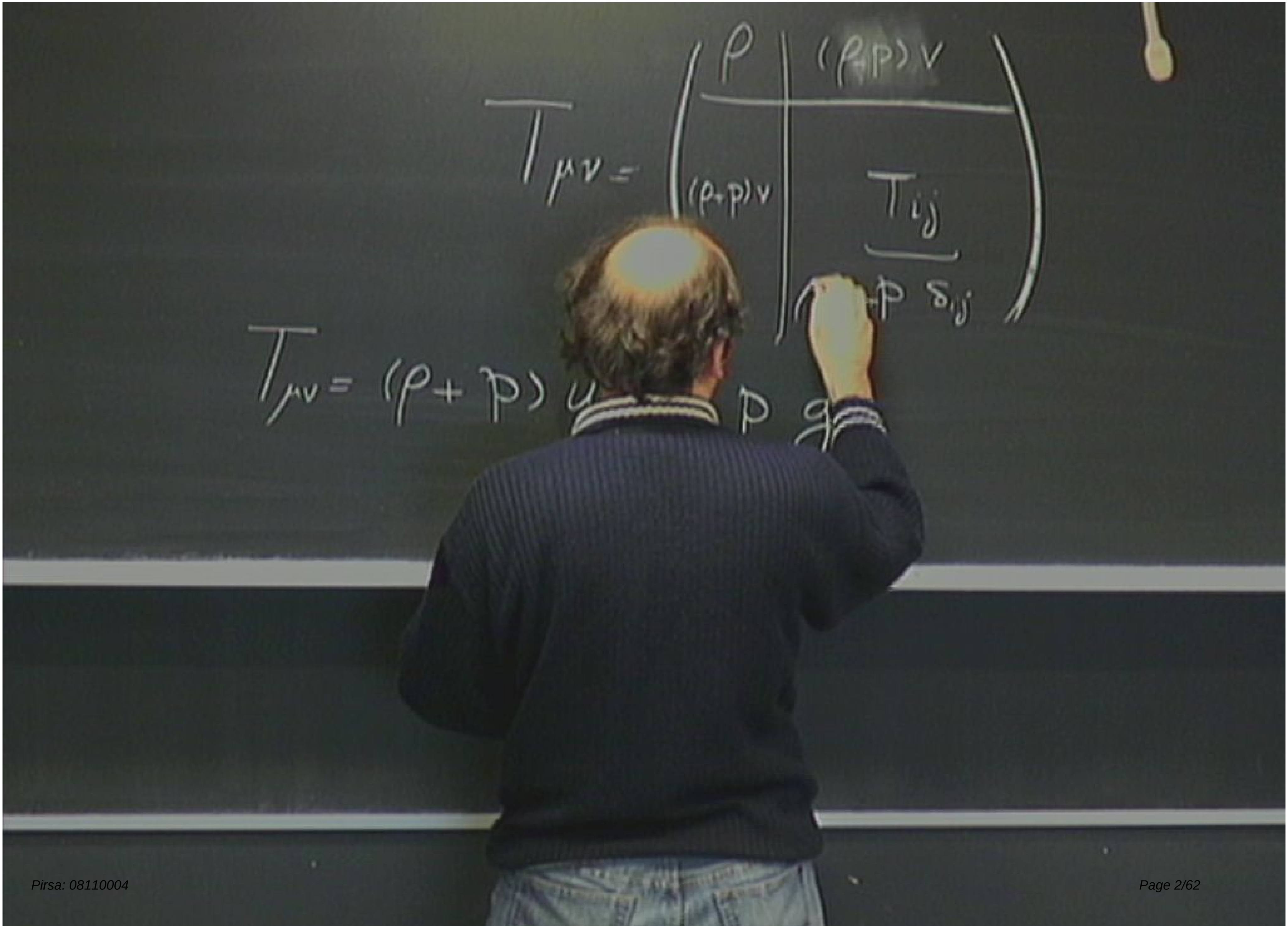
Title: Astrophysics and Cosmology through Problems - 10A

Date: Nov 06, 2008 10:00 AM

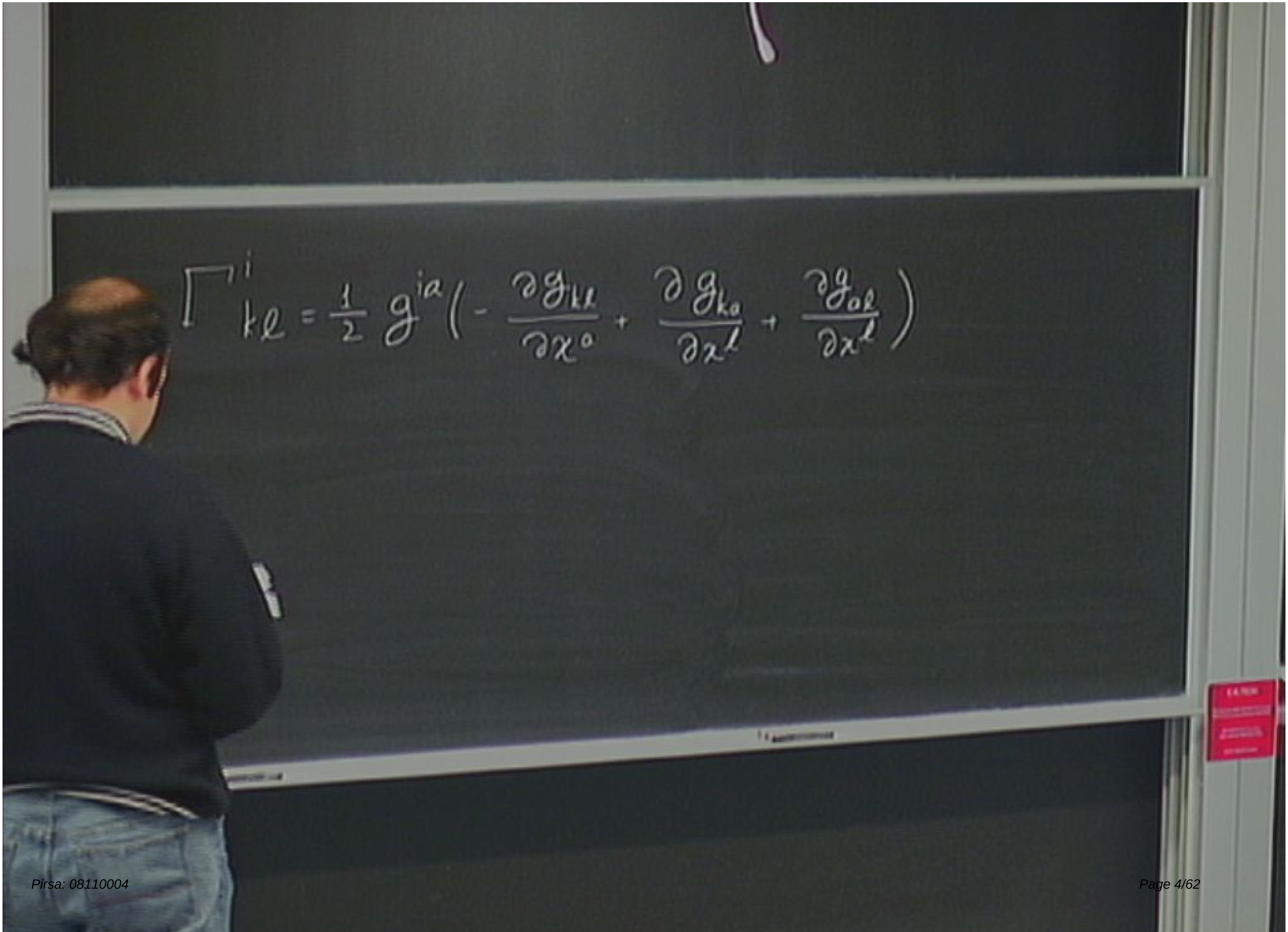
URL: http://pirsa.org/08110004

Abstract: This course is aimed at advanced undergraduate and beginning graduate students, and is inspired by a book by the same title, written by Padmanabhan. Each session consists of solving one or two pre-determined problems, which is done by a randomly picked student. While the problems introduce various subjects in Astrophysics and Cosmology, they do not serve as replacement for standard courses in these subjects, and are rather aimed at educating students with hands-on analytic/numerical skills to attack new problems.


$T_{\mu\nu} = \left(\begin{array}{c|c} \rho & (\rho+p)v \\ \hline (\rho+p)v & T_{ij} = p\,\delta_{ij} \end{array} \right)$
$T_{\mu\nu} = (\rho + p)\,u \quad p\,g$

$$T_{\mu\nu} = \left(\begin{array}{c|c} \rho & (\rho+p)\,v_i \\ \hline (\rho+p)\,v_i & \underbrace{T_{ij}}_{\rho v_i v_j + p\,\delta_{ij}} \end{array}\right)$$

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p\, g_{\mu\nu}$$


$\Gamma^{i}_{k\ell} = \frac{1}{2} g^{ia}\left(- \frac{\partial g_{k\ell}}{\partial x^{a}} + \frac{\partial g_{ka}}{\partial x^{\ell}} + \frac{\partial g_{a\ell}}{\partial x^{\ell}} \right)$

$$\Gamma^{i}_{k\ell} = \frac{1}{2} g^{ia}\left(- \frac{\partial g_{k\ell}}{\partial x^{a}} + \frac{\partial g_{ka}}{\partial x^{\ell}} + \frac{\partial g_{a\ell}}{\partial x^{\ell}} \right)$$

$$\frac{du^{i}}{d\lambda} + \Gamma^{i}_{k\ell} u^{k} u^{\ell} = 0$$

$$\Gamma^{i}{}_{k\ell} = \frac{1}{2} g^{ia} \left(- \frac{\partial g_{k\ell}}{\partial x^{a}} + \frac{\partial g_{ka}}{\partial x^{\ell}} + \frac{\partial g_{a\ell}}{\partial x^{\ell}} \right)$$

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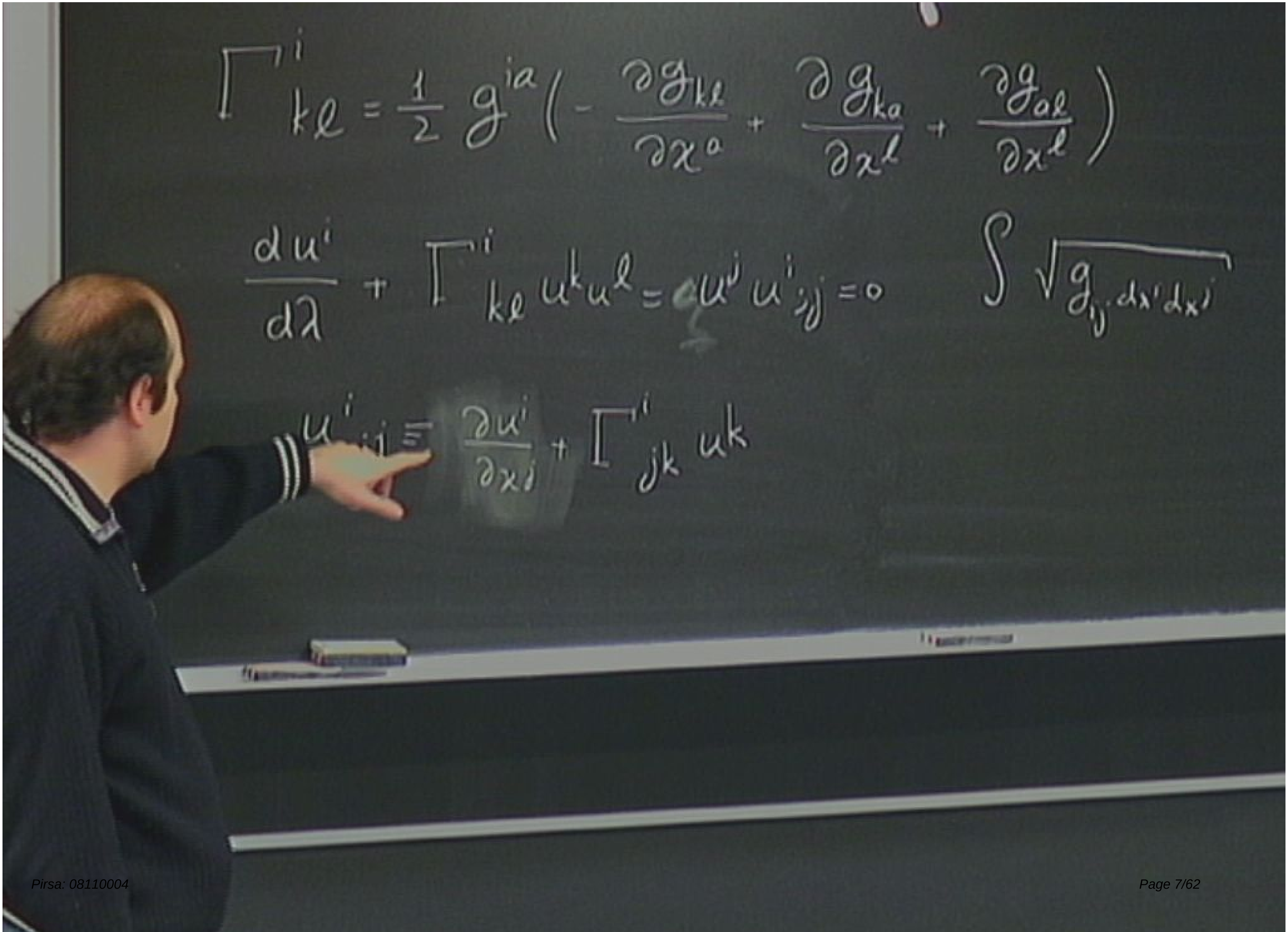
$$\int \sqrt{g_{ij} \, dx^{i} dx^{j}}$$



$\Gamma^i_{k\ell} = \frac{1}{2} g^{ia}\left(- \frac{\partial g_{k\ell}}{\partial x^a} + \frac{\partial g_{ka}}{\partial x^\ell} + \frac{\partial g_{a\ell}}{\partial x^\ell} \right)$
$\frac{du^i}{d\lambda} + \Gamma^i_{k\ell} u^k u^\ell = u^j u^i_{;j} = 0$
$\int \sqrt{g_{ij} dx^i dx^j}$
$u^i_{;j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{jk} u^k$

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{ia}\left(- \frac{\partial g_{k\ell}}{\partial x^a} + \frac{\partial g_{ka}}{\partial x^\ell} + \frac{\partial g_{a\ell}}{\partial x^\ell} \right)$$

$$\frac{du^i}{d\lambda} + \Gamma^i_{k\ell} u^k u^\ell = u^j u^i_{;j} = 0 \qquad \int \sqrt{g_{ij}\, dx^i dx^j}$$

$$\boxed{u^j = \frac{dx^j}{d\lambda}} \qquad u^i_{;j} \equiv \frac{\partial u^i}{\partial x^j} + \Gamma^i_{jk} u^k$$

$$u^j \times \frac{\partial u^i}{\partial x^j} = \frac{du^i}{d\lambda}$$

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{ia}\left(- \frac{\partial g_{k\ell}}{\partial x^a} + \frac{\partial g_{ka}}{\partial x^\ell} + \frac{\partial g_{a\ell}}{\partial x^\ell} \right)$$

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$$\frac{du^i}{d\lambda} + \Gamma^i_{k\ell} u^k u^\ell = u^j u^i_{;j} = 0 \qquad \boxed{L = \int \sqrt{g_{ij}\, dx^i dx^j}}$$

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$$\boxed{L = \int \sqrt{g_{ij}\, dx^i dx^j}}$$

$$\boxed{u^j = \frac{dx^j}{d\lambda}} \qquad \boxed{u^i{}_{;j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{jk} u^k}$$

$$A^\mu = (1, 0, 0, 0)$$

$$A^\mu A$$

$$u^j \times \frac{\partial u^i}{\partial x^j} = \frac{du^i}{d\lambda}$$

$$\Gamma^i_{k\ell} = \frac{1}{2} g^{ia}\left(- \frac{\partial g_{k\ell}}{\partial x^a} + \frac{\partial g_{ka}}{\partial x^\ell} + \frac{\partial g_{a\ell}}{\partial x^\ell} \right)$$

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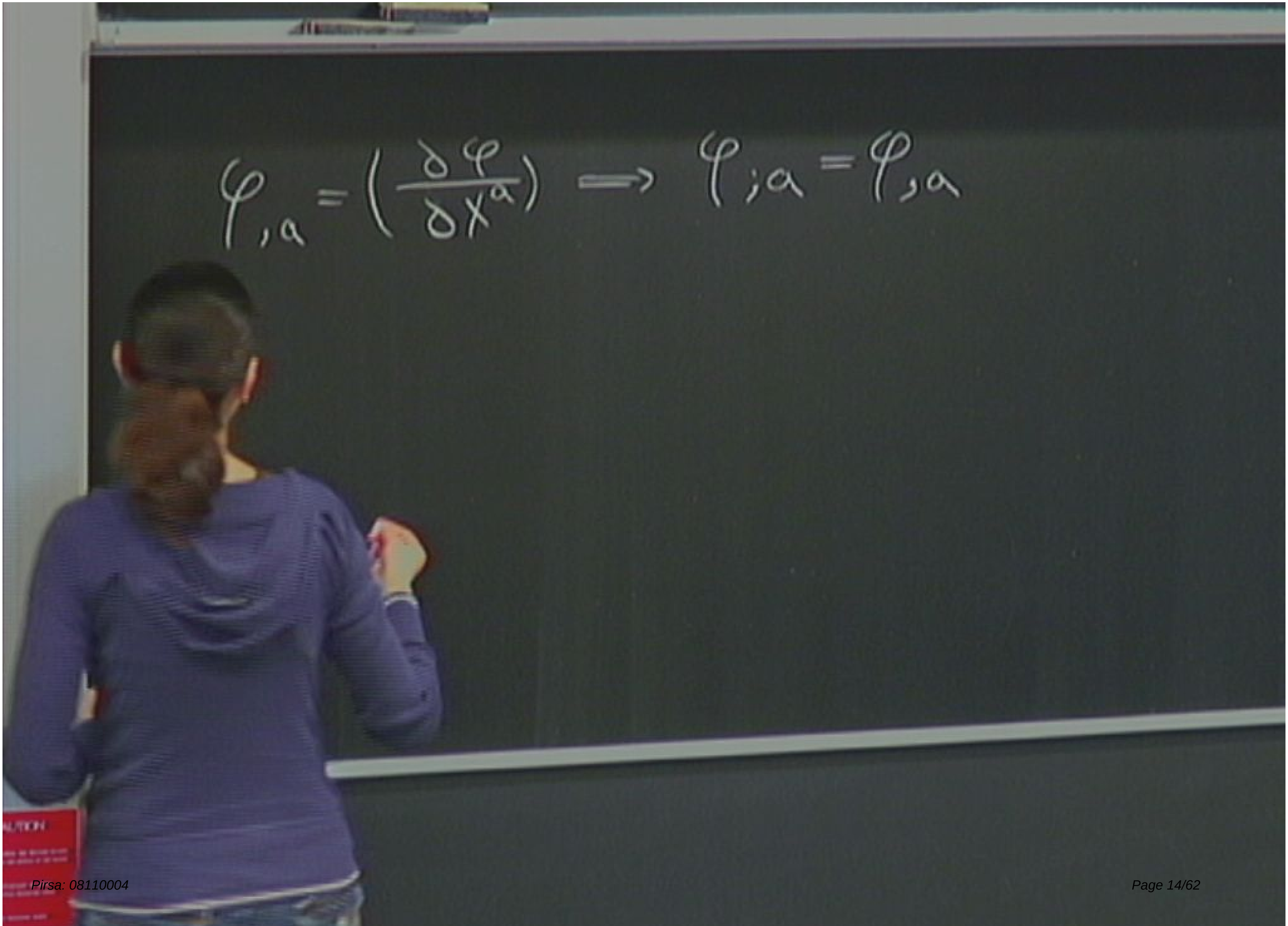
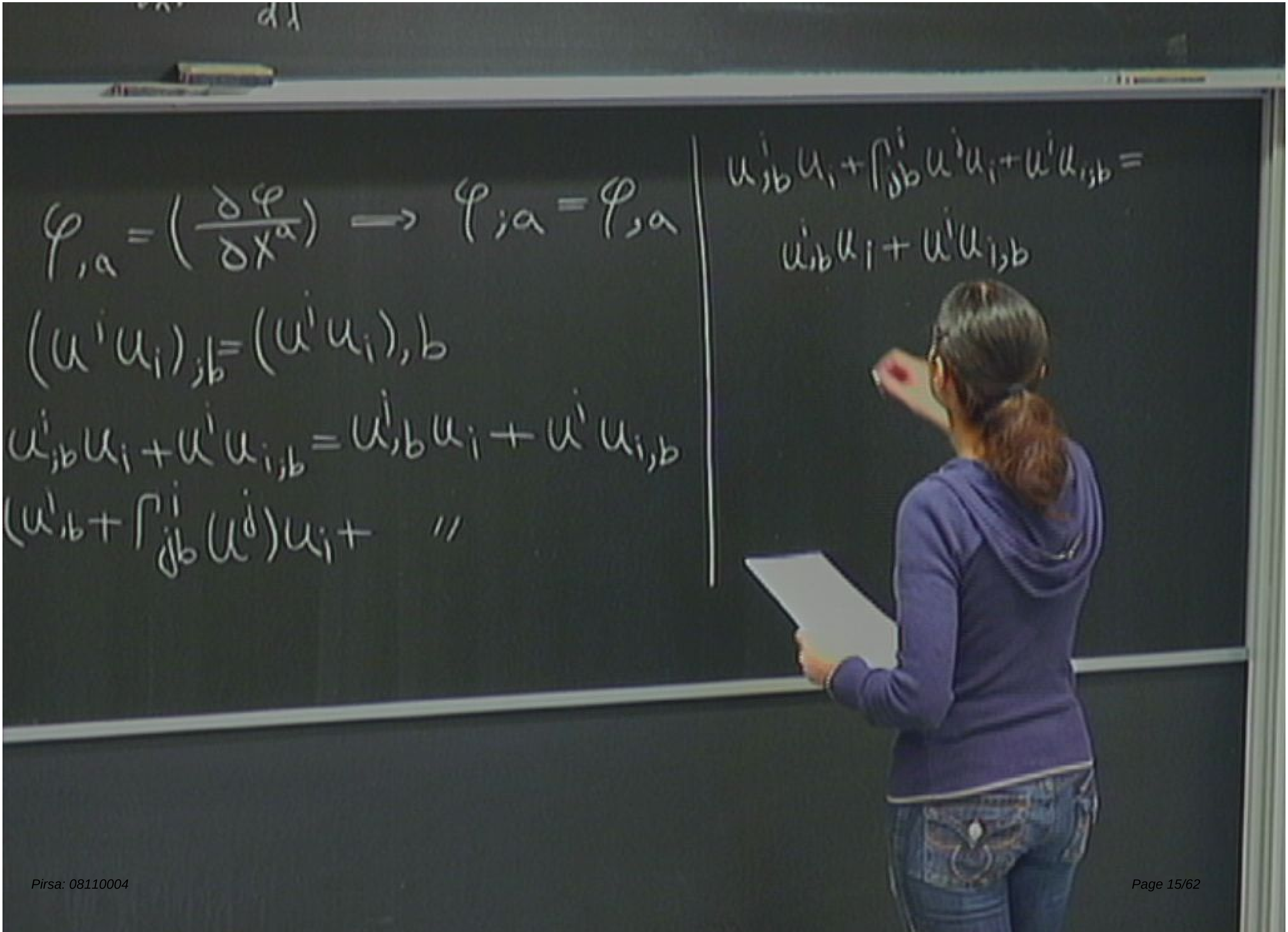

$\varphi_{,a} = \left(\frac{\partial \varphi}{\partial x^a}\right) \Longrightarrow \varphi_{;a} = \varphi_{,a}$


$\varphi_{,a} = \left(\frac{\partial \varphi}{\partial x^a}\right) \longrightarrow \varphi_{;a} = \varphi_{,a}$
$(u^i u_i)_{;b} = (u^i u_i)_{,b}$
$u^i_{;b} u_i + u^i u_{i;b} = u^i_{,b} u_i + u^i u_{i,b}$
$(u^i_{,b} + \Gamma^i_{jb} (u^j)) u_i +$ //
$u^i_{;b} u_i + \Gamma^i_{jb} u^j u_i + u^i u_{i;b} =$
$u^i_{,b} u_i + u^i u_{i,b}$

$$\varphi_{,a} = \left(\frac{\partial \varphi}{\partial x^a}\right) \Longrightarrow \varphi_{;a} = \varphi_{,a}$$

$$(u^i u_i)_{;b} = (u^i u_i)_{,b}$$

$$u^i_{;b} u_i + u^i u_{i;b} = u^i_{,b} u_i + u^i u_{i,b}$$

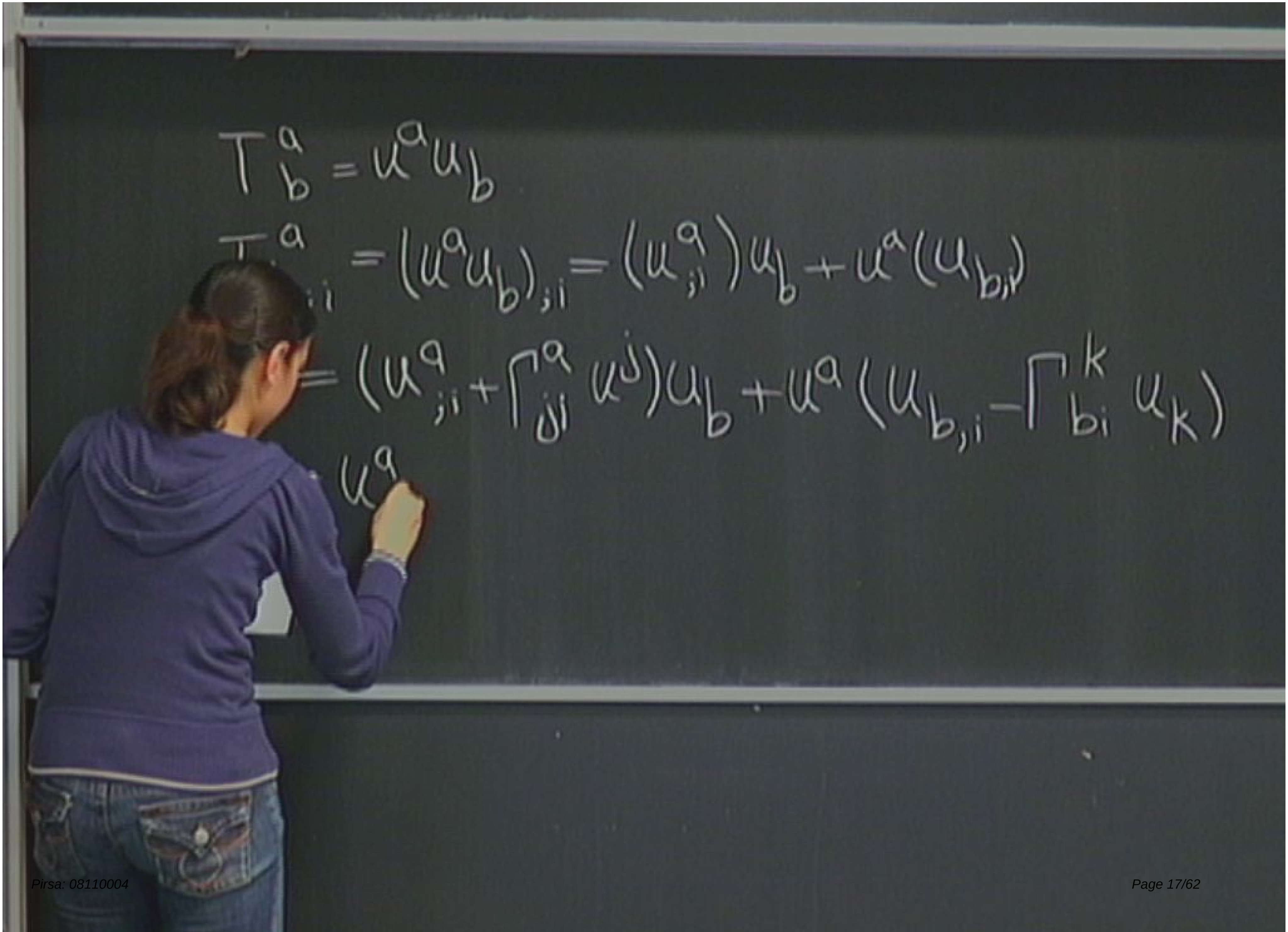
$$(u^i_{,b} + \Gamma^i_{jb} u^j) u_i + \quad ''$$

$$\cancel{u^i_{,b} u_i} + \Gamma^i_{jb} u^j u_i + u^i u_{i;b} =$$

$$\cancel{u^i_{,b} u_i} + u^i u_{i,b}$$

$$\Longrightarrow \cancel{u^i} u_{i;b} + \Gamma^j_{ib} \cancel{u^i} u_j = \cancel{u^i} u_{i,b}$$

$$u_{i;b} = u_{i,b} - \Gamma^j_{ib} u_j$$


$T^a_{\ b} = u^a u_b$
$T^a_{\ b;i} = (u^a u_b)_{;i} = (u^a_{\ ;i}) u_b + u^a (u_{b;i})$
$= (u^a_{\ ,i} + \Gamma^a_{ji} u^j) u_b + u^a (u_{b,i} - \Gamma^k_{bi} u_k)$
$= u^a$

$$T^{a}_{\ b} = u^{a} u_{b}$$

$$T^{a}_{\ b;i} = (u^{a} u_{b})_{;i} = (u^{a}_{\ ;i}) u_{b} + u^{a} (u_{b;i})$$

$$= (u^{a}_{\ ,i} + \Gamma^{a}_{\ ji} u^{j}) u_{b} + u^{a} (u_{b,i} - \Gamma^{k}_{\ bi} u_{k})$$

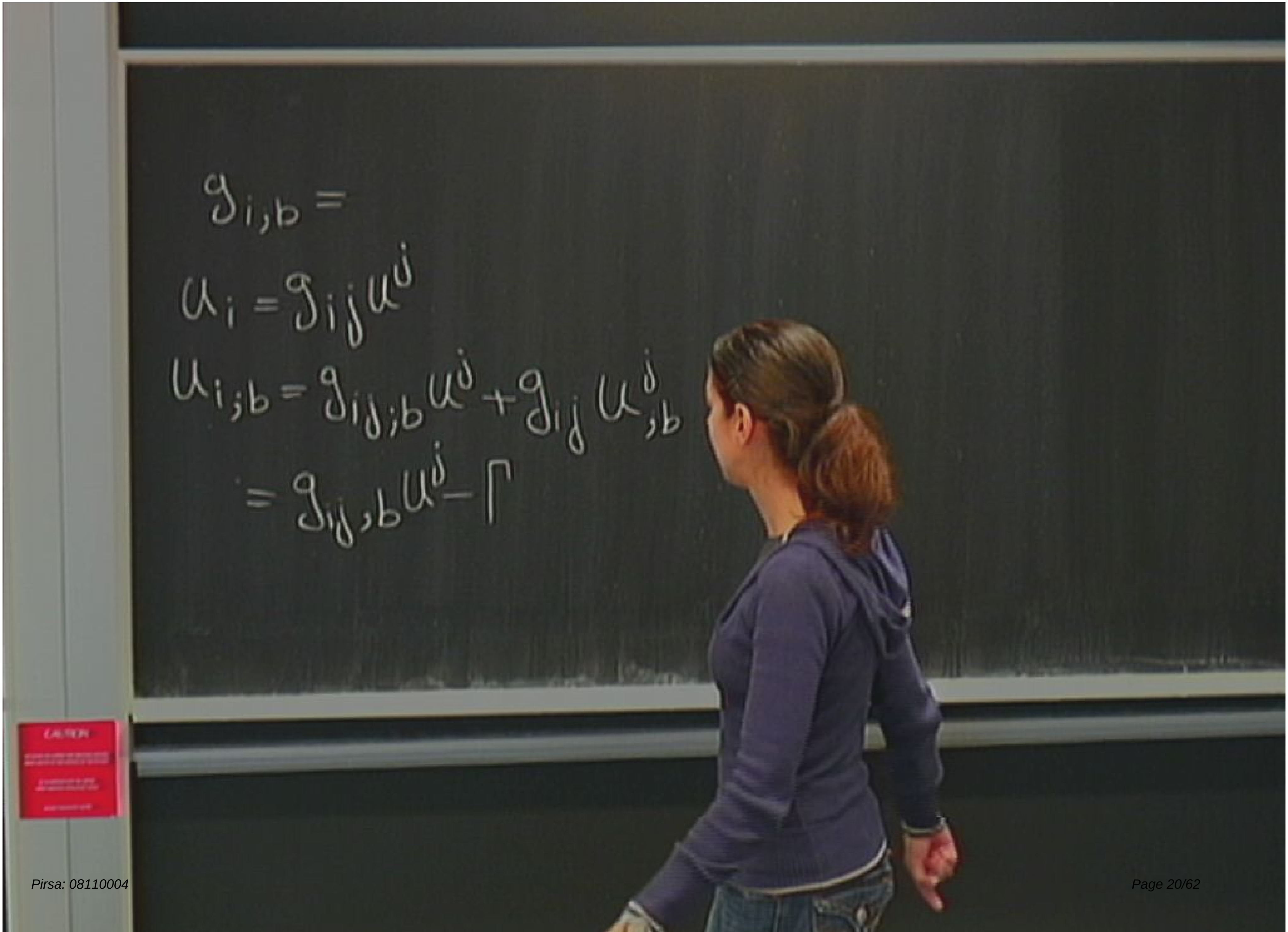
$$= u^{a}_{\ ,i} u_{b}$$

$$T^a{}_b = u^a u_b$$

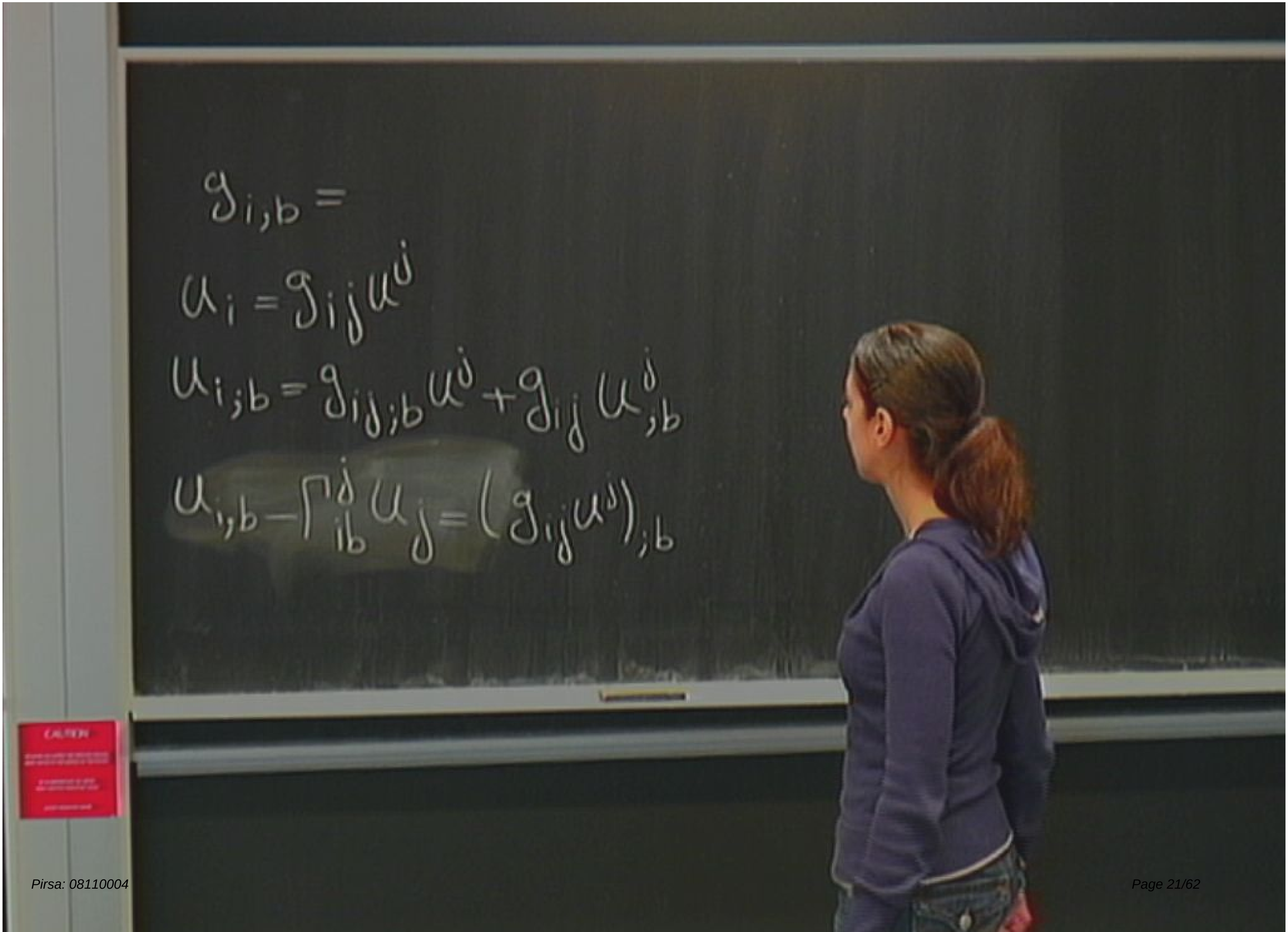
$$T^a{}_{b;i} = (u^a u_b)_{;i} = (u^a{}_{;i})\, u_b + u^a (u_{b;i})$$

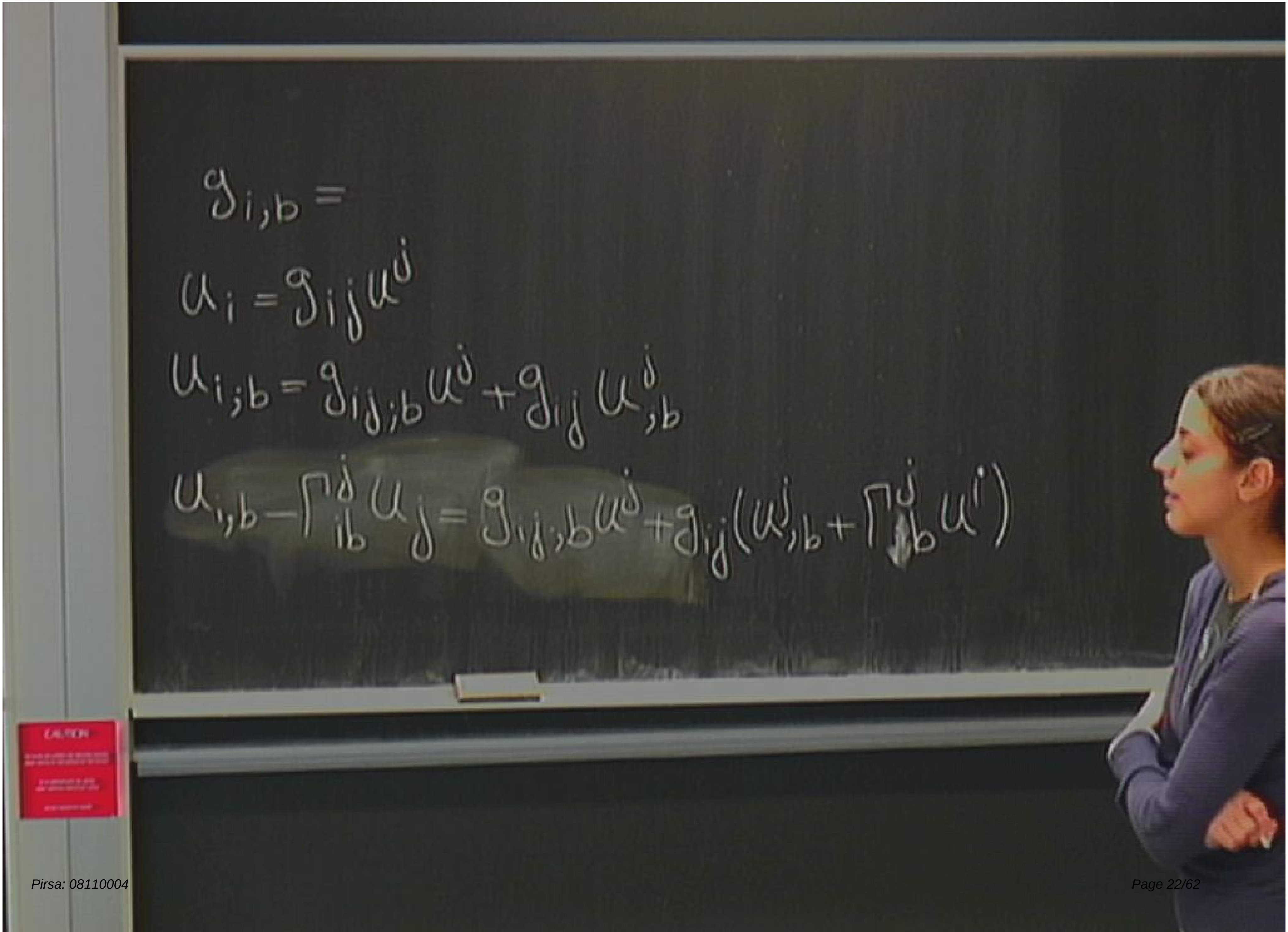
$$= (u^a{}_{,i} + \Gamma^a_{ji} u^j)\, u_b + u^a (u_{b,i} - \Gamma^k_{bi} u_k)$$

$$= u^a{}_{,i}\, u_b$$



$g_{i,b} =$
$u_i = g_{ij} u^j$
$u_{i;b} = g_{ij;b} u^j + g_{ij} u^j_{,b}$
$= g_{ij,b} u^j - \Gamma$


$g_{i,b} =$
$u_i = g_{ij} u^j$
$u_{i;b} = g_{ij;b} u^j + g_{ij} u^j_{,b}$
$u_{i,b} - \Gamma^j_{ib} u_j = (g_{ij} u^j)_{;b}$
CAUTION


$g_{i,b} =$
$u_i = g_{ij} u^j$
$u_{i;b} = g_{ij;b} u^j + g_{ij} u^j_{,b}$
$u_{i,b} - \Gamma^j_{ib} u_j = g_{ij,b} u^j + g_{ij}(u^j_{,b} + \Gamma^j_{ib} u^i)$

$$g_{i,b} =$$

$$u_i = g_{ij} u^j$$

$$u_{i;b} = g_{ij;b} u^j + g_{ij} u^j_{;b}$$

$$u_{i,b} - \Gamma^j_{ib} u_j = g_{ij,b} u^j + g_{ij}(u^j_{,b} + \Gamma^j_{ib} u^i)$$

$$g_{ij} u^j_{;b} = (u_{i;b}$$

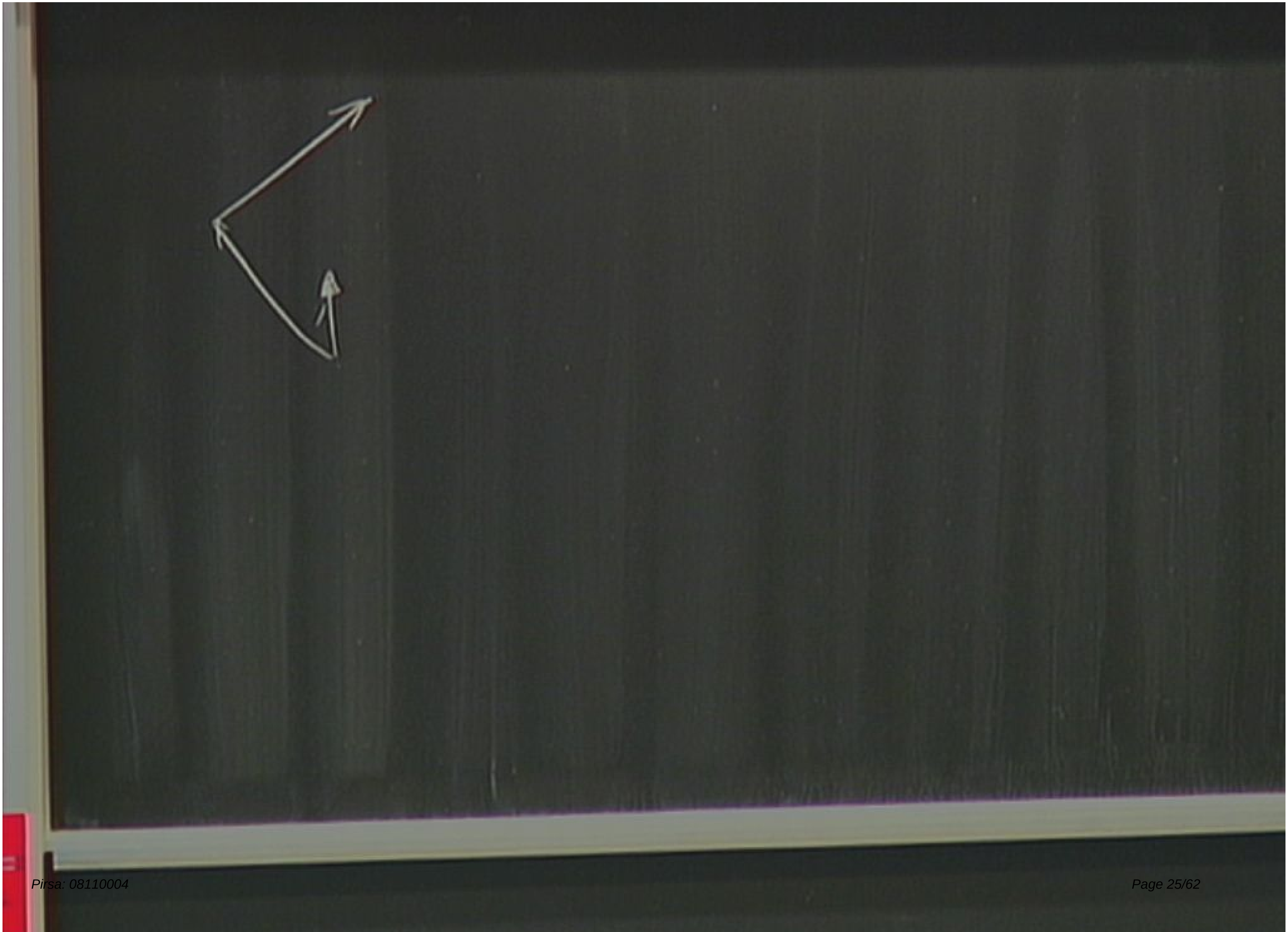
$$= (g_{ij} u^j)_{;b}$$

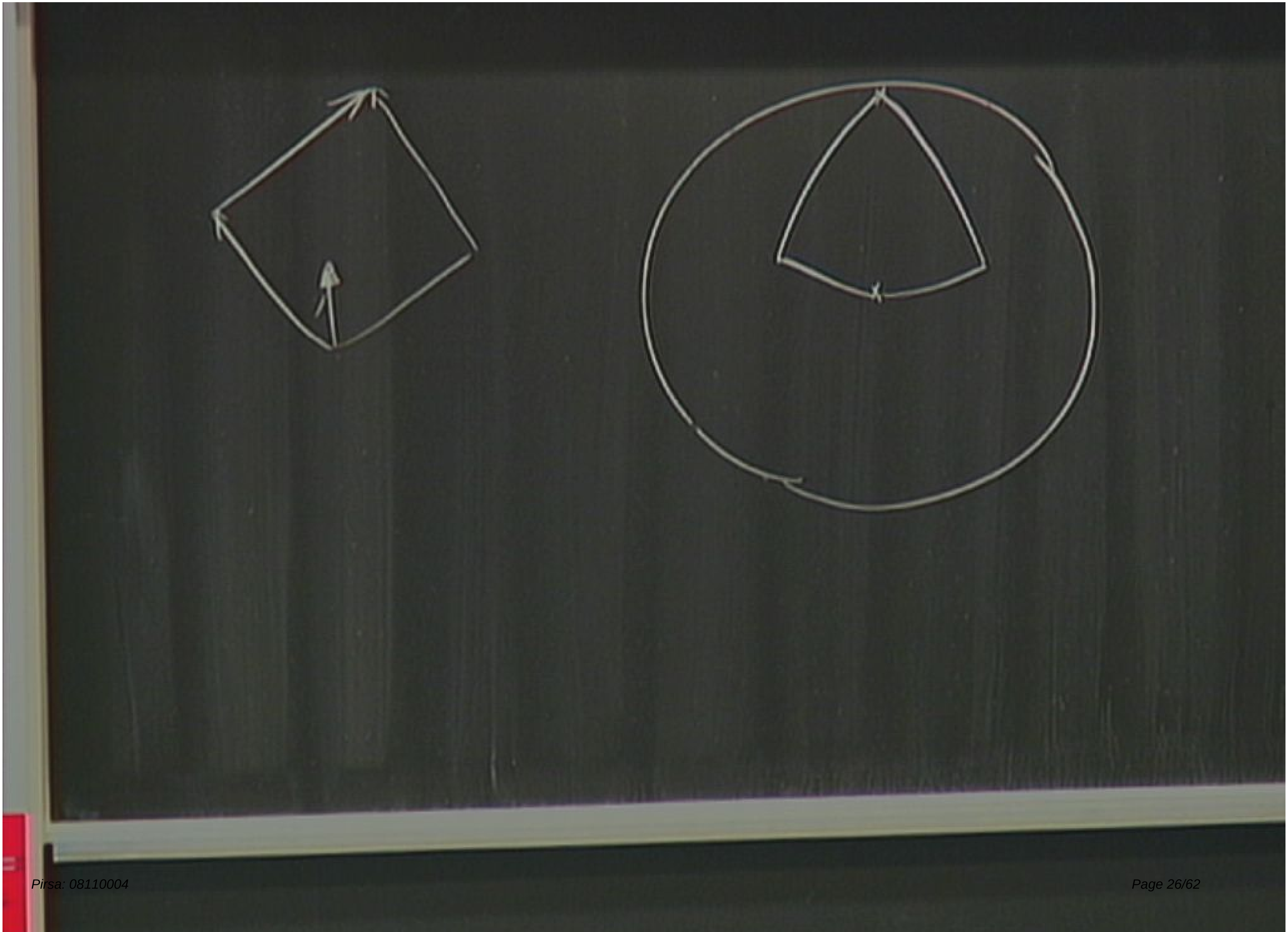
$$A^a_{;b} = A^a_{,b} + \Gamma^a_{mb} A^m$$

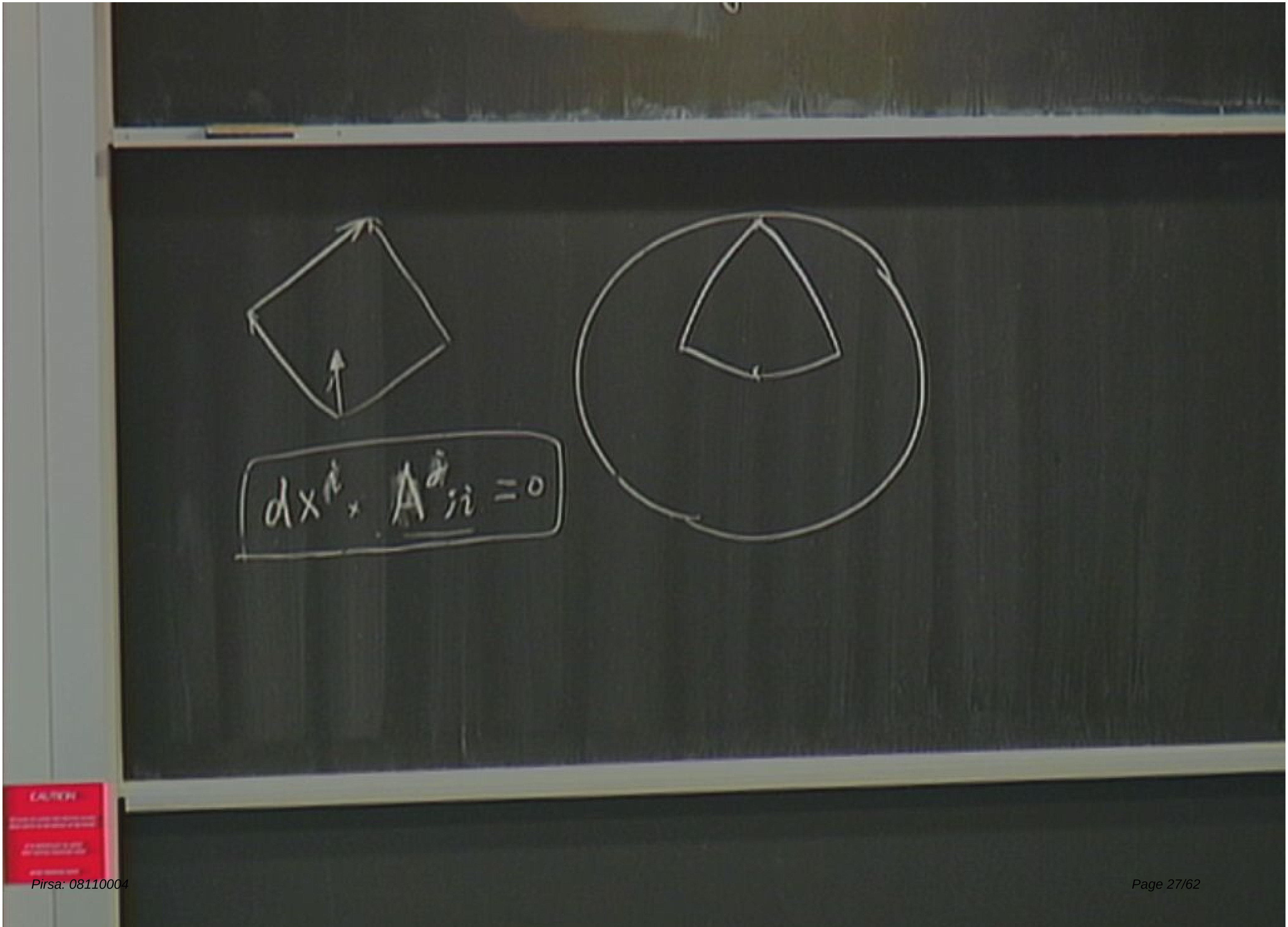
$$A^a_{;b;c} = (A^a_{;b})_{;c} + \Gamma^a_{nc} A^n_{;b} - \Gamma$$

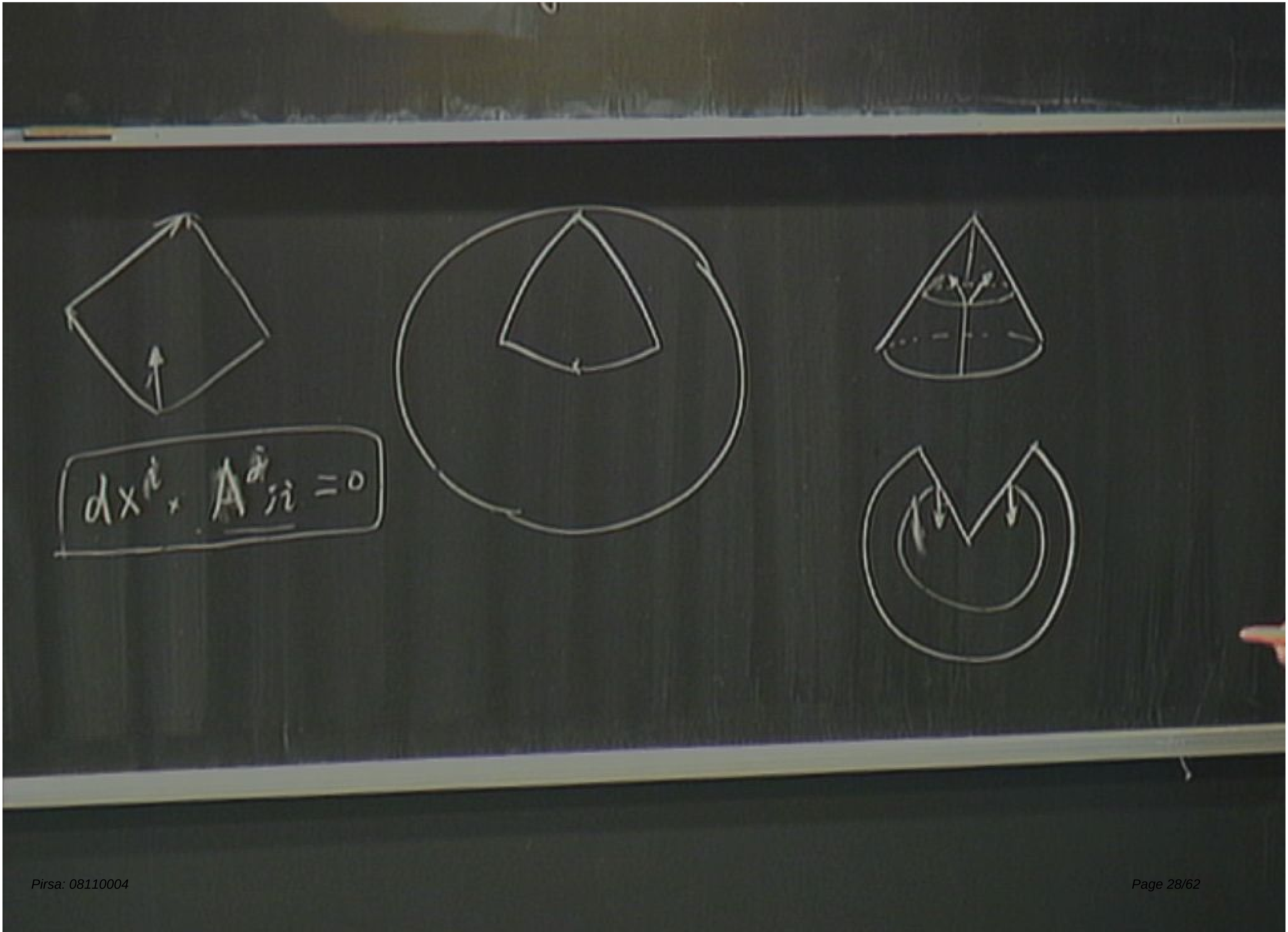
$$A^a_{;b;c} - A^a_{;c;b} = -\left(\Gamma^a_{mc,b} - \Gamma^a_{mb,c} + \Gamma^a_{nb}\Gamma^n_{mc} + \Gamma^a_{nc}\Gamma^n_{mb}\right) A^m$$

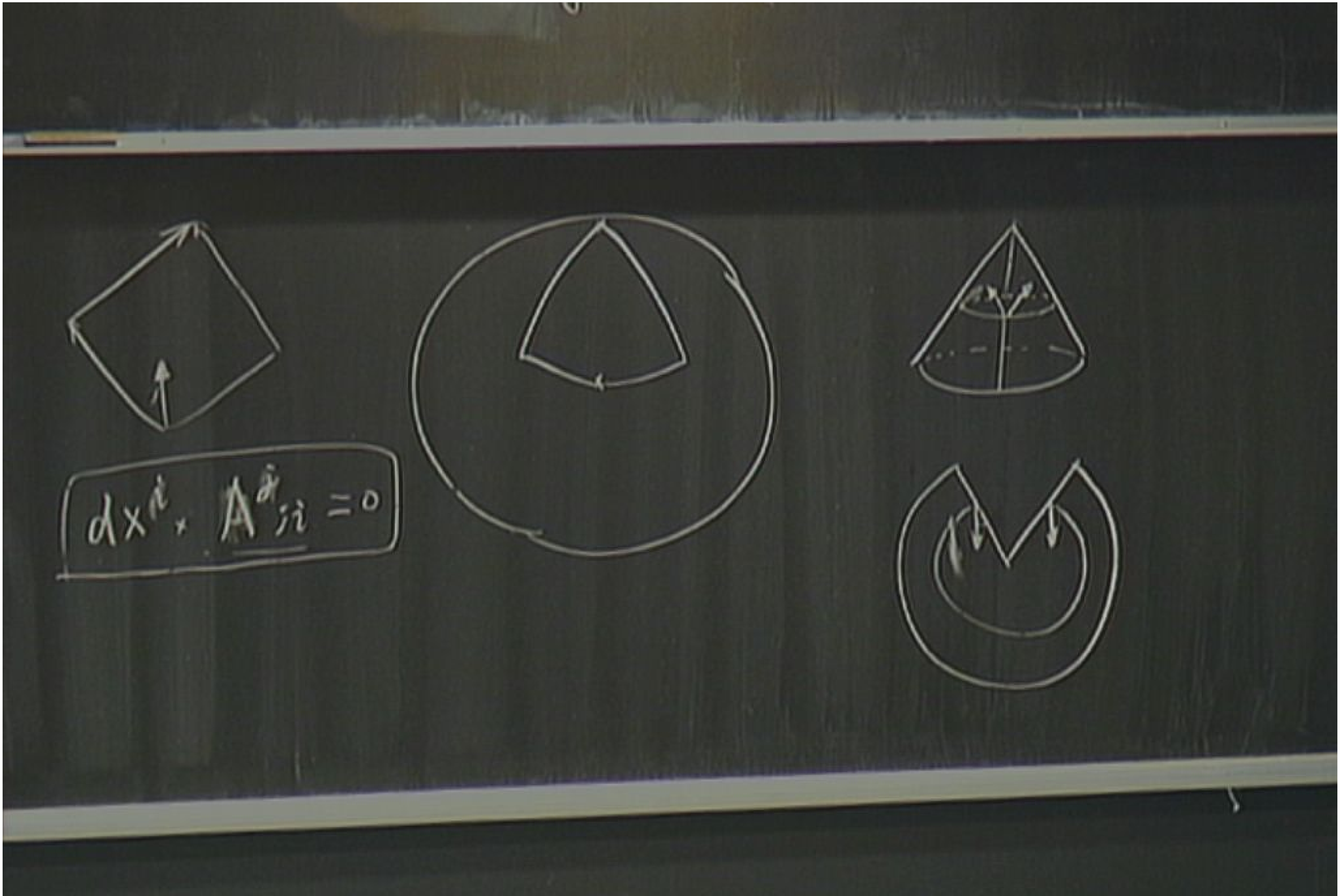
$$= -R^a_{mbc} A^m$$

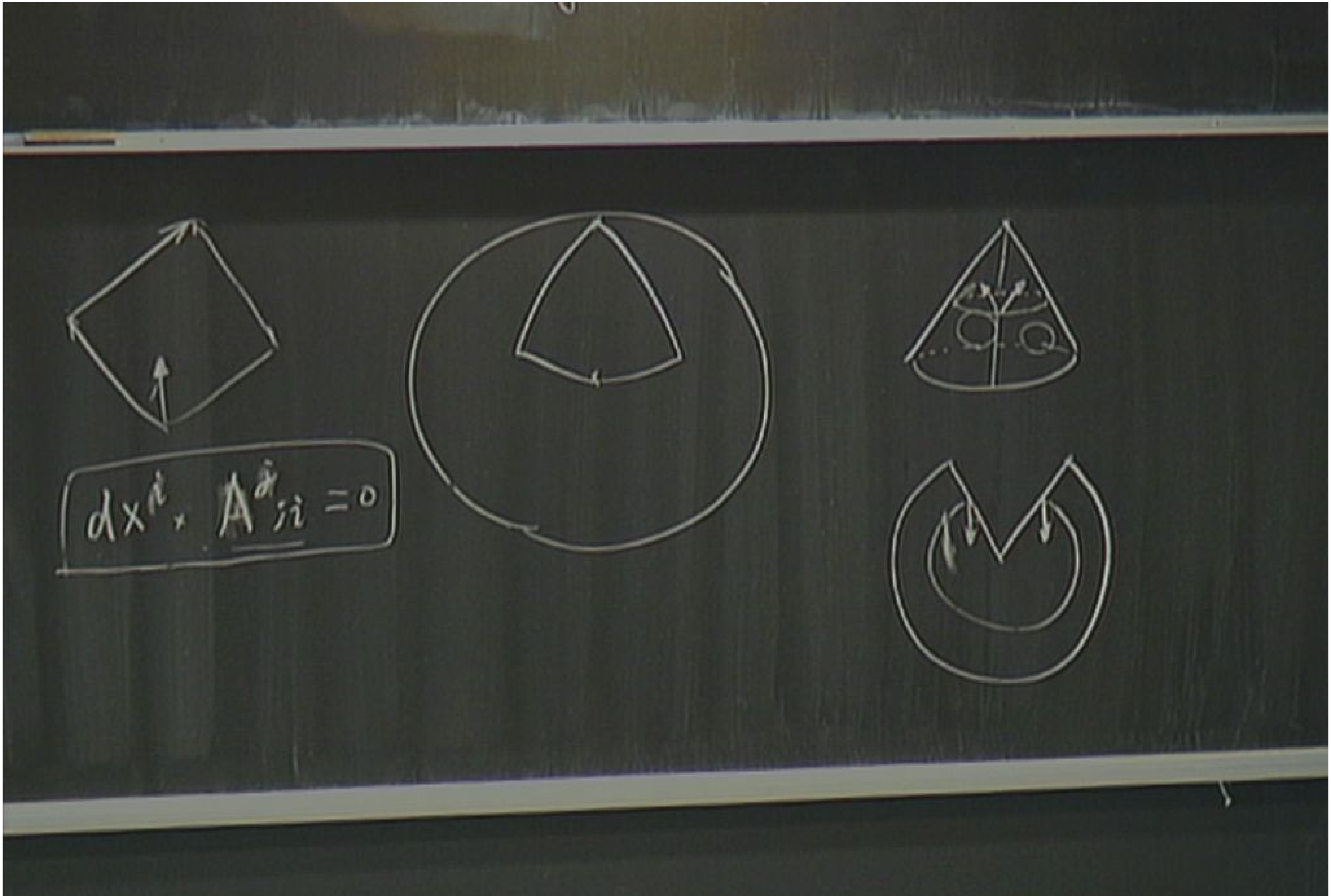





dx^i ∧ A^j_{ji} = 0


$dx^i \cdot A^j{}_{;i} = 0$


$dx^{i} \cdot A^{a}{}_{ji} = 0$



Static isotropic gravitational field:

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2\,d\varphi^2$$

$\theta = cst = \frac{\pi}{2}$

$B(r) = \left(1 - \frac{2GM}{r}\right)$

$A(r) = \left(1 - \frac{2GM}{r}\right)^{-1}$

The equations of free fall:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda} = 0$$

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$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right)$$

The only non-zero comp. of Γ :

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right)$$

The only non-zero comp. of Γ :

$$\Gamma^{r}_{rr} = \frac{1}{2A} \frac{dA}{dr}$$

Static isotropic gravitational field:

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2 \overset{\sin^2\theta}{} d\varphi^2 \;\; - r^2\,d\theta^2$$

$\theta = cst = \frac{\pi}{2}$

$B(r) = \left(1 - \frac{2GM}{r}\right)$

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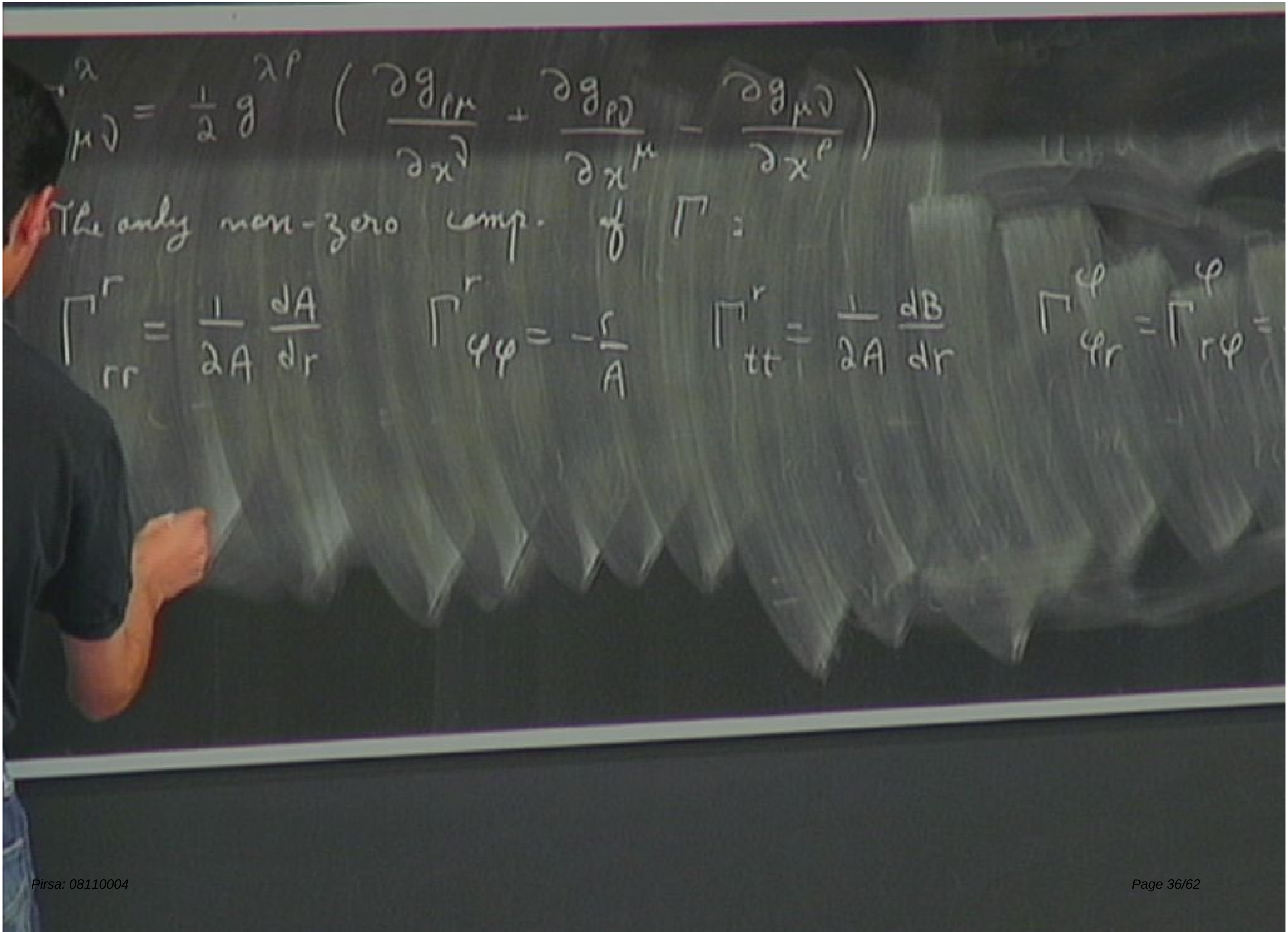
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The only non-zero comp. of Γ :
$\Gamma^{r}_{rr} = \frac{1}{2A}\frac{dA}{dr}$
$\Gamma^{r}_{\varphi\varphi} = -\frac{r}{A}$
$\Gamma^{r}_{tt} = \frac{1}{2A}\frac{dB}{dr}$
$\Gamma^{\varphi}_{\varphi r} = \Gamma^{\varphi}_{r\varphi} =$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right)$$

The only non-zero comp. of Γ:

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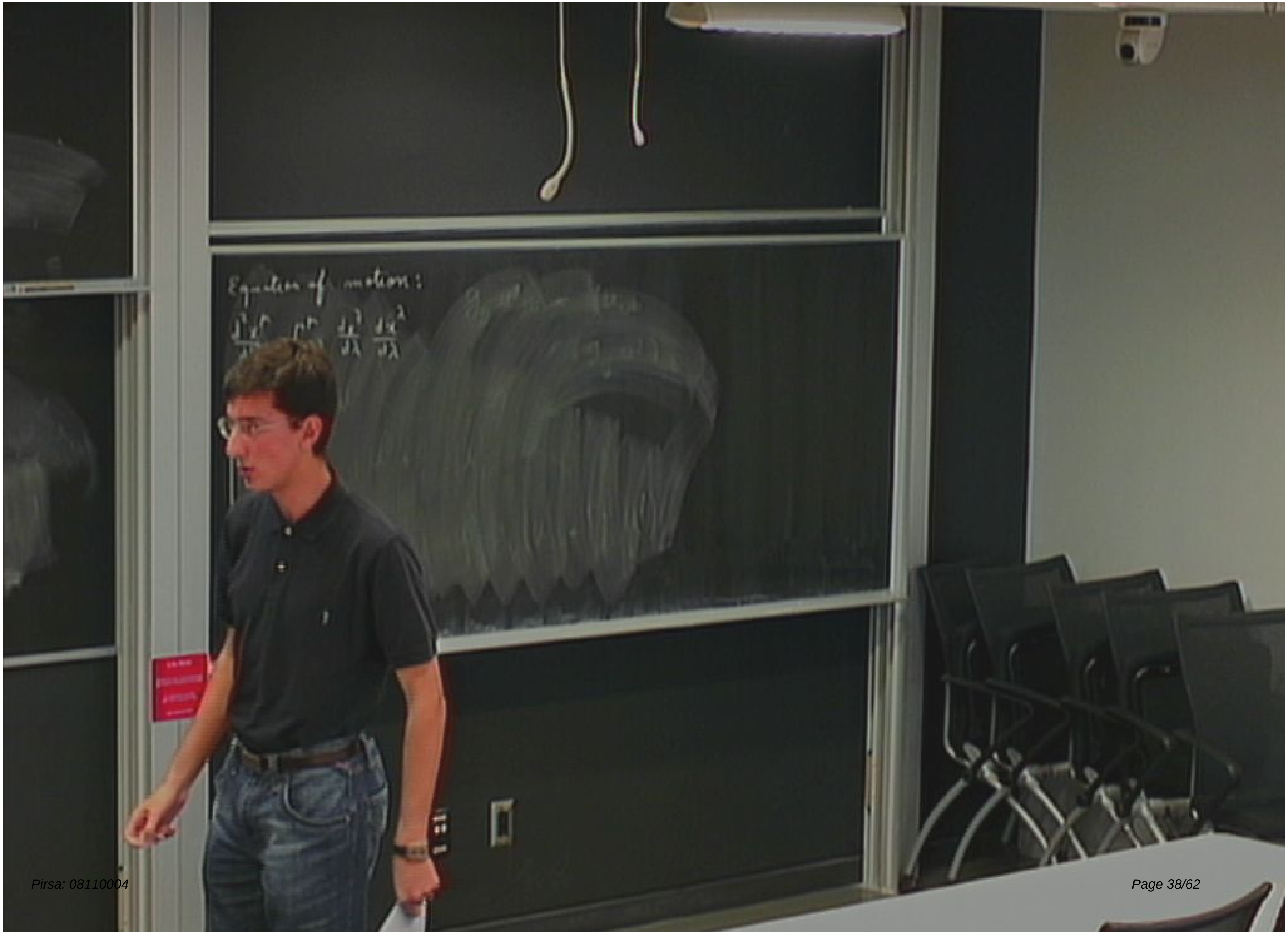
$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{1}{2B}\frac{dB}{dr}$$


Equation of motion:

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$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\lambda} \frac{dx^\lambda}{d\lambda}$$

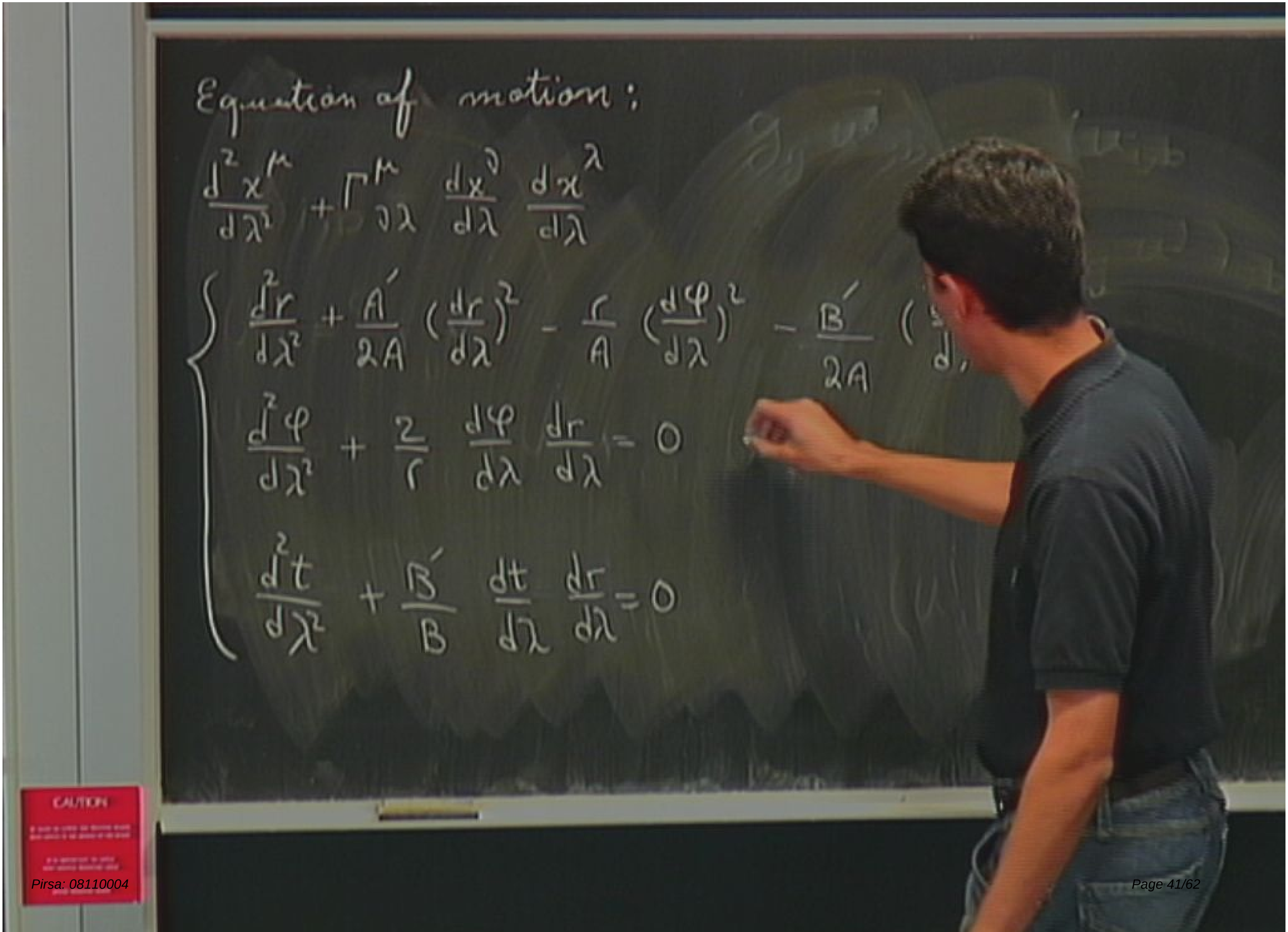
Equation of motion:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\lambda} \frac{dx^\lambda}{d\lambda}$$

$$\frac{d^2 r}{d\lambda^2} + \frac{A'}{2A} \left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{A} \left(\frac{d\varphi}{d\lambda}\right)^2 - \frac{B'}{2A} \left(\frac{dt}{d\lambda}\right)^2 = 0$$

$$\frac{d^2 \varphi}{d\lambda^2} + \frac{2}{r} \frac{d\varphi}{d\lambda} \frac{dr}{d\lambda} = 0$$

$$\frac{d^2 t}{d\lambda^2}$$


Equation of motion:
$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda}$
$\frac{d^2r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{A}\left(\frac{d\varphi}{d\lambda}\right)^2 - \frac{B'}{2A}\left(\frac{d}{d}\right.$
$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = 0$
$\frac{d^2t}{d\lambda^2} + \frac{B'}{B}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$

Equation of motion:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda} \qquad \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\, g_{\mu\nu}$$

$$\begin{cases} \dfrac{d^2r}{d\lambda^2} + \dfrac{A'}{2A}\left(\dfrac{dr}{d\lambda}\right)^2 - \dfrac{r}{A}\left(\dfrac{d\varphi}{d\lambda}\right)^2 - \dfrac{B'}{2A}\left(\dfrac{dt}{d\lambda}\right)^2 = 0 \\[2ex] \dfrac{d^2\varphi}{d\lambda^2} + \dfrac{2}{r}\dfrac{d\varphi}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \\[2ex] \dfrac{d^2t}{d\lambda^2} + \dfrac{B'}{B}\dfrac{dt}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \end{cases}$$

Equation of motion:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda}$$

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} g_{\mu\nu} \qquad d\lambda = dS$$

$$\begin{cases} \dfrac{d^2r}{d\lambda^2} + \dfrac{A'}{2A}\left(\dfrac{dr}{d\lambda}\right)^2 - \dfrac{r}{A}\left(\dfrac{d\varphi}{d\lambda}\right)^2 - \dfrac{B'}{2A}\left(\dfrac{dt}{d\lambda}\right)^2 = 0 \\[2ex] \dfrac{d^2\varphi}{d\lambda^2} + \dfrac{2}{r}\dfrac{d\varphi}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \\[2ex] \dfrac{d^2t}{d\lambda^2} + \dfrac{B'}{B}\dfrac{dt}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \end{cases}$$

Equation of motion:

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu}$$

$d\lambda = ds \longrightarrow d\lambda = \alpha\, ds$

$B' = \frac{dB}{dr}$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda}$$

$$\begin{cases} \dfrac{d^2r}{d\lambda^2} + \dfrac{A'}{2A}\left(\dfrac{dr}{d\lambda}\right)^2 - \dfrac{r}{A}\left(\dfrac{d\varphi}{d\lambda}\right)^2 - \dfrac{B'}{2A}\left(\dfrac{dt}{d\lambda}\right)^2 = 0 \\ \dfrac{d^2\varphi}{d\lambda^2} + \dfrac{2}{r}\dfrac{d\varphi}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \\ \dfrac{d^2t}{d\lambda^2} + \dfrac{B'}{B}\dfrac{dt}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \dfrac{d}{d\lambda}\left(\ln\dfrac{dt}{d\lambda} + \ln B\right) = 0 \rightarrow \ln\dfrac{dt}{d\lambda} + \ln B \end{cases}$$

Equation of motion:

$$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu}$$

$$d\lambda = dS \longrightarrow d\lambda = \alpha\, ds \qquad B' = \frac{dB}{dr}$$

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda}$$

$$\begin{cases} \dfrac{d^2r}{d\lambda^2} + \dfrac{A'}{2A}\left(\dfrac{dr}{d\lambda}\right)^2 - \dfrac{r}{A}\left(\dfrac{d\varphi}{d\lambda}\right)^2 - \dfrac{B'}{2A}\left(\dfrac{dt}{d\lambda}\right)^2 = 0 \\ \dfrac{d^2\varphi}{d\lambda^2} + \dfrac{2}{r}\dfrac{d\varphi}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \\ \dfrac{d^2t}{d\lambda^2} + \dfrac{B'}{B}\dfrac{dt}{d\lambda}\dfrac{dr}{d\lambda} = 0 \longrightarrow \dfrac{d}{d\lambda}\left(\ln\dfrac{dt}{d\lambda} + \ln B\right) = 0 \rightarrow \ln\dfrac{dt}{d\lambda} + \ln B = 0 \end{cases}$$

$$\rightarrow \boxed{\frac{dt}{d\lambda} = \frac{1}{B} \qquad (1)}$$

Static isotropic gravitational field:

$$ds^2 = B(r)\,dt^2 - A(r)\,dr^2 - r^2 \overset{\sin^2\theta}{} d\varphi^2 \quad - r^2 d\theta^2$$

$\theta = cst = \frac{\pi}{2}$

$B(r) = \left(1 - \frac{2GM}{r}\right)$

$A(r) = \left(1 - \frac{2GM}{r}\right)^{-1}$

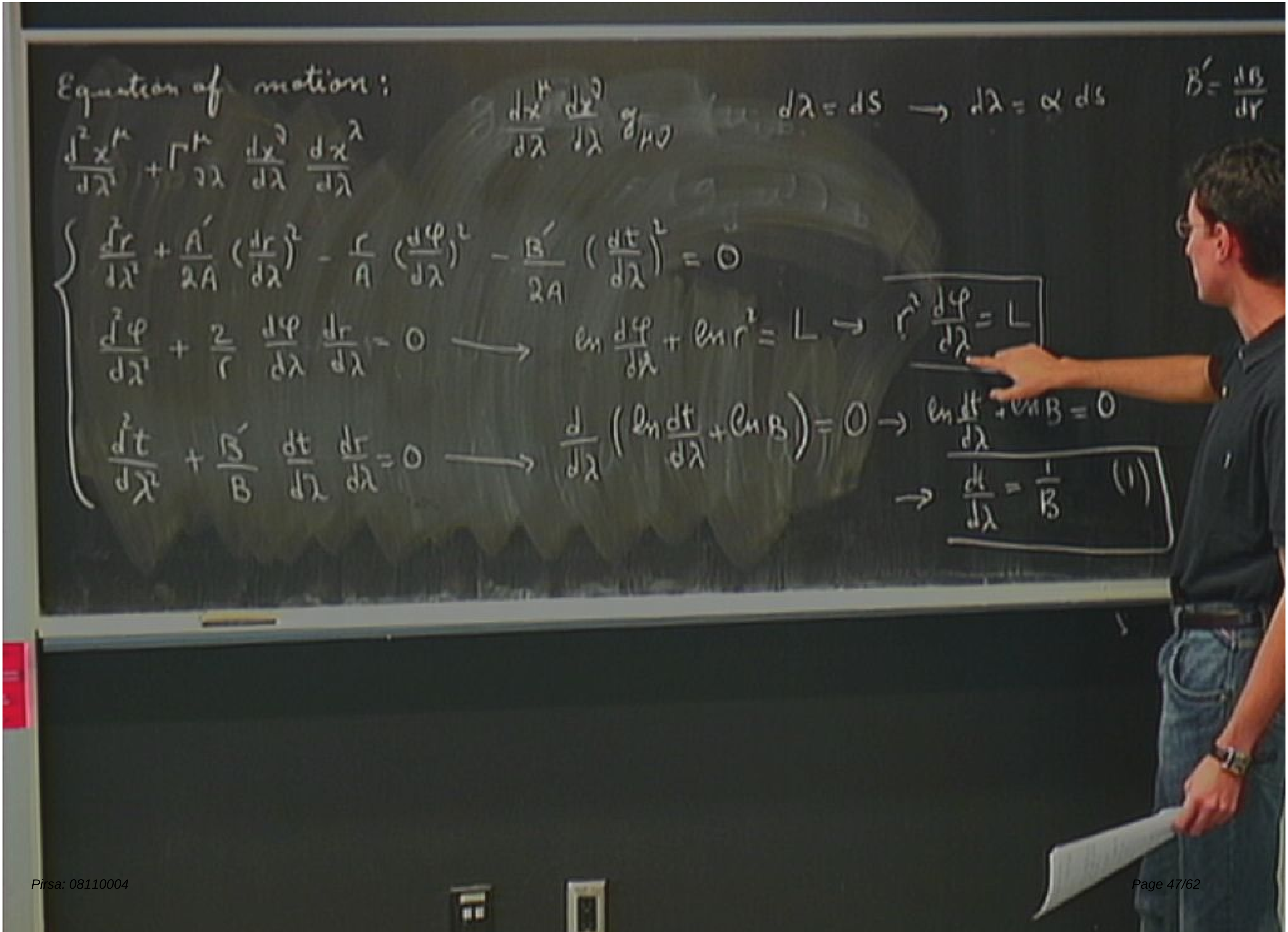
The equations of free fall:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\lambda} \frac{dx^\lambda}{d\lambda} = 0$$

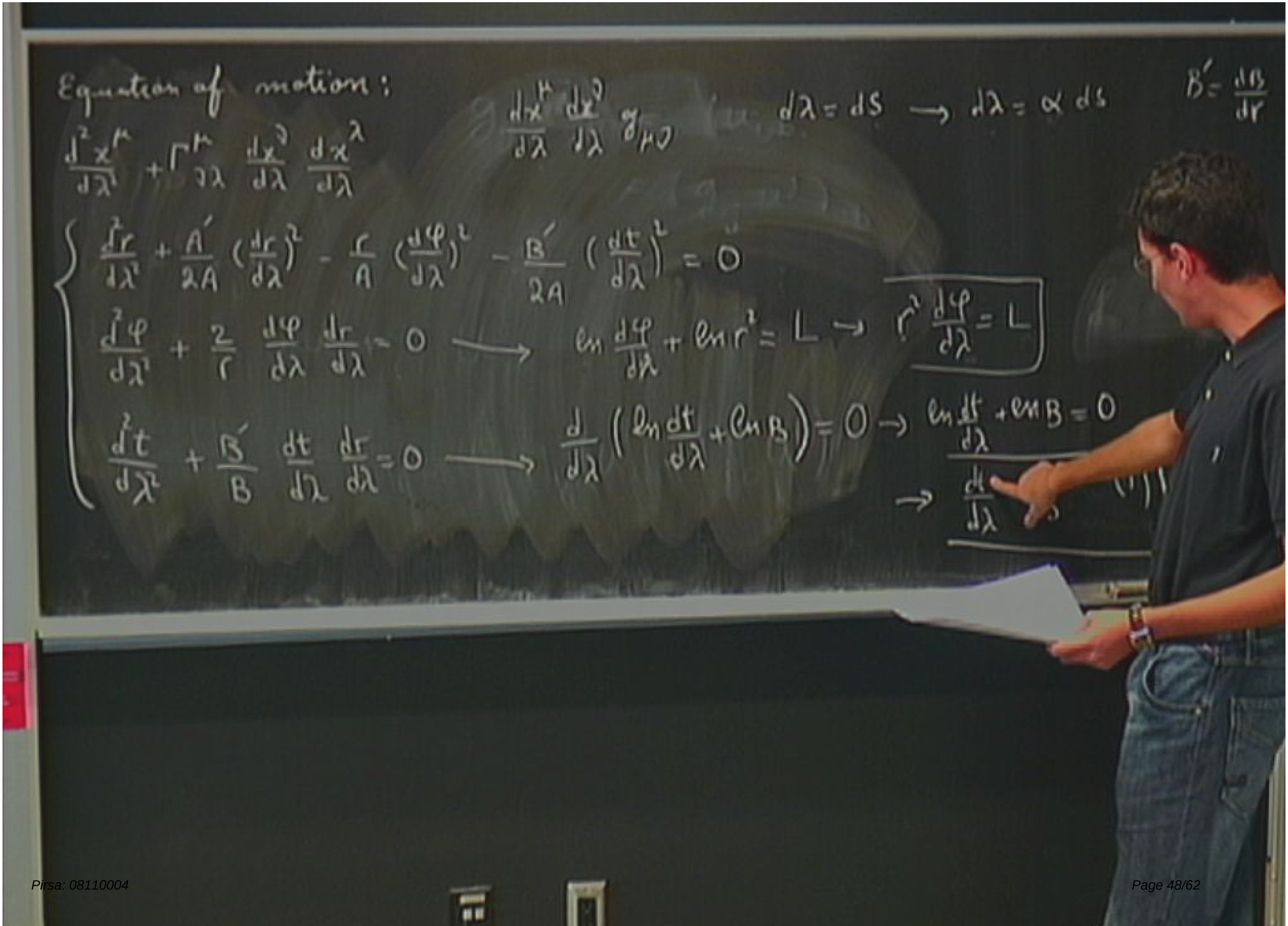
$$\int \Big[B(r) \left(\frac{dt}{d\lambda}\right)^2 \cdots \Big] d\lambda$$

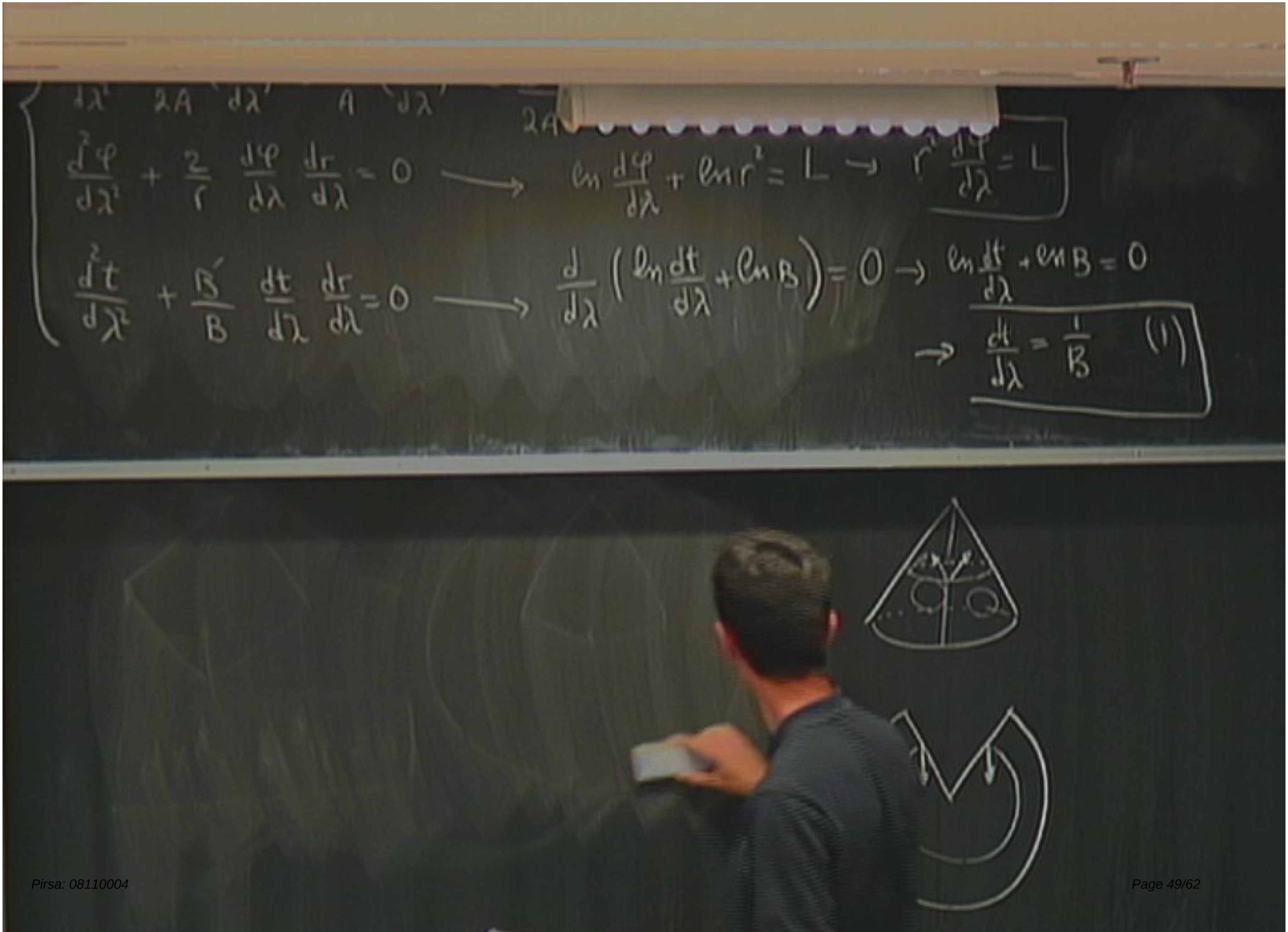
$\mathcal{L}(x^\mu, \dot{x}^\mu)$

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\lambda}\left[\frac{\partial \mathcal{L}}{\partial \frac{dx^\mu}{d\lambda}}\right]$$



Equation of motion:
$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda}$
$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu}$
$d\lambda = ds \longrightarrow d\lambda = \alpha\, ds$
$B' = \frac{dB}{dr}$
$\frac{d^2r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{A}\left(\frac{d\varphi}{d\lambda}\right)^2 - \frac{B'}{2A}\left(\frac{dt}{d\lambda}\right)^2 = 0$
$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \ln\frac{d\varphi}{d\lambda} + \ln r^2 = L \to \boxed{r^2\frac{d\varphi}{d\lambda} = L}$
$\frac{d^2t}{d\lambda^2} + \frac{B'}{B}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \frac{d}{d\lambda}\left(\ln\frac{dt}{d\lambda} + \ln B\right) = 0 \to \ln\frac{dt}{d\lambda} + \ln B = 0$
$\to \boxed{\frac{dt}{d\lambda} = \frac{1}{B} \quad (1)}$


Equation of motion:
$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda}$
$\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}g_{\mu\nu}$
$d\lambda = ds \longrightarrow d\lambda = \alpha\, ds$
$B' = \frac{dB}{dr}$
$\frac{d^2r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{r}{A}\left(\frac{d\varphi}{d\lambda}\right)^2 - \frac{B'}{2A}\left(\frac{dt}{d\lambda}\right)^2 = 0$
$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \ln\frac{d\varphi}{d\lambda} + \ln r^2 = L \rightarrow \boxed{r^2\frac{d\varphi}{d\lambda} = L}$
$\frac{d^2t}{d\lambda^2} + \frac{B'}{B}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \frac{d}{d\lambda}\left(\ln\frac{dt}{d\lambda} + \ln B\right) = 0 \rightarrow \ln\frac{dt}{d\lambda} + \ln B = 0$
$\rightarrow \frac{dt}{d\lambda}$


$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \ln\frac{d\varphi}{d\lambda} + \ln r^2 = L \to \boxed{r^2\frac{d\varphi}{d\lambda} = L}$
$\frac{d^2t}{d\lambda^2} + \frac{B'}{B}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \frac{d}{d\lambda}\left(\ln\frac{dt}{d\lambda} + \ln B\right) = 0 \to \ln\frac{dt}{d\lambda} + \ln B = 0$
$\to \boxed{\frac{dt}{d\lambda} = \frac{1}{B} \quad (1)}$

$$\frac{d^2r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3A} + \frac{B'}{2AB^2} = 0 \qquad \times\; 2A\,\frac{dr}{d\lambda}$$

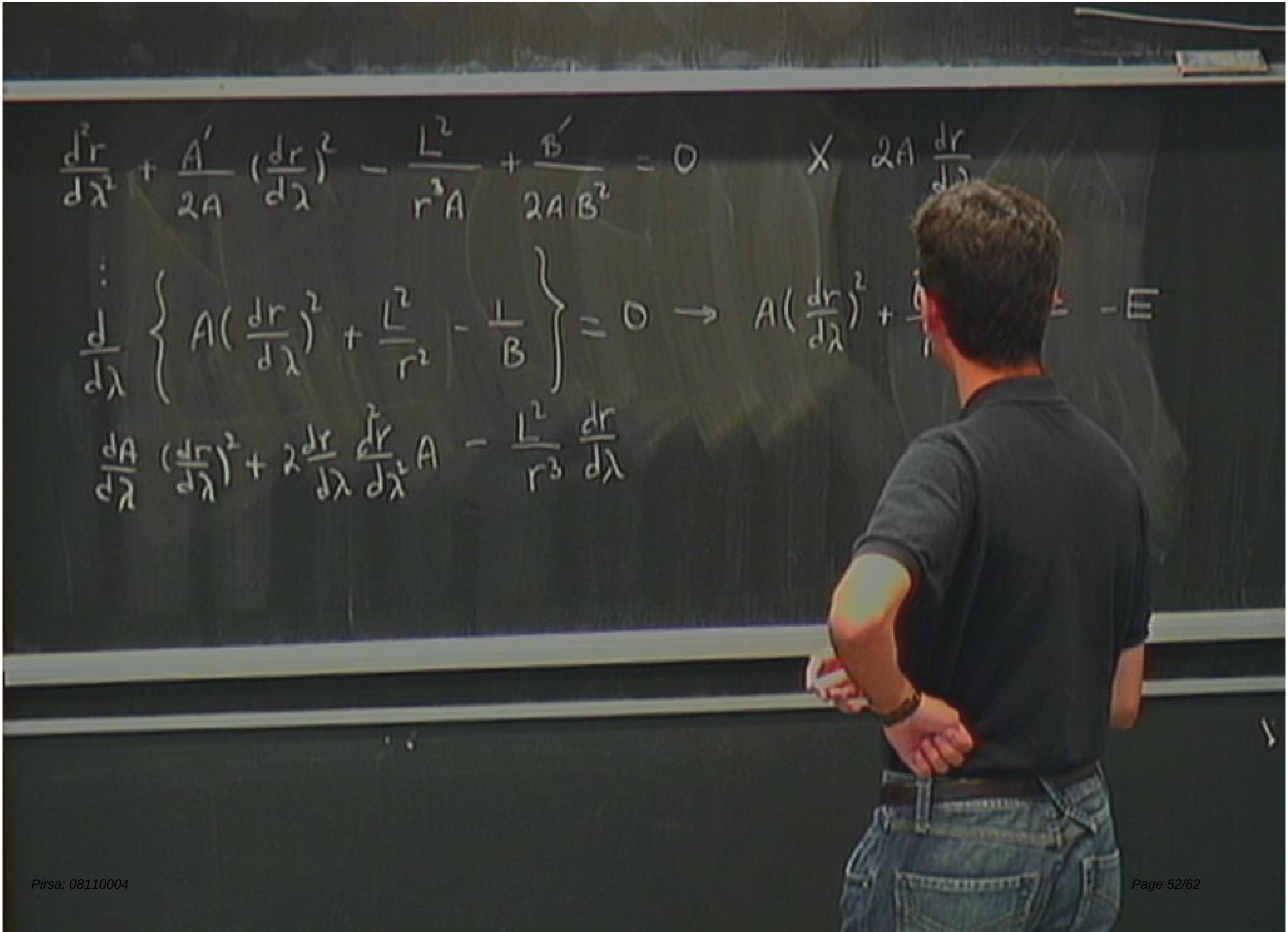
$$\vdots$$

$$\frac{d}{d\lambda}\left\{ A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} \right\} = 0$$

$$\frac{d^2r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3 A} + \frac{B'}{2AB^2} = 0 \qquad \times\ 2A\frac{dr}{d\lambda}$$

$$\vdots$$

$$\frac{d}{d\lambda}\left\{ A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} \right\} = 0 \rightarrow A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} = -E$$



$\frac{d^2r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3A} + \frac{B'}{2AB^2} = 0 \quad \times \; 2A\frac{dr}{d\lambda}$
$\frac{d}{d\lambda}\left\{ A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} \right\} = 0 \longrightarrow A\left(\frac{dr}{d\lambda}\right)^2 +$
$-E$
$\frac{dA}{d\lambda}\left(\frac{dr}{d\lambda}\right)^2 + 2\frac{dr}{d\lambda}\frac{d^2r}{d\lambda^2}A - \frac{L^2}{r^3}\frac{dr}{d\lambda}$

$$\frac{d^2 r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3 A} + \frac{B'}{2AB^2} = 0 \qquad \times\ 2A\frac{dr}{d\lambda}$$

$$\vdots$$

$$\frac{d}{d\lambda}\left\{ A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} \right\} = 0 \rightarrow A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} = -E$$

$$\frac{dA}{d\lambda}\left(\frac{dr}{d\lambda}\right)^2 + 2\frac{dr}{d\lambda}\frac{d^2 r}{d\lambda^2}A - \frac{L^2}{r^3}\frac{dr}{d\lambda} + \frac{1}{B^2}\frac{dB}{d\lambda} = 0$$

$$\frac{dA}{d\lambda} = \frac{dA}{dr}\frac{dr}{d\lambda} \qquad \frac{dB}{d\lambda} \downarrow \frac{dB}{dr}\frac{dr}{d\lambda}$$

$$\frac{d^2r}{d\lambda^2} + \frac{A'}{2A}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^3A} + \frac{B'}{2AB^2} = 0 \qquad \times \; 2A\frac{dr}{d\lambda}$$

$$\vdots$$

$$\frac{d}{d\lambda}\left\{ A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} \right\} = 0 \longrightarrow A\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2} - \frac{1}{B} = -E$$

$$\frac{dA}{d\lambda}\left(\frac{dr}{d\lambda}\right)^2 + 2\frac{dr}{d\lambda}\frac{d^2r}{d\lambda^2}A - \frac{L^2}{r^3}\frac{dr}{d\lambda} + \frac{1}{B^2}\frac{dB}{d\lambda} = 0$$

$$\frac{dB}{d\lambda} \Downarrow \frac{dB}{dr}\frac{dr}{d\lambda}$$

$$\frac{dA}{dr}\frac{dr}{d\lambda}$$

$$u^j \frac{\partial u^i}{\partial x^j} = \frac{du^i}{d\lambda}$$

$$\delta \int \sqrt{g}\, dx\, dx \qquad ds^2 = c^2 dt^2 \ldots$$

$$S = \int \mathcal{L}\, d\lambda = \int \left(B\dot{t}^2 - A\dot{r}^2 - r^2\dot{\varphi}^2 \right) d\lambda$$

$$\text{Const.} \; \frac{\partial \mathcal{L}}{\partial \dot{t}} = B\dot{t} \qquad B\dot{t}^2 - A\dot{r}^2 - r^2\dot{\varphi}^2 = 1$$

$$\text{Const} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = -r^2\dot{\varphi}$$

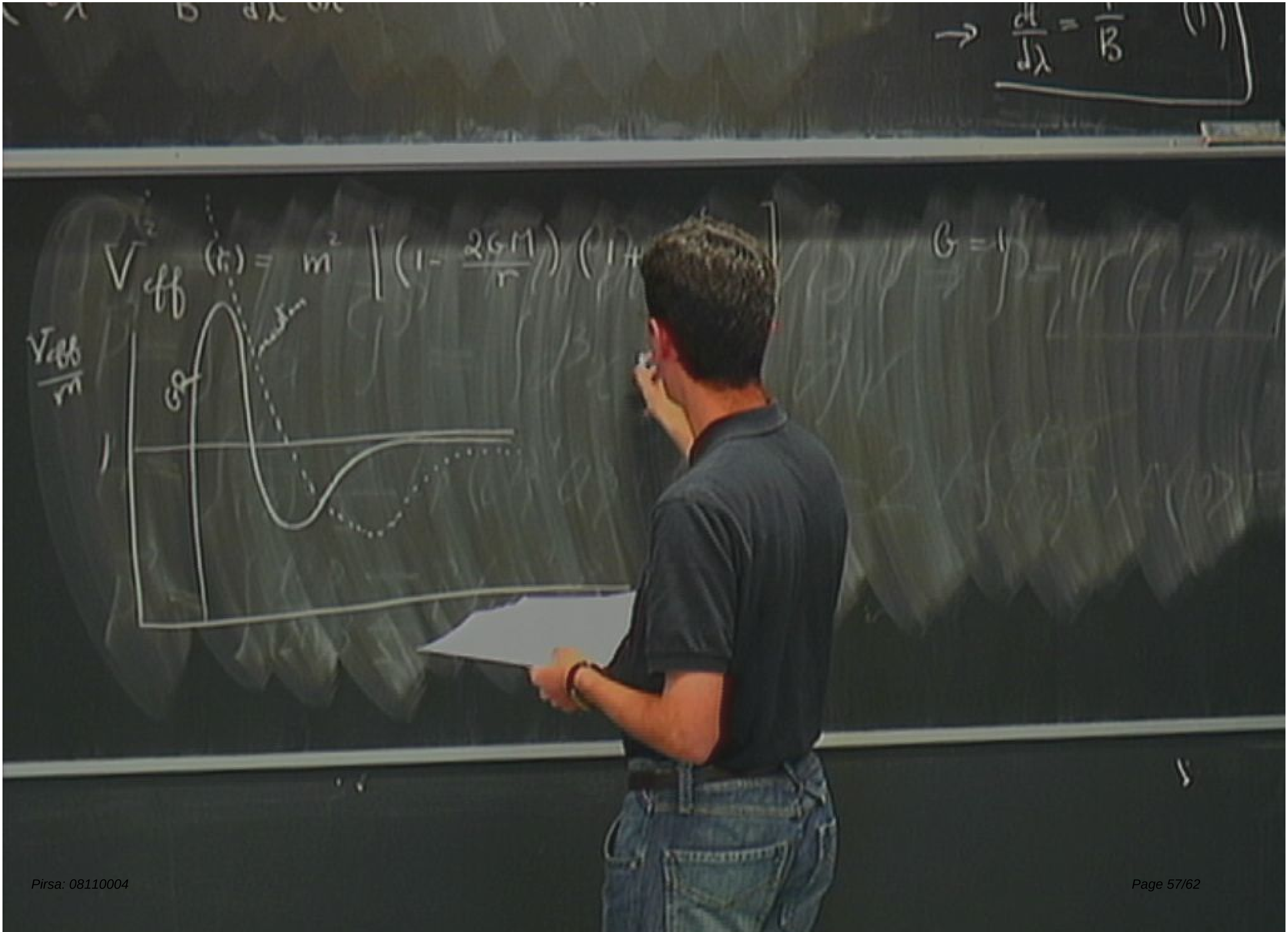
$$S = \int \mathcal{L}\, d\lambda = \int (B\dot{t}^2 - A\dot{r}^2 - r^2\dot{\varphi}^2)\, d\lambda$$

$$M = \text{Const} = \frac{\partial \mathcal{L}}{\partial \dot{t}} = B\dot{t}$$

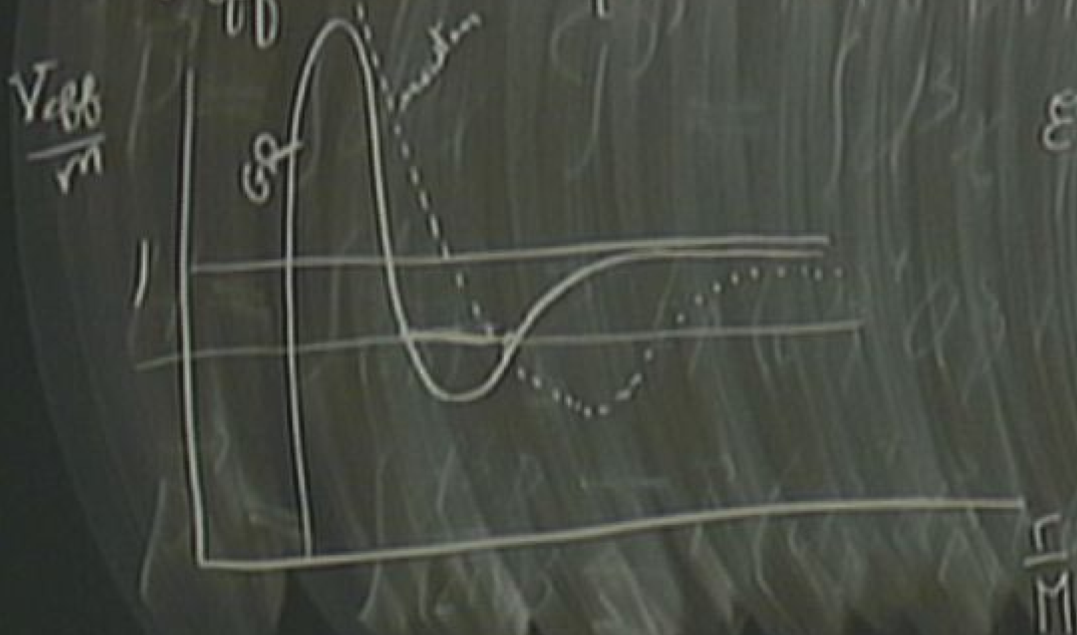
$$\boxed{B\dot{t}^2 - A\dot{r}^2 - r^2\dot{\varphi}^2 = 1}$$

$$N = \text{Const} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = -r^2\dot{\varphi}$$

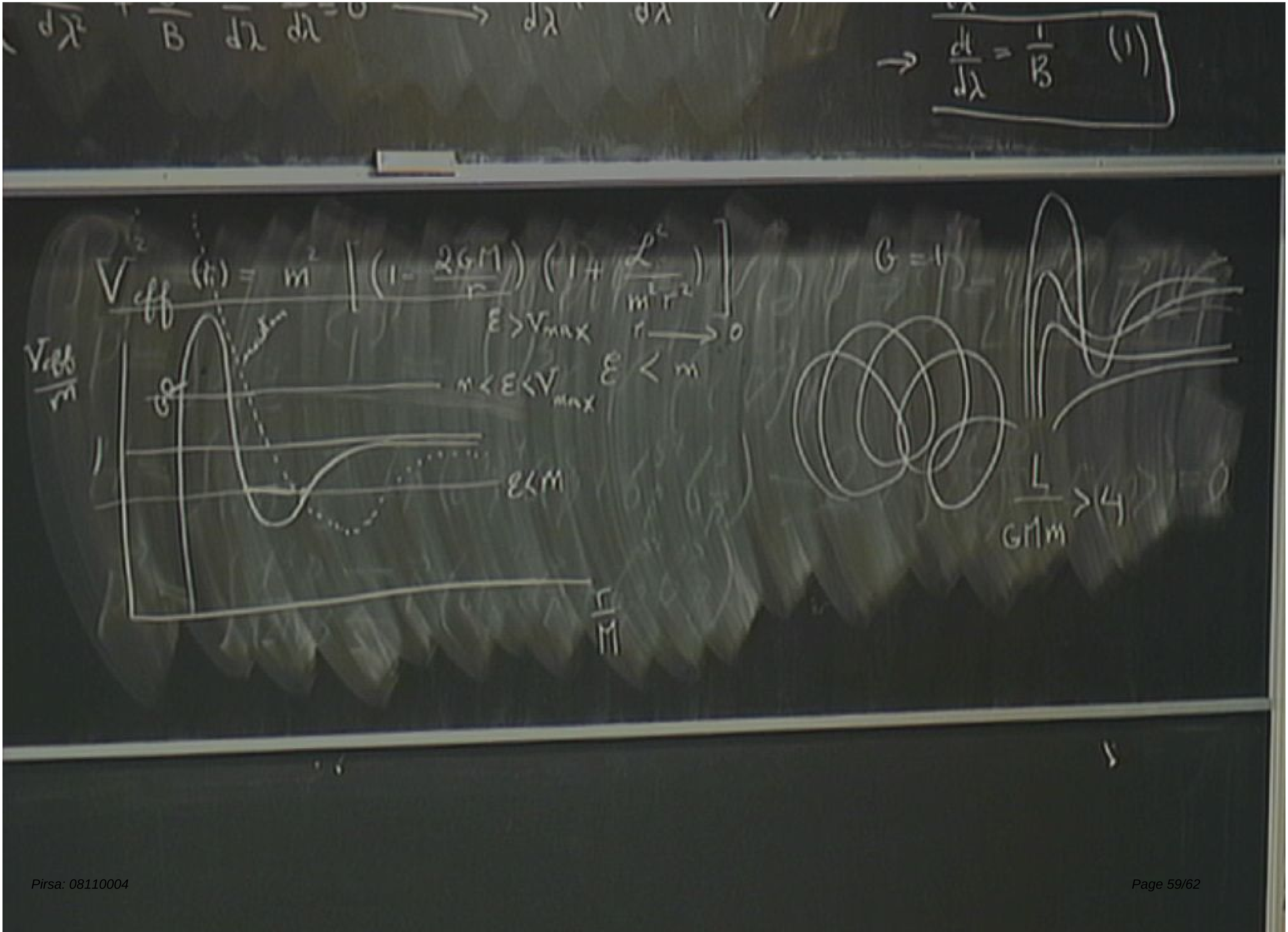
$$\left(\frac{M}{B}\right)^2 \left[B - A\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\varphi}{dt}\right)^2\right] = 1$$



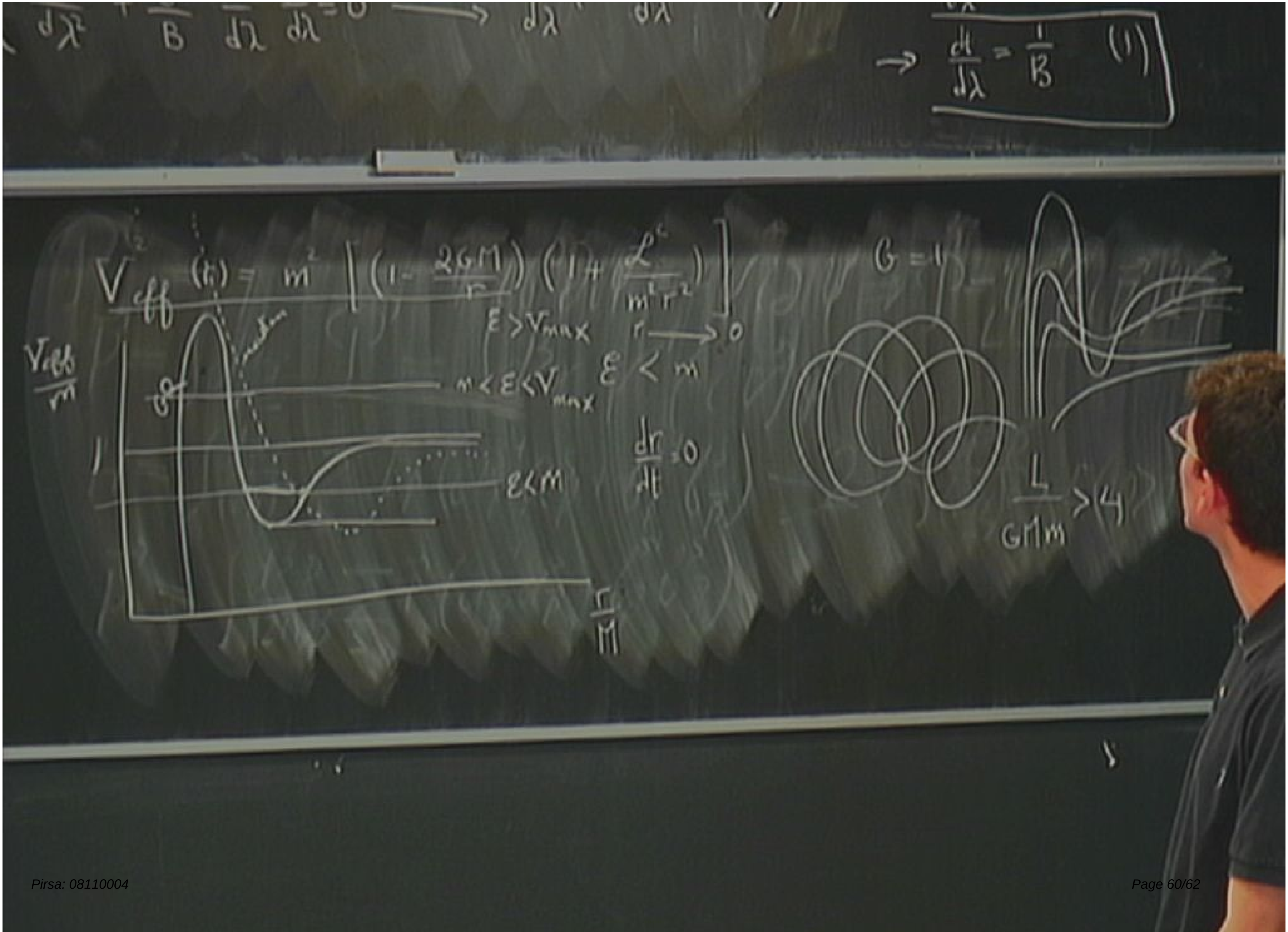
$$V^2_{eff}(r) = m^2 \left[\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\mathcal{L}^2}{m^2 r^2}\right)\right] \qquad G = 1$$

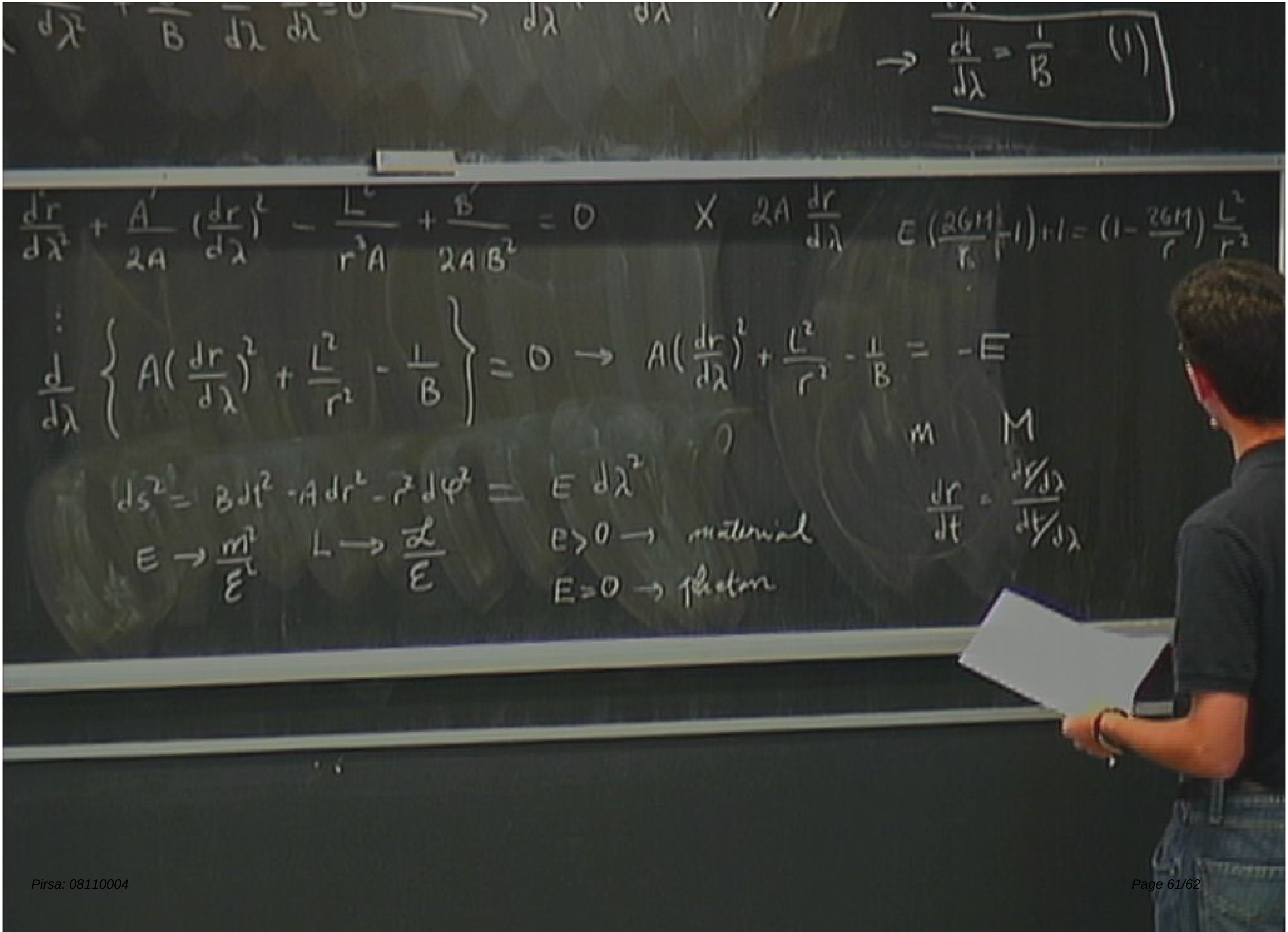


$\mathcal{E} < m$

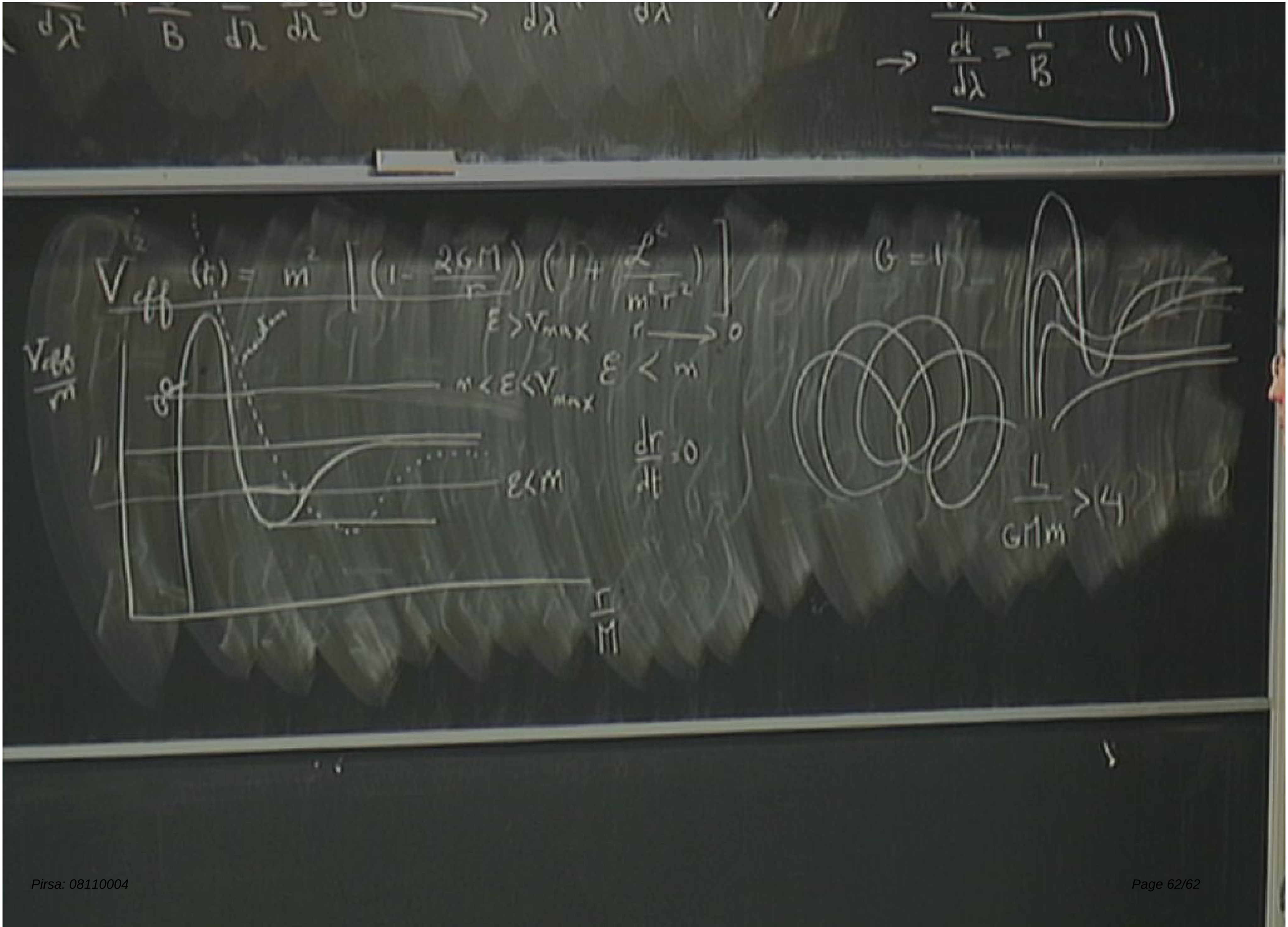


d/dλ = 1/B (1)
V_eff (r) = m² [(1 - 2GM/r)(1 + ℒ²/m²r²)]
G = 1
E > V_max   r → 0
m < E < V_max   E < m
V_eff / m
E < M
r / M
L / GMm > √(?)4


dt/dλ = 1/B (1)
V_eff(r) = m² [(1 − 2GM/r)(1 + L²/m²r²)]
G = 1
E > V_max
r → 0
m < E < V_max
E < m
dr/dt = 0
E<M
L/GMm > 4
V_eff/m
r/M



dt/dλ = 1/B (1)
d²r/dλ² + A′/2A (dr/dλ)² − L²/r³A + B′/2AB² = 0
X 2A dr/dλ
d/dλ { A(dr/dλ)² + L²/r² − 1/B } = 0 → A(dr/dλ)² + L²/r² − 1/B = −E
m M
ds² = B dt² − A dr² − r² dφ² = E dλ²
dr/dt = (dr/dλ)/(dt/dλ)
E → m²/ε² L → ℒ/ε
E>0 → material
E=0 → photon



$\frac{dt}{d\lambda} = \frac{1}{B}$ (1)
$V_{eff}(r) = m^2 \left[\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\mathcal{L}^2}{m^2 r^2}\right)\right]$
$G = 1$
$E > V_{max}$ $r \longrightarrow 0$
$m < E < V_{max}$ $E < m$
$\frac{dr}{dt} = 0$
$E < m$
$\frac{V_{eff}}{m}$
$\frac{r}{M}$
$\frac{L}{GMm} > 4$