Title: Generating Tree Amplitudes in N=4 SYM and N=8 Super-Gravity

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Abstract: TBA

Generating Tree Amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG

Dan Freedman MIT

Fall 2008

- EFK2 to appear w/ H. Elvang and M. Kiermaier
- EFK1 arXiv:0808.1720 w/ H. Elvang and M. Kiermaier
- BEF arXiv:0805.0757 w/ M. Bianchi and H. Elvang

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I. Introduction and Preview

Study of n-point on-shell tree amplitudes in

$\mathcal{N} = 4$ SYM $\quad A_n(1, 2, \ldots, n)$ all processes - 16 states
$\mathcal{N} = 8$ SG $\quad M_n(1, 2, \ldots, n)$ all processes - 256 states

Amplitudes classified within N^{k}MHV sectors— A_n with k-ve helicity and $n - k$ +ve helicity gluons and all amplitudes related by SUSY WI's. Idem for M_n in SG.

Simplest amplitudes in MHV sector: (2 -ve helicity lines), and become more complex with increasing k in $\mathcal{N} = 4$ and increasing k, n in $\mathcal{N} = 8$

Our Focus: Generating Functions which package all n-point amplitudes in each N^kMHV sector:

$$F_n^k(p_i, \eta_{ia}) \qquad i = 1, 2, .., n \qquad a = 1, ..., \mathcal{N}$$

p_i^μ are momenta, "spinorized" in terms of $|i\rangle$, $|i]$

η_{ia} are $n \times \mathcal{N}$ Grassmann variables

Generating functions encode state dependence of all N^kMHV n-point amplitudes in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories.

$$\langle \text{any amplitude} \rangle = \prod \frac{\partial}{\partial \eta_{ia}} F_n^k$$

Grassmann derivative of order $4k + 8$ in N^k sector of $\mathcal{N} = 4$ and $8k + 16$ in $\mathcal{N} = 8$.

Generating Functions answers interesting questions:
1. Precise characterization of MHV and N^{k}MHV sectors
MHV sector of N=4 contains 8-pt. function of +ve helicity gluinos

$$\langle F^a_+(1)\, F^b_+(2)\, F^c_+(3)\, F^d_+(4)\, F^e_+(5)\, F^f_+(6)\, F^g_+(7)\, F^h_+(8)\rangle$$

2. Gen. Fn. counts distinct processes in each sector,
$\mathcal{N}=4$ # partitions of $4k+8$ with $n_{max}=4$
$\mathcal{N}=8$ # partitions of $8k+16$ with $n_{max}=8$

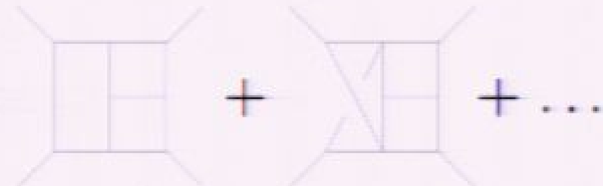
	MHV	NMHV
$\mathcal{N}=4$	15	34
$\mathcal{N}=8$	186	919

Results:

$\mathcal{N}=4$ MHV generating function Nair 1988
NMHV " " GGK 0407027
Further development and applications (TBA) in BEF, EFK1
N^kMHV Proved validity of underlying recursion relations
+ gen. fns. EFK1, EFK2

$\mathcal{N}=8$ MHV generating function + applics. BEF
NMHV " " works for many 6-point amplitudes,
but not all. Worse for $n > 6$
Fails for n-graviton amplitudes for $n \geq 12$
Repair is possible but impractical for applications.

Motivation Conjectured UV finiteness of $\mathcal{N}=8$ SG.

4-graviton amplitude is finite at 3-loop order. 

Bern, Carrasco, Dixon, Johansson, Kosower, Roiban 2007

4-loop calculation well underway.

Two questions:
1. How can work at tree level help with a program already at 4-loop level?
2. How can work in $\mathcal{N}=4$ SYM be relevant to $\mathcal{N}=8$ SG.

Answers:
1. Direct calc. of Feynman diagrams is a poor approach
Far too many dias. with complicated vertices.
Instead, obtain tree amplitudes from recursion relations.
Compute loops from (generalized) unitarity cuts.
One must sum over all intermediate states which can propagate on cut lines.
These unitarity sums are easily done using generating functions.

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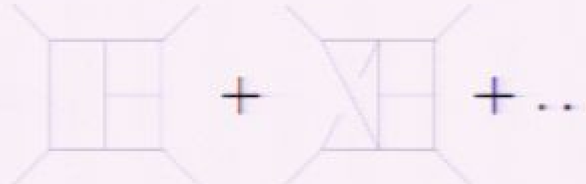

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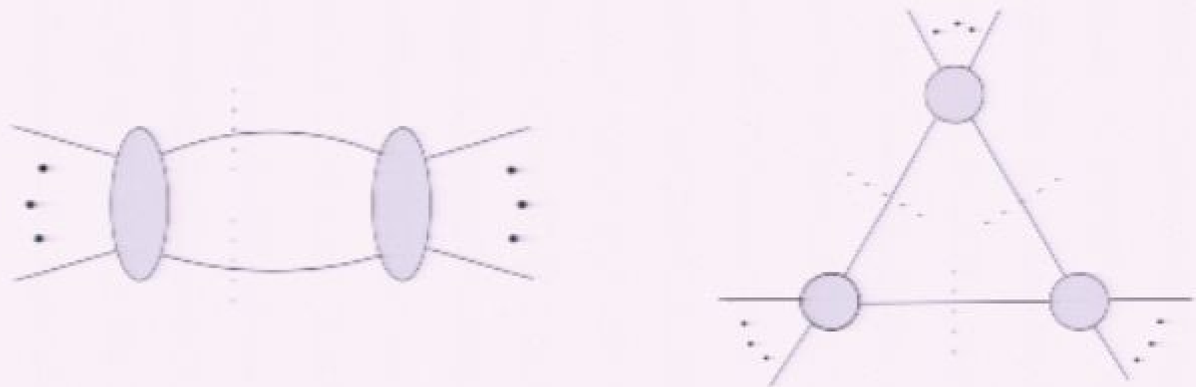
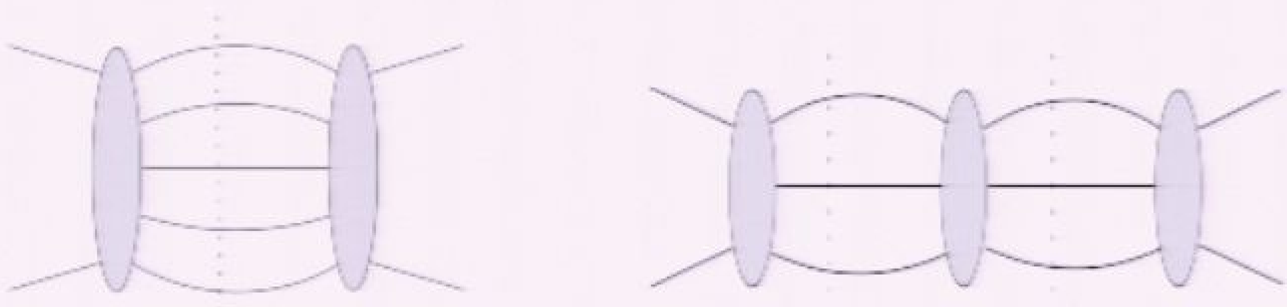
In BEF we computed unitarity sums with MHV subamplitudes.



In EFK1 we did sums which include NMHV and antiNMHV subamps. up to 4-loop order:



2. Why trees in $\mathcal{N}=4$ are relevant to $\mathcal{N}=8$.

1:1 map: compatible with SUSY BEF

$$\mathcal{N}=8 \leftrightarrow (\mathcal{N}=4)_L \otimes (\mathcal{N}=4)_R$$

. $256 \quad 16 \times 16$

. $|SG\rangle_{ij} = |SYM\rangle_i \otimes |SYM\rangle_j$

KLT relations for tree amplitudes give map dynamical content:

$$\begin{aligned} M_4(1234) &= s_{12} A_4(1234) A_4(1243) \\ M_n(12\ldots n) &= \sum_{perms} k_n\, A_n(\Pi_L)\, A_n(\Pi_R) \\ k_n &= \prod s_{ij} \qquad n-3 \quad \text{factors.} \end{aligned}$$

One can express any helicity sum in $\mathcal{N}=8$ as product of helicity sums in $(\mathcal{N}=4)_L \otimes (\mathcal{N}=4)_R$. Bern et al 2008

II. Spinor-Helicity Formalism

Spinor-helicity methods —

a. any 4-momentum can be "spinorized" (presented as a bi-spinor)

$$p^\mu \quad \leftrightarrow \quad p^{\dot\alpha\beta} \;\; (=\sigma_\mu^{\dot\alpha\beta} p^\mu)$$

b. any null vector is a rank 1 bi-spinor

$$p^2 = 0 \quad \Longrightarrow \quad p^{\dot\alpha\beta} = \tilde\lambda_p^{\dot\alpha} \lambda_p^\beta$$

Better notation: $\tilde\lambda^{\dot\alpha} \leftrightarrow |p\rangle \quad \lambda_p \leftrightarrow |p]$

c. Lorentz invariant inner products – (formed with $\epsilon^{\alpha\beta}$)

$$\langle i\, j\rangle = -\langle j\, i\rangle \qquad [i\, j] = [j\, i]$$

d. Other features of notation: $p_1 + p_2 + \ldots = |1\rangle[1| + |2\rangle[2| + \ldots$

. $\langle i\,|p_j|k\,] = \langle i\,j\rangle[j\,k]$ $\quad p_i \cdot p_j = \langle ij\rangle[ij]$

Schouten identity: $\langle ij\rangle\langle k| + \langle jk\rangle\langle i| + \langle ki\rangle\langle j| = 0$

Momentum conservation: $\sum p_i^\mu = 0 \leftrightarrow \sum |i\rangle[i| = 0$

e. In textbooks, e.g. Srednicki 2007

λ_p, $\tilde\lambda_p$ or $|p]$, $|p\rangle$ are left, right-handed sols. of Dirac eqtn.

Related by $\tilde\lambda = \lambda^*$. BUT it is very useful to consider *complex* null momenta *defined* by $p^{\alpha\dot\beta} = \lambda^\alpha\tilde\lambda^{\dot\beta} \;\leftrightarrow\; |p\rangle[p|$ with *independent* spinors $|p\rangle,\ |p]$.

f. n-gluon MHV Parke–Taylor 1986 amplitudes

$$A_n(1^-2^-3^+\dots n^+) = \frac{\langle 1\,2\rangle^4}{cyc(1,n)}, \qquad cyc(1,n) = \prod_1^n \langle i, i+1\rangle$$

Purely holomorphic, (depends on $|i\rangle$, not on $|i]$)
Simple expression agrees with sum of many Feynman diagrams.
(8-gluon amplitude has ≈ 200 dias.)

III. The $\mathcal{N} = 4$ theory

8 boson + 8 fermion states in SU(4) irreps, $a, b, \ldots = 1, .., 4$

$B_+(p),\ F_+^a(p),\ B^{ab}(p) = \frac{1}{2}\epsilon^{abcd} B_{cd}(p),\ F_d^-(p),\ \ B^-(p)$

Use ϵ^{abcd} to raise all indices.

New uniform notation for annihilation operators -

$A(p),\quad A^a(p),\quad A^{ab}(p),\qquad\qquad A^{abc}(p),\ \ A^{abcd}$

Helicity and bose-fermi statistics is determined by # of indices.

Supercharges: $\tilde{Q}_a = \epsilon_{\dot{\alpha}} \tilde{Q}_a^{\dot{\alpha}}$, $\quad Q^a = \epsilon^{\alpha} Q_{\alpha}^a$, $\quad a = 1, .., 4$
act on annihilators as:

$$[\tilde{Q}_a \, , \, A(p)] = 0 \qquad [Q^a \, , \, A(p)] = [\epsilon \, p] \, A^a(p)$$
$$[\tilde{Q}_a \, , \, A^b(p)] = \langle \epsilon \, p \rangle \delta_a^b A(p) \qquad [Q^a \, , \, A^b(p)] = [\epsilon \, p] \, A^{ab}(p)$$

etc. etc.

$\tilde{Q}_a$ raises helicity, eliminates an SU(4) index, involves $\langle \epsilon \, p \rangle$
Q^a lowers helicity, adds an SU(4) index, involves $[\epsilon \, p]$.

Pattern of SU(4) indices on any amplitude gives quick and useful information.

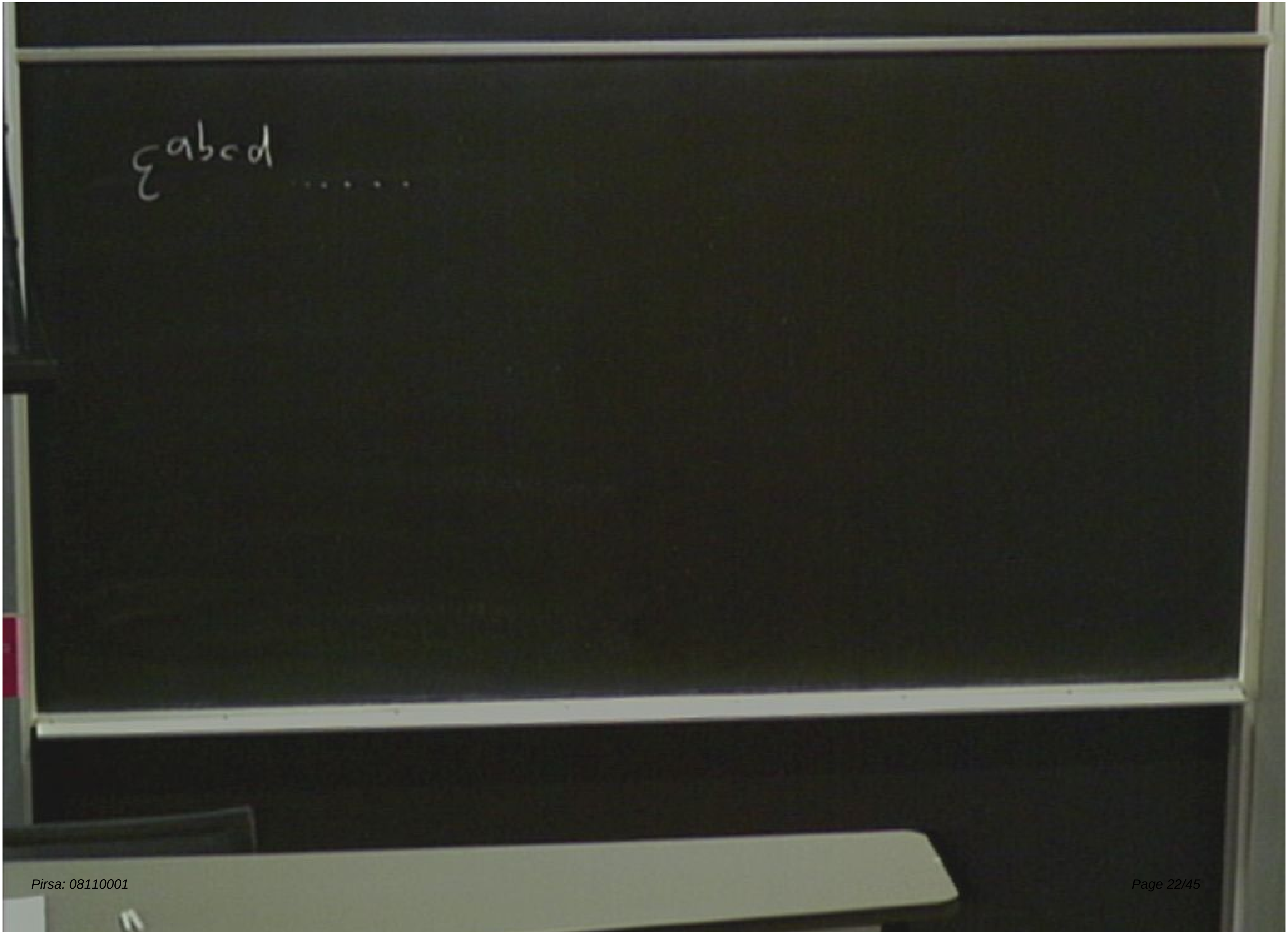
SU(4) invariance $\Longrightarrow$ States of any non-vanishing amplitude carry total of $4k+4$ indices. Each index value $a=1,2,3,4$ occurs $k+2$ times.

Examples:

$\langle A^{12}(1)\, A(2)\, A^{123}(3)\, A(4)\, A^{4}(5)\, A^{34}(6)\rangle$ is MHV.

$\langle A^{1234}(1)\, A(2)\, A^{1234}(3)\, A(4)\, A(5)\, A^{1234}(6)\rangle$ is NMHV.

$\langle A^{1234}(1)\, A^{12}(2)\, A^{34}(3)\, A(4)\, A^{13}(5)\rangle = 0$ 10 indices


ε^{abcd} ......

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IV. MHV Generating Function

The Nair generating function is:

$$F_n(|i\rangle, \eta_{ia}) = \delta^{(8)}(\sum_{i=1}^{n} |i\rangle \eta_{ia})/cyc(1,n).$$

It is Lorentz invariant, SU(4) invariant, and cyclically symmetric.

A δ-function of m variables $\theta_1, ..., \theta_m$ is a product $\prod_i \theta_i$. We have a $\delta^{(8)}$with 8 arguments $Z_a^{\dot\alpha} = \sum_i \tilde\lambda_i^{\dot\alpha} \eta_{ia}$. Thus

$$\delta^{(8)}(Z_a^{\dot\alpha}) = \frac{1}{16} \prod_{a=1}^{4} \sum_{i,j} \langle i\, j\rangle \eta_{ia}\eta_{ja}.$$

Any product of 8 $\partial/\partial\eta_{jb}$ gives a c-number of form $\langle\ \rangle\langle\ \rangle\langle\ \rangle\langle\ \rangle$.

Associate particles of $\mathcal{N} = 4$ SYM with $\partial/\partial\eta$'s:

$A(i) \leftrightarrow 1$

$A^a(i) \leftrightarrow D_i^a = \partial/\partial\eta_{ia} = \partial_i^a$

$A^{ab}(i) \leftrightarrow D_i^{ab} = \partial_i^a\partial_i^b$

$A^{abc}(i) \leftrightarrow D_i^{abc} = \partial_i^a\partial_i^b\partial_i^c$

$A^{abcd}(i) \leftrightarrow D_i = \partial_i^1\partial_i^2\partial_i^3\partial_i^4$

Any amplitudes is obtained by applying corresponding 8th order diff.op. to $\mathcal{F}_n$. Examples:

i. $\langle A^{1234}\, A^{1234}\, A.....A\rangle = \partial_1^1\partial_1^2\partial_1^3\partial_1^4\partial_2^1\partial_2^2\partial_2^3\partial_2^4\ \delta^{(8)}(\ \ )/cyc(1,n)$

. $= \langle 1\,2\rangle^4/cyc(1,n)$

ii. $\langle A^{1234}\, A^1\, A^2\, A^{34}\, A...A\rangle = D_1 D_2^1 D_3^2 D_4^{34}\delta^{(8)}(\ \ )/cyc(1,n)$

. $= \langle 1\,2\rangle\langle 1\,3\rangle\langle 1\,4\rangle^2/cyc(1,n)$

The particular $\langle\rangle\langle\rangle\langle\rangle\langle\rangle$ in any amplitude is called its spin factor.

Easy to calculate spin factors using Wick contraction rule:

$$\hat{\partial}_i^a...\hat{\partial}_j^b = \pm\delta^{ab}\langle i\, j\rangle$$

If an amplitude vanishes by SU(4), its spin factor will automatically vanish.

Supercharges: $\tilde{Q}_a = \sum_{i=1}^{n} \langle \epsilon\, i \rangle \eta_{ia}$, $\quad Q^a = \sum_{j=1}^{n} [j\, \epsilon] \partial_j^a$

It can (easily) be shown that:

1. correct SUSY algebra $[\tilde{Q}, Q] = 0$, since $\sum p_i^\mu = 0$
2. trfs $[\tilde{Q}_a\,,\, D_i^{\cdots}]$ are isomorphic to $[\tilde{Q}_a\,,\, A_i^{\cdots}]$
3. Correct WI's, both formal and concrete:

Formal: $\tilde{Q}_a \mathcal{F}_n(|i\rangle, \eta) = 0$

Concrete: $\tilde{Q}_a (\partial/\partial\eta)^9 \mathcal{F}_n(|i\rangle, \eta) = 0$

Moving $\tilde{Q}$ to the right $\rightarrow$ gives same relation among amplitudes as in QFT WI. Similar for Q^a WI's.

V. MHV Generating Function – N=8 SG

Extension to MHV sector of $\mathcal{N}=8$ SG: BEF

i. The function

$$w_n(12...n) = M_n(1^-2^-3^+...n^+)/\langle 1\,2\rangle^8$$

has full Bose symmetry.

ii. The generating function is

$$\Omega_n(|i\rangle, |i], \eta_{iA}) = w_n(...)\,\delta^{(16)}(\sum_{i=1}^{16} |i\rangle \eta_{iA}).$$

. Now 8n Grassmann variables.

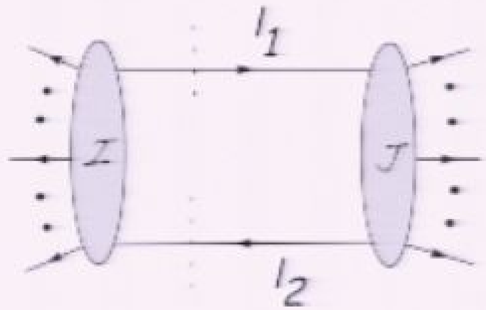
iii. All previous properties hold if we associate

$B(i) \leftrightarrow 1, \quad B_+^A(i) \leftrightarrow \partial/\partial\eta_{iA}, \ldots\ldots, B^{12345678}(i) \leftrightarrow \partial_i^1 ... \partial_i^8$

iv. 1:1 corresp. MHV processes in N=8 $\leftrightarrow$ partitions of 16 with $n_{\max} \leq 8$ $\exists$ 186 distinct processes!

VI. Intermediate state helicity sum

Example: One-loop MHV amplitude



Use product of two MHV generating functions to compute intermediate state sum of unitarity cut:

$$\frac{1}{cyc(I)\, cyc(J)}\; D^{(4)}_{l_1}\, D^{(4)}_{l_2} \left[\delta^{(8)}(I)\, \delta^{(8)}(J)\right]$$

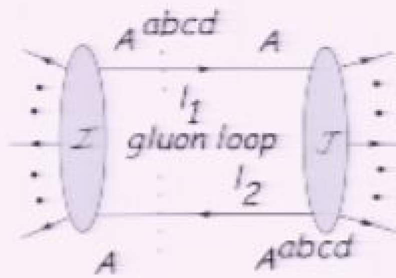
D_{l_1} and D_{l_2} distribute themselves between $\delta^{(8)}(I)$ and $\delta^{(8)}(J)$. This automatically takes care of the sum over states.

Let's see how →

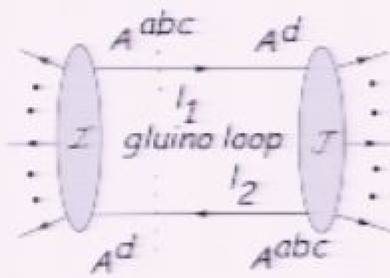
How it works:

$$D_{l_1}{}^{(4)} D_{l_2}{}^{(4)} \left[\delta^{(8)}(I)\,\delta^{(8)}(J)\right]$$

$$= \left[D_{l_1}{}^{abcd}\,\delta^{(8)}(I)\right]\left[D_{l_2}{}^{abcd}\,\delta^{(8)}(J)\right] \quad \longleftarrow$$



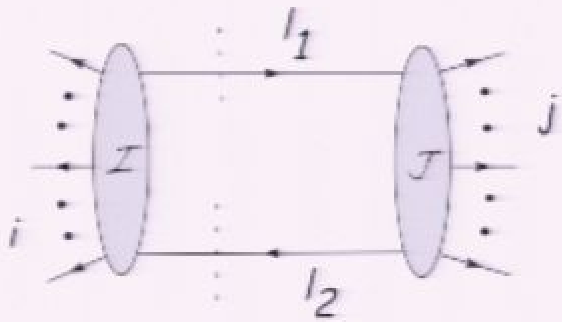
$$-4\left[D_{l_1}{}^{abc} D_{l_2}{}^{d}\,\delta^{(8)}(I)\right]\left[D_{l_2}{}^{abc} D_{l_1}{}^{d}\,\delta^{(8)}(J)\right] \quad \longleftarrow$$



$$+\,6\ \text{(scalar loop)} - 4\ \text{(gluino loop)} + \text{(gluon loop)}\,.$$

D_{l_1} and D_{l_2} distribute themselves between $\delta^{(8)}(I)$ and $\delta^{(8)}(J)$.
This does takes care of the intermediate state sum, but it is awkward.

PRACTICAL evaluation of the spin sum: $D^{(4)}_{l_1}\, D^{(4)}_{l_2} \left[\delta^{(8)}(I_a)\, \delta^{(8)}(J_a)\right]$



$$I_a = |l_1\rangle\eta_{1a} - |l_2\rangle\eta_{2a} + \textstyle\sum_{\text{ext } i} |i\rangle\eta_{ia}$$
$$J_a = -|l_1\rangle\eta_{1a} + |l_2\rangle\eta_{2a} + \textstyle\sum_{\text{ext } j} |j\rangle\eta_{ja}$$

Use δ-function identity $\delta^{(8)}(I_a)\, \delta^{(8)}(J_a) = \delta^{(8)}(I_a + J_a)\, \delta^{(8)}(J_a)$ and note that

- $\delta^{(8)}(I_a + J_a) = \delta^{(8)}(\text{ext})$ is independent of loop momenta.
- $\delta^{(8)}(J_a) = 2^{-4} \prod_{a=1}^{4} \sum_{j,j' \in J} \langle jj' \rangle \eta_{ja}\eta_{j'a} = \prod_{a=1}^{4} \left(\langle l_1 l_2 \rangle \eta_{1a}\eta_{2a} + \dots \right)$.

So

$$D^{(4)}_{l_1}\, D^{(4)}_{l_2} \left[\delta^{(8)}(I_a)\, \delta^{(8)}(J_a)\right] = \delta^{(8)}(\text{ext})\, D^{(4)}_{l_1}\, D^{(4)}_{l_2}\, \delta^{(8)}(J_a) = \delta^{(8)}(\text{ext})\, \langle l_1 l_2 \rangle^4 .$$

Put in $1/cyc(I)cyc(J)$ to get generating function for the cut 1-loop dia.
Can apply D_i for external lines of any MHV process.

VII. N^kMHV Amplitudes from MHV vertex expansion

We use MHV-vertex expansion (CSW 0403047)which constructs N^kMHV amplitudes from MHV sub-amplitudes. Advantages:

i. physicists like diagrams

ii. $\exists$ many fewer MHV vertex dias. than Feynman dias.

iii. MHV vertices are (slightly modified) Parke-Taylor amplitudes, very simple!

iv. expansion is easily coded.

Full justification and extension to $\mathcal{N} = 4$ SYM requires RECURSION RELATIONS.

Inspired by method of BCFW 2005, but based on different shift of momenta Risager 0508206, B-BDIPR 0509016

1. Risager shift at NMHV level, shift of 3 external momenta p_1, p_2, p_3, specified by spinor shift:

$|\hat{1}\rangle = |1\rangle$, same for $|\hat{2}\rangle$, $|\hat{3}\rangle$

$|\hat{1}] = |1] + z\langle 2\,3\rangle |X]$, with $|\hat{2}]$, $|\hat{3}]$ cylic.

Shift of square $|i]$ only with *reference spinor* $|X]$.

Shifted momenta $\hat{p}_i = |i\rangle[\hat{i}|$ – rank 1, hence null.

Overall momentum is conserved by Schouten.

For amplitudes in $\mathcal{N} = 4$, one must choose a shift of 3 lines which carry a common SU(4) index value $a = 1, 2, 3, 4$. BEF, EFK1.

2. Analytically continued tree amplitudes $A_n(z)$ have only simple poles where shifted Feynman propagators vanish.

If $A_n(z) \to 0$ as $z \to \infty$ for all choices of $|X]$,

$$0 = \oint \frac{A_n(z)}{z} \implies A_n(0) = -\sum_{z_I \neq 0} \mathrm{Res} \frac{A_n(z_I)}{z_I}$$

3. Result is recursion relation

$$A_n(0) = \sum_I A_{n_1}(z_I) \frac{1}{P_I^2} A_{n_2}(z_I), \qquad n_1 + n_2 = n + 2$$

i. for 2-line BCFW shift the subamplitudes include both MHV×MHV and antiMHV×NMHV, undesirable.

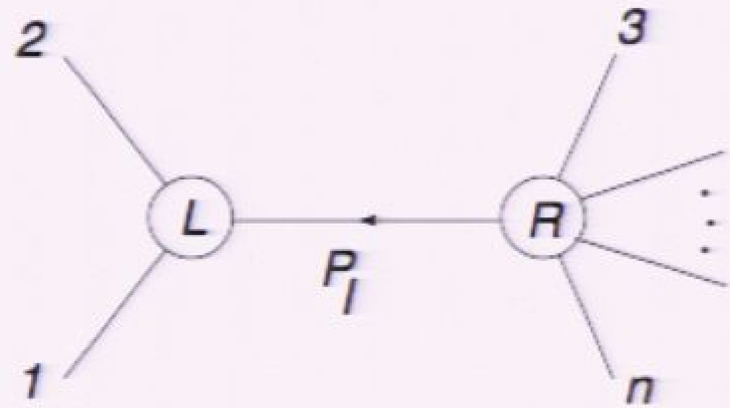
ii. for 3-line shift, the only subdias. are MHV!

iii. Individ. dias. depend on $|X]$, but sum over all dias. does not!

4. Extension to all NMHV amplitudes in $\mathcal{N} = 4$. Package amplitudes in generating function.

Georgiou, Glover, Khoze 2004, BEF, EFK1

The generating function for each diagram is the product of gen. fn. for the MHV subdias. L, R, with common η_{la} for internal line. Apply $D_I^{(4)}$ to account for particles on each side of int. line.



$$A_{n,I} = D_I^{(4)} \frac{\delta(L)}{cyc(L)} \frac{1}{P_I^2} \frac{\delta(R)}{cyc(R)}$$

Evaluate $D_I^{(4)}$.... and sum over dias. to get complete gen fn.

$$\mathcal{F}_n^{NMHV} = \delta^{(8)}(\sum_{ext} |i\rangle\eta_{ia}) \sum_{dias.} \frac{\prod_{b=1}^{4} \sum_j' \langle P_I\, j\rangle \eta_{jb}}{cyc(L) P_I^2 cyc(R)}$$

where $\sum_j'$ means sum only over external particle in subamp., and $|P_I\rangle = P_I|X]$ ← the CSW prescription.

$$\varepsilon^{abcd} \; \ldots\ldots$$

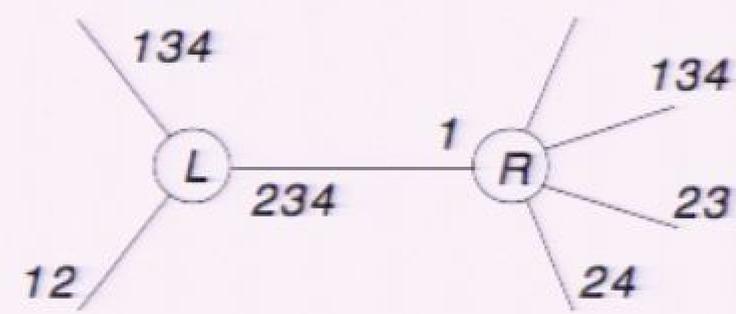
$$P_{12} = |1\rangle[1 + |2\rangle[2$$

$$\mathcal{F}_n^{NMHV} = \delta^{(8)}\left(\sum_{ext} |i\rangle \eta_{ia}\right) \sum_{dias.} \frac{\prod_{b=1}^{4} \sum_j' \langle P_I\, j\rangle \eta_{jb}}{cyc(L) P_I^2 cyc(R)}$$

a. # dias. $= n(n-3)/2 << \#$ Feynman dias.
Compare with Drummond-Henn 2008 , dual conformal symmetry, Fewer dias., $(n-3)(n-4)/2$, but each dia. is more complicated.

b. Each term in $\mathcal{F}_n^{NMHV}$ is a product of 12 η_{ia}'s.
Any NMHV amplitude $= D^{(12)} \mathcal{F}_n^{NMHV}$

c. In practice, we specify SU(4) indices and calculate spin factors directly. This procedure is easily coded.
$|X]$-independence has been checked in many examples.



d. The form above is applied to NMHV unitarity sums in EFK1.

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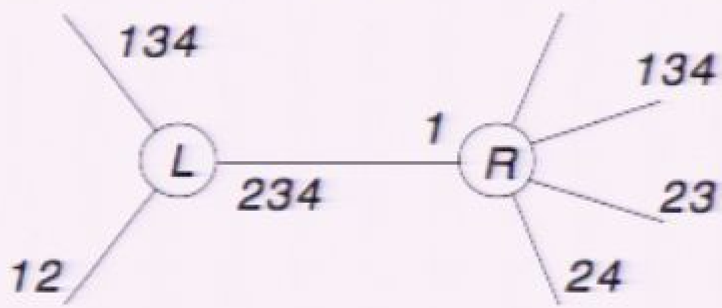

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5. Extension to general level N^{k}MHV– EFK2

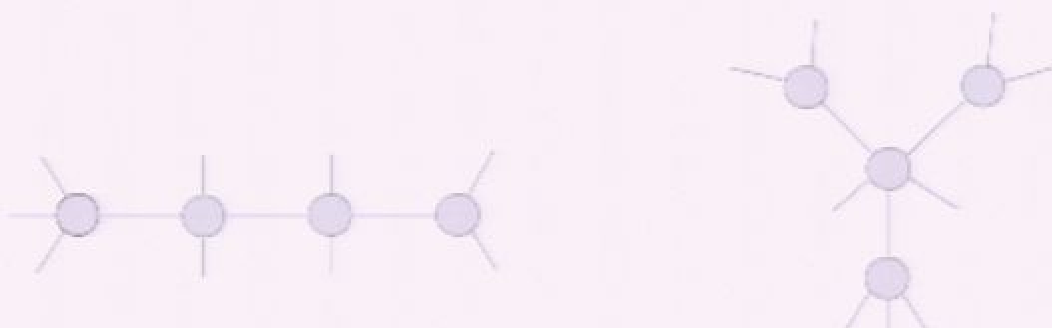
a. Use <u>all-line shift</u>

$$|\hat{i}\rangle = |i\rangle \qquad\qquad |\hat{i}] = |i] + zc_i|X]$$

Momentum is conserved if $\sum_i c_i|i\rangle = 0$.
Two conditions on n complex c_i.

b. All-line shift simplifies the proof of MHV vertex expansion, but an intricate inductive argument involving both n, k is needed.

c. MHV vertex diagrams have various topologies.
Example: N^{3}MHV all dias. with 4 vertices.

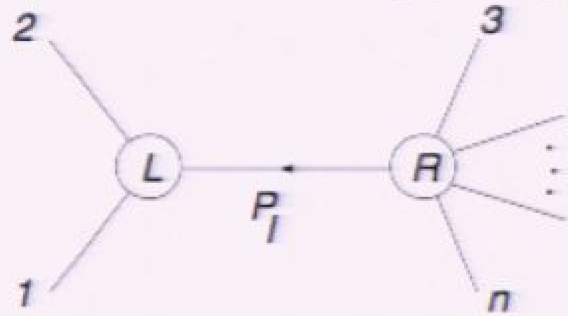


d. Numerical checks of $|X]$- independence for several N^2MHV amplitudes. Analytic checks for special N^3MHV amplitudes with only 1 or 2 dias.

e. Surprise: Every dia. for an N^kMHV process vanishes as $1/z^k$ as $z \to \infty$ for
i. all-line shifts
ii. any $(k+2)$-line common-index shift. $k+2$ is minimal number of shifted lines to obtain only MHV subdias. from recursion relation.

VIII. NMHV Generating Function for N=8 SG

Natural extension $\mathcal{N} = 4 \;\rightarrow\; \mathcal{N} = 8$ gives generating function:



$$\Omega_n^{NMHV} = \delta^{(16)}\Big(\sum_i |i\rangle \eta_{ia}\Big) \sum_{\text{dias.}} \frac{w_{n_1}(L) w_{n_2}(R)}{P_I^2} \prod_{A=1}^{8} \sum_{j \in L}^{\prime} \langle P_I\, j\rangle \eta_{jA}$$

Sum of products of 24 η_{iA} $A = 1, \ldots, 8$

One might hope that any $M_n = D^{(24)} \Omega_n^{NMHV}$

Check $|X]$-independence of many amplitudes and compare with KLT formula:

$$M_n = \sum_{\text{perms}} (s_{ij})^{n-3} A_n A_n'$$

1. Many 6-point amplitudes pass the test, but results for several depend on $|X]$. e.g, a 6-scalar amplitude.

2. Can shift the KLT formula in these cases and find that they behave as z^p, $p = 0, 1$ for large z. $\exists$ "pole at ∞" for generic $|X]$.

3. Find special values of $|X]$ for which KLT result $M_6(z) \to 0$. For these values result from generating function agrees with KLT! The method works, but auxiliary computation to check large z behavior is cumbersome.

4. 3-line shift of KLT for n-graviton amplitudes. Find numerically $M_n(- - - + \ldots +) \sim z^{n-12}$. Thus even n-graviton amplitudes are badly behaved for $n \geq 12$.

IX. Other approaches

1. 2-line *supershift* in generating function A-HCK, BHT 2008
Combine BCFW shift + shift of η variable to leave supercharge
$\tilde{Q}_a = \sum_i |i\rangle \eta_{ia}$ invariant.
Many advantages! But extension to k-line shifts of $|\ ]$'s unclear.

2. Supershift recursion relation solved by Drummond-Henn, 2008
N^kMHV tree amplitudes constructed from dual superconformal invariants.
Probably fundamental, but new and more complicated invariants are needed at each level k.

X. Conclusions, I

1. $\mathcal{N} = 4$ Generating Functions based on MHV vertex expansion proven and constructed for all N^{k}MHV amplitudes.

2. $\mathcal{N} = 8$ MHV gen. fn. works well, but $\exists$ difficulties for NMHV.

3. Practical applications to multiplet sums on unitarity cuts of ℓ-loop amplitudes for $1 \leq \ell \leq 4$.

Omitted topics:

i. cute analogy between spin factors and CFT correlators.

ii. E(7,7) low energy theorems for $\mathcal{N} = 8$. Also A-HCK

iii. results on antiMHV and antiNMHV generating functions.

Open questions:

a. improve treatment of $\mathcal{N} = 8$ SG?

b. Is result $M_n^{NMHV} \sim z^{n-12}$ relevant to UV behavior?

Conclusions, II

These projects were **fun!**
I don't care what anybody else says!