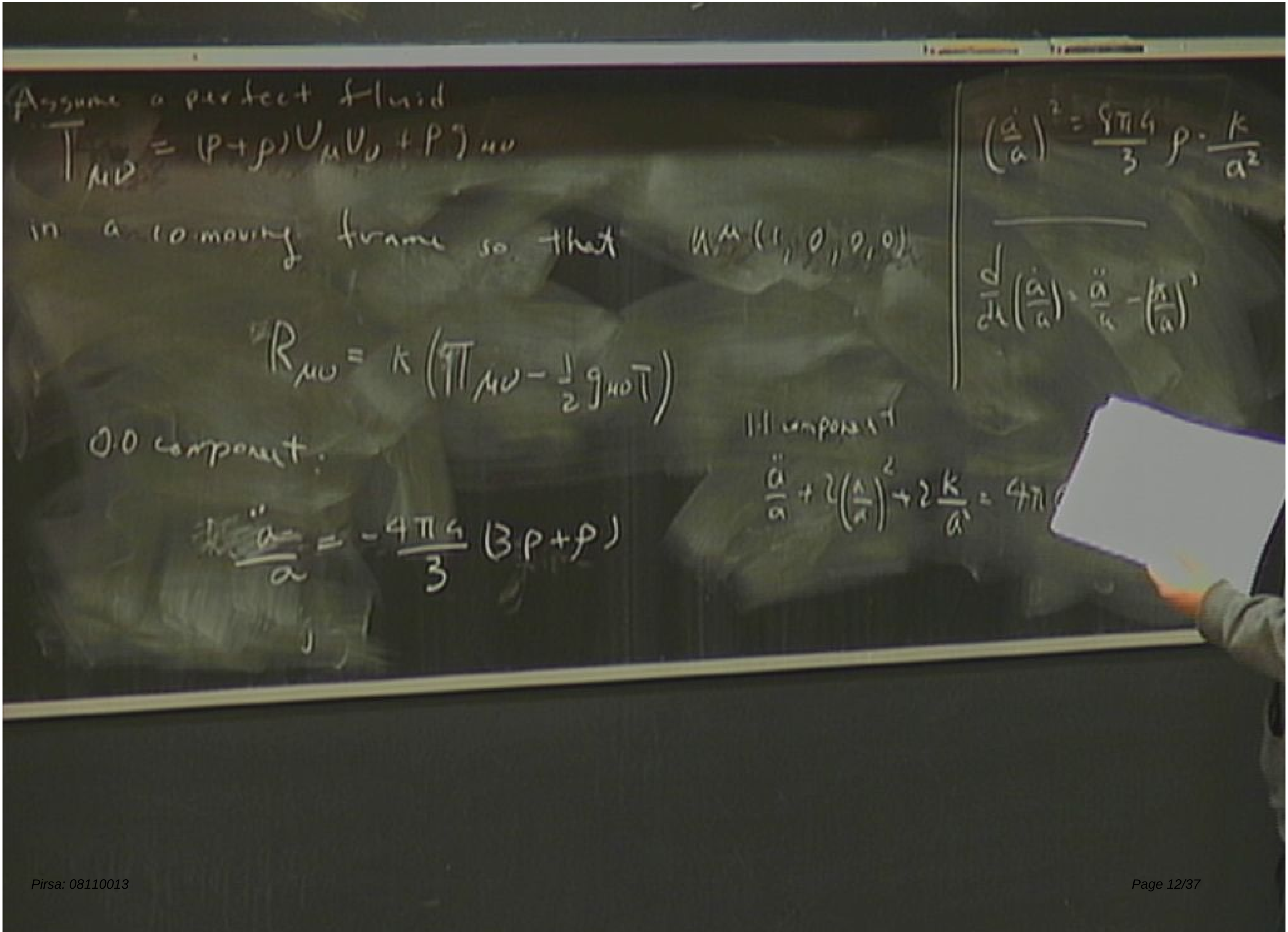
0.0 component:
ä/a = −4πG/3 (3p+ρ)
1.1 component
(ȧ/a)^2 =



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1.1 component
$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi$
$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$
$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2$

Assume a perfect fluid

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p\, g_{\mu\nu}$$

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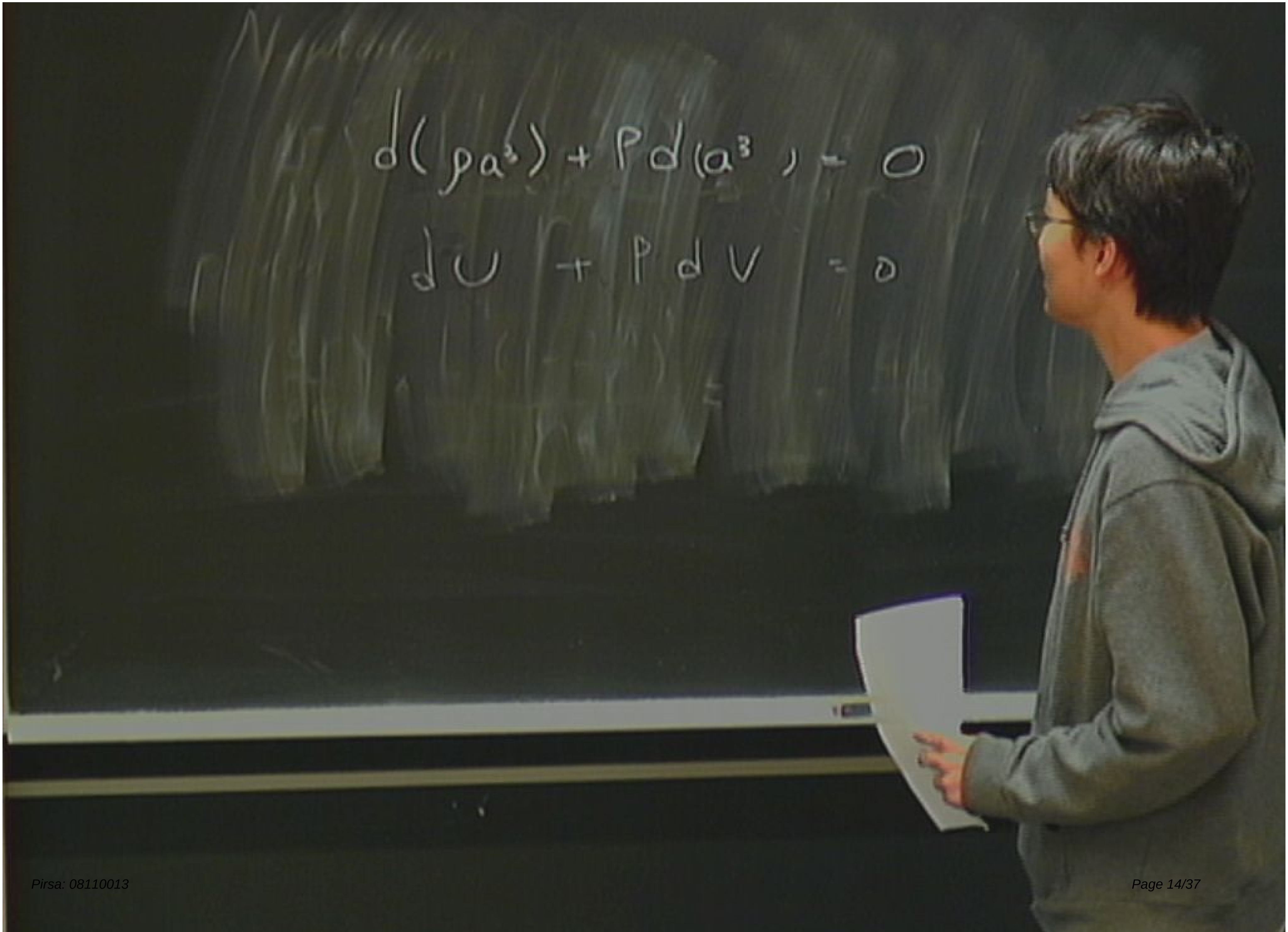
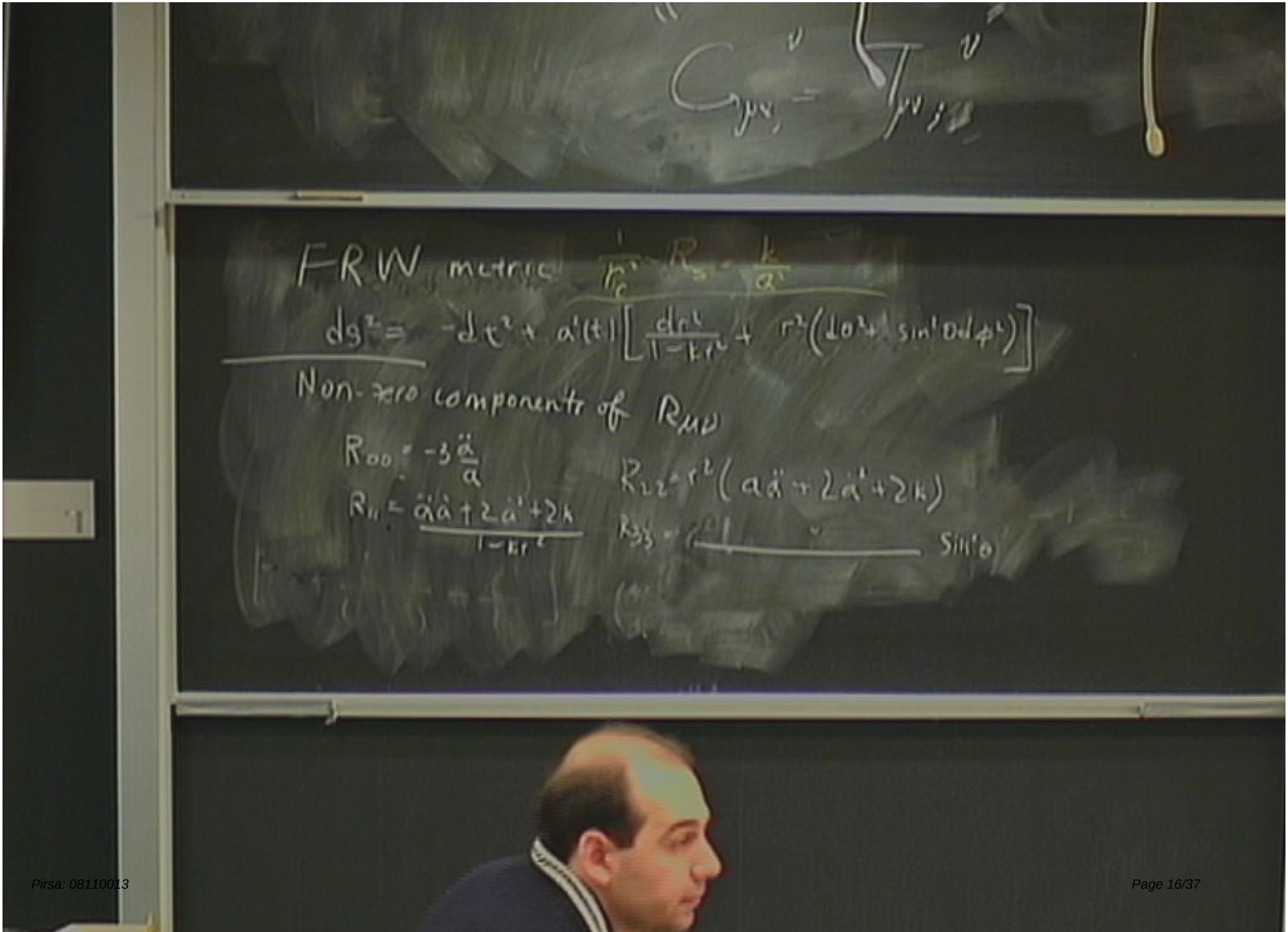

$d(\rho a^3) + P\,d(a^3) = 0$
$dU + P\,dV = 0$

$$d(\rho a^3) + P\, d(a^3) = 0$$

$$dU + P\, dV = 0$$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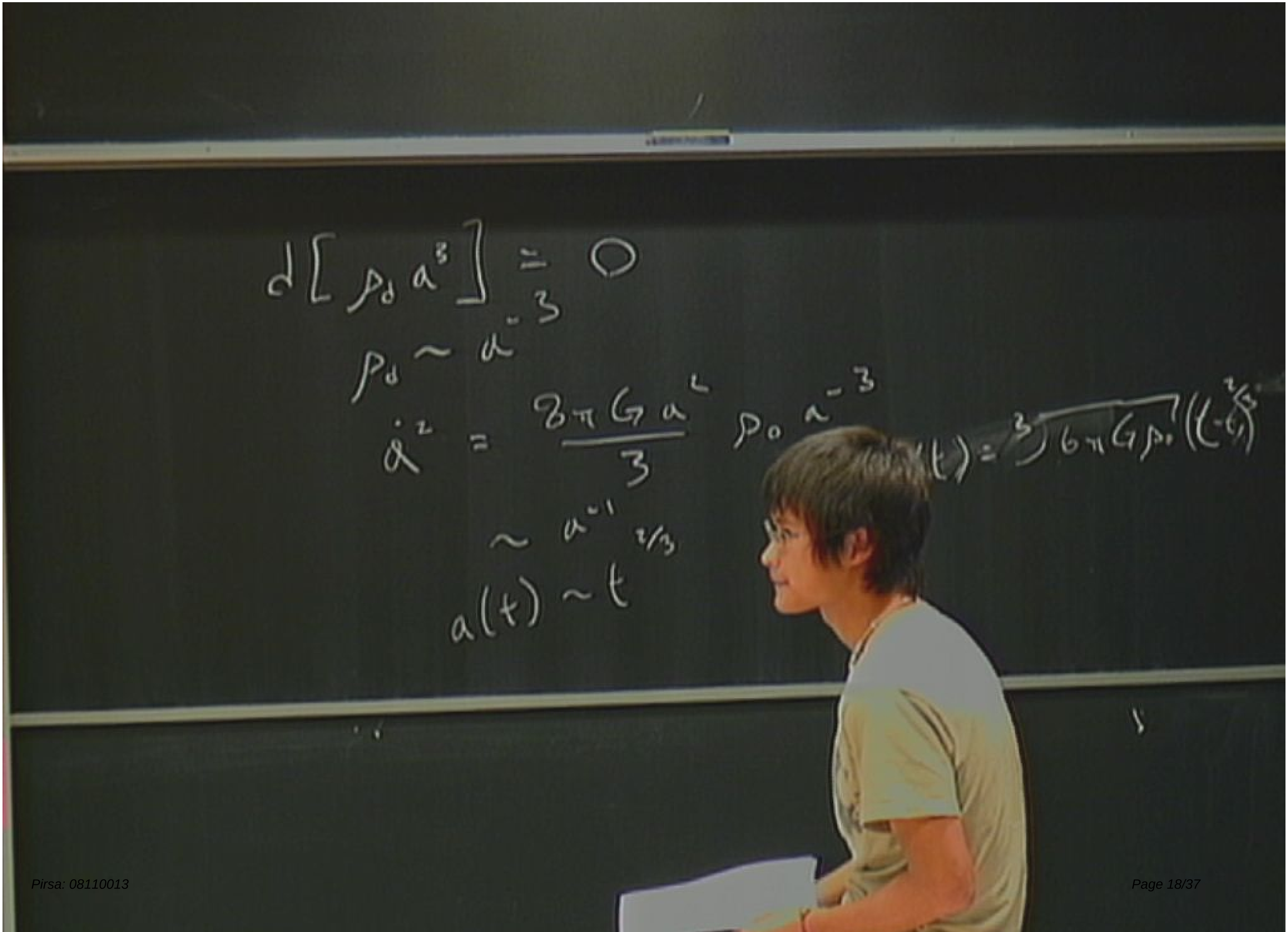
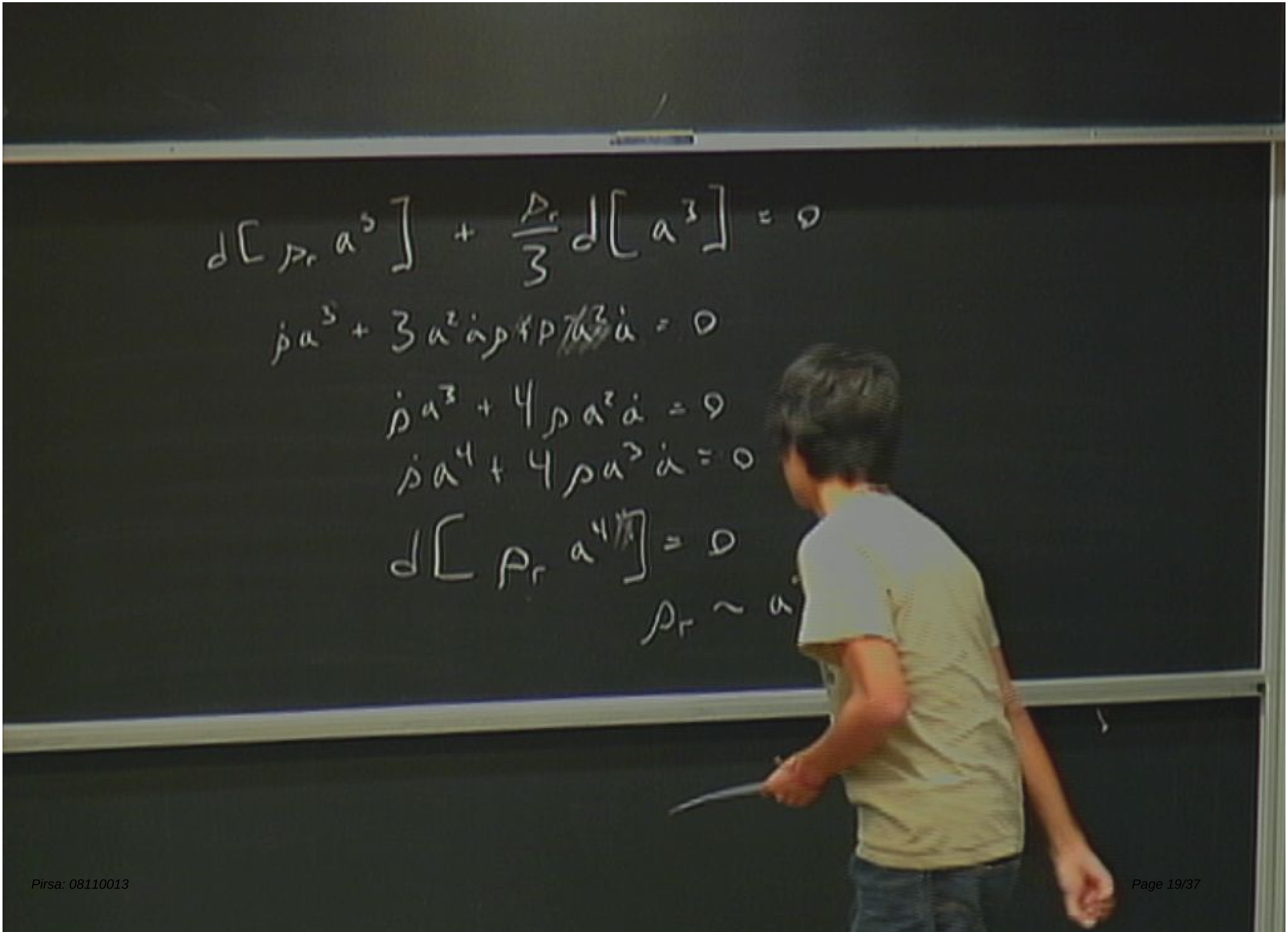
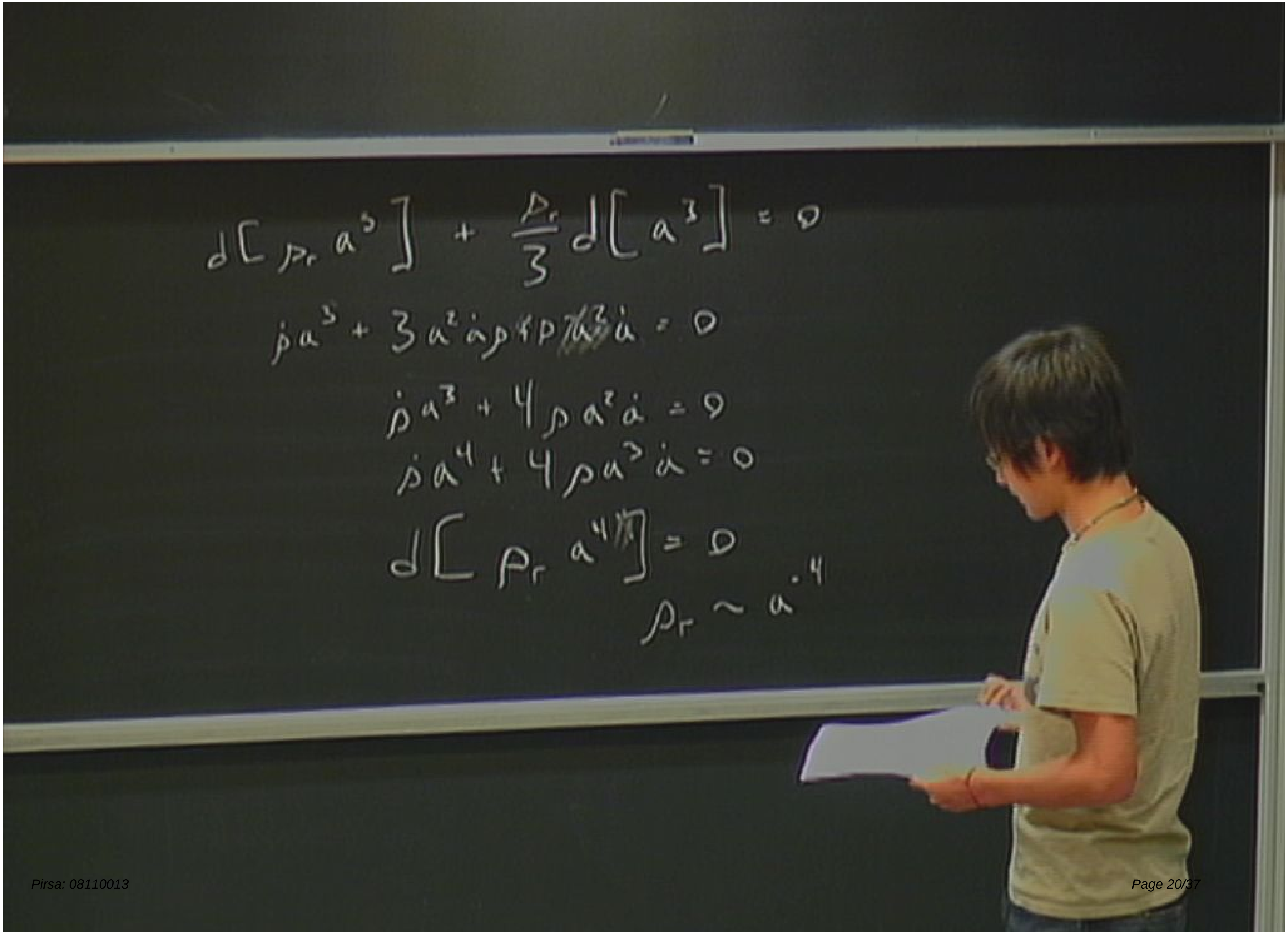
$$G_{\mu\nu;}{}^{\nu} \overset{0}{=} T_{\mu\nu;}{}^{\nu} = 0$$

$$\frac{1}{r_c^2} \sim R_s = \frac{k}{a^2}$$


$d[\rho_d a^3] = 0$
$\rho_d \sim a^{-3}$
$\dot{a}^2 = \frac{8\pi G a^2}{3} \rho_0 a^{-3}$
$\sim a^{-1}$
$a(t) \sim t^{2/3}$


$d[\rho_r a^3] + \frac{\rho_r}{3} d[a^3] = 0$
$\dot{\rho} a^3 + 3a^2 \dot{a} \rho + \rho a^2 \dot{a} = 0$
$\dot{\rho} a^3 + 4 \rho a^2 \dot{a} = 0$
$\dot{\rho} a^4 + 4 \rho a^3 \dot{a} = 0$
$d[\rho_r a^4] = 0$
$\rho_r \sim a$



$d[\rho_r a^3] + \frac{\rho_r}{3} d[a^3] = 0$
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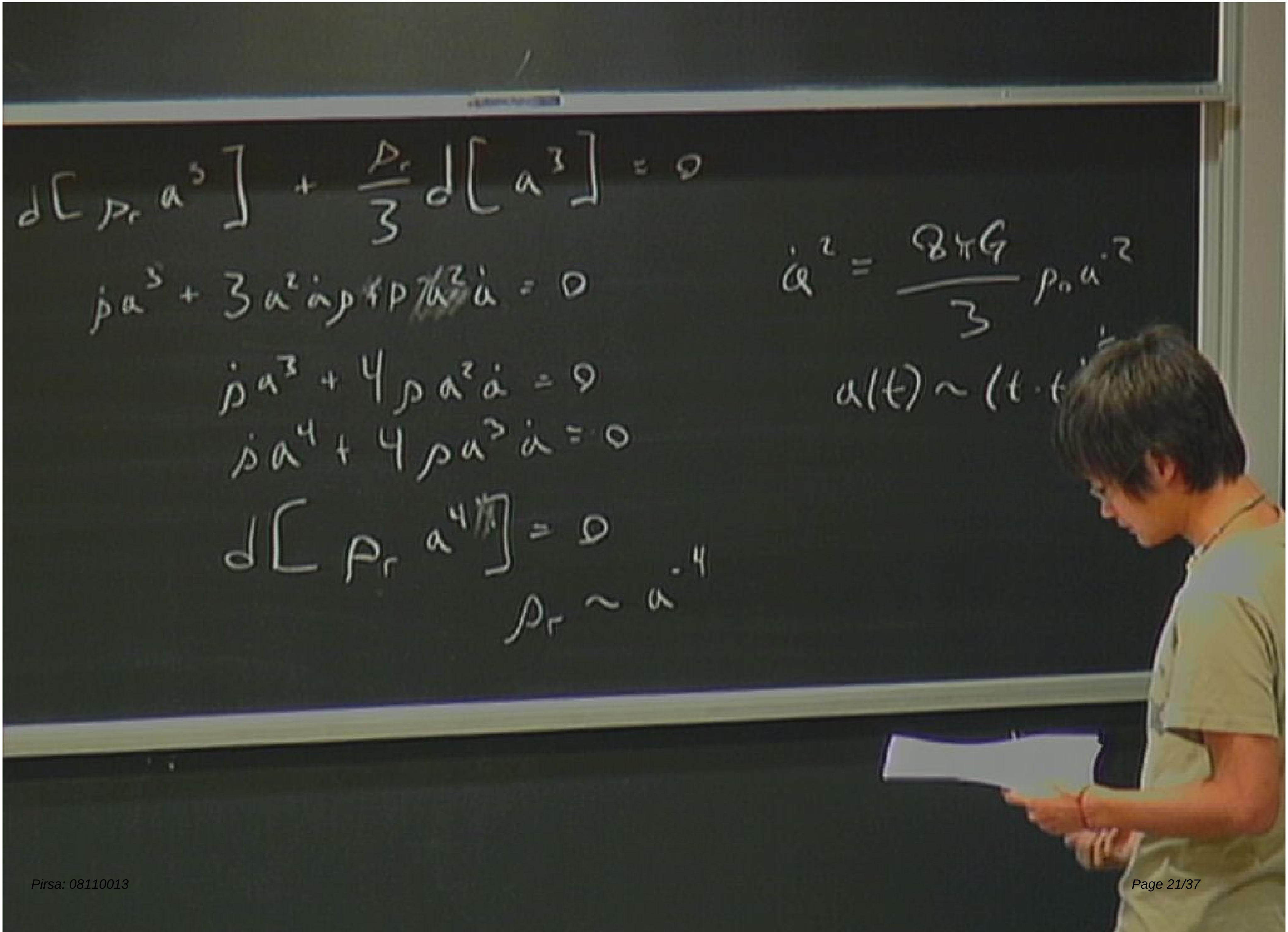
$d[\rho_r a^4] = 0$
$\rho_r \sim a^{-4}$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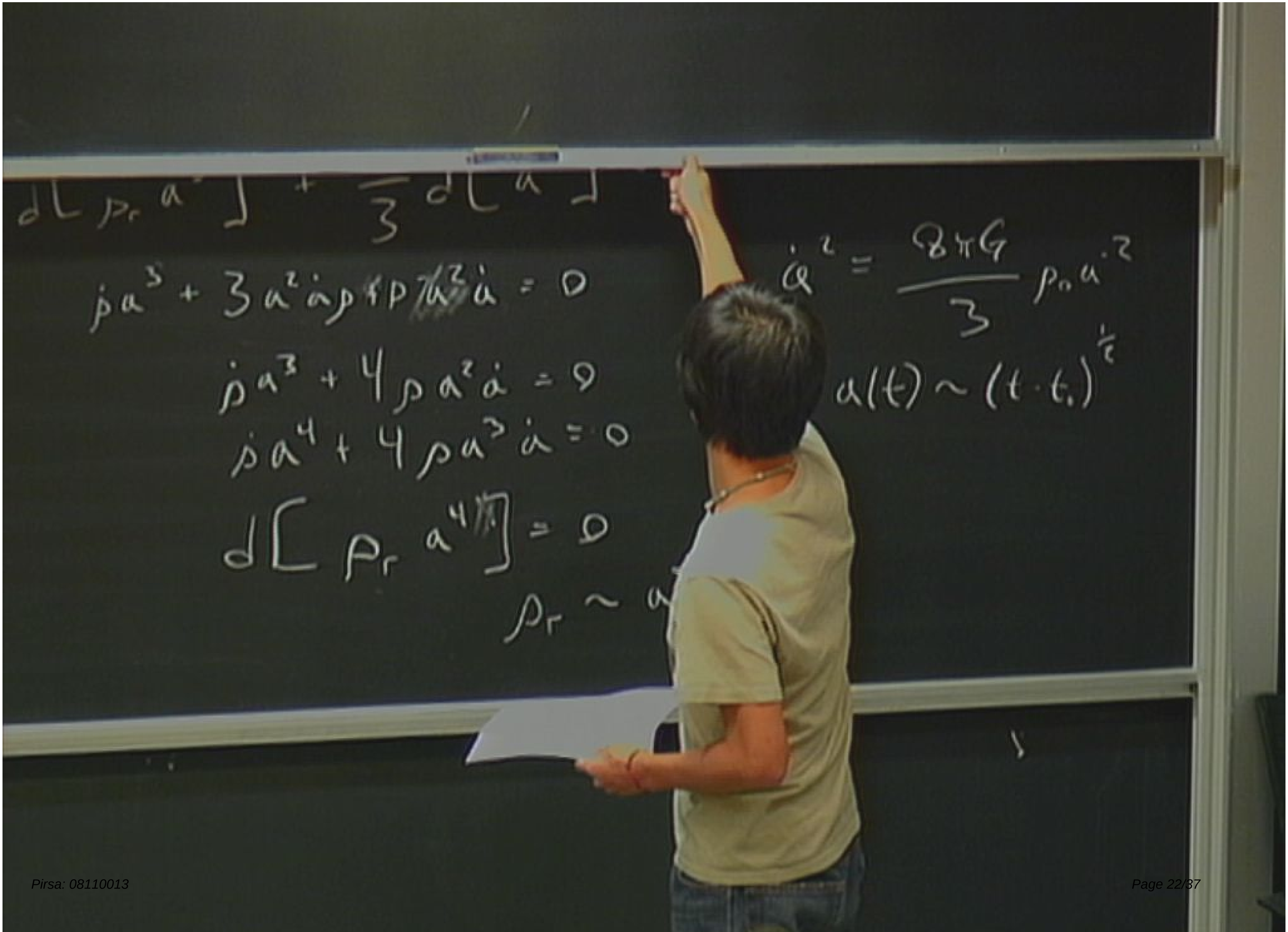
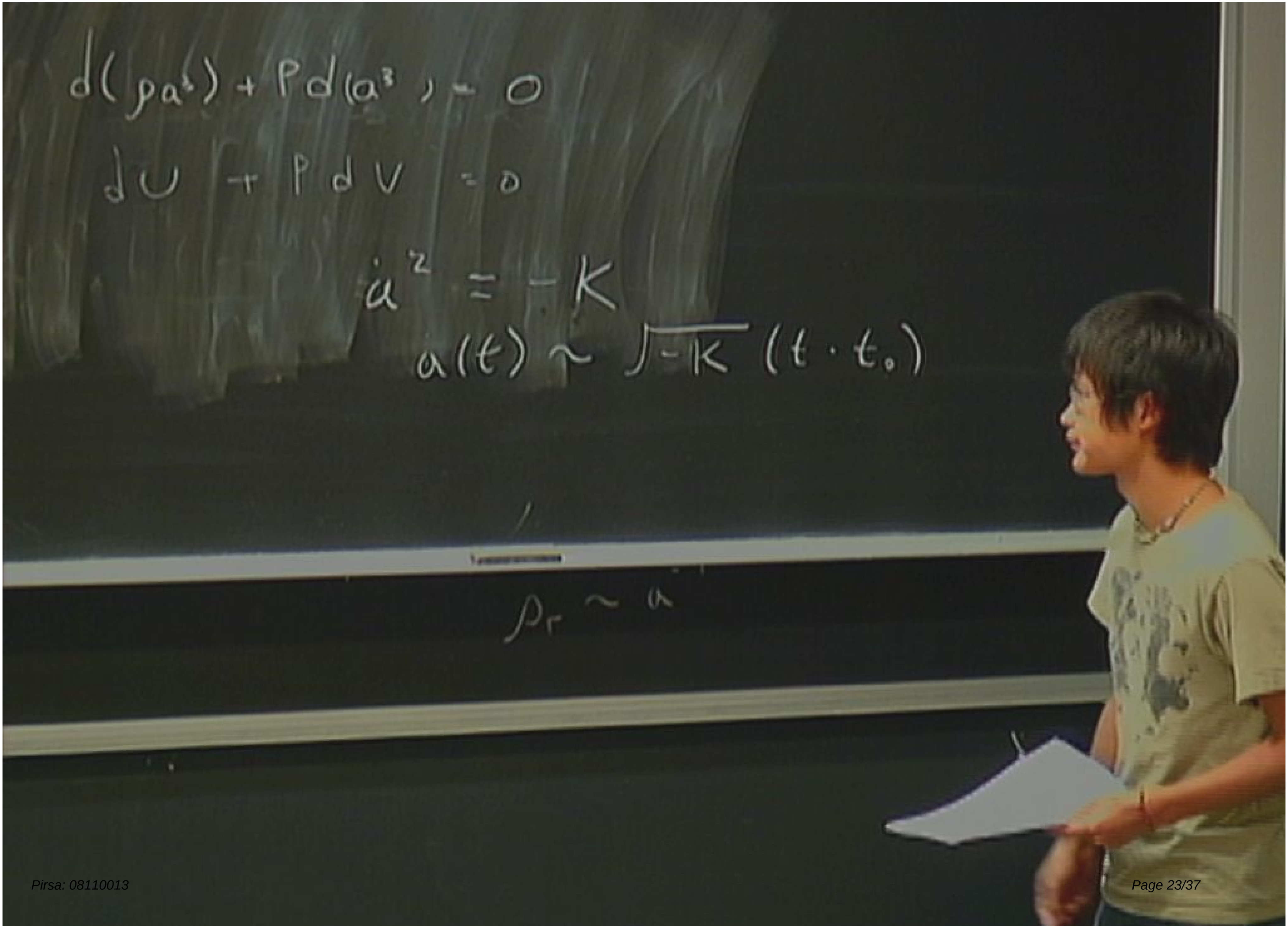
$\dot{\rho} a^4 + 4 \rho a^3 \dot{a} = 0$
$d[\rho_r a^4] = 0$
$\rho_r \sim a^{-4}$
$\dot{a}^2 = \frac{8\pi G}{3} \rho_0 a^{-2}$
$a(t) \sim (t - t$



$\dot\rho a^3 + 3a^2\dot a\rho + p\,3a^2\dot a = 0$
$\dot\rho a^3 + 4\rho a^2\dot a = 0$
$\dot\rho a^4 + 4\rho a^3\dot a = 0$
$d[\rho_r a^4] = 0$
$\rho_r \sim a$
$\dot a^2 = \frac{8\pi G}{3}\rho_0 a^{-2}$
$a(t) \sim (t-t_0)^{\frac{1}{2}}$


$d(\rho a^3) + P d(a^3) = 0$
$dU + P dV = 0$
$\dot{a}^2 = -K$
$a(t) \sim \sqrt{-K}\,(t - t_0)$
$\rho_r \sim a$

$$d(\rho a^3) + P\,d(a^3) = 0$$

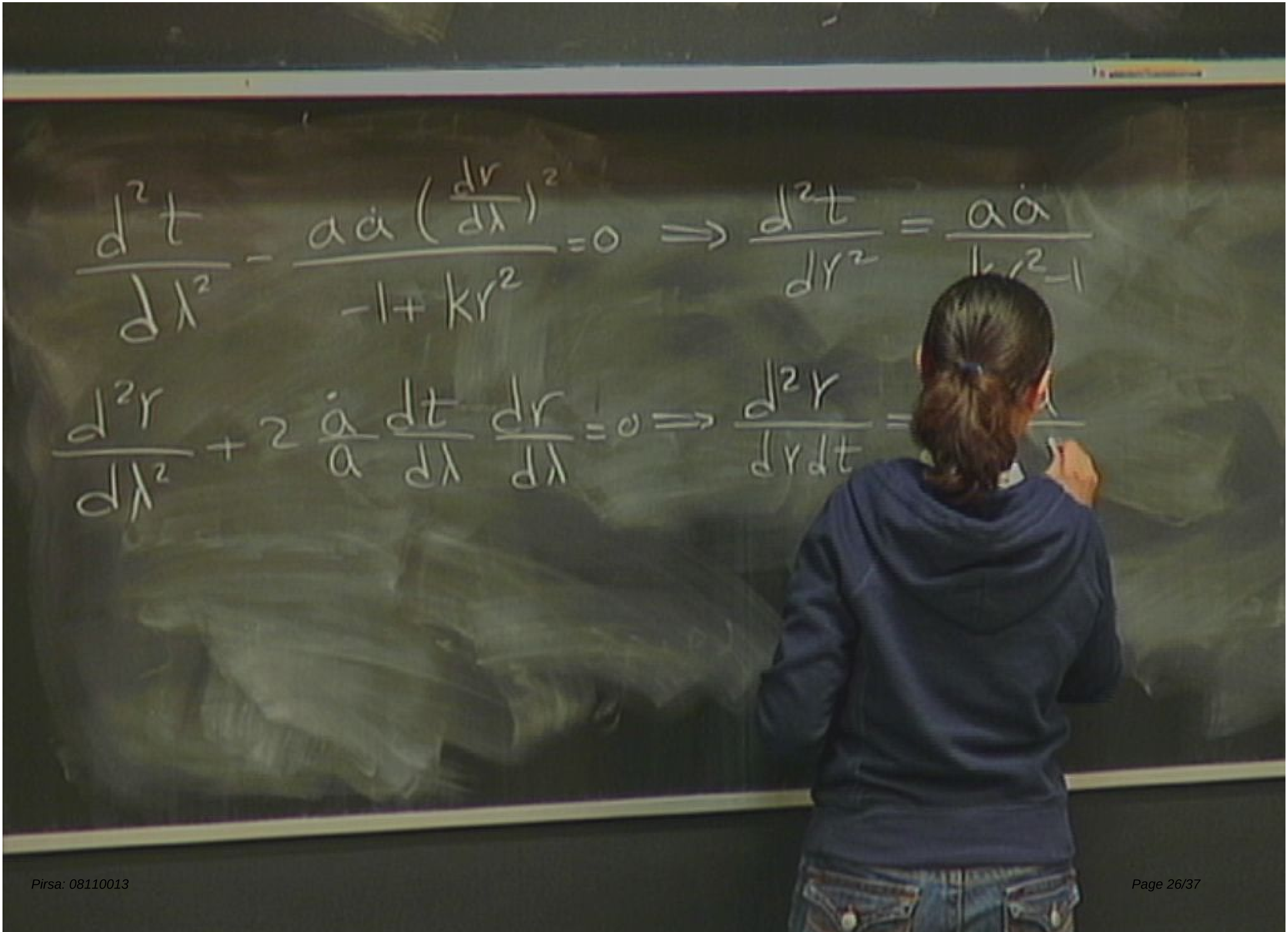
$$dU + P\,dV = 0$$

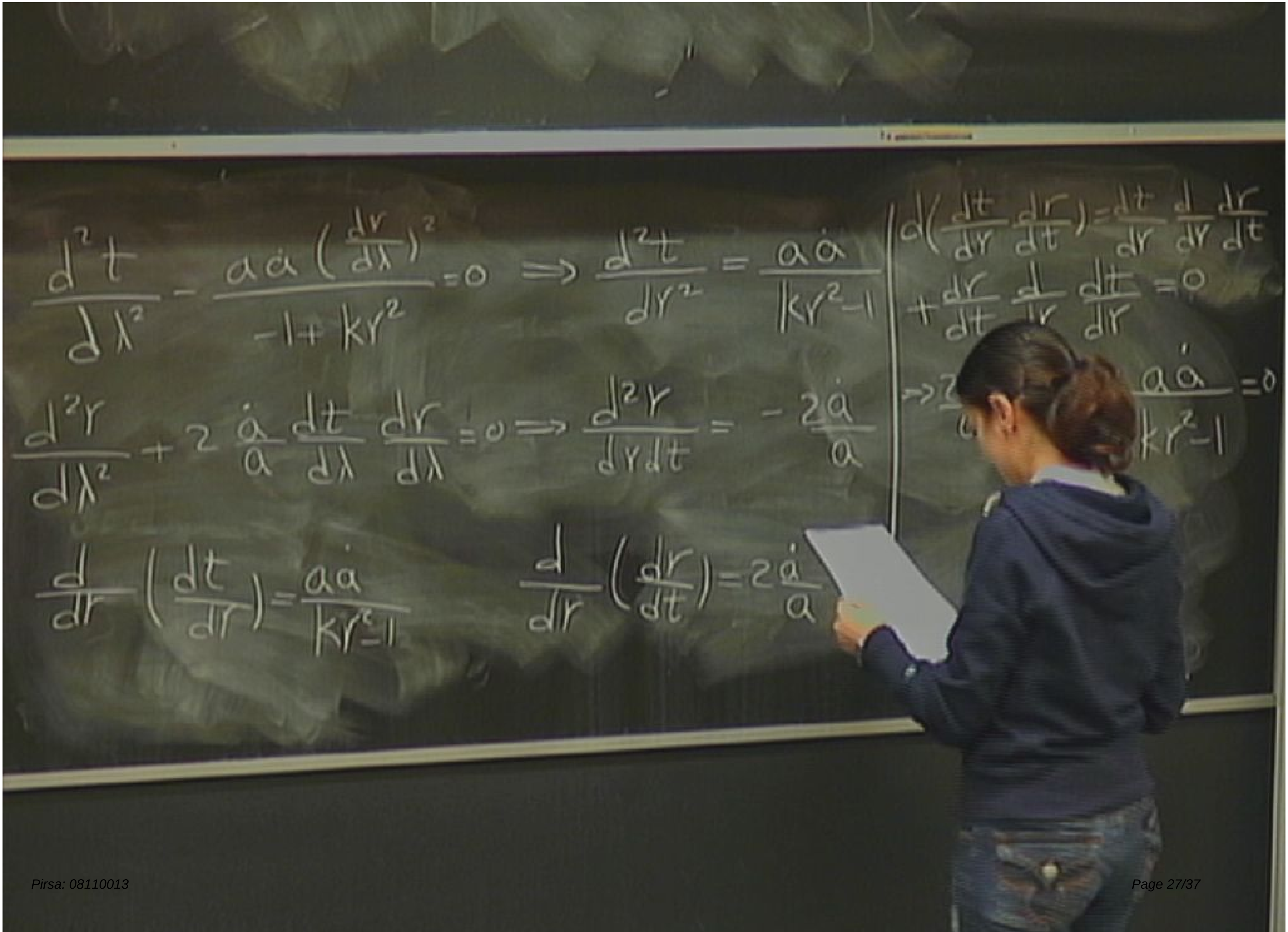
$$\dot{a}^2 = -K$$

$$a(t) \sim \pm\sqrt{-K}\,(t - t_0)$$

$$\frac{d^2 t}{d\lambda^2} - \frac{a\dot{a}\left(\frac{dr}{d\lambda}\right)^2}{-1+kr^2} = 0$$

$$\frac{d^2 r}{d\lambda^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$


$\frac{d^2t}{d\lambda^2} - \frac{a\dot{a}\left(\frac{dr}{d\lambda}\right)^2}{-1+kr^2} = 0 \Rightarrow \frac{d^2t}{dr^2} = \frac{a\dot{a}}{kr^2-1}$
$\frac{d^2r}{d\lambda^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \Rightarrow \frac{d^2r}{dr\,dt} =$



$\frac{d^2t}{d\lambda^2} - \frac{a\dot{a}\left(\frac{dr}{d\lambda}\right)^2}{-1+kr^2} = 0 \Longrightarrow \frac{d^2t}{dr^2} = \frac{a\dot{a}}{kr^2-1}$
$\frac{d^2r}{d\lambda^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \Longrightarrow \frac{d^2r}{dr\,dt} = -2\frac{\dot{a}}{a}$
$\frac{d}{dr}\left(\frac{dt}{dr}\right) = \frac{a\dot{a}}{kr^2-1}$
$\frac{d}{dr}\left(\frac{dr}{dt}\right) = -2\frac{\dot{a}}{a}$
$d\left(\frac{dt}{dr}\frac{dr}{dt}\right) = \frac{dt}{dr}\frac{d}{dr}\frac{dr}{dt} + \frac{dr}{dt}\frac{d}{dr}\frac{dt}{dr} = 0$

$$\frac{d^2t}{d\lambda^2} - \frac{a\dot a \left(\frac{dr}{d\lambda}\right)^2}{-1+kr^2} = 0 \Longrightarrow \frac{d^2t}{dr^2} = \frac{a\dot a}{kr^2-1}$$

$$\frac{d^2r}{d\lambda^2} + 2\frac{\dot a}{a}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \Longrightarrow \frac{d^2r}{dr\,dt} = -2\frac{\dot a}{a}$$

$$\frac{d}{dr}\left(\frac{dt}{dr}\right) = \frac{a\dot a}{kr^2-1} \qquad \frac{d}{dr}\left(\frac{dr}{dt}\right) = -2\frac{\dot a}{a}$$

$$d\left(\frac{dt}{dr}\frac{dr}{dt}\right) = \frac{dt}{dr}\frac{d}{dr}\frac{dr}{dt} + \frac{dr}{dt}\frac{d}{dr}\frac{dt}{dr} = 0$$

$$\Longrightarrow 2\frac{\cancel{\dot a}}{a}\frac{1}{V} + V\frac{a\cancel{\dot a}}{kr^2-1} = 0$$

$$\Longrightarrow \frac{2}{a} = -V^2\frac{a}{kr^2-1}$$

$$\Longrightarrow V^2 = \frac{-2\left(kr^2-1\right)}{a^2}$$

$$\frac{d^2t}{d\lambda^2} - \frac{a\dot{a}\left(\frac{dr}{d\lambda}\right)^2}{-1+kr^2} = 0 \Longrightarrow \frac{d^2t}{dr^2} = \frac{a\dot{a}}{kr^2-1}$$

$$\frac{d^2r}{d\lambda^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \Longrightarrow \frac{d^2r}{dr\,dt} = -2\frac{\dot{a}}{a}$$

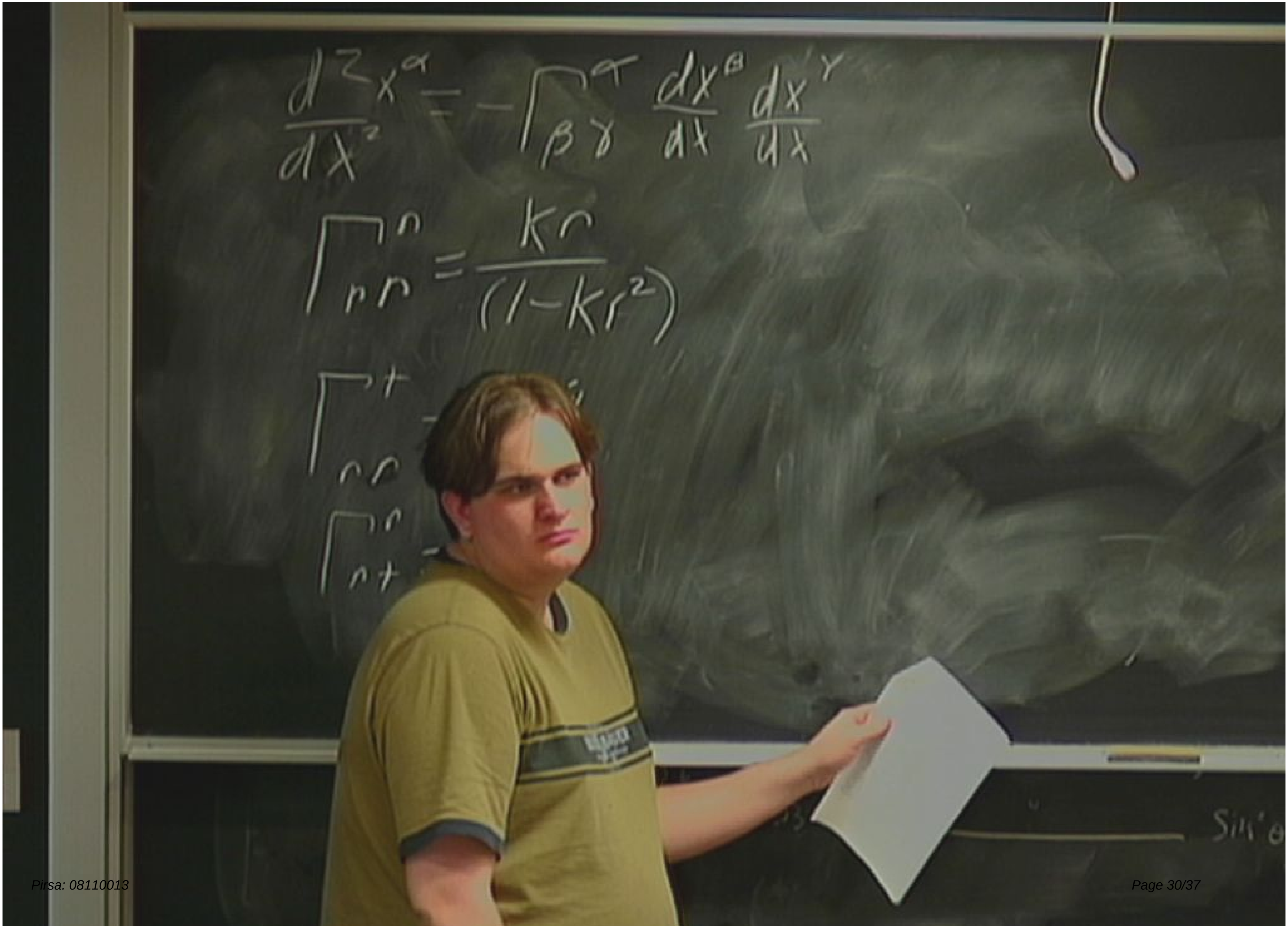
$$\frac{d}{dr}\left(\frac{dt}{dr}\right) = \frac{a\dot{a}}{kr^2-1} \qquad \frac{d}{dr}\left(\frac{dr}{dt}\right) = -2\frac{\dot{a}}{a}$$

$$d\left(\frac{dt}{dr}\frac{dr}{dt}\right) = \frac{dt}{dr}\frac{d}{dr}\frac{dr}{dt} + \frac{dr}{dt}\frac{d}{dr}\frac{dt}{dr} = 0$$

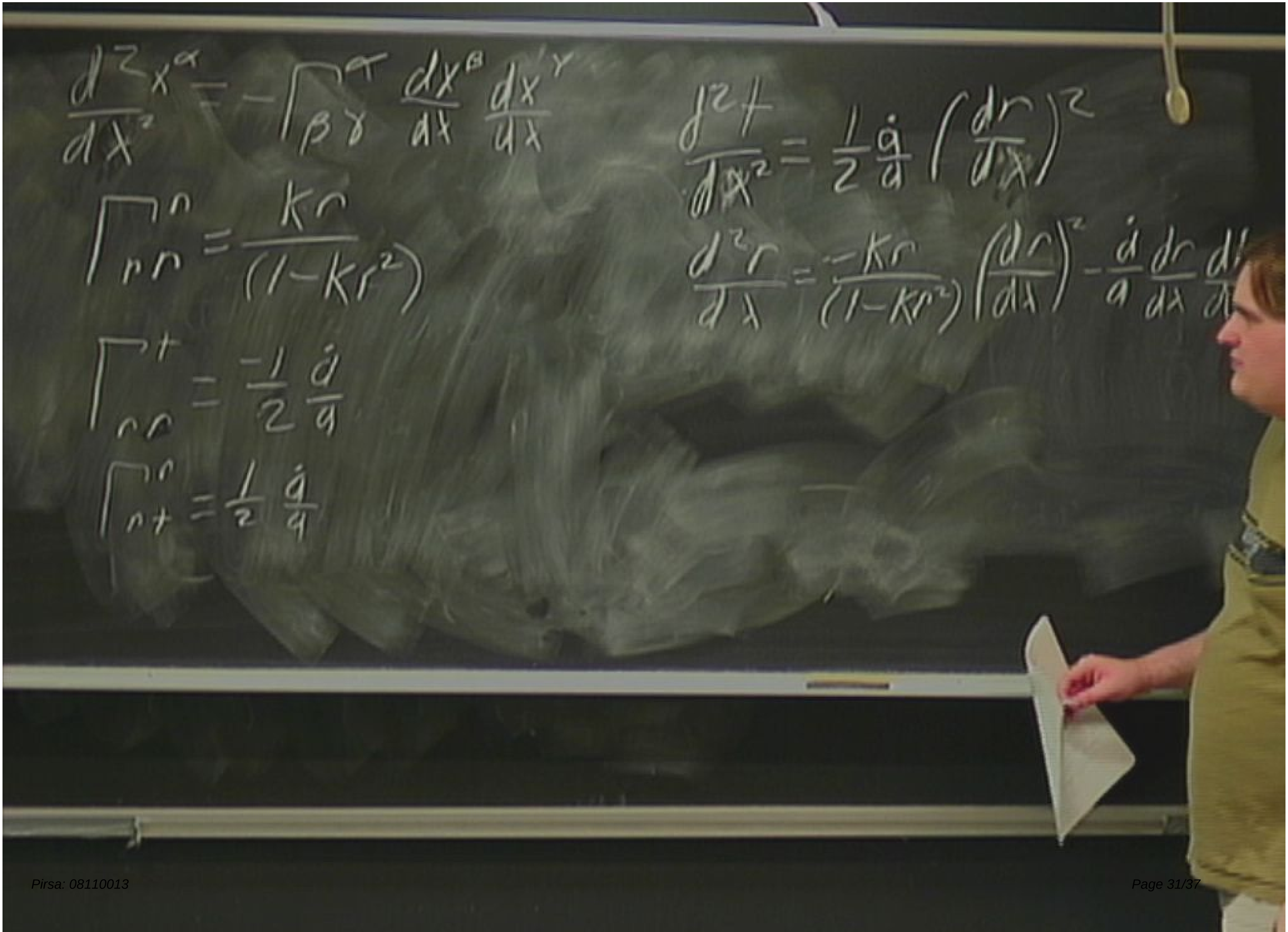
$$\Rightarrow 2\frac{\cancel{\dot{a}}}{a}\frac{1}{V} + V\frac{a\cancel{\dot{a}}}{kr^2-1} = 0$$

$$\Rightarrow \frac{2}{a} = +V^2\frac{a}{kr^2-1}$$

$$\Rightarrow V^2 = \frac{+2(kr^2-1)}{a^2}$$



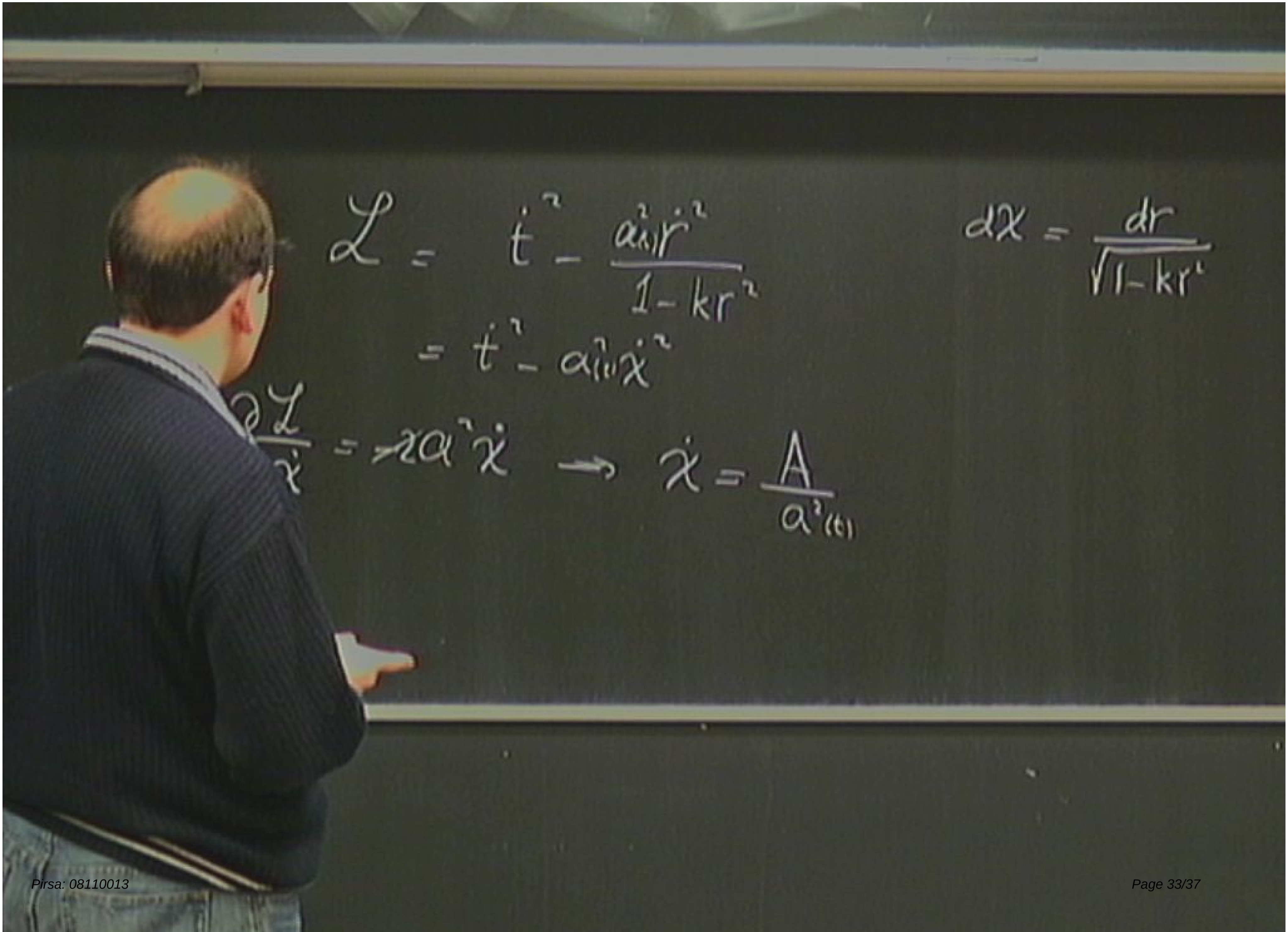
$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}$
$\Gamma^r_{rr} = \frac{kr}{(1-kr^2)}$

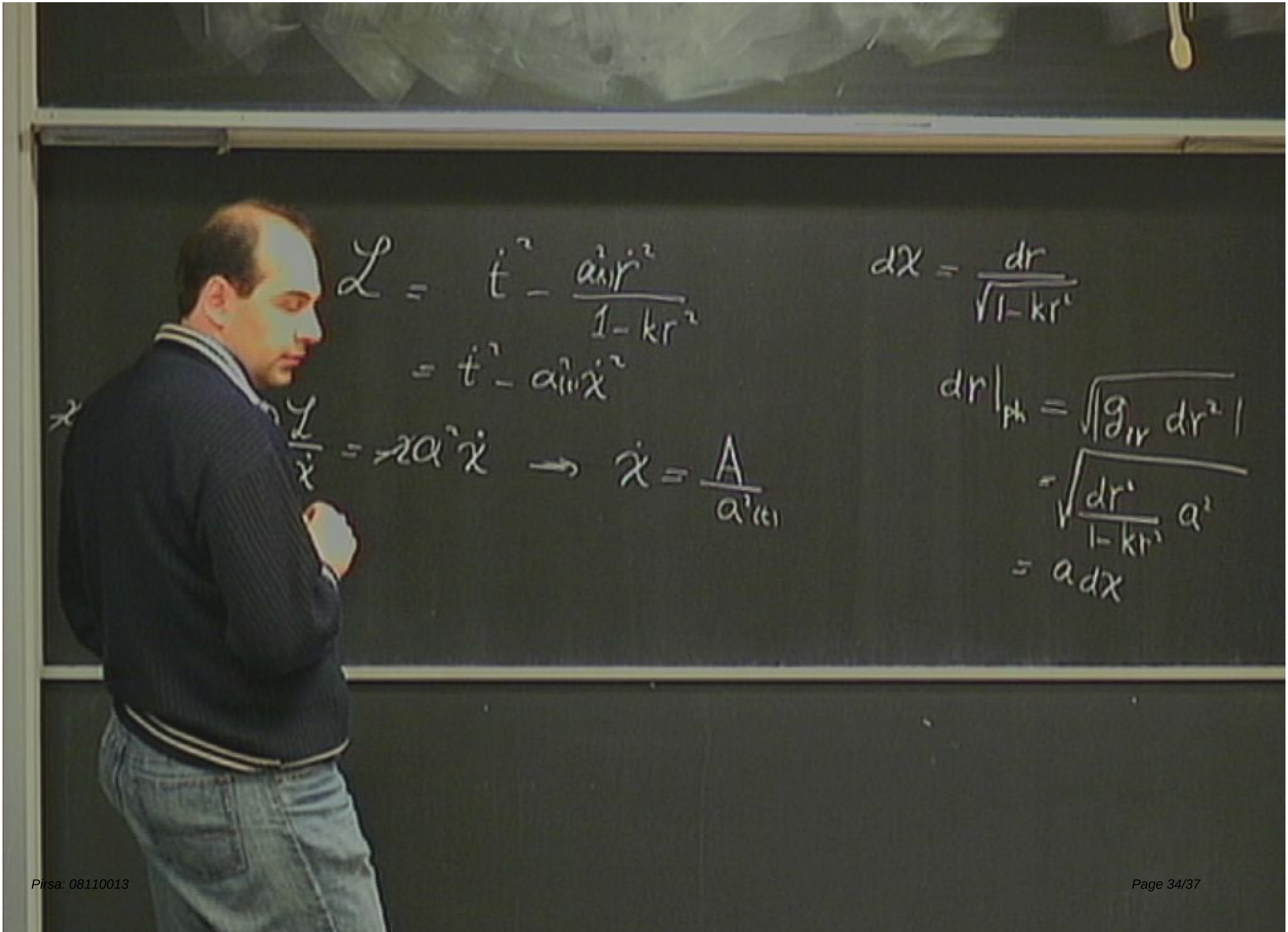

$\frac{d^2 x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda}$
$\Gamma^r_{rr} = \frac{kr}{(1-kr^2)}$
$\Gamma^t_{rr} = \frac{-1}{2} \frac{\dot a}{a}$
$\Gamma^r_{rt} = \frac{1}{2} \frac{\dot a}{a}$
$\frac{d^2 t}{d\lambda^2} = \frac{1}{2} \frac{\dot a}{a} \left(\frac{dr}{d\lambda}\right)^2$
$\frac{d^2 r}{d\lambda^2} = \frac{-kr}{(1-kr^2)} \left(\frac{dr}{d\lambda}\right)^2 - \frac{\dot a}{a} \frac{dr}{d\lambda} \frac{dt}{d\lambda}$

$$\frac{d^2 r}{dt^2} = -\frac{\dot a}{a}\frac{dr}{dt}$$

$$\dot v = -\frac{\dot a}{a} v \Rightarrow v = \frac{v_0}{a}$$

$$s = \frac{k}{a^2}$$


$\mathcal{L} = \dot{t}^2 - \frac{a^2(t)\dot{r}^2}{1-kr^2}$
$= \dot{t}^2 - a^2(t)\dot{x}^2$
$\frac{\partial \mathcal{L}}{\partial \dot{x}} = -2a^2\dot{x} \longrightarrow \dot{x} = \frac{A}{a^2(t)}$
$dx = \frac{dr}{\sqrt{1-kr^2}}$



$\mathcal{L} = \dot{t}^2 - \frac{a^2(t)\dot{r}^2}{1-kr^2}$
$= \dot{t}^2 - a^2(t)\dot{\chi}^2$
$\frac{\partial \mathcal{L}}{\partial \dot{\chi}} = -2a^2\dot{\chi} \longrightarrow \dot{\chi} = \frac{A}{a^2(t)}$
$d\chi = \frac{dr}{\sqrt{1-kr^2}}$
$dr|_{ph} = \sqrt{|g_{rr}\, dr^2|}$
$= \sqrt{\frac{dr^2}{1-kr^2}\, a^2}$
$= a\, d\chi$

$$\boxed{p^\mu = m\frac{dx^\mu}{d\tau}}$$

$$\mathcal{L} = \dot{t}^2 - \frac{a^2(t)\dot{r}^2}{1-kr^2}$$

$$= \dot{t}^2 - a^2(t)\dot{\chi}^2$$

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

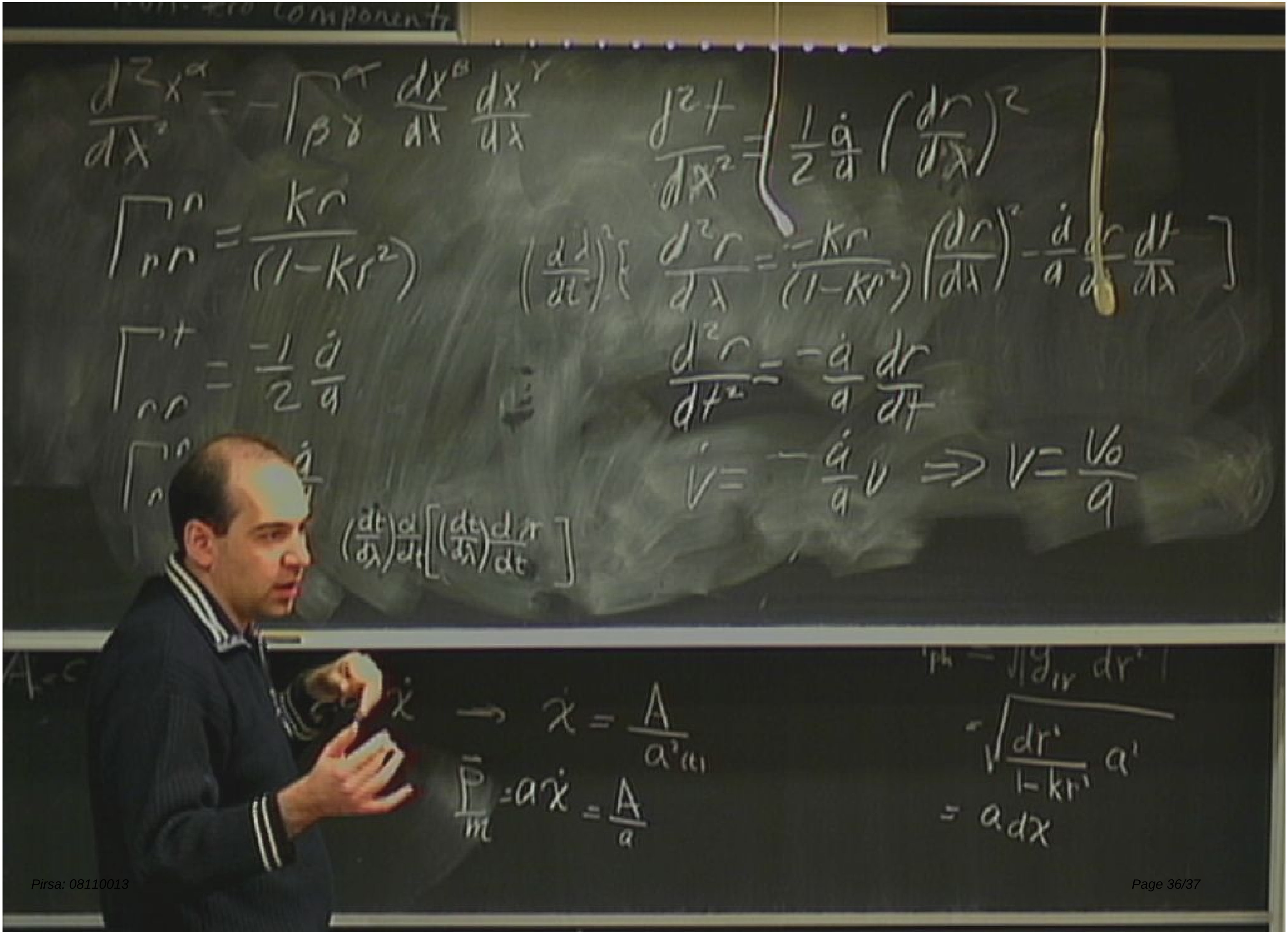
$$dr|_{ph} = \sqrt{|g_{rr}\, dr^2|}$$

$$= \sqrt{\frac{dr^2}{1-kr^2}\, a^2}$$

$$= a\, d\chi$$

$$A = \text{const} = \frac{\partial \mathcal{L}}{\partial \dot{\chi}} = 2a^2\dot{\chi} \longrightarrow \dot{\chi} = \frac{A}{a^2(t)}$$

$$\frac{\vec{p}}{m} = a\dot{\chi} = \frac{A}{a}$$


components
$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}$
$\frac{d^2t}{d\lambda^2} = \frac{1}{2}\frac{\dot a}{a}\left(\frac{dr}{d\lambda}\right)^2$
$\Gamma^r_{rr} = \frac{kr}{(1-kr^2)}$
$\frac{d^2r}{d\lambda^2} = \frac{-kr}{(1-kr^2)}\left(\frac{dr}{d\lambda}\right)^2 - \frac{\dot a}{a}\frac{dr}{d\lambda}\frac{dt}{d\lambda}$
$\Gamma^t_{rr} = \frac{-1}{2}\frac{\dot a}{a}$
$\frac{d^2r}{dt^2} = -\frac{\dot a}{a}\frac{dr}{dt}$
$\dot v = -\frac{\dot a}{a}v \Rightarrow v = \frac{v_0}{a}$
$\dot x = \frac{A}{a^2(t)}$
$\frac{P}{m} = a\dot x = \frac{A}{a}$
$= a\,dx$

$$\Gamma^{r}_{rr} = -\frac{1}{2}\frac{\dot a}{a}$$

$$\Gamma^{r}_{rt} = \frac{1}{2}\frac{\dot a}{a}$$

$$\left(\frac{dt}{d\lambda}\right)\frac{d}{dt}\left[\left(\frac{dt}{d\lambda}\right)\frac{dr}{dt}\right]$$

$$\frac{d^2 r}{dt^2} = -\frac{\dot a}{a}\frac{dr}{dt}$$

$$\dot v = -\frac{\dot a}{a} v \Rightarrow v = \frac{v_0}{a}$$

$$\boxed{p^\mu = m\frac{dx^\mu}{d\tau}}$$

$$\mathcal{L} = \dot t^2 - \frac{a^2(t)\dot r^2}{1-kr^2}$$

$$= \dot t^2 - a^2(t)\dot\chi^2$$

$$d\chi = \frac{dr}{\sqrt{1-kr^2}}$$

$$dr|_{ph} = \sqrt{|g_{rr}\,dr^2|} = \sqrt{\frac{dr^2}{1-kr^2}a^2} = a\,d\chi$$

$$A = \text{const} = \frac{\partial \mathcal{L}}{\partial \dot\chi} = \cancel{2}a^2\dot\chi \longrightarrow \dot\chi = \frac{A}{a^2(t)}$$

$$\boxed{\frac{p}{m} = a\dot\chi = \frac{A}{a}}$$