Title: Quantum Field Theory 1 - Lecture 9A

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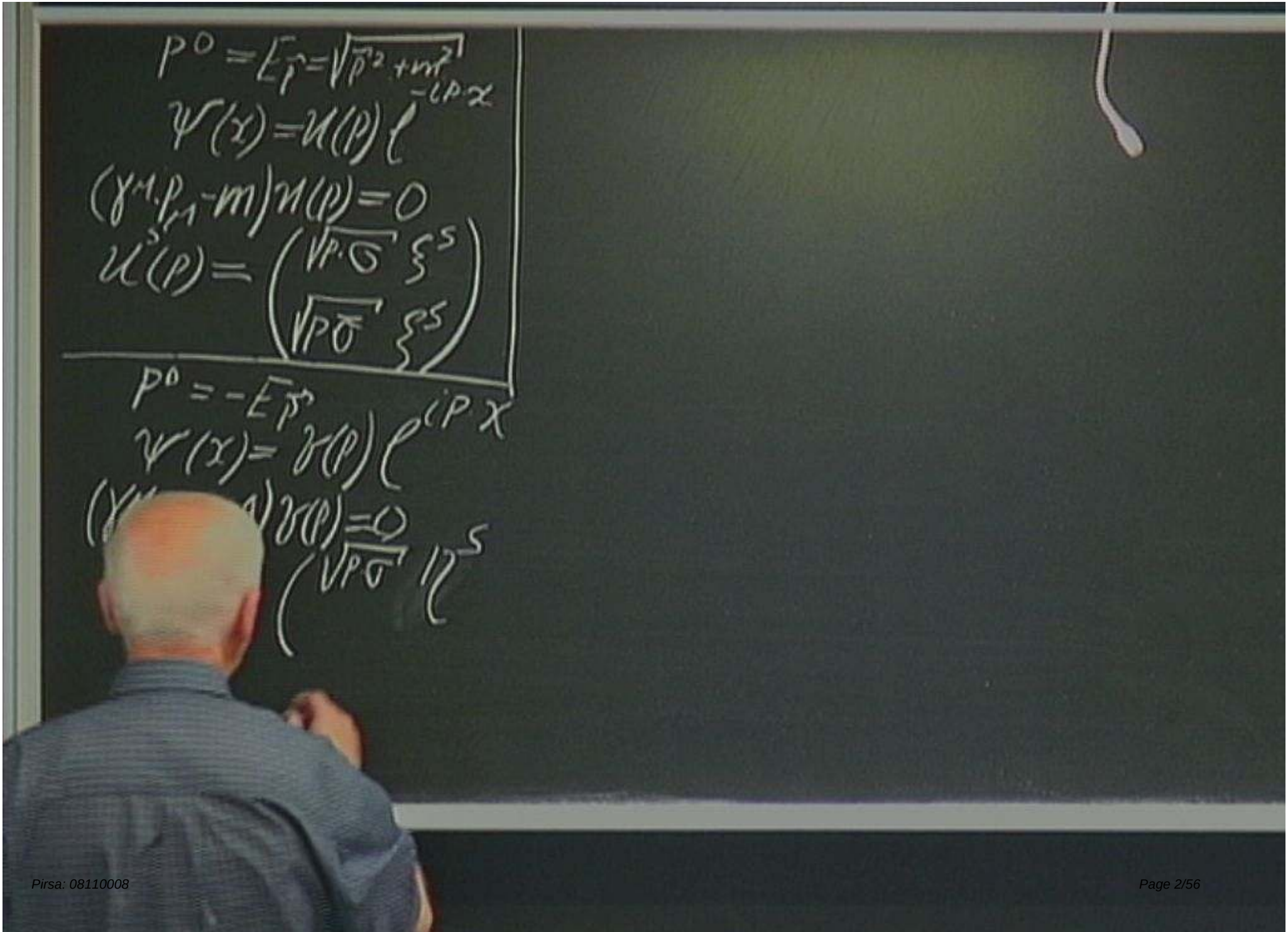
Abstract: Quantum Field Theory I course taught by Volodya Miransky of the University of Western Ontario


$p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$
$\psi(x) = u(p) e^{-ip\cdot x}$
$(\gamma^\mu p_\mu - m) u(p) = 0$
$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix}$
$p^0 = -E_p$
$\psi(x) = v(p) e^{ip x}$
$\begin{pmatrix} \sqrt{p\sigma}\,\eta^s \end{pmatrix}$

$$\psi(x) = u(p)\, e^{-ip\cdot x}$$

$$(\gamma^\mu p_\mu - m)\, u(p) = 0$$

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix}$$

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$$\psi(x) = v(p)\, e^{ip\cdot x}$$

$$(\gamma^\mu p_\mu + m)\, v(p) = 0$$

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^{-s} \\ -\sqrt{p\bar{\sigma}}\,\eta^s \end{pmatrix}$$

Quantization of Dirac Field

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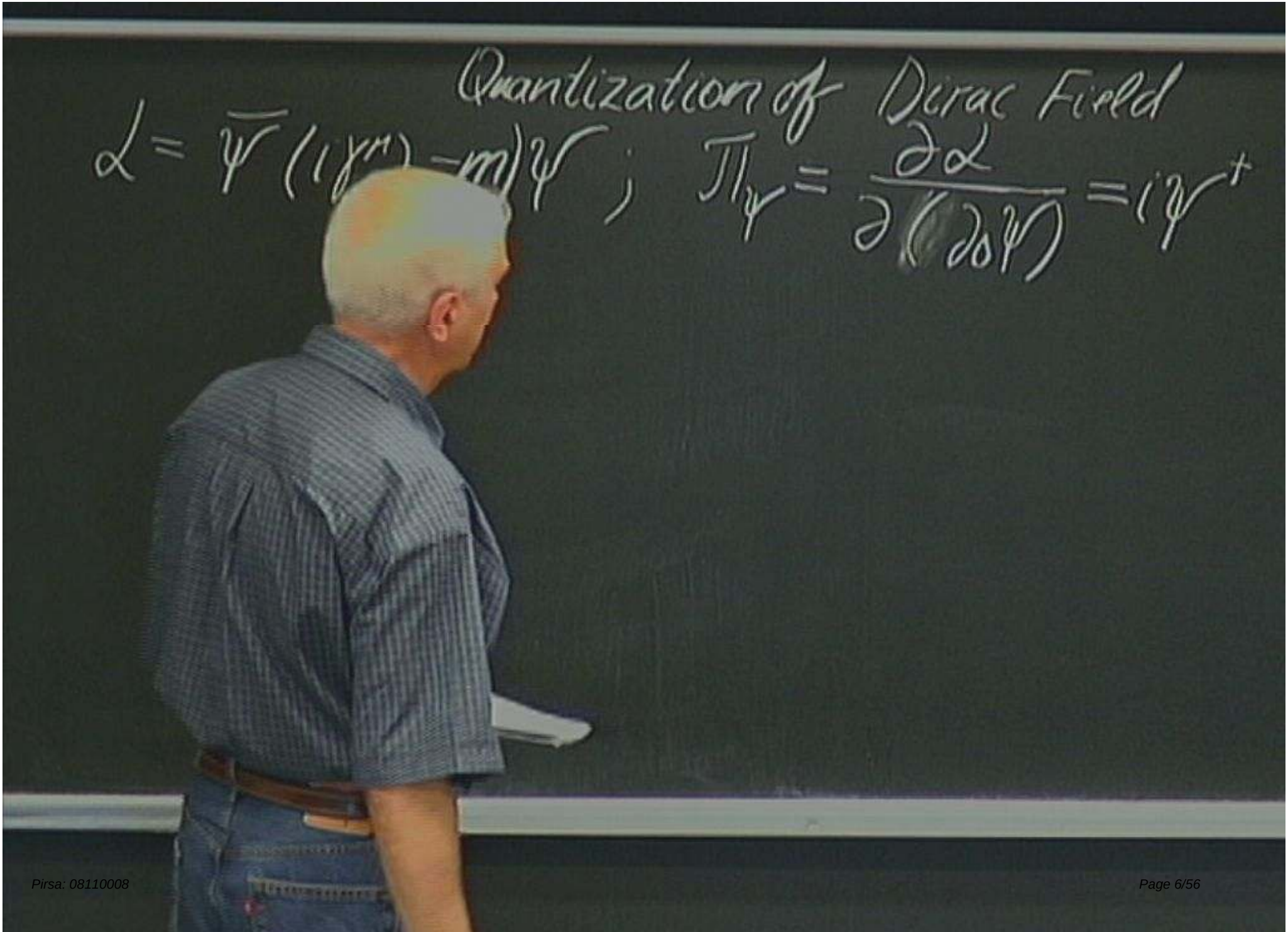
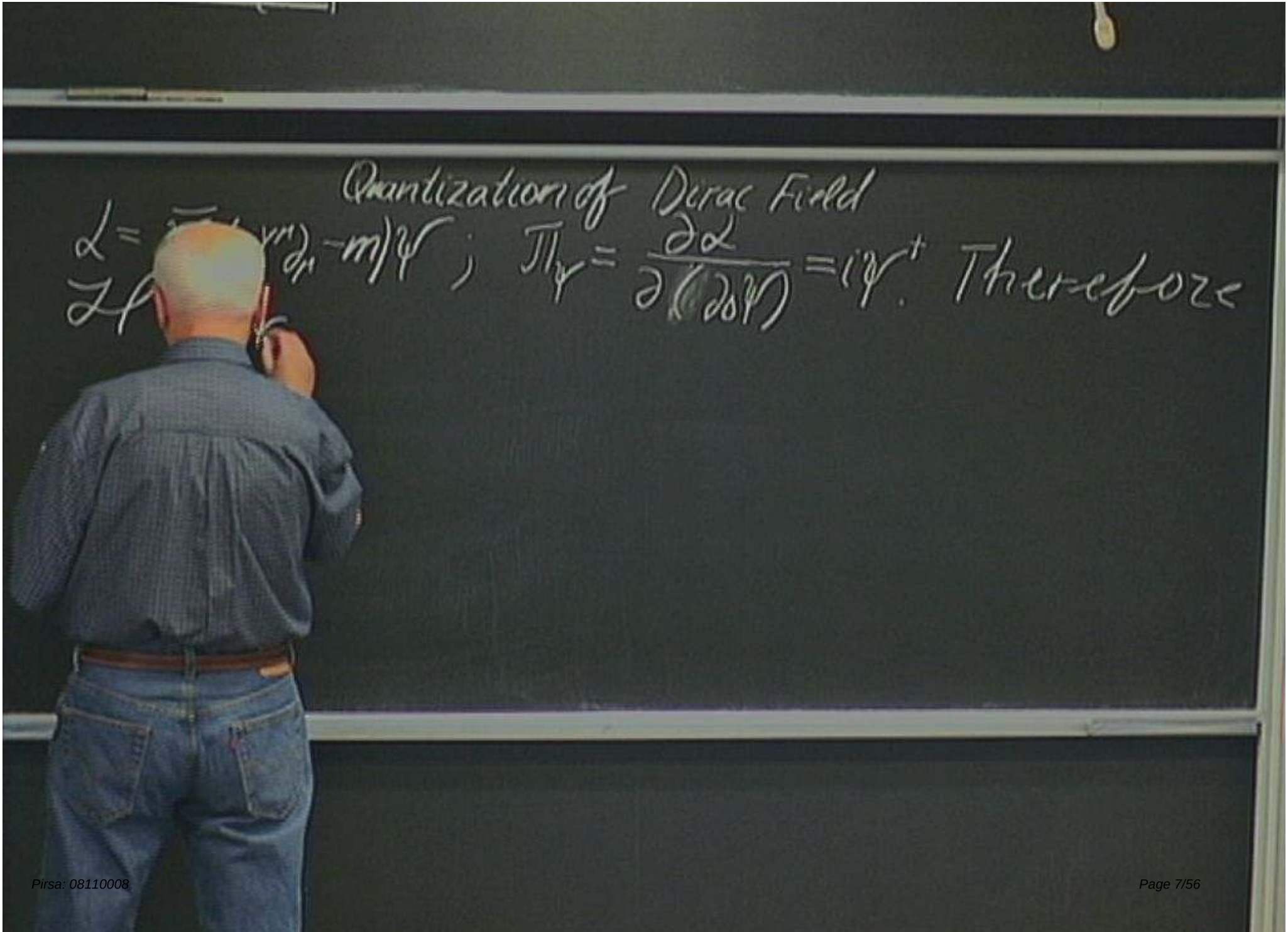
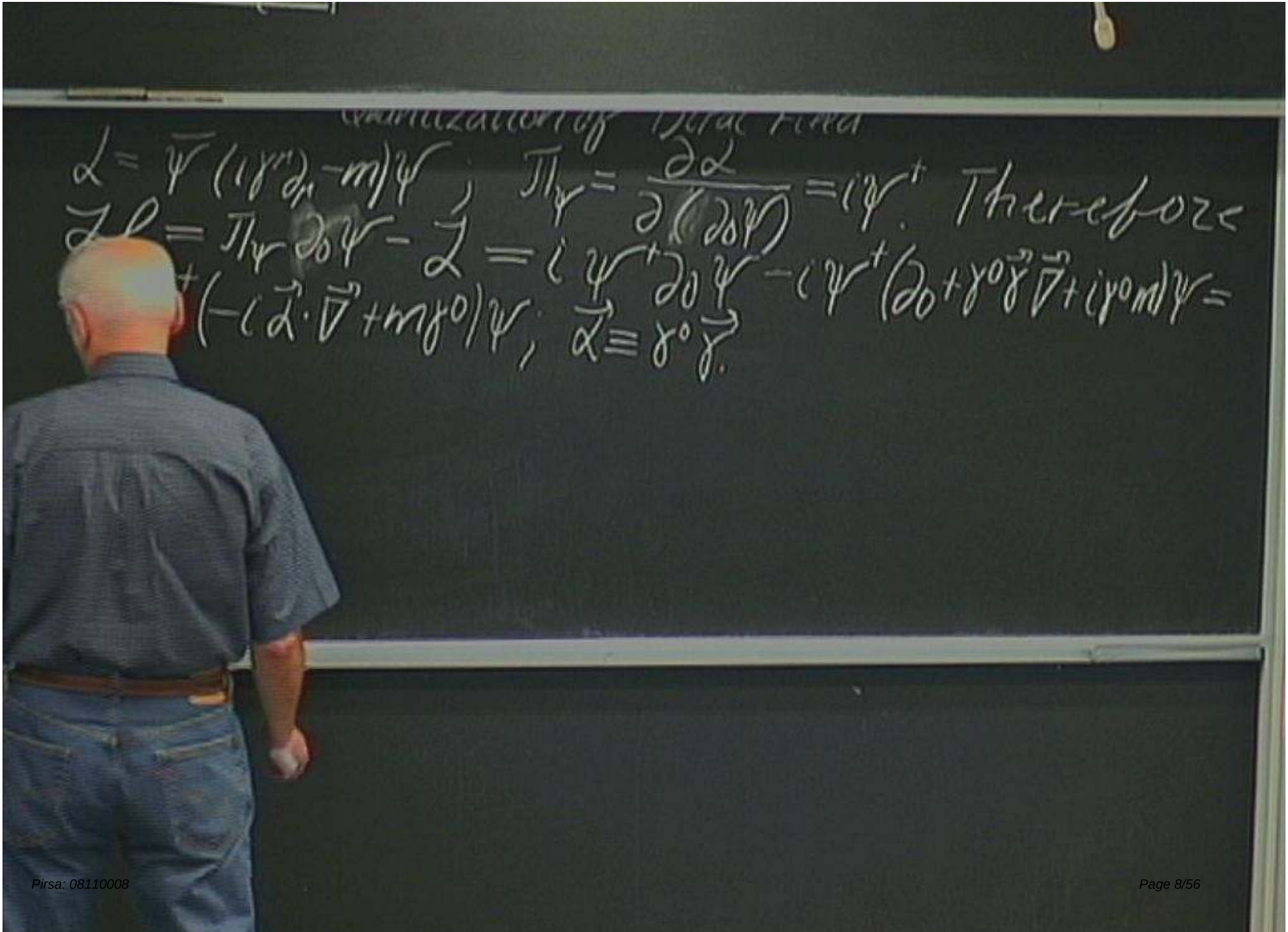
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi ;$$


Quantization of Dirac Field
$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$; $\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi^+$


Quantization of Dirac Field
$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$; $\Pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi^\dagger$ Therefore

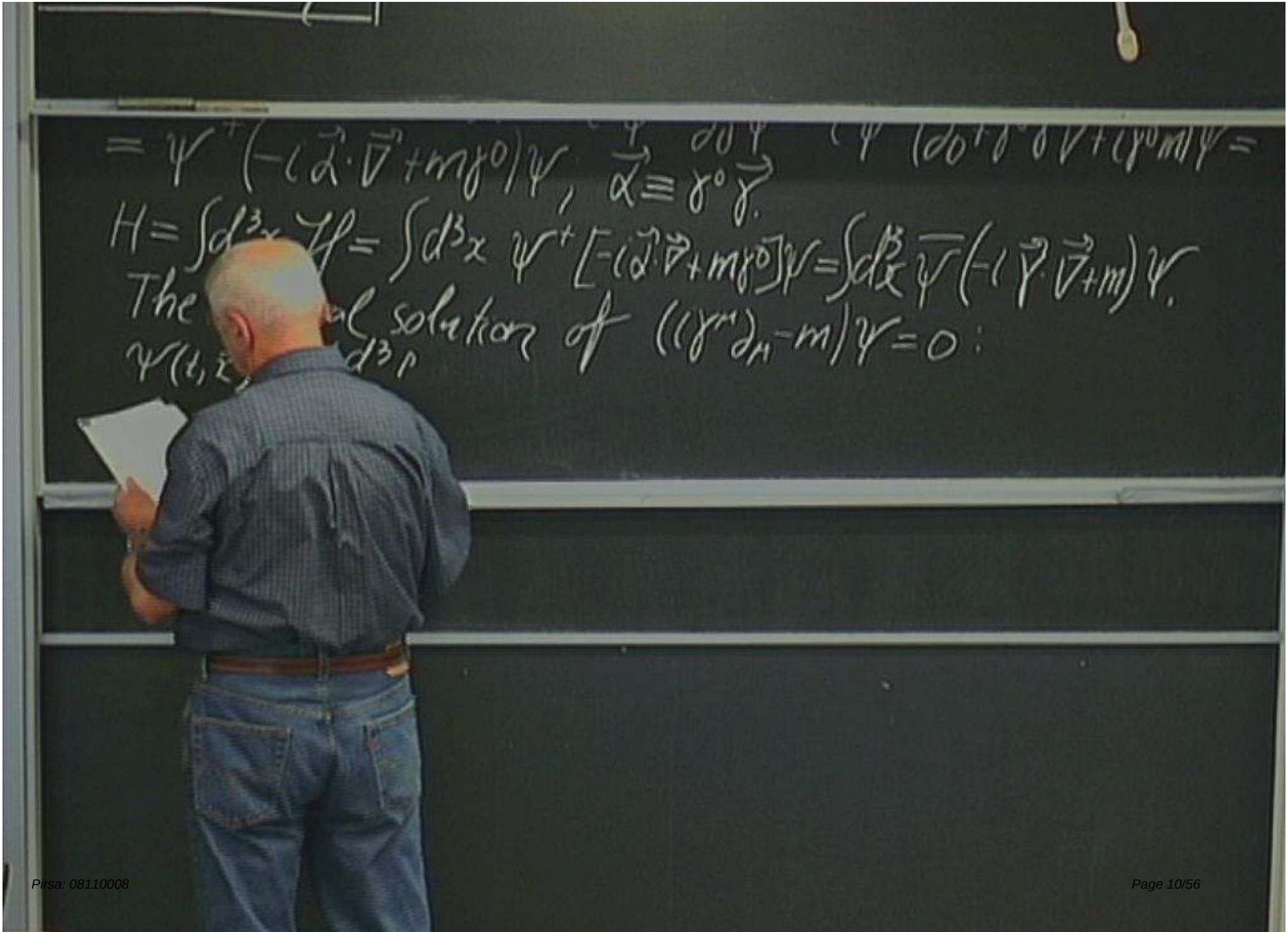

Quantization of Dirac Field
$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$, $\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi^\dagger$. Therefore
$\mathcal{H} = \pi_\psi \partial_0 \psi - \mathcal{L} = i\psi^\dagger \partial_0 \psi - i\psi^\dagger(\partial_0 + \gamma^0\vec{\gamma}\cdot\vec{\nabla} + i\gamma^0 m)\psi =$
$(-i\vec{\alpha}\cdot\vec{\nabla} + m\gamma^0)\psi$, $\vec{\alpha} \equiv \gamma^0\vec{\gamma}$.

$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$, $\pi_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} = i\psi^\dagger$. Therefore

$\mathcal{H} = \pi_\psi \partial_0\psi - \mathcal{L} = i\psi^\dagger\partial_0\psi - i\bar{\psi}(\partial_0 + \gamma^0\vec{\gamma}\cdot\vec{\nabla} + i\gamma^0 m)\psi =$

$= \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + m\gamma^0)\psi$, $\vec{\alpha} \equiv \gamma^0\vec{\gamma}$.

$H = \int d^3x\, \mathcal{H} = \int d^3x\, \psi^\dagger[-i\vec{\alpha}\cdot\vec{\nabla} + m\gamma^0]\psi = \int d^3x\, \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi$.


$= \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + m\gamma^0)\psi$, $\vec{\alpha} \equiv \gamma^0\vec{\gamma}$.
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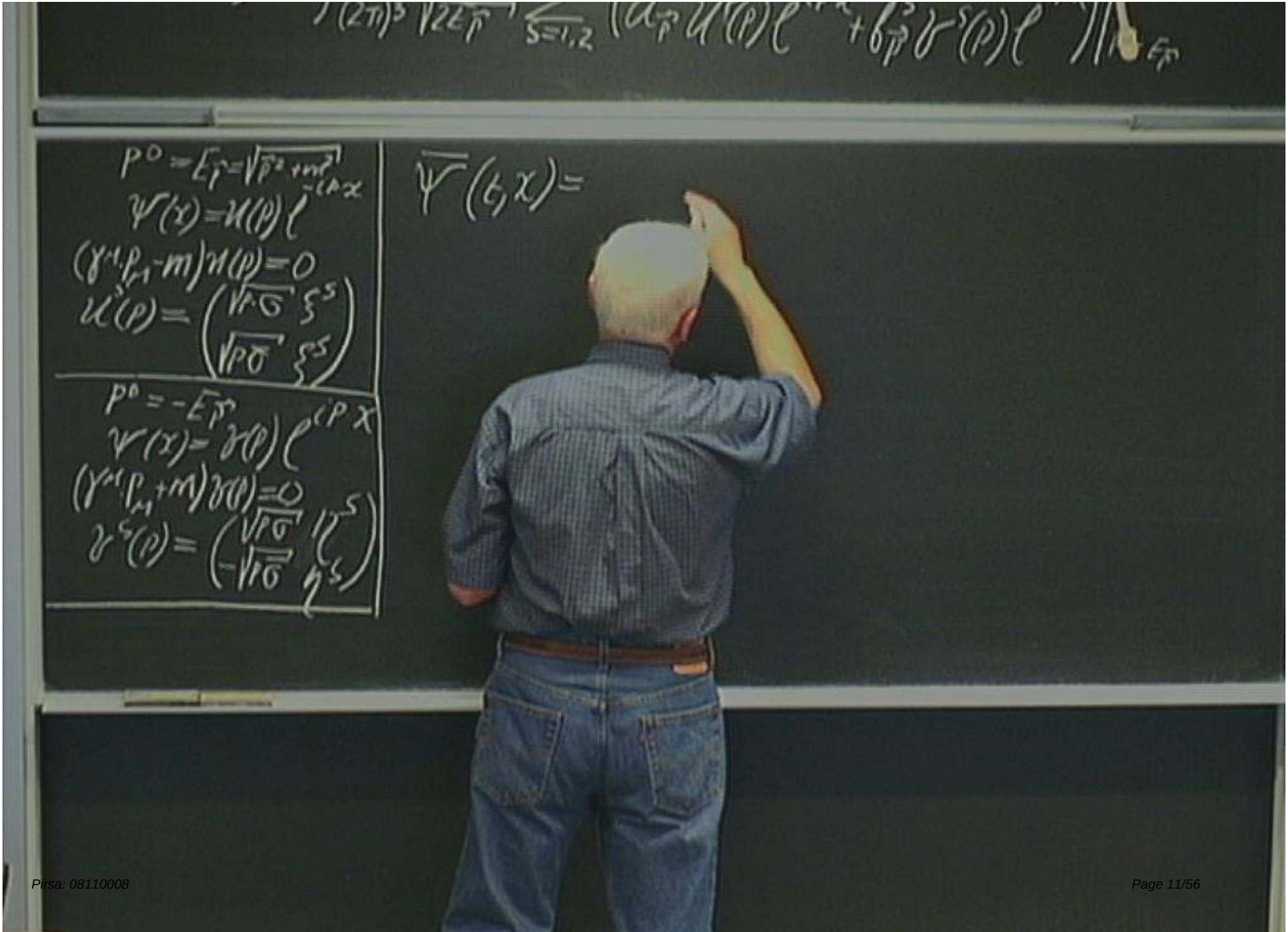
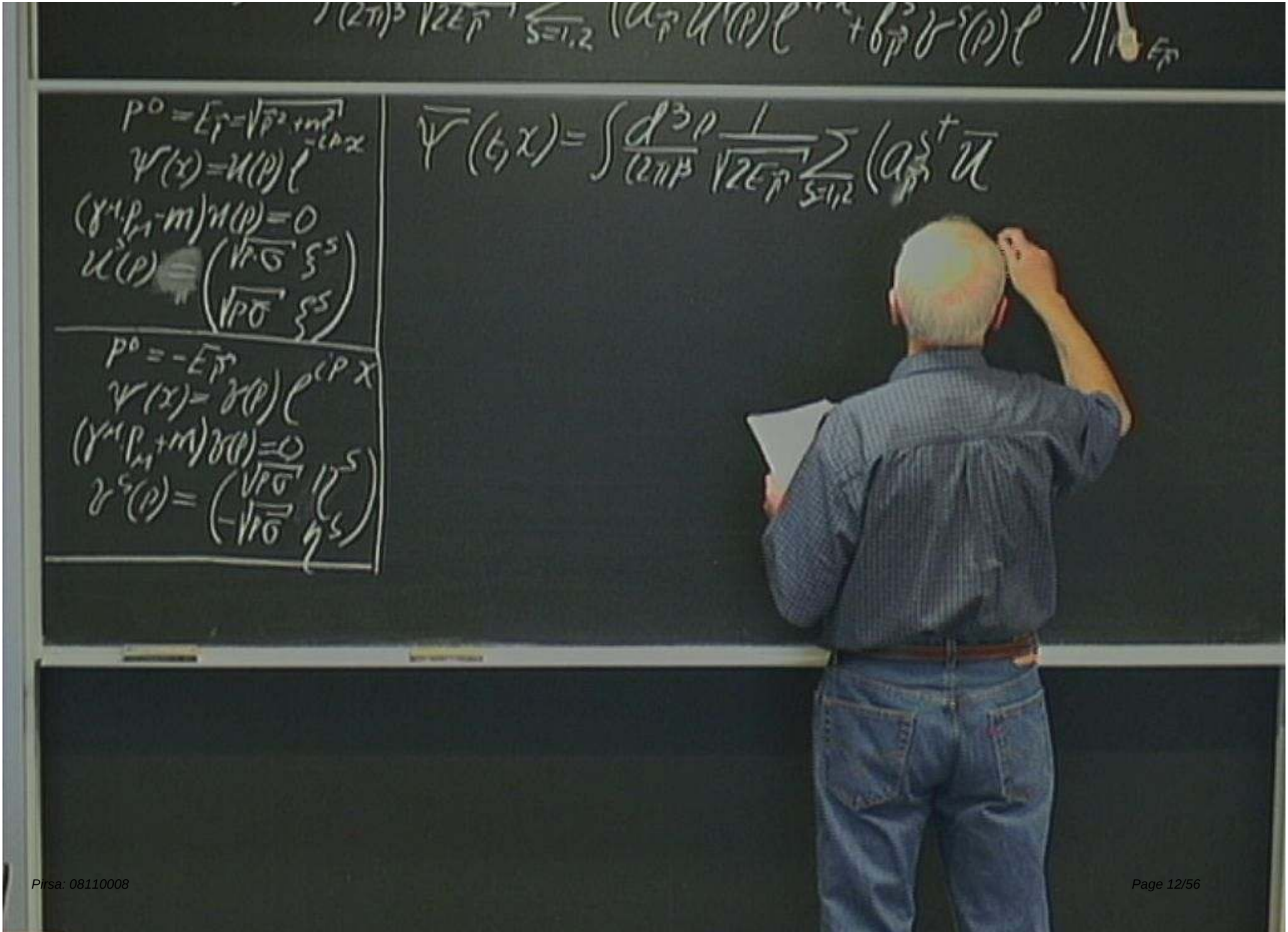
The
solution of $((\gamma^\mu\partial_\mu - m)\psi = 0$:
$\psi(t,\vec{x})$
d^3p


$\sum_{s=1,2}$
$p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$
$\psi(x) = u(p) e^{-ipx}$
$(\gamma^\mu p_\mu - m) u(p) = 0$
$u^s(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix}$
$p^0 = -E_p$
$\psi(x) = v(p) e^{ipx}$
$(\gamma^\mu p_\mu + m) v(p) = 0$
$v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\,\eta^s \\ -\sqrt{p\bar{\sigma}}\,\eta^s \end{pmatrix}$
$\bar{\psi}(t,x) =$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$(\gamma^\mu p_\mu + m) v(p) = 0$
$v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\,\eta^s \\ -\sqrt{p\bar{\sigma}}\,\eta^s \end{pmatrix}$
$\overline{\psi}(t,x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} (a_p^{s\dagger} \bar{u}$

$$\frac{1}{(2\pi)^3\sqrt{2E_{\vec p}}}\sum_{s=1,2}\left(a_{\vec p}\,u(p)e^{ipx}+b^{s}_{\vec p}\,v^{s}(p)e^{\dots}\right)\Big|_{E_{\vec p}}$$

$p^0 = E_{\vec p} = \sqrt{\vec p^2 + m^2}$

$\psi(x) = u(p)\,e^{-ipx}$

$(\gamma^\mu p_\mu - m)\,u(p) = 0$

$u^s(p) = \begin{pmatrix} \sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar\sigma}\,\xi^s \end{pmatrix}$

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$(\gamma^\mu p_\mu + m)\,v(p) = 0$

$v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\,\eta^s \\ -\sqrt{p\bar\sigma}\,\eta^s \end{pmatrix}$

$$\bar\psi(t,x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\sum_{s=1,2}\left(a^{s\dagger}_{\vec p}\,\bar u^s(p)e^{ipx} + b^{s\dagger}_{\vec p}\,\bar v^s e^{-ipx}\right)\Big|_{p^0=E_{\vec p}}$$

$$\frac{1}{(2\pi)^3\sqrt{2E_{\vec p}}}\sum_{s=1,2}\left(a_{\vec p}\,u^s(p)\,e^{ipx}+b^{s\dagger}_{\vec p}\,v^s(p)\,e^{-ipx}\right)\Big|_{p^0=E_{\vec p}}$$

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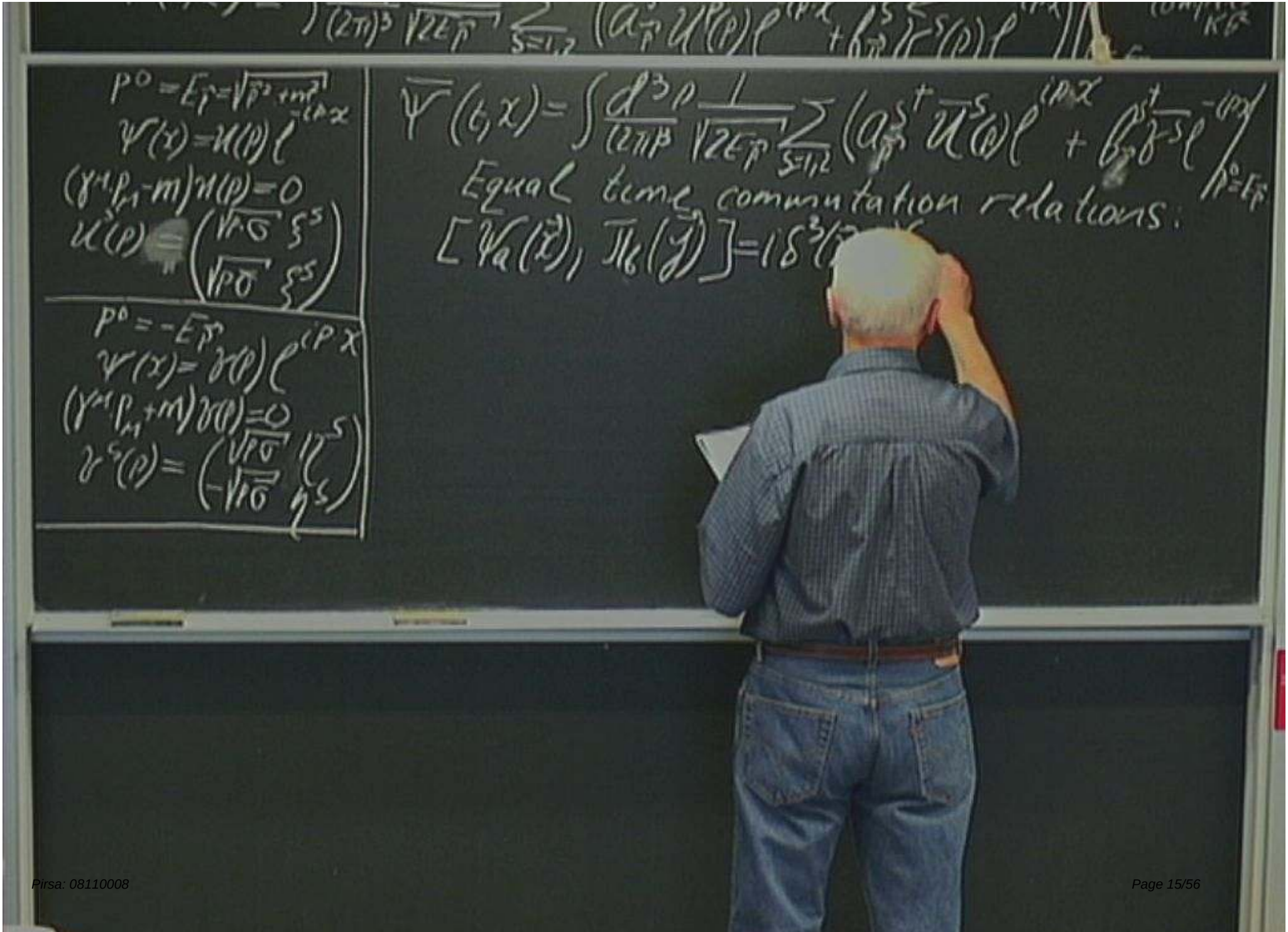
$p^0 = E_p = \sqrt{\vec{p}^2 + m^2}$
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$\Psi(t,x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(a_p^s u^s(p) e^{ipx} + b_p^{s\dagger} v^s(p) e^{-ipx} \right)_{p^0 = E_p}$
Equal time commutation relations:
$[\Psi_a(\vec{x}), \Pi_b(\vec{y})] = i\delta^3(\vec{x} - \vec{y})$

$p^0 = E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

$\psi(x) = u(p) e^{-ipx}$

$(\gamma^\mu p_\mu - m) u(p) = 0$

$u^s(p) = \begin{pmatrix} \sqrt{p\sigma}\, \xi^s \\ \sqrt{p\bar{\sigma}}\, \xi^s \end{pmatrix}$

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$v^s(p) = \begin{pmatrix} \sqrt{p\sigma}\, \eta^s \\ -\sqrt{p\bar{\sigma}}\, \eta^s \end{pmatrix}$

$$\hat{\psi}(t,x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(a^s_{\vec{p}}\, u^s(p) e^{ipx} + b^{s\dagger}_{\vec{p}}\, v^s(p) e^{-ipx} \right)_{p^0 = E_{\vec{p}}}$$

Equal time commutation relations:

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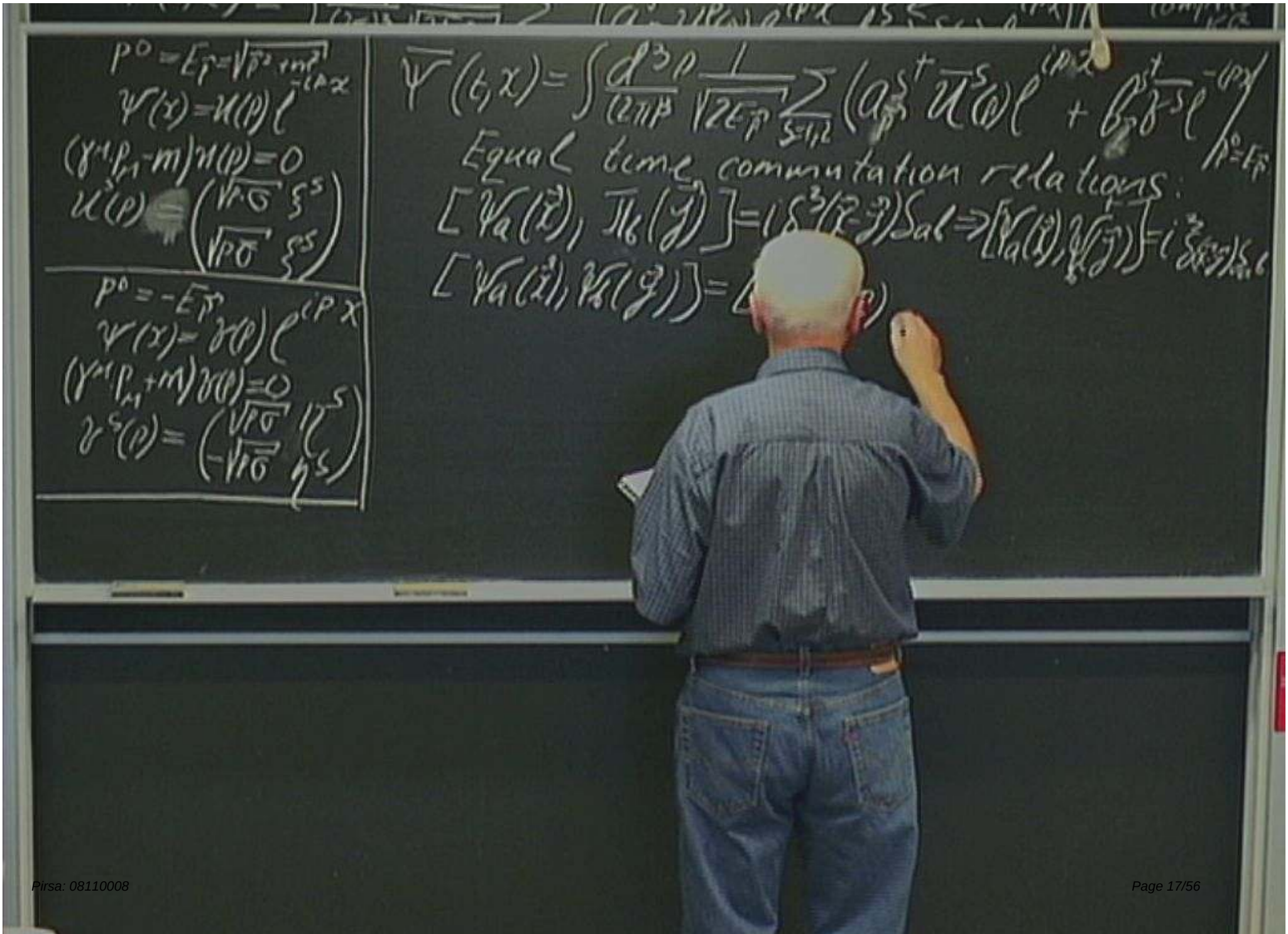
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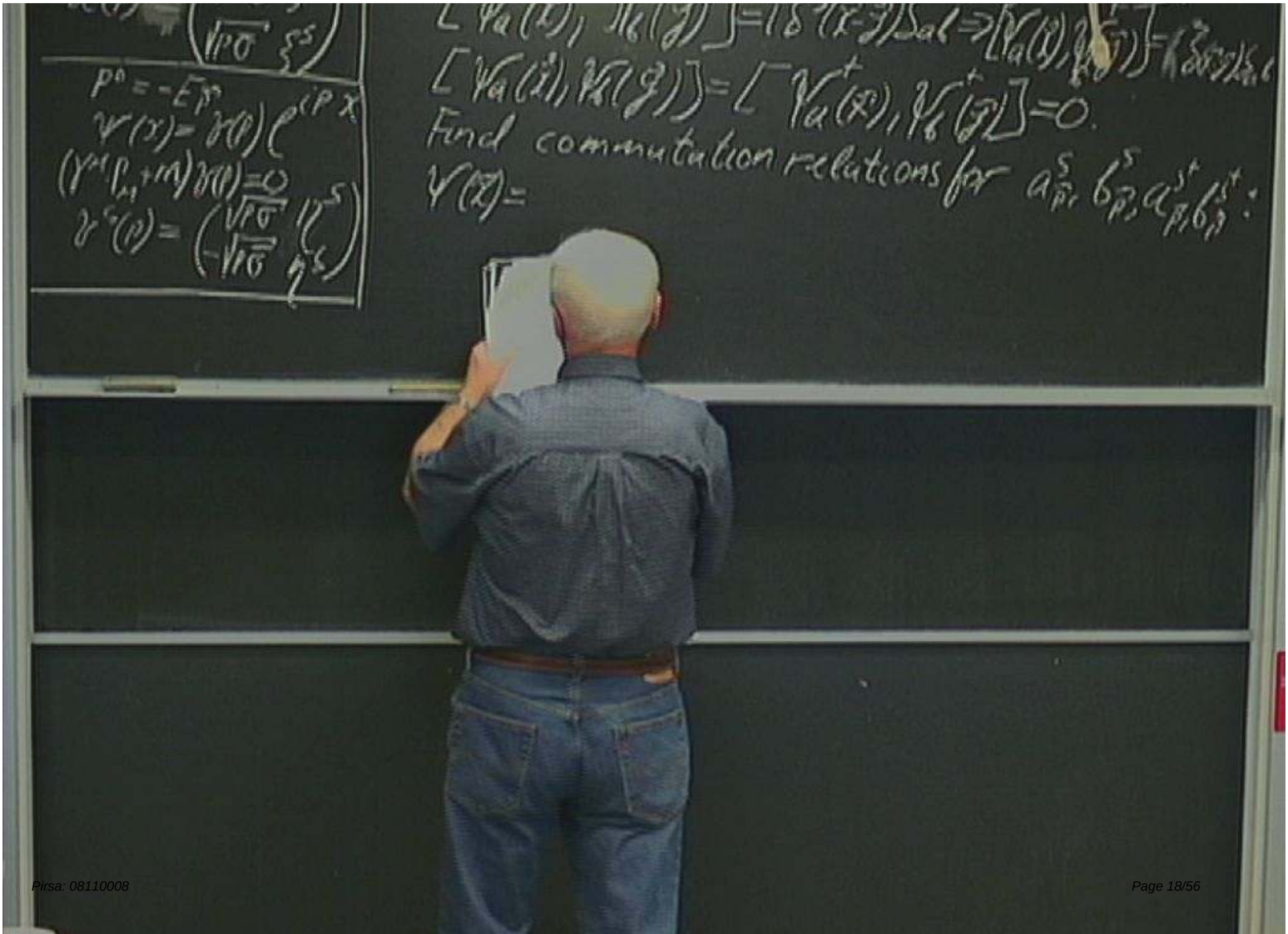
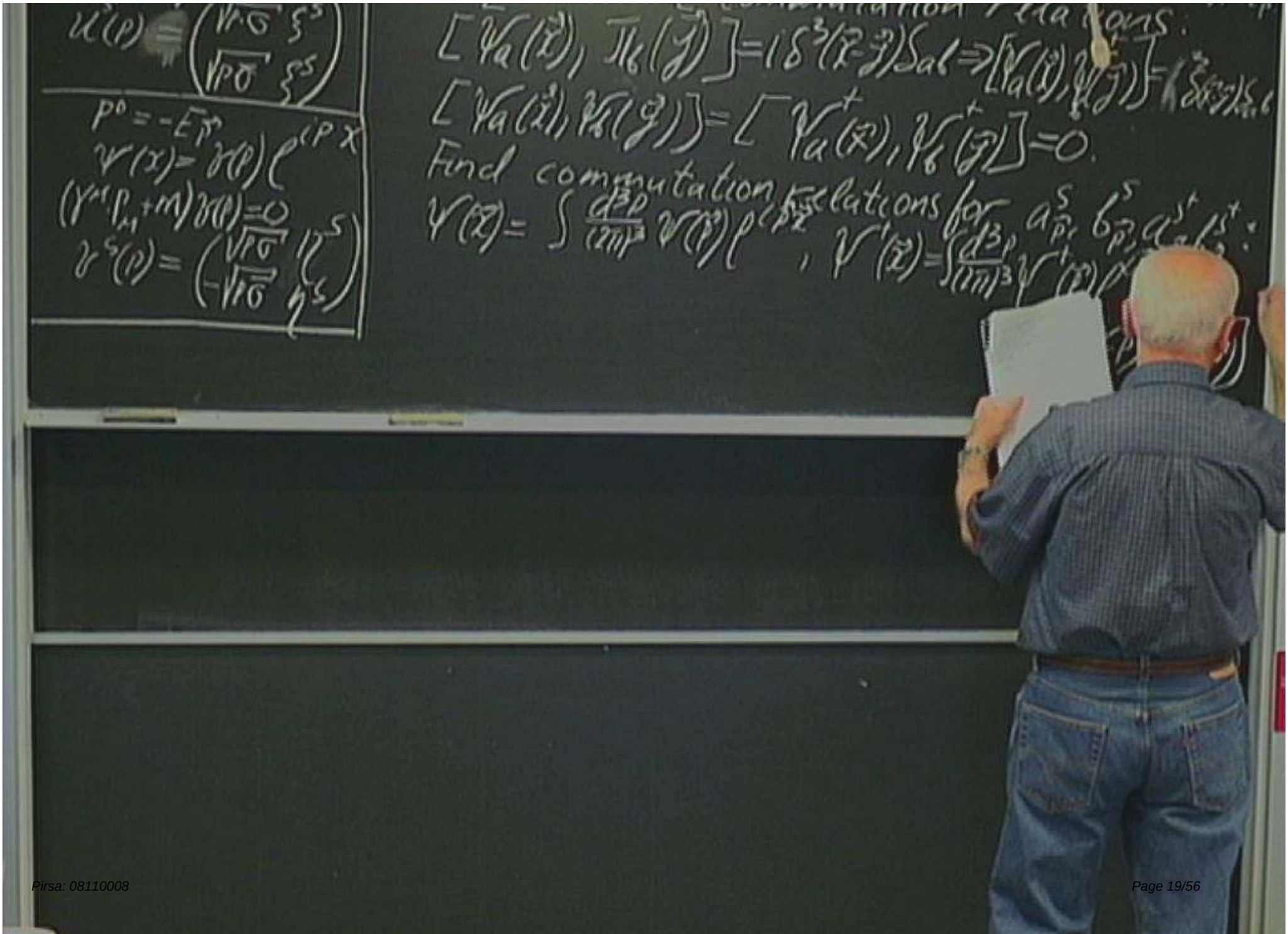
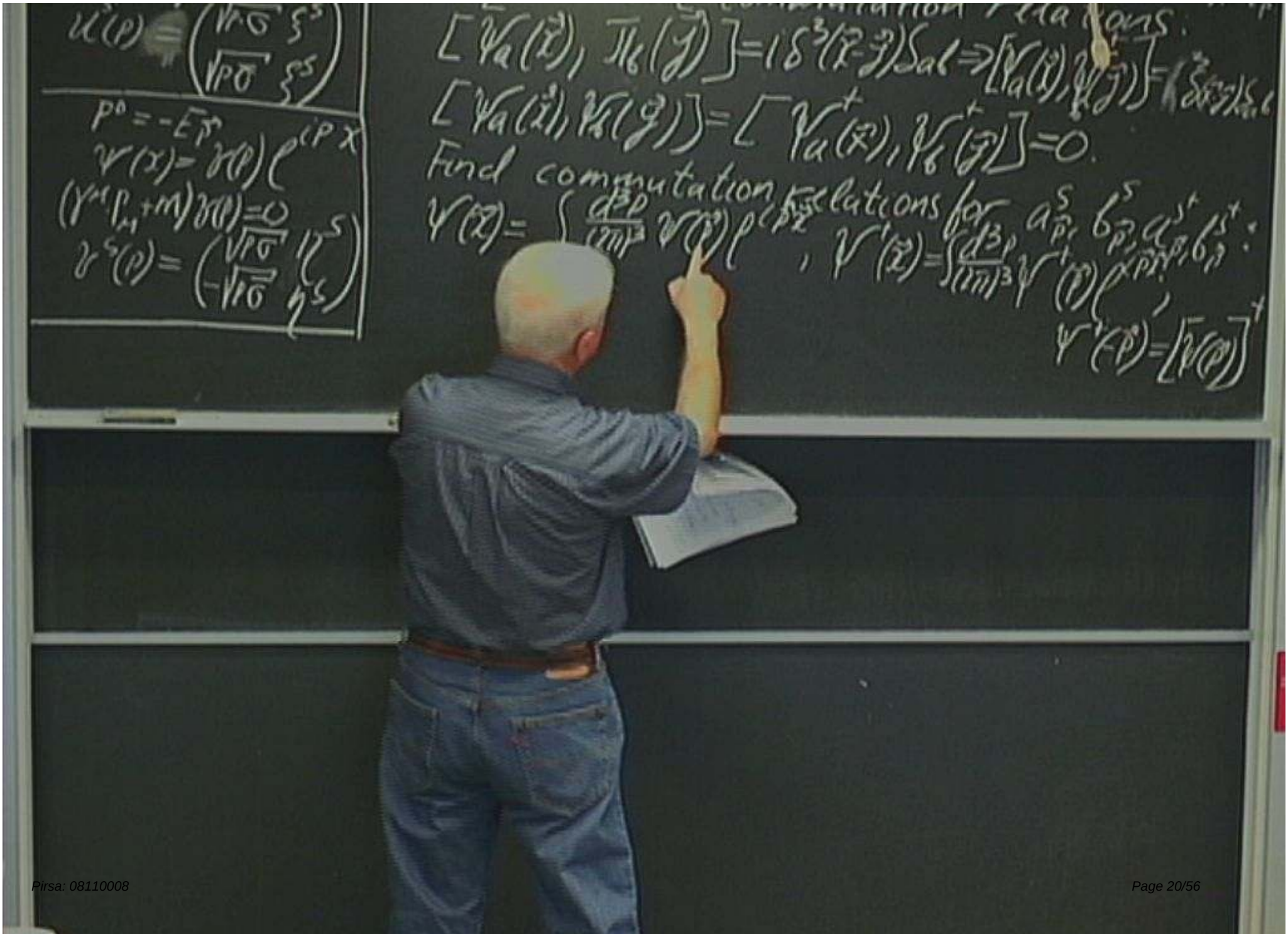
$p^0 = E_p$
Equal time commutation relations:
$[\psi_a(\vec{x}), \pi_b(\vec{y})] = i\delta^3(\vec{x}-\vec{y})\delta_{ab} \Rightarrow [\psi_a(\vec{x}), \psi_b(\vec{y})] = i\delta^3(\vec{x}-\vec{y})\delta_{ab}$
$[\psi_a(\vec{x}), \psi_b(\vec{y})] = 0$


Find commutation relations for
$\psi(x) =$


$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}$
$p^0 = -E_p$
$\psi(x) = a(p)\,e^{ipx}$
$(\gamma^\mu p_\mu + m)\,a(p) = 0$
$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix}$
$[\psi_a(\vec x), \pi_b(\vec y)] = i\delta^3(\vec x - \vec y)\delta_{ab} \Rightarrow [\psi_a(\vec x), \psi_b(\vec y)] = [\psi_a(\vec x), \psi_b(\vec y)] = 0$
$[\psi_a(\vec x), \psi_b(\vec y)] = [\psi_a^+(\vec x), \psi_b^+(\vec y)] = 0$.
Find commutation relations for
$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\, a(p)\, e^{ipx}$, $\psi^+(x) = \int \frac{d^3p}{(2\pi)^3}\, a^+(p)\, e^{-ipx}$


Find commutation relations for
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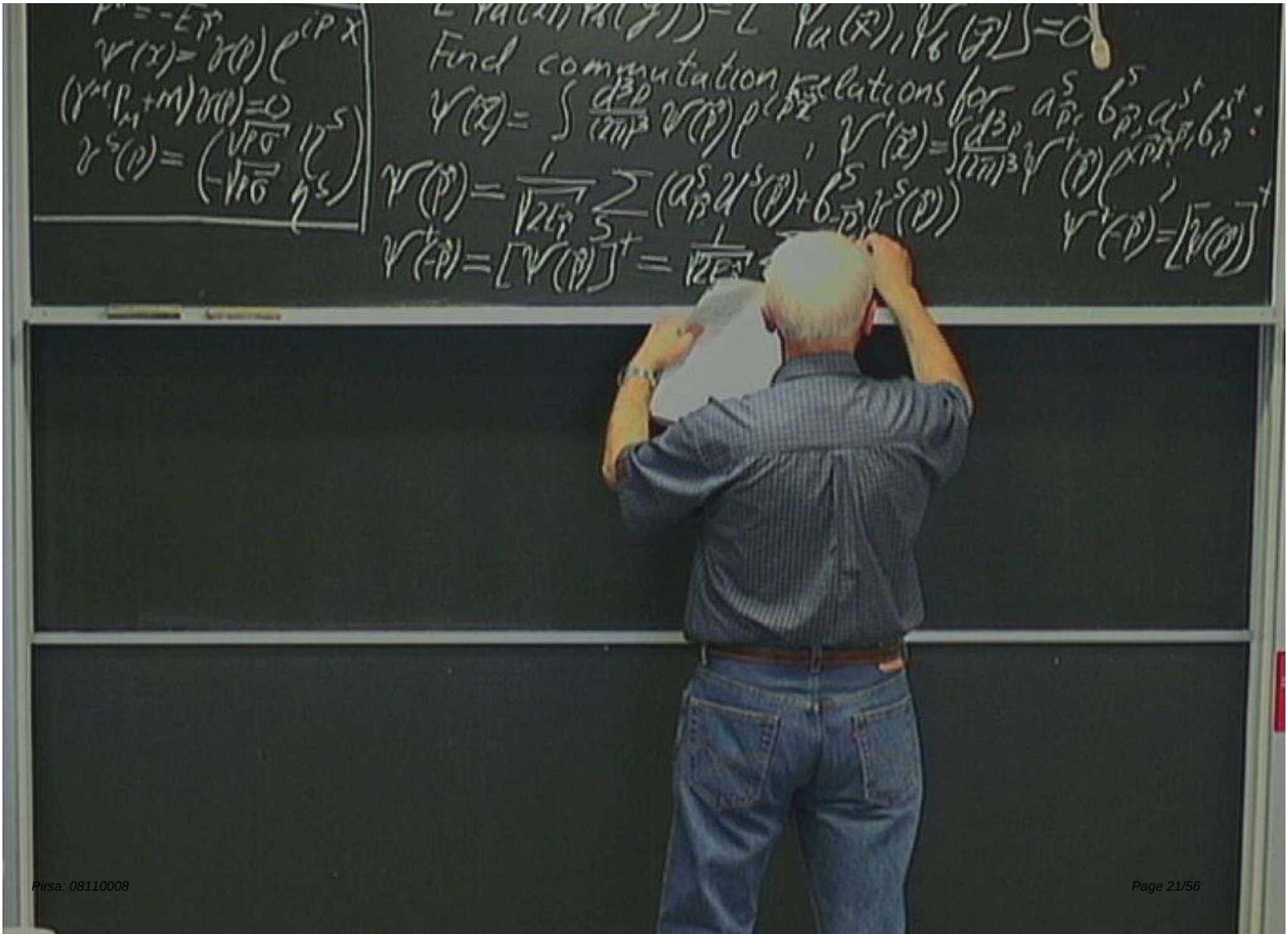
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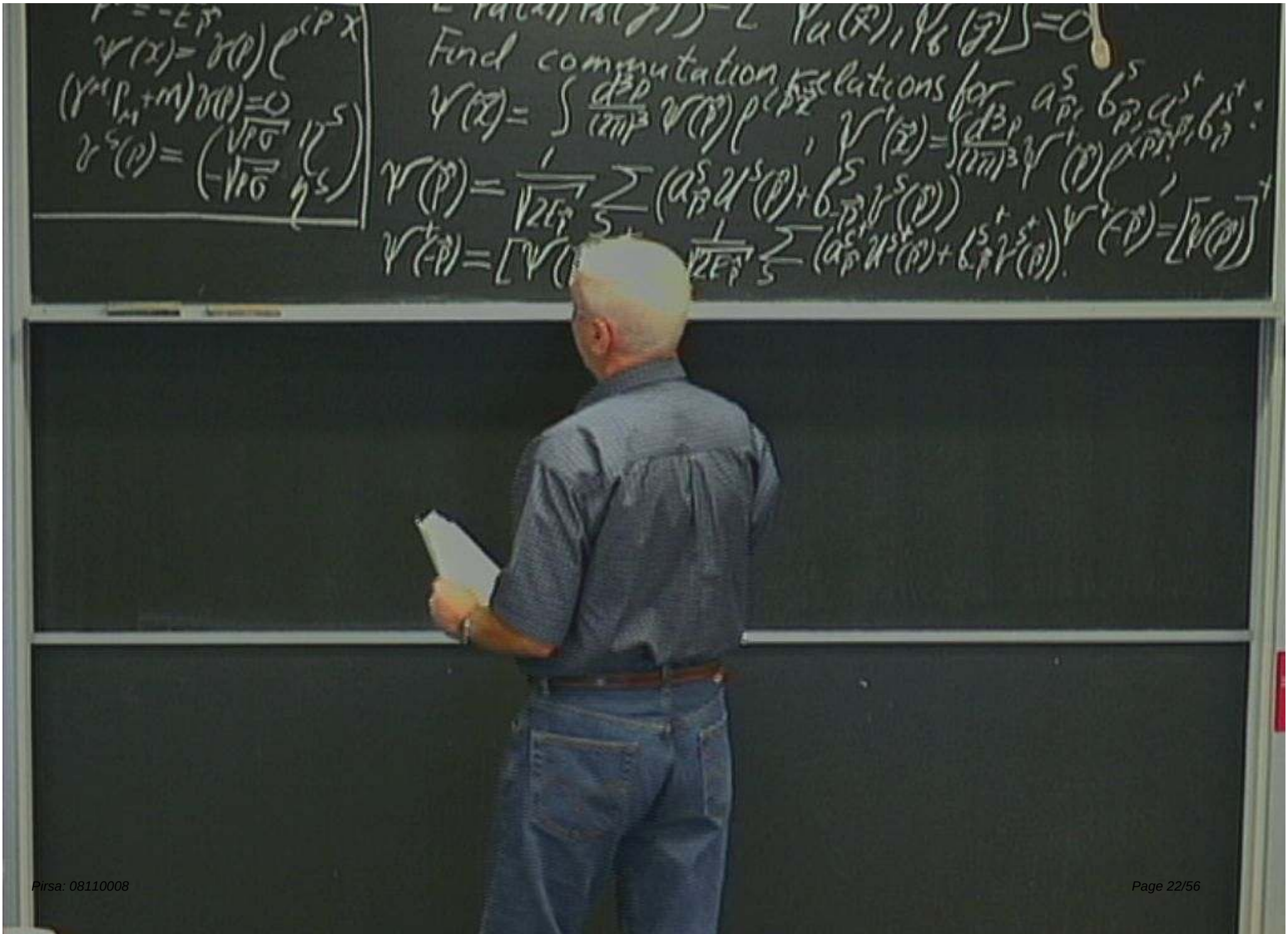
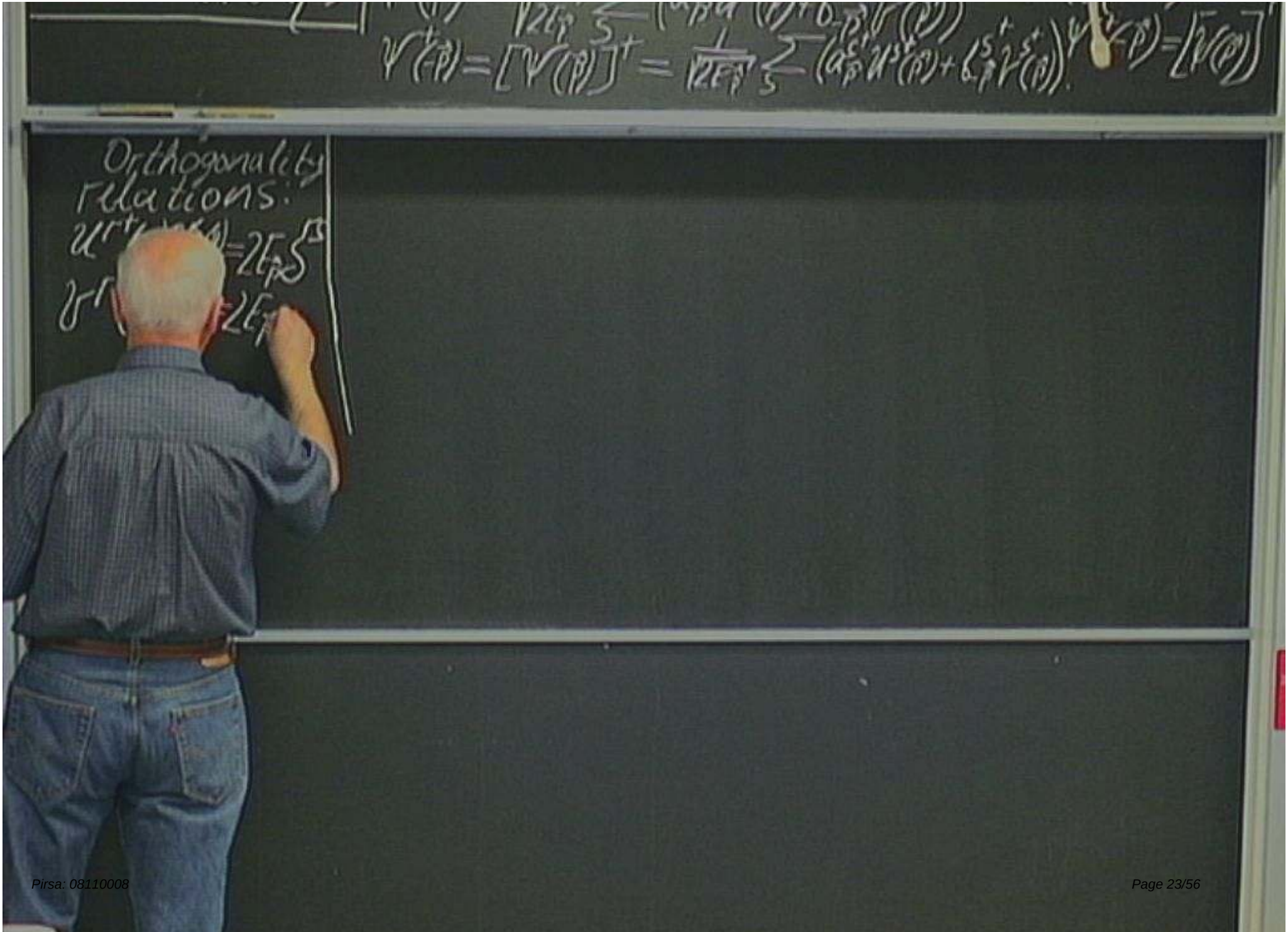
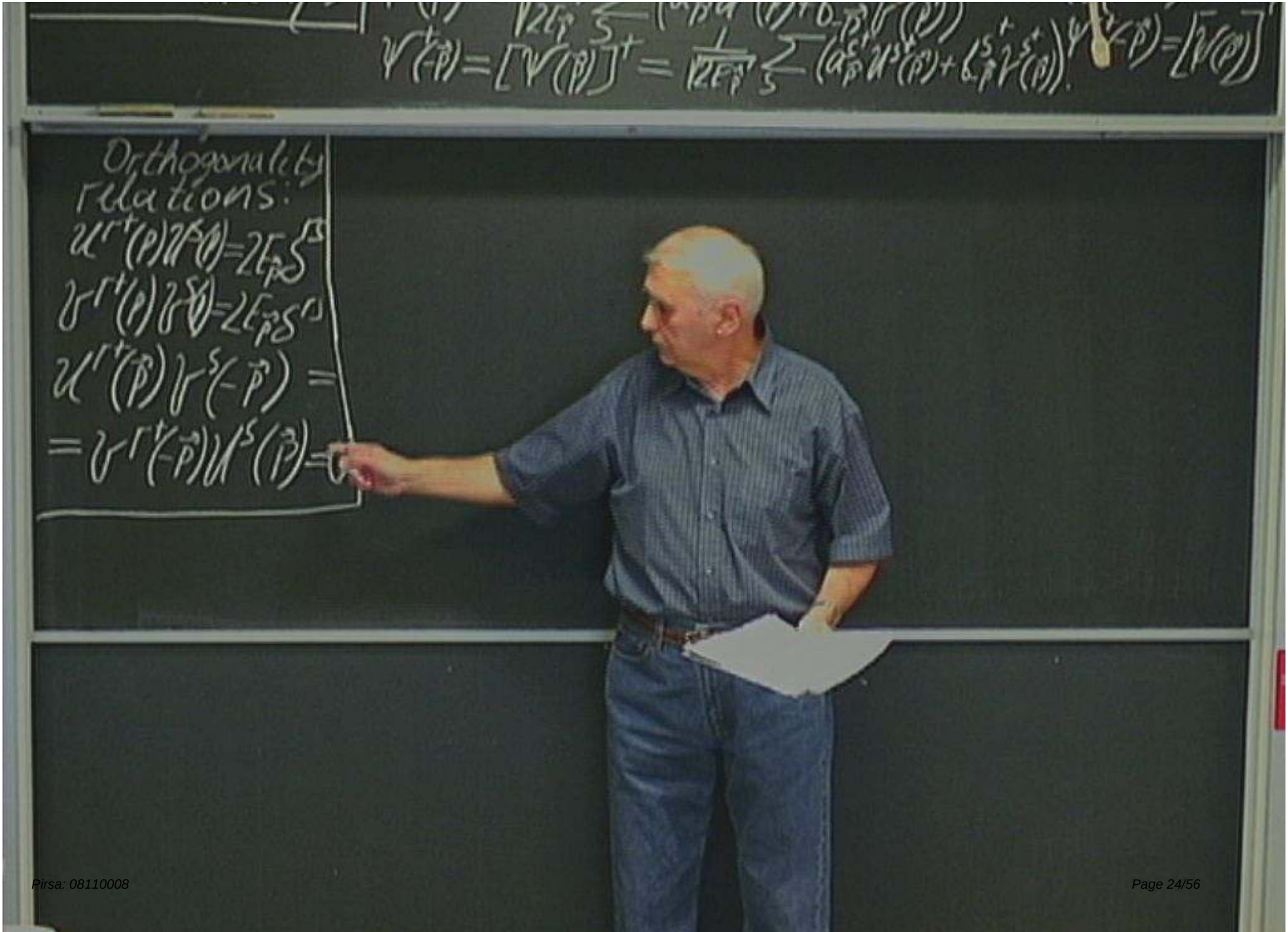

Find commutation relations for


Orthogonality relations: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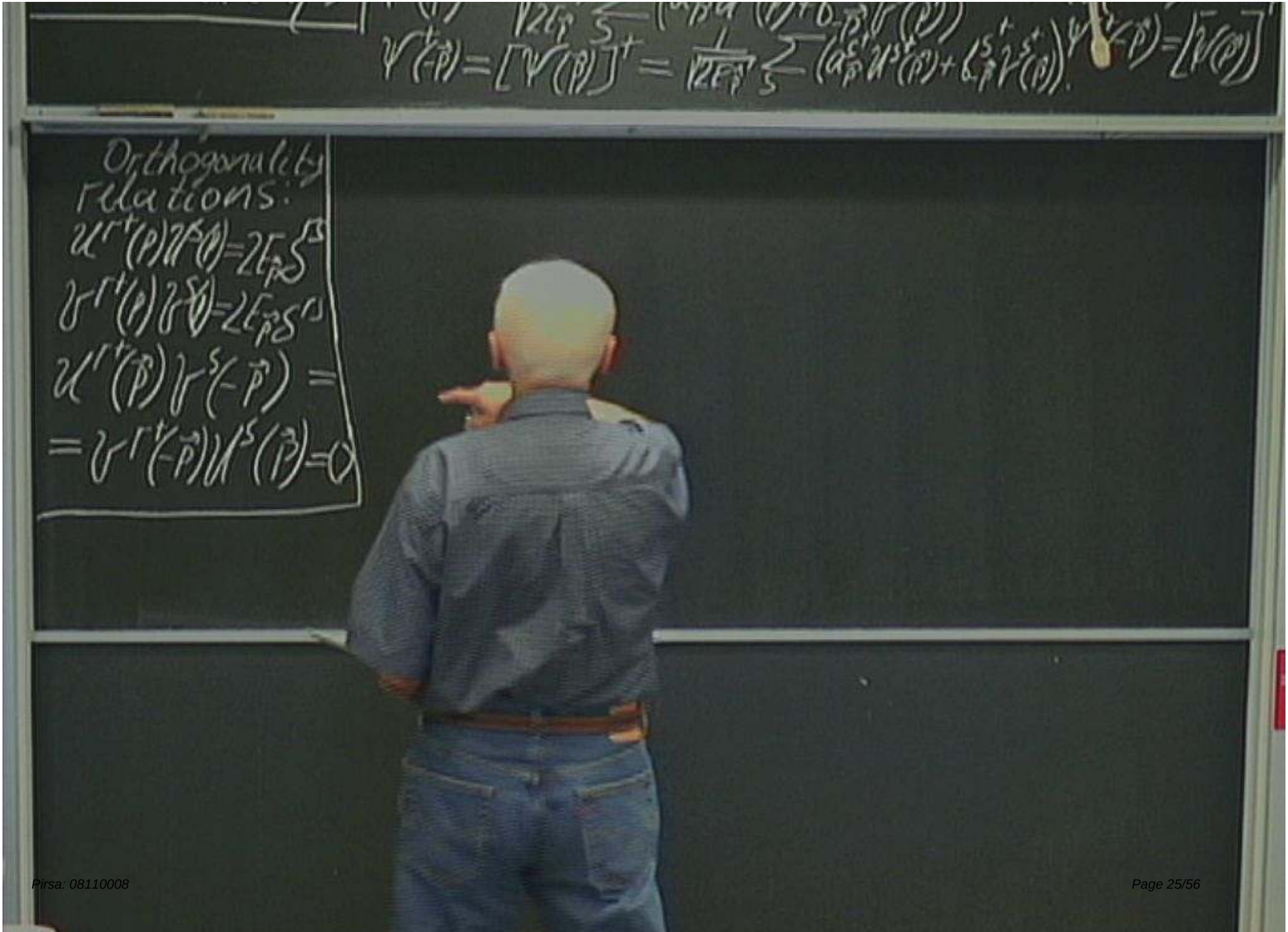
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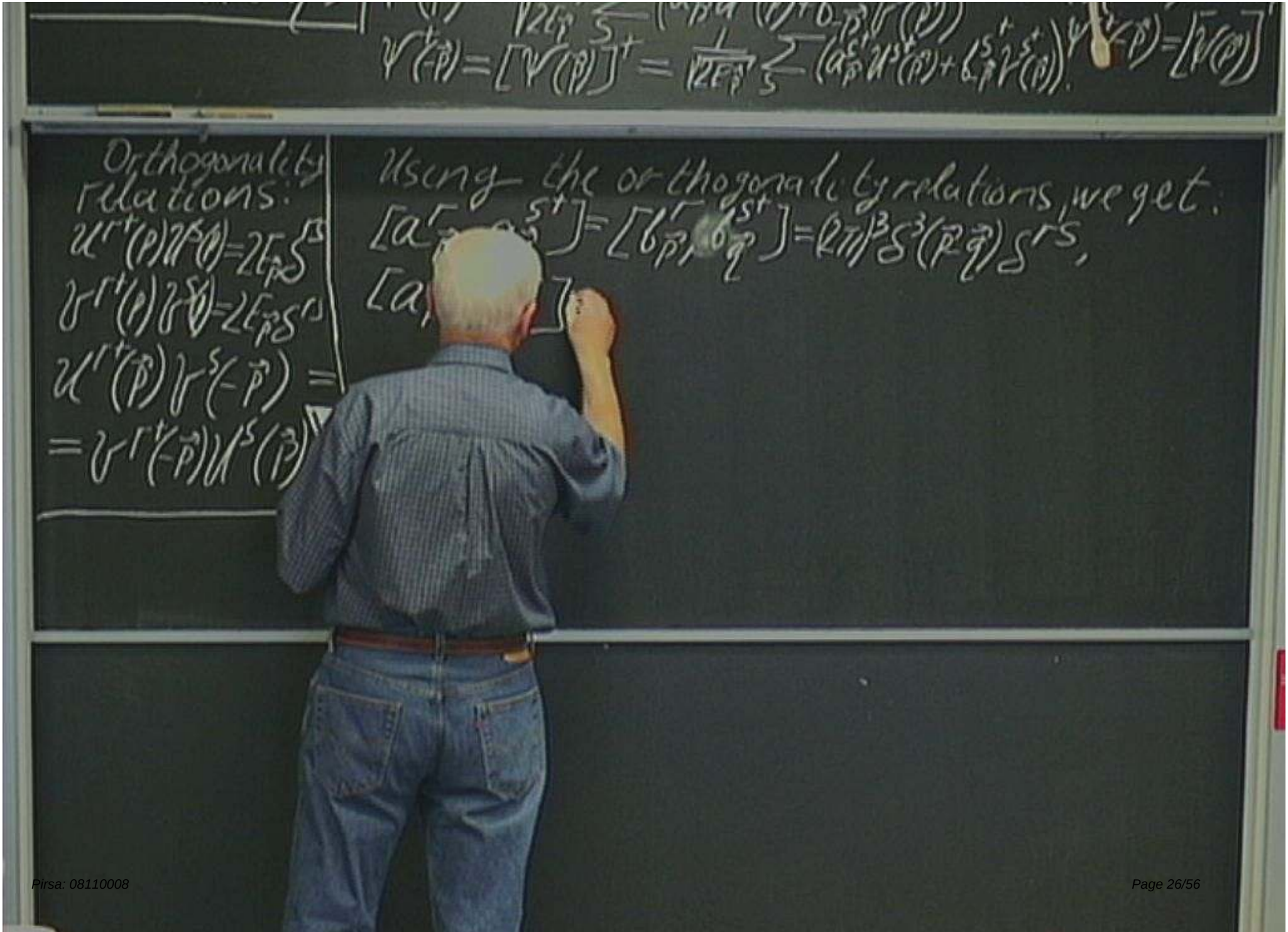
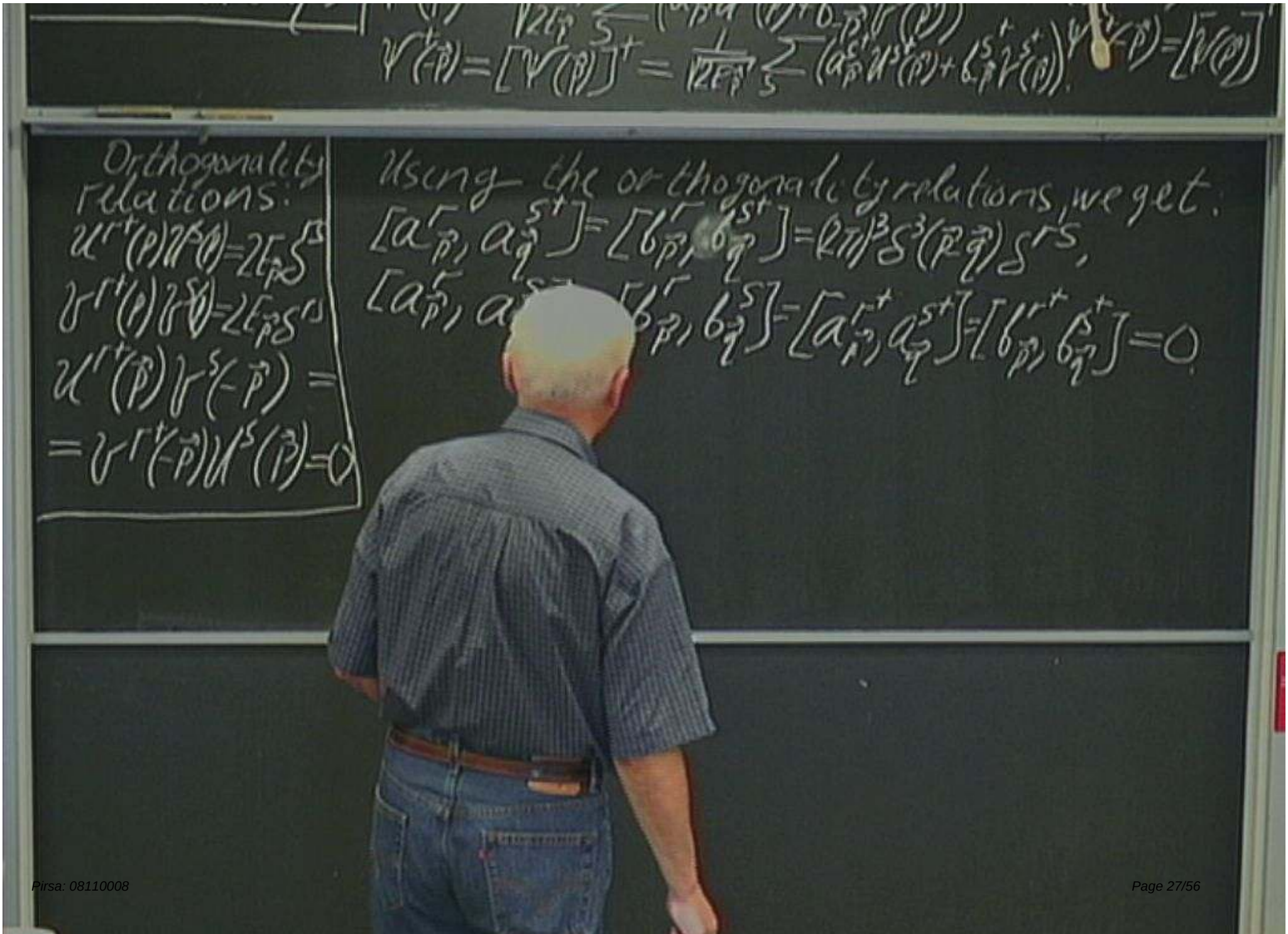

Orthogonality relations:
Using the orthogonality relations, we get.


Orthogonality relations:
Using the orthogonality relations, we get:

$$\psi(-\vec{p}) = [\psi(\vec{p})]^{+} = \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s} \left(a^{s+}_{\vec{p}} u^{s}(\vec{p}) + b^{s}_{\vec{p}} v^{s+}(\vec{p}) \right) \qquad \psi(-\vec{p}) = [\psi(\vec{p})]$$

Orthogonality relations:

$$u^{r+}(\vec{p}) u^{s}(\vec{p}) = 2E_{\vec{p}} \delta^{rs}$$

$$v^{r+}(\vec{p}) v^{s}(\vec{p}) = 2E_{\vec{p}} \delta^{rs}$$

$$u^{r+}(\vec{p}) v^{s}(-\vec{p}) = v^{r+}(-\vec{p}) u^{s}(\vec{p}) = 0$$

Using the orthogonality relations, we get:

$$[a^{r}_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^{r}_{\vec{p}}, b^{s+}_{\vec{q}}] = (2\pi)^{3} \delta^{3}(\vec{p} - \vec{q}) \delta^{rs},$$

$$[a^{r}_{\vec{p}}, a^{s}_{\vec{q}}] = [b^{r}_{\vec{p}}, b^{s}_{\vec{q}}] = [a^{r+}_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^{r+}_{\vec{p}}, b^{s+}_{\vec{q}}] = 0.$$

$$\psi(-\vec{p}) = [\psi(\vec{p})]^{+} = \frac{1}{\sqrt{2E_p}} \sum_s \left(a^{s+}_{\vec{p}}\, u^s(\vec{p}) + b^{s}_{\vec{p}}\, v^s(\vec{p})\right) \qquad \psi^{+}(-\vec{p}) = [\psi(\vec{p})]$$

Orthogonality relations:

$$u^{r+}(\vec{p})\,u^s(\vec{p}) = 2E_{\vec{p}}\,\delta^{rs}$$

$$v^{r+}(\vec{p})\,v^s(\vec{p}) = 2E_{\vec{p}}\,\delta^{rs}$$

$$u^{r+}(\vec{p})\,v^s(-\vec{p}) = v^{r+}(-\vec{p})\,u^s(\vec{p}) = 0$$

Using the orthogonality relations, we get:

$$[a^r_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^r_{\vec{p}}, b^{s+}_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\,\delta^{rs},$$

$$[a^r_{\vec{p}}, a^s_{\vec{q}}] = [b^r_{\vec{p}}, b^s_{\vec{q}}] = [a^{r+}_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^{r+}_{\vec{p}}, b^{s+}_{\vec{q}}] = 0.$$

Write H in terms of these operators using ... ality relations.

$$\psi^+(-\vec{p}) = [\psi(\vec{p})]^+ = \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^{s+}_{\vec{p}} u^s(\vec{p}) + b^{s}_{\vec{p}} v^s(\vec{p}) \right) \qquad \psi^+(-\vec{p}) = [\psi(\vec{p})]$$

Orthogonality relations:

$$u^{r+}(\vec{p}) u^s(\vec{p}) = 2E_{\vec{p}} \delta^{rs}$$

$$v^{r+}(\vec{p}) v^s(\vec{p}) = 2E_{\vec{p}} \delta^{rs}$$

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Using the orthogonality relations, we get:

$$[a^r_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^r_{\vec{p}}, b^{s+}_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \delta^{rs},$$

$$[a^r_{\vec{p}}, a^s_{\vec{q}}] = [b^r_{\vec{p}}, b^s_{\vec{q}}] = [a^{r+}_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^{r+}_{\vec{p}}, b^{s+}_{\vec{q}}] = 0.$$

Write H in terms of these operators using the orthogonality relations:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\vec{p}} a^{s+}_{\vec{p}} a^s_{\vec{p}} - E_{\vec{p}} b^{s+}_{\vec{p}} b^s_{\vec{p}} \right)$$

$$\psi^+(-\vec{p}) = [\psi(\vec{p})]^+ = \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^{s+}_{\vec{p}} u^s(\vec{p}) + b^s_{\vec{p}} v^s(\vec{p})\right) \qquad \psi^+(-\vec{p}) = [\psi(\vec{p})]$$

Orthogonality relations:

$$u^{r+}(\vec{p}) u^s(\vec{p}) = 2E_{\vec{p}} \delta^{rs}$$

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$$u^{r+}(\vec{p}) v^s(-\vec{p}) =$$

$$= v^{r+}(-\vec{p}) u^s(\vec{p}) = 0$$

Using the orthogonality relations, we get:

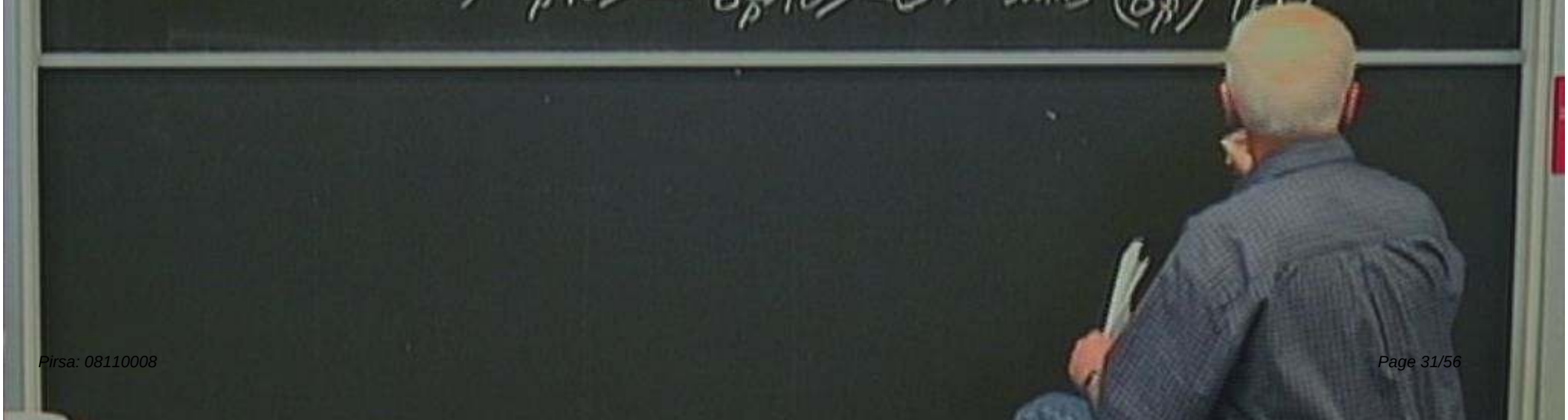
$$[a^r_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^r_{\vec{p}}, b^{s+}_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \delta^{rs},$$

$$[a^r_{\vec{p}}, a^s_{\vec{q}}] = [b^r_{\vec{p}}, b^s_{\vec{q}}] = [a^{r+}_{\vec{p}}, a^{s+}_{\vec{q}}] = [b^{r+}_{\vec{p}}, b^{s+}_{\vec{q}}] = 0.$$

Write H in terms of these operators using the orthogonality relations:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\vec{p}} a^{s+}_{\vec{p}} a^s_{\vec{p}} - E_{\vec{p}} b^{s+}_{\vec{p}} b^s_{\vec{p}}\right)$$

"vacuum" $|0\rangle$, $a^s_{\vec{p}}|0\rangle = b^s_{\vec{p}}|0\rangle = 0$; states $(b^{s+}_{\vec{p}})^n|0\rangle$



$$\psi(-\vec{p}) = [\psi(\vec{p})]^\dagger = \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^{s\dagger}_{\vec{p}}\, u^s(\vec{p}) + b^s_{\vec{p}}\, v^s(\vec{p})\right) \qquad \bar{\psi}(-\vec{p}) = [\bar{\psi}(\vec{p})]$$

Orthogonality relations:

$$u^{r\dagger}(\vec{p})\, u^s(\vec{p}) = 2E_{\vec{p}}\,\delta^{rs}$$

$$v^{r\dagger}(\vec{p})\, v^s(\vec{p}) = 2E_{\vec{p}}\,\delta^{rs}$$

$$u^{r\dagger}(\vec{p})\, v^s(-\vec{p}) = v^{r\dagger}(-\vec{p})\, u^s(\vec{p}) = 0$$

Using the orthogonality relations, we get:

$$[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}] = [b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q})\,\delta^{rs},$$

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Write H in terms of these operators using the orthogonality relations:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\vec{p}}\, a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - E_{\vec{p}}\, b^{s\dagger}_{\vec{p}} b^s_{\vec{p}}\right)$$

There is ground state $|0\rangle$ ("vacuum"), $a^s_{\vec{p}}|0\rangle = b^s_{\vec{p}}|0\rangle = 0$; states $(b^{s\dagger}_{\vec{p}})^n|0\rangle$ have $E(\vec{p}) < 0$ in this theory.

P. Jordan and E. Wigner: replace commutation relations by anticommutation ones!

P. Jordan and E. Wigner: replace commutation relations by anticommutation ones: $\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \psi_a(\vec{x})\psi_b^+(\vec{y}) + \psi_b^+(\vec{y})\psi_a(\vec{x}) = \delta(\vec{x}-\vec{y})\delta_{ab}$

$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi_a^+(\vec{x}), \psi_b(\vec{y})\} = 0.$

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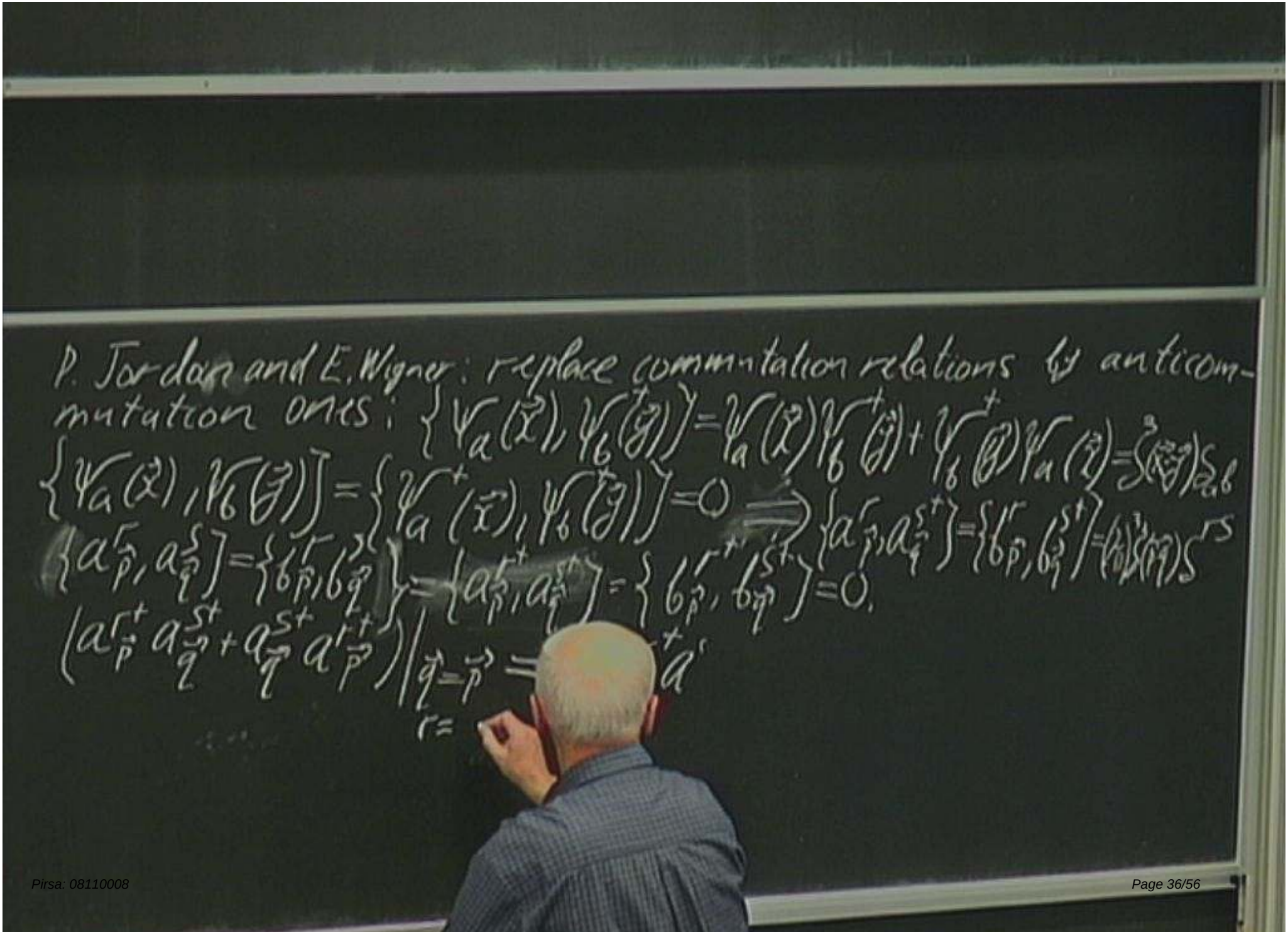
$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi_a^+(\vec{x}), \psi_b^+(\vec{y})\} = 0 \Longrightarrow$


P. Jordan and E. Wigner: replace commutation relations by anticommutation ones:
$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \psi_a(\vec{x})\psi_b^+(\vec{y}) + \psi_b^+(\vec{y})\psi_a(\vec{x}) = \delta(\vec{x}-\vec{y})\delta_{ab}$
$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi_a^+(\vec{x}), \psi_b^+(\vec{y})\} = 0 \Rightarrow$
$\{a^r_{\vec{p}}, a^{s+}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s+}_{\vec{q}}\} = \{a^{r+}_{\vec{p}}, a^s_{\vec{q}}\} = \{b^{r+}_{\vec{p}}, b^s_{\vec{q}}\} = 0$
$\{a^r_{\vec{p}}, a^{s+}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s+}_{\vec{q}}\} = \delta_{\vec{p}\vec{q}}\delta^{rs}$
$\left(a^{r+}_{\vec{p}} a^{s+}_{\vec{q}} + a^{s+}_{\vec{q}} a^{r+}_{\vec{p}}\right)\Big|_{\vec{q}=\vec{p}} = a^{r+}$
$r =$

P. Jordan and E. Wigner: replace commutation relations by anticommutation ones: $\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \psi_a(\vec{x})\psi_b^+(\vec{y}) + \psi_b^+(\vec{y})\psi_a(\vec{x}) = \delta(\vec{x}-\vec{y})\delta_{ab}$

$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi_a^+(\vec{x}), \psi_b(\vec{y})\} = 0 \Longrightarrow \{a_{\vec{p}}^r, a_{\vec{q}}^{s+}\} = \{b_{\vec{p}}^r, b_{\vec{q}}^{s+}\} = \delta_{ab}\delta(\vec{p}-\vec{q})\delta^{rs}$

$\{a_{\vec{p}}^r, a_{\vec{q}}^s\} = \{b_{\vec{p}}^r, b_{\vec{q}}^s\} = \{a_{\vec{p}}^{r+}, a_{\vec{q}}^{s+}\} = \{b_{\vec{p}}^{r+}, b_{\vec{q}}^{s+}\} = 0.$

$\left(a_{\vec{p}}^{r+} a_{\vec{q}}^{s+} + a_{\vec{q}}^{s+} a_{\vec{p}}^{r+}\right)\Big|_{\substack{\vec{q}=\vec{p} \\ r=s}} = 2a_{\vec{p}}^{r+} a_{\vec{p}}^{r+} = 0. \quad ; \quad a^{n+}$

P. Jordan and E. Wigner: replace commutation relations by anticommutation ones: $\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \psi_a(\vec{x})\psi_b(\vec{y}) + \psi_b(\vec{y})\psi_a(\vec{x}) = \delta(\vec{x}-\vec{y})\delta_{ab}$

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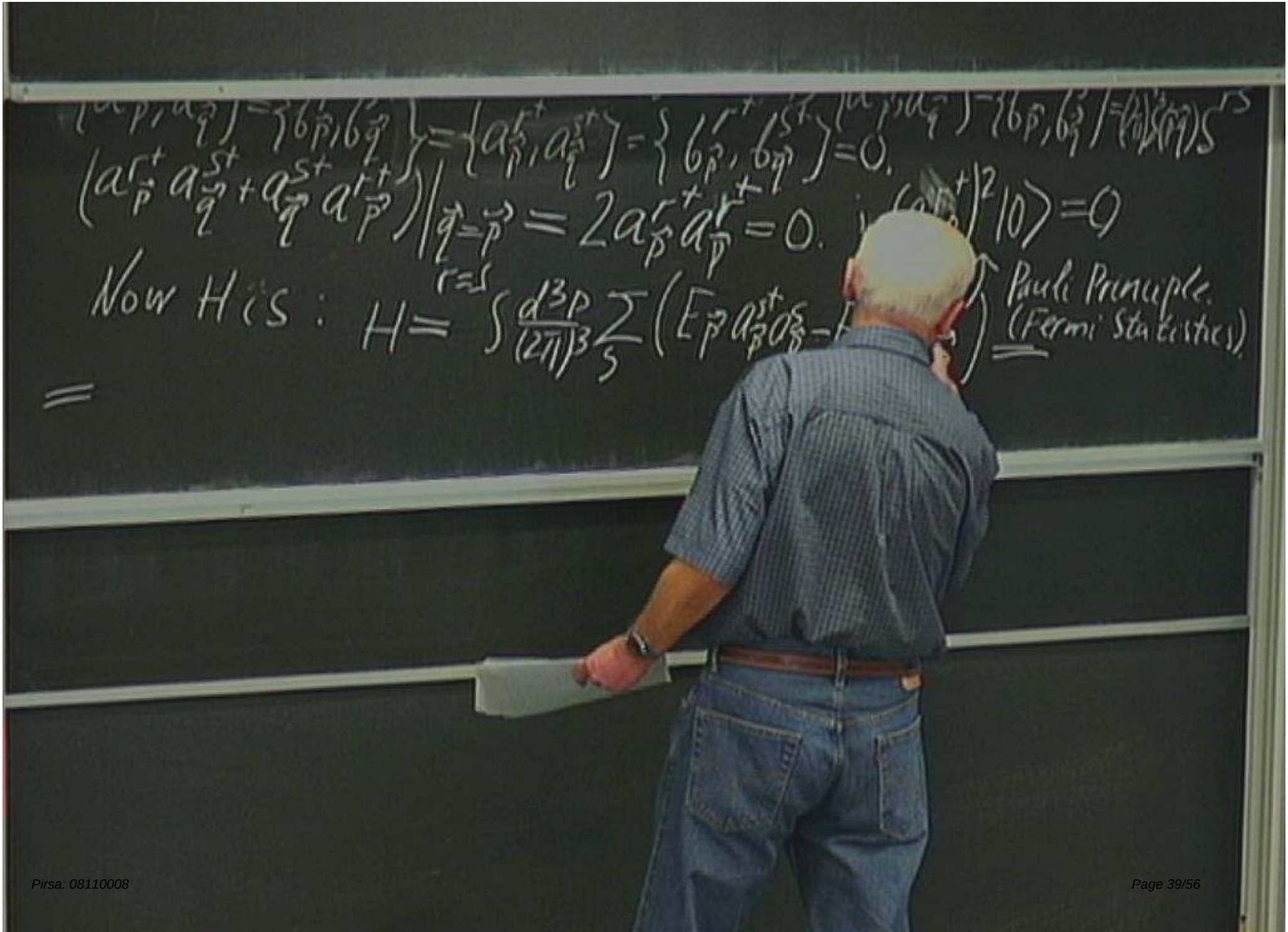
$\{a^r_{\vec{p}}, a^s_{\vec{q}}\} = \{b^r_{\vec{p}}, b^s_{\vec{q}}\} = \{a^{r+}_{\vec{p}}, a^{s+}_{\vec{q}}\} = \{b^{r+}_{\vec{p}}, b^{s+}_{\vec{q}}\} = 0.$

$\left(a^{r+}_{\vec{p}} a^{s+}_{\vec{q}} + a^{s+}_{\vec{q}} a^{r+}_{\vec{p}}\right)\Big|_{\substack{\vec{q}=\vec{p} \\ r=s}} = 2a^{r+}_{\vec{p}} a^{r+}_{\vec{p}} = 0. \quad ; \quad (a^{r+}_{\vec{p}})^2 |0\rangle = 0$

↑ Pauli Principle. (Fermi Statistics)


Now His:
Pauli Principle.
(Fermi Statistics)

$\{a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = \{a^r_{\vec{p}}, b^s_{\vec{q}}\} = \{a^{r\dagger}_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = 0.$

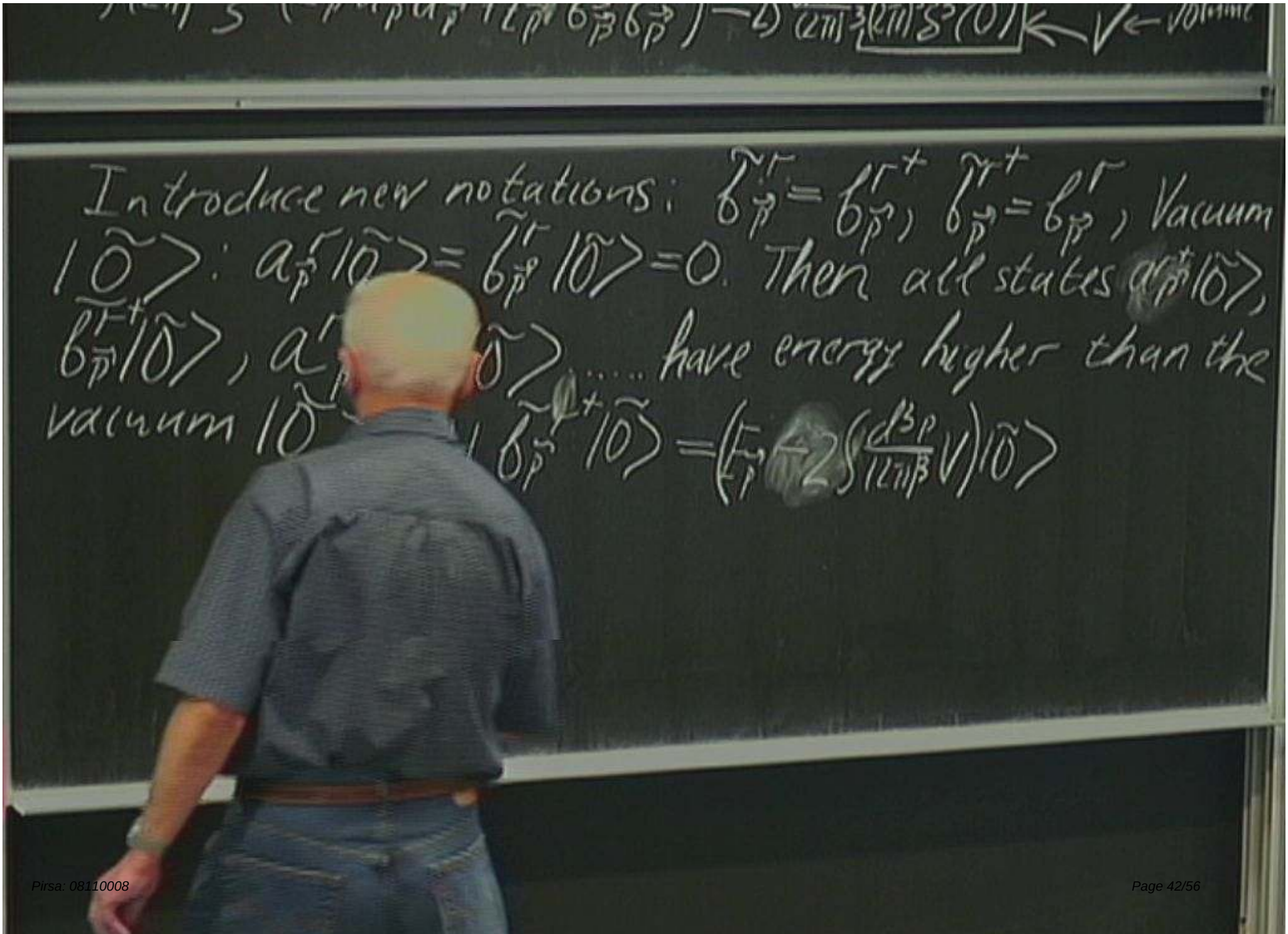
$\left(a^r_{\vec{p}}\, a^{s\dagger}_{\vec{q}} + a^{s\dagger}_{\vec{q}}\, a^r_{\vec{p}}\right)\Big|_{\vec{q}=\vec{p}} = 2a^r_{\vec{p}}\, a^{r\dagger}_{\vec{p}} = 0. \quad ; \quad (a^{r\dagger}_{\vec{p}})^2 |0\rangle = 0$

Now H is: $H = \int \frac{d^3p}{(2\pi)^3} \sum_{r=s} \left(E_{\vec{p}}\, a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - E_{\vec{p}}\, b^s_{\vec{p}} b^{s\dagger}_{\vec{p}} \right)$ ← Pauli Principle (Fermi Statistics)

$= \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\vec{p}}\, a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + E_{\vec{p}}\, b^s_{\vec{p}} b^{s\dagger}_{\vec{p}} \right) - \delta \int \frac{d^3p}{(2\pi)^3} \underbrace{(2\pi)^3 \delta^3(0)}_{V}$ ← volume

$\delta(\vec{p}-\vec{p}')\,\delta(\vec{p}-\vec{p}')\to \delta(0)\,\delta(0)\to V\cdot\delta(0)\leftarrow V\leftarrow$ volume

Introduce new notations: $\tilde{b}^r_{\vec{p}} = b^{r+}_{\vec{p}}$, $\tilde{b}^{r+}_{\vec{p}} = b^r_{\vec{p}}$


Introduce new notations: $\tilde{b}^{r}_{\vec{p}} = b^{r+}_{\vec{p}}$, $\tilde{b}^{r+}_{\vec{p}} = b^{r}_{\vec{p}}$, Vacuum
$|\tilde{0}\rangle$: $a^{r}_{\vec{p}}|\tilde{0}\rangle = \tilde{b}^{r}_{\vec{p}}|\tilde{0}\rangle = 0$. Then all states $a^{r+}_{\vec{p}}|\tilde{0}\rangle$,
$b^{r+}_{\vec{p}}|\tilde{0}\rangle$, a' ... have energy higher than the
vacuum $|\tilde{0}\rangle$ ... $\tilde{b}^{r+}_{\vec{p}}|\tilde{0}\rangle = (E_r + \int \frac{d^3p}{(2\pi)^3} V)|\tilde{0}\rangle$

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$\delta^3(0) \leftarrow V \leftarrow$ volume

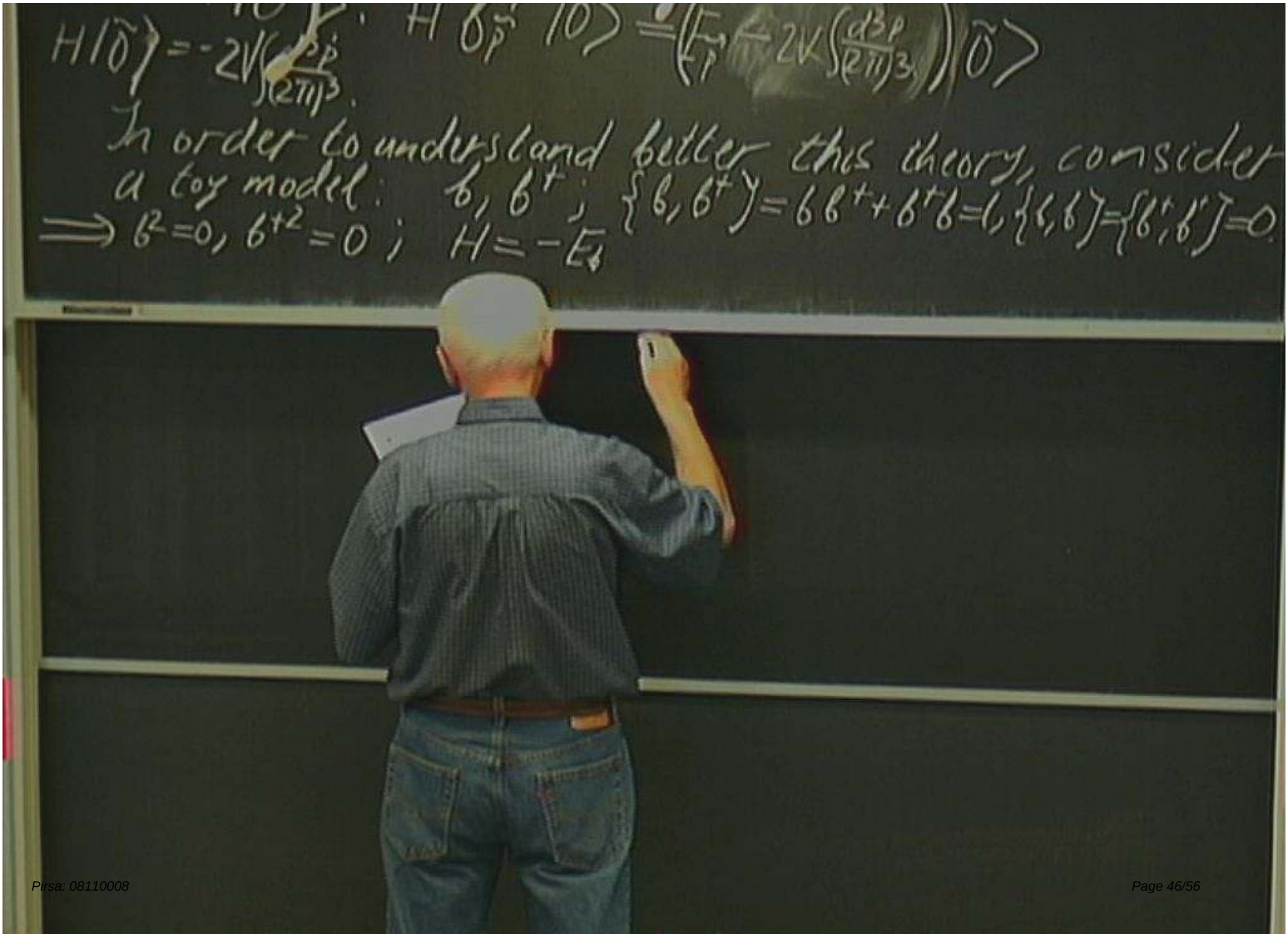
Introduce new notations: $\tilde{b}^r_{\vec{p}} = b^{r+}_{\vec{p}}$, $\tilde{b}^{r+}_{\vec{p}} = b^r_{\vec{p}}$, Vacuum $|\tilde{0}\rangle$: $a^r_{\vec{p}}|\tilde{0}\rangle = \tilde{b}^r_{\vec{p}}|\tilde{0}\rangle = 0$. Then all states $b^{r+}_{\vec{p}}|\tilde{0}\rangle$, $a^{r+}_{\vec{p}} b^{s+}_{\vec{q}}|\tilde{0}\rangle$ ..... have energy higher than vacuum $|\tilde{0}\rangle$: $H b^{r+}_{\vec{p}}|\tilde{0}\rangle = \left(E_r + 2\int \left(\frac{d^3p}{(2\pi)^3} V\right)\right)|\tilde{0}\rangle$

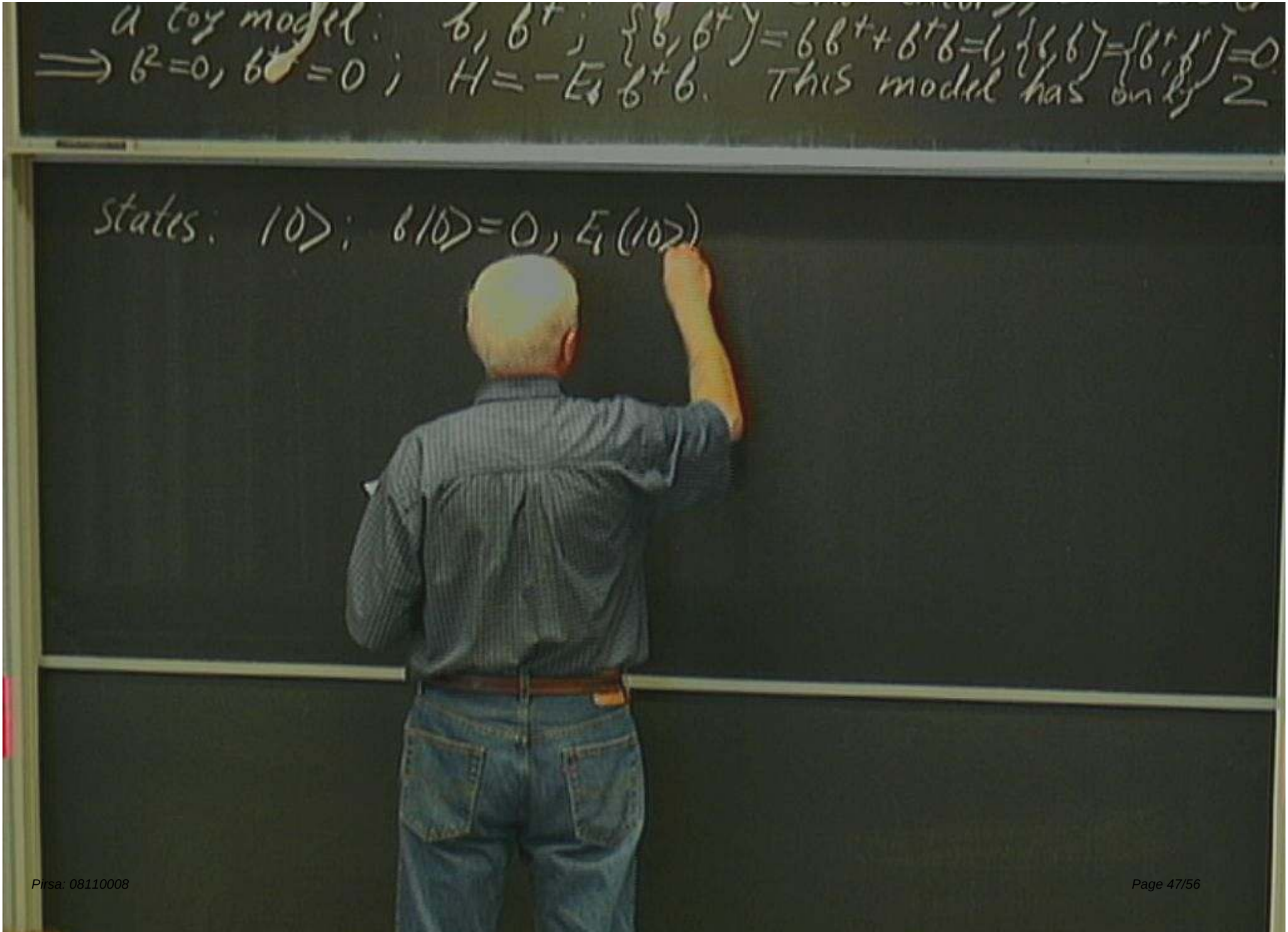
$H|\tilde{0}\rangle = -2V\int d^3$

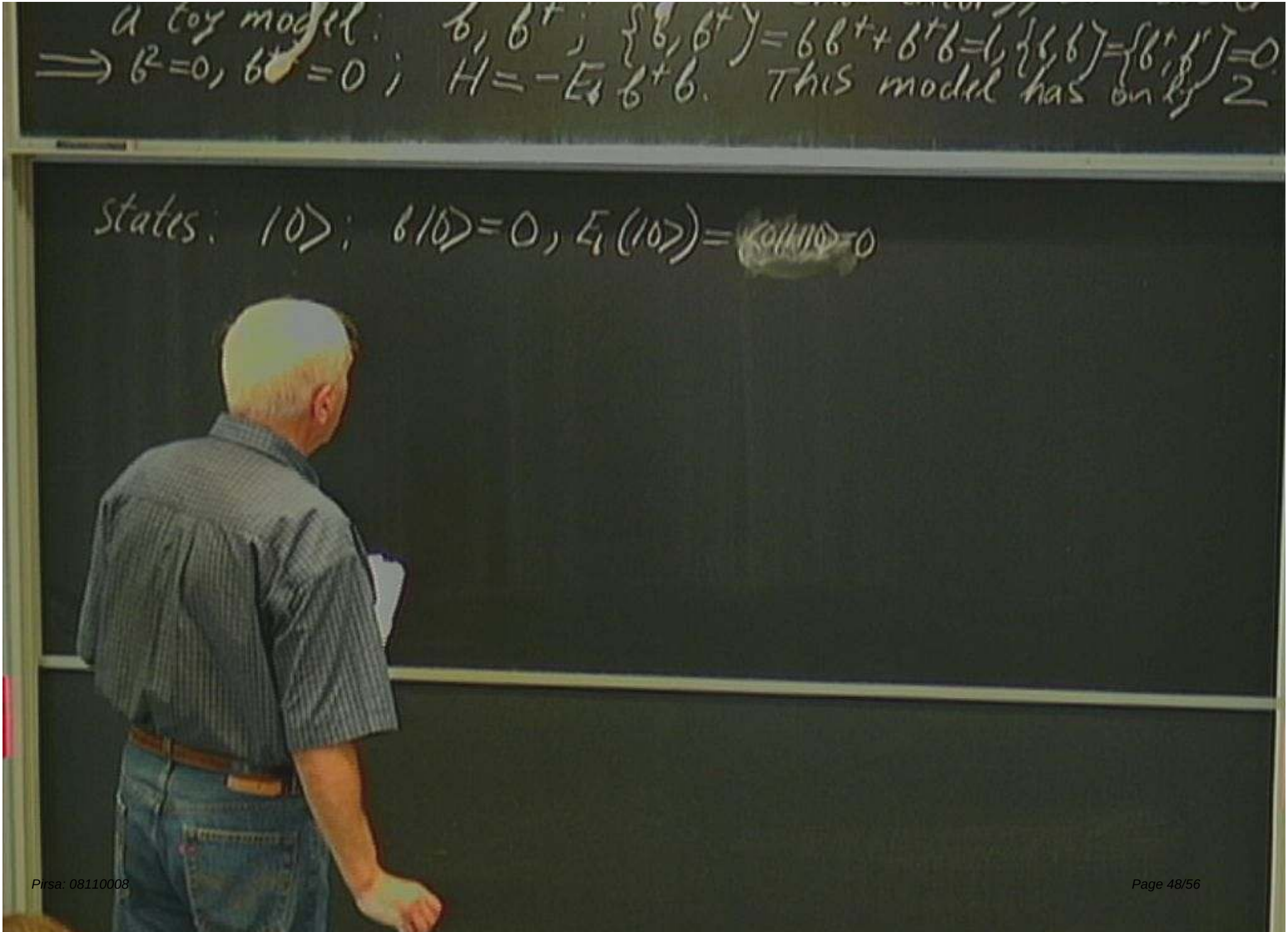
$\frac{1}{2}\delta^3(0)$ ← V ← volume

Introduce new notations: $\tilde{b}^r_{\vec{p}} = b^{r+}_{\vec{p}}$, $\tilde{b}^{r+}_{\vec{p}} = b^r_{\vec{p}}$, Vacuum $|\tilde{0}\rangle$: $a^r_{\vec{p}}|\tilde{0}\rangle = \tilde{b}^r_{\vec{p}}|\tilde{0}\rangle = 0$. Then all states $a^{r+}_{\vec{p}}|\tilde{0}\rangle$, $b^{r+}_{\vec{p}}|\tilde{0}\rangle$, $a^{r+}_{\vec{p}} b^{s+}_{\vec{q}}|\tilde{0}\rangle$ ..... have energy higher than the vacuum $|\tilde{0}\rangle$: $H b^{r+}_{\vec{p}}|\tilde{0}\rangle = \left(E_r + 2V\int\frac{d^3p}{(2\pi)^3}\right)|\tilde{0}\rangle$

$H|\tilde{0}\rangle = -2V\int\frac{d^3p}{(2\pi)^3}$.


H|0⟩ = -2V ∫ d³p/(2π)³ ...
In order to understand better this theory, consider
a toy model: b, b⁺ ; {b, b⁺} = bb⁺ + b⁺b = 1, {b, b} = {b⁺, b⁺} = 0
⟹ b² = 0, b⁺² = 0 ; H = -E


a toy model: b, b^+; $\{b, b^+\} = bb^+ + b^+b = 1$, $\{b, b\} = \{b^+, b^+\} = 0$
$\Rightarrow b^2 = 0$, $b^{+2} = 0$; $H = -E_1 b^+ b$. This model has only 2
states. $|0\rangle$; $b|0\rangle = 0$, $E_1(|0\rangle)$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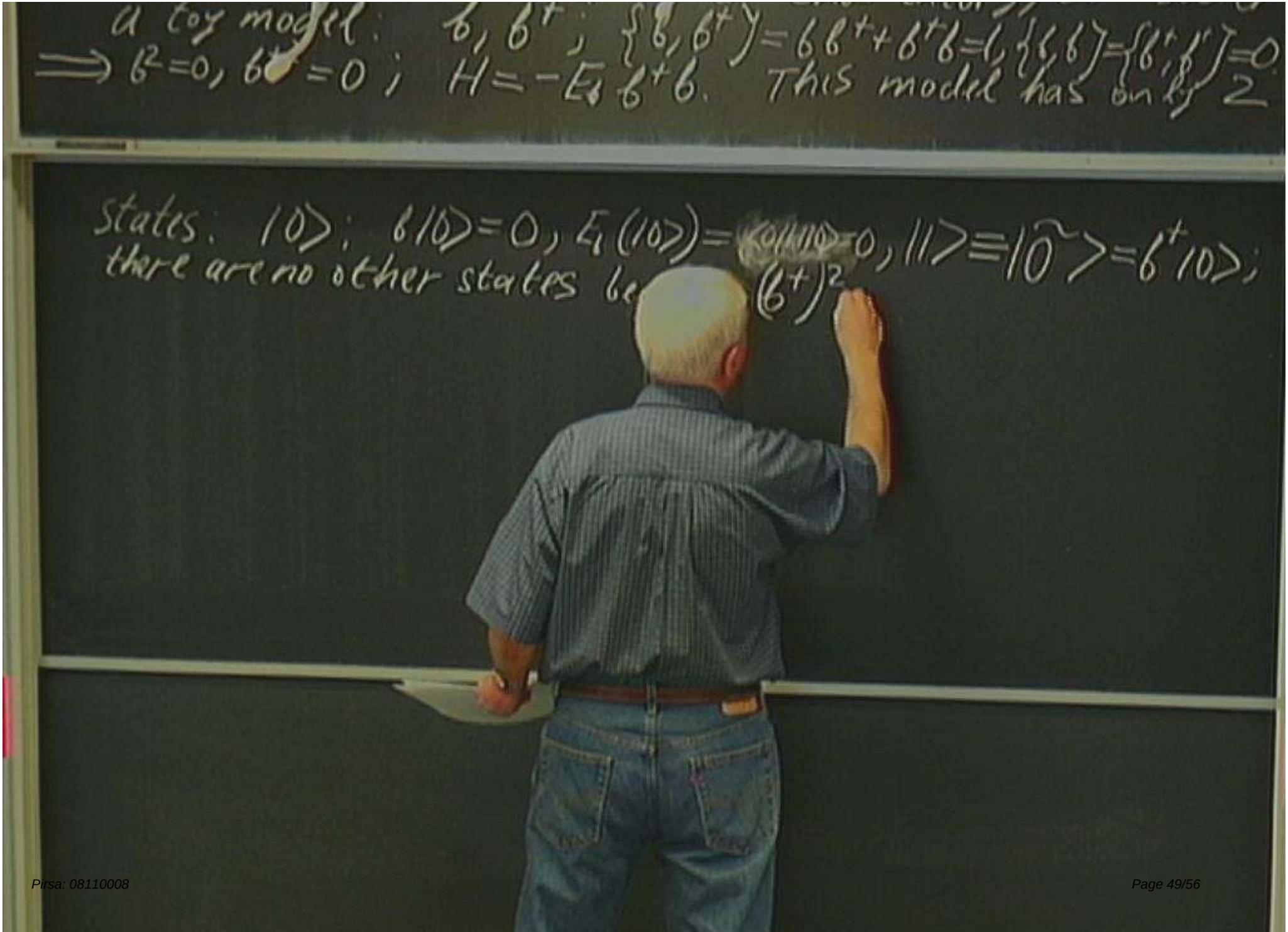
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states: $|0\rangle$; $b|0\rangle = 0$, $E_0|0\rangle = E_0|0\rangle$, $|1\rangle \equiv |0\tilde{}\rangle = b^+|0\rangle$;
there are no other states b
$(b^+)^2$

A toy model: b, b^+; $\{b, b^+\} = bb^+ + b^+b = 1$, $\{b, b\} = \{b^+, b^+\} = 0$.

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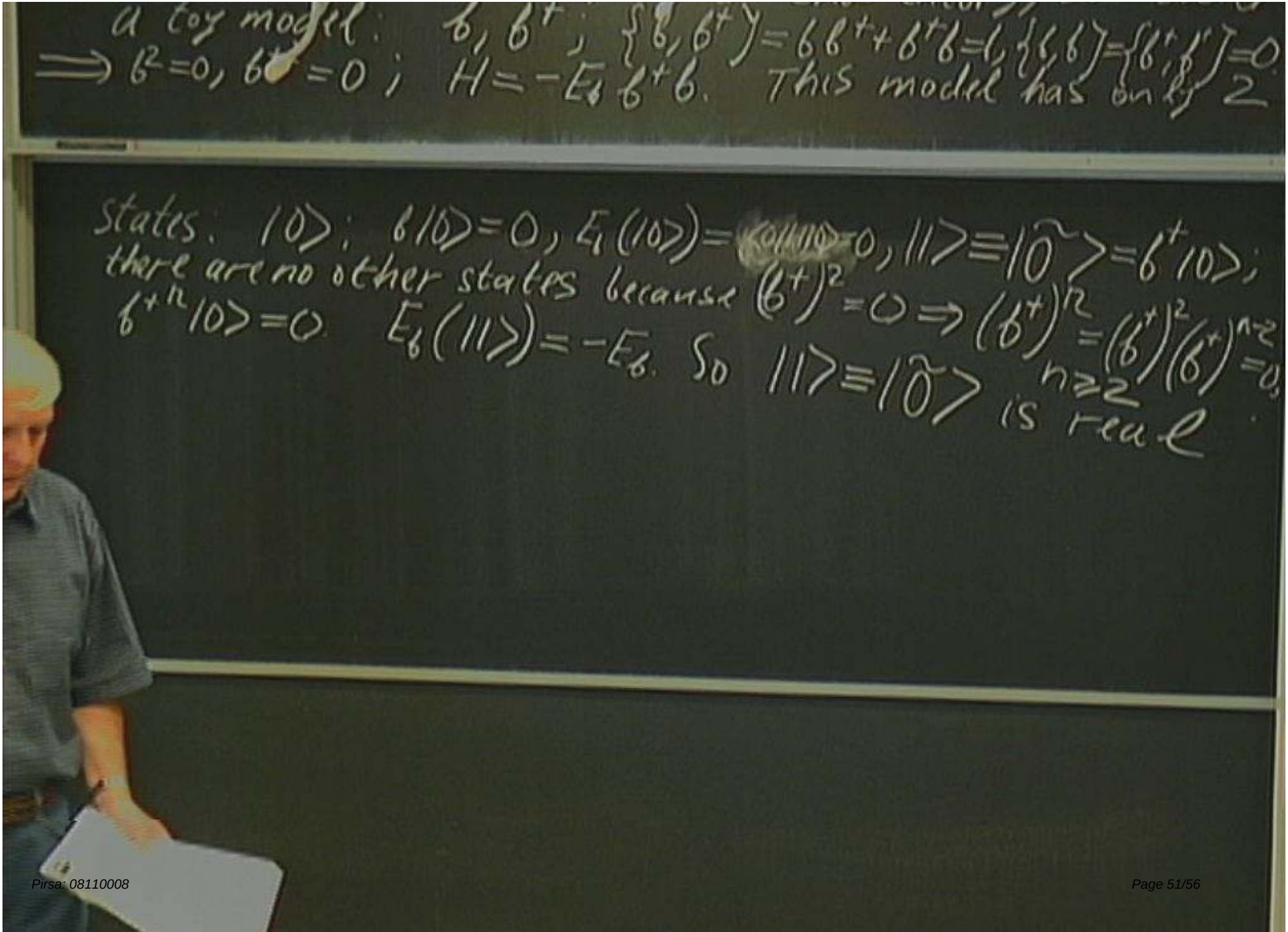
$b^{+n}|0\rangle = 0$.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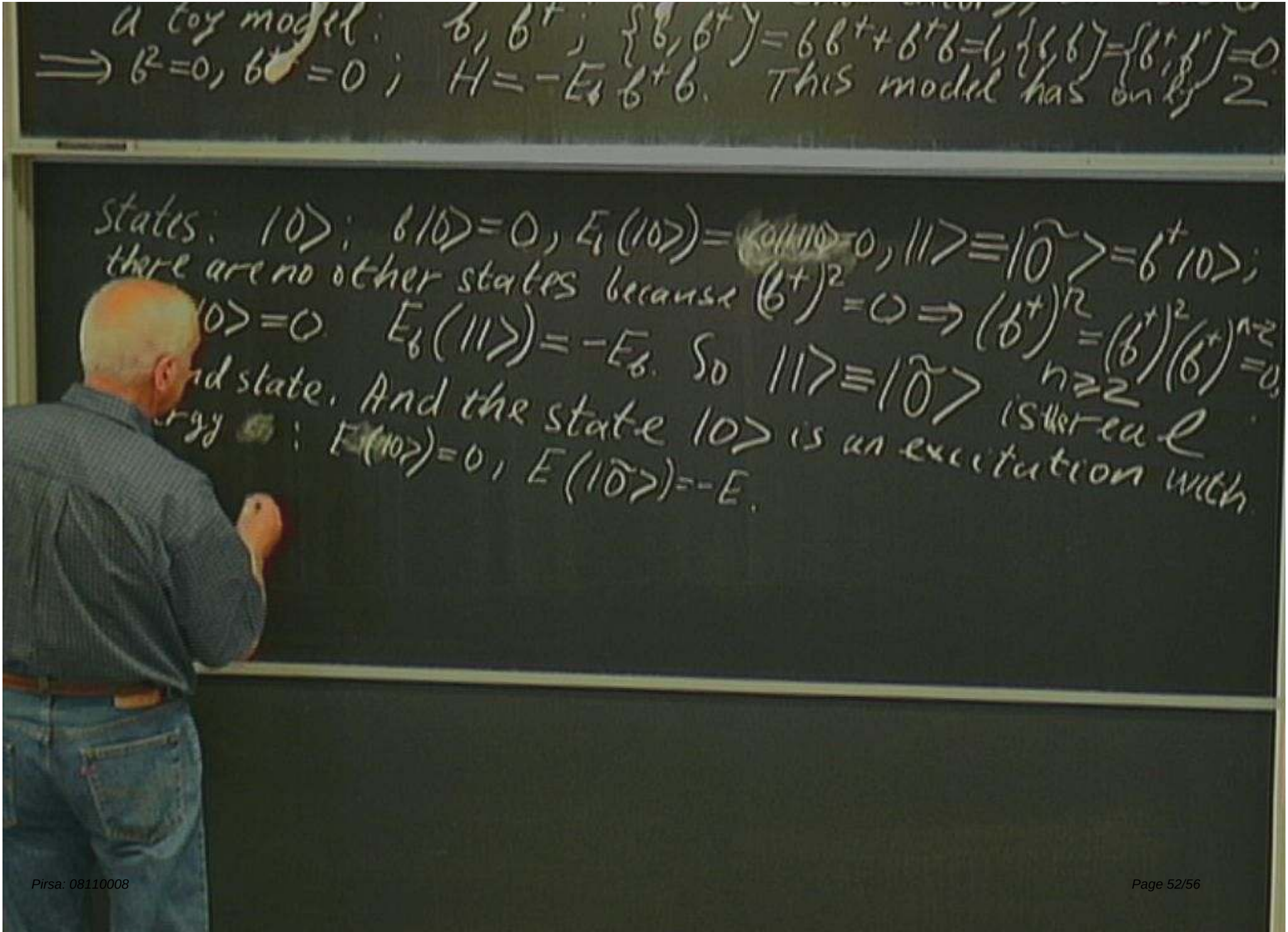
$b^{+n}|0\rangle = 0$. $E_b(|1\rangle) = -E_b$. So $|1\rangle \equiv |\tilde{0}\rangle$ is real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$b^{+n}|0\rangle = 0$. $E_b(|1\rangle) = -E_b$. So $|1\rangle \equiv |\tilde{0}\rangle$ is the ground state. And the state $|0\rangle$ is an excitation with energy E_b: $E_b(|0\rangle) = 0$, $E_b(|\tilde{0}\rangle) = -E_b$.

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a toy model: b, b^+; $\{b, b^+\} = bb^+ + b^+b = 1$, $\{b, b\} = \{b^+, b^+\} = 0$.

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As before, I could introduce $\tilde{b} = b^+$ and $\tilde{b}^+ = b$; then $\tilde{b}|0\rangle = 0$ and $H = -E_b b^+ b = E_b b b^+ - E_b = E_b \tilde{b}^+ \tilde{b} - E_b$