Title: Seeking Symmetries of SIC-POVMs

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Abstract: By definition, SIC-POVMs are symmetric in the sense that the magnitude of the inner product between any pair of vectors is constant. All known constructions are based on additional symmetries, mainly with respect to the Weyl-Heisenberg group. Analyzing solutions for small dimensions, Zauner has identified an additional symmetry of order three and conjectured that these symmetries can be used to construct SIC-POVMs for all dimensions. Appleby has confirmed that all numerical solutions of Renes et al. indeed have that additional symmetry. This leads to the main questions addressed in the talk: Do all SIC-POVMs necessarily possess these symmetries, or can we construct SIC-POVMs without or with other symmetries?

Seeking SICs:
An Intense Workshop on Quantum Frames and Designs
October 26–30, 2008

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Seeking Symmetries of SIC-POVMs

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The Problem: Equiangular Lines in Complex Space

The General Problem

Find m normalized vectors $\{\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(m)}\} \subset \mathbb{C}^d$ such that the modulus of the inner product between any pair of vectors is constant, i.e.

$$|\langle \boldsymbol{v}^{(i)} | \boldsymbol{v}^{(j)} \rangle|^2 = \begin{cases} 1 & \text{for } i = j, \\ c & \text{for } i \neq j \end{cases}$$

Special Case

Find d^2 normalized vectors $\{\boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(d^2)}\} \subset \mathbb{C}^d$ such that the modulus of the inner product between any pair of vectors is constant, i.e.

$$|\langle \boldsymbol{v}^{(i)} | \boldsymbol{v}^{(j)} \rangle|^2 = \begin{cases} 1 & \text{for } i = j, \\ 1/(d+1) & \text{for } i \neq j \end{cases}$$

Related Problems

Complex Spherical 2-Designs

The integral of any degree-two polynomial over the complex sphere in $\mathbb{C}^d$ can be computed as finite average, i.e.

$$\frac{1}{\mu(\mathbb{C}S^{d-1})}\int_{g\in\mathbb{C}S^{d-1}} f(g)d\mu(g) = \frac{1}{m}\sum_{i=1}^{m} f(\boldsymbol{v}^{(i)})$$

if $m = d^2$ and the vectors $\boldsymbol{v}^{(i)}$ are equiangular lines.

Banach Spaces [König & Tomczak-Jaegermann 94]

The projection constant

$$\lambda(E) = \sup_{X\supseteq E}\inf_{P}\{\|P\| : P\colon X \to E \text{ is linear projection onto } E\}$$

of a complex d-dimensional normed space E is maximal iff a set of d^2 equiangular lines exists.

Most General Approach

use $2d$ real variables per vector

$$\boldsymbol{v}^{(j)} = \left(a_1^{(j)} + ib_1^{(j)}, \ldots, a_d^{(j)} + ib_d^{(j)}\right),$$

where $i^2 = -1$.

$d = 2$, $m = d^2 = 4$

$$\begin{aligned}
\boldsymbol{v}^{(1)} &= \left(a_1^{(1)} + ib_1^{(1)}, a_2^{(1)} + ib_2^{(1)}\right) \\
\boldsymbol{v}^{(2)} &= \left(a_1^{(2)} + ib_1^{(2)}, a_2^{(2)} + ib_2^{(2)}\right) \\
\boldsymbol{v}^{(3)} &= \left(a_1^{(3)} + ib_1^{(3)}, a_2^{(3)} + ib_2^{(3)}\right) \\
\boldsymbol{v}^{(4)} &= \left(a_1^{(4)} + ib_1^{(4)}, a_2^{(4)} + ib_2^{(4)}\right)
\end{aligned}$$

already too complicated for $d = 3$ and $m > 4$

Symmetries of SIC-POVMs

SIC-POVM

$$\mathcal{S} = \{P_1, \ldots, P_{d^2}\} \quad \text{where } P_i^2 = P_i,\ P_i = P_i^\dagger,\ \mathrm{tr}(P_i) = 1$$

unitary symmetry U acts on $\mathcal{S}$:

$$U P_i U^\dagger = P_{\pi(i)}$$

- permutation representation of the symmetry group $A(\mathcal{S})$

$$A(\mathcal{S}) \to S_{d^2}, U \mapsto \pi(U)$$

- for SIC-POVMs, the kernel corresponds to global phases
  $\Rightarrow$ projective representation of the permutation group

Open problem: Efficient way of computing $A(\mathcal{S})$.

Special Symmetries of SIC-POVMs

For $U \in A(\mathcal{S})$, the number $f(U)$ of fixed points i, i.e. $UP_iU^\dagger = P_i$ is given by

$$f(U) = |\operatorname{tr}(U)|^2.$$

[Zauner, Satz 2.34]

- **transitive symmetry group:**
  The SIC-POVM is a single orbit under $A(\mathcal{S})$, i.e. $P_i = U_i P_1 U_i^\dagger$.
- **regular symmetry (sub)group:**
  Up to phases, there is a unique element U_i with $P_i = U_i P_1 U_i^\dagger$.

candidates for regular symmetry groups are nice unitary error bases (UEBs)

Weyl-Heisenberg Group

- Generators: $H_d := \langle X, Z \rangle$

  where $X := \sum_{j=0}^{d-1} |j+1\rangle \langle j|$ and $Z := \sum_{j=0}^{d-1} \omega_d^j |j\rangle \langle j|$

  $(\omega_d := \exp(2\pi i/d))$

- Relations:

$$(\omega_d^c X^a Z^b)\left(\omega_d^{c'} X^{a'} Z^{b'}\right) = \omega_d^{a'b-b'a}\left(\omega_d^{c'} X^{a'} Z^{b'}\right)(\omega_d^c X^a Z^b)$$

- Basis:

$$H_d/\zeta(H_d) = \left\{X^a Z^b : a, b \in \{0, \ldots, d-1\}\right\} \cong \mathbb{Z}_d \times \mathbb{Z}_d$$

  trace-orthogonal basis of all $d \times d$ matrices

Jacobi Group (or Clifford Group)

- automorphism group of the Heisenberg group H_d, i.e.

$$\forall T \in J_d : T^\dagger H_d T = H_d$$

- the action of J_d on H_d modulo phases corresponds to the symplectic group $SL(2, \mathbb{Z}_d)$, i.e.

$$T^\dagger X^a Z^b T = \omega_d^c X^{a'} Z^{b'} \qquad \text{where } \begin{pmatrix} a' \\ b' \end{pmatrix} = \tilde{T} \begin{pmatrix} a \\ b \end{pmatrix}, \tilde{T} \in SL(2, \mathbb{Z}_d)$$

$\Longrightarrow$ homomorphism $J_d \to SL(2, \mathbb{Z}_d)$

- J_d is generated by the discrete Fourier transform and a diagonal matrix "with quadratic phases" (depends on d odd or even)

Zauner's Conjecture

[G. Zauner, Dissertation, Universität Wien, 1999]

Conjecture:
For every dimension $d \geq 2$ there exists a SIC-POVM whose elements are the orbit of a rank-one operator E_0 under the Heisenberg group H_d.
What is more, E_0 commutes with an element T of the Jacobi group J_d.
The action of T on H_d modulo the center has order three.

support for this conjecture:

- algebraic solutions by [Zauner 99, Appleby 05] for $d = 2, 3, 4, 5, 7, 19$ (only prime powers)
- numerical evidence by [Renes et al. 04] for $d \leq 45$
- our algebraic solutions for $d = 6, 8, 9, 10, 11, 12, 13, 15$ [Grassl 04–06]

Constructing SIC-POVMs

Ansatz 1:

SIC-POVM that is the orbit under H_d, i.e.,

$$|\phi_{a,b}\rangle := X^a Z^b |\phi_0\rangle$$

$$|\langle\phi_{a,b}|\phi_{a',b'}\rangle|^2 = \begin{cases} 1 & \text{for } (a,b) = (a',b'), \\ 1/(d+1) & \text{for } (a,b) \neq (a',b') \end{cases}$$

$$|\phi_0\rangle = \sum_{j=0}^{d-1} (x_{2j} + i x_{2j+1}) |j\rangle ,$$

($x_0, \ldots, x_{2d-1}$ are real variables, $x_1 = 0$)

$\Longrightarrow$ polynomial equations for $2d-1$ variables, but already too complicated for $d = 6$

Constructing SIC-POVMs (cntd.)

Ansatz 2:

SIC-POVM that is the orbit under H_d,

additionally:

$|\phi_0\rangle$ lies in a (degenerate) ℓ-dimensional eigenspace of some $T \in J_d$

$$|\phi_0\rangle = \sum_{j=0}^{\ell-1} (x_{2j} + i x_{2j+1}) |b_j\rangle ,$$

where $|b_j\rangle$, $j = 1, \ldots, \ell$ is the basis of that eigenspace

$\Longrightarrow$ reduced number of variables

$\Longrightarrow$ better chances to compute algebraic solutions

Strategy Supported by MAGMA

1. Find a suitable non-trivial automorphism $T \in J_d$ with a small eigenspace.
   - use the homomorphism $J_d \to SL(2, \mathbb{Z}_d)$
   - use the conjugacy classes of $SL(2, \mathbb{Z}_d)$
2. Construct the system of polynomial equations.
3. Try to solve the resulting system of polynomial equations.
4. Construct number field which contains some solutions.
5. Find a real subfield.
6. Compute automorphisms of solutions.

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Example: $d = 6$

here: $d = 6$, $\ell = 3$, i.e., only 5 variables

$\Longrightarrow$ algebraic solutions computed using MAGMA

- 144 complex solutions for the real variables
  $\Longrightarrow$ only the real solutions are valid
- in total 96 "different" such SIC-POVMs, but all these SIC-POVMs are related by complex conjugation or a global basis change

Example: $d = 12$

solutions in number field $\mathbb{Q}(\sqrt{2}, \sqrt{13}, \theta_1, \theta_2, i, \omega_3)$ of degree 64 generated by

$$\theta_1 := \sqrt{\sqrt{13}-1}, \quad \theta_2 := \sqrt{\sqrt{13}+3}, \quad i^2 = -1, \quad \omega_3 := \exp 2\pi i/3$$

Coordinates of the (not normalized) initial vector $|\psi_{12}\rangle = \sum_{i=1}^{12} v_i\,|i\rangle$

$$\begin{aligned}
v_1 &= 16\\
v_2 &= \Big(((\sqrt{26}+\sqrt{2}-\sqrt{13}-1)\theta_1 + (\sqrt{26}-5\sqrt{2}-2\sqrt{13}+10))\theta_2\\
&\qquad +((-\sqrt{26}-3\sqrt{2}+2\sqrt{13}+6)\theta_1 + (4\sqrt{2}-4))\Big)i\\
&\qquad +((-\sqrt{13}-1)\theta_1 + (\sqrt{26}-5\sqrt{2}))\theta_2 + (\sqrt{26}+3\sqrt{2})\theta_1 + 4\\
v_3 &= ((4\sqrt{2}-8)\theta_1 - 4\sqrt{26} - 4\sqrt{2} + 4\sqrt{13} + 4)i\\
v_4 &= (((4\sqrt{2}-4)\theta_1 - 4\sqrt{2}+8)\theta_2 + (8\sqrt{2}-8))i + (-4\theta_1 - 4\sqrt{2})\theta_2 + 8
\end{aligned}$$

$$
\begin{aligned}
v_5 &= (-2\sqrt{26} - 6\sqrt{2})\theta_1 - 8 \\
v_6 &= \Big(((\sqrt{26} - \sqrt{2} - \sqrt{13} + 1)\theta_1 + (2\sqrt{2} - 4))\theta_2 \\
&\qquad + ((-2\sqrt{2} + 4)\theta_1 + (2\sqrt{26} + 2\sqrt{2} - 2\sqrt{13} - 2))\Big)i \\
&\quad + ((-\sqrt{13} + 1)\theta_1 + 2\sqrt{2})\theta_2 + 2\sqrt{2}\theta_1 + 2\sqrt{13} + 2 \\
v_7 &= (16\sqrt{2} - 16)i \\
v_8 &= \Big(((\sqrt{26} + \sqrt{2} - \sqrt{13} - 1)\theta_1 + (\sqrt{26} - 5\sqrt{2} - 2\sqrt{13} + 10))\theta_2 \\
&\qquad + ((\sqrt{26} + 3\sqrt{2} - 2\sqrt{13} - 6)\theta_1 - 4\sqrt{2} + 4)\Big)i \\
&\quad + ((-\sqrt{13} - 1)\theta_1 + (\sqrt{26} - 5\sqrt{2}))\theta_2 + (-\sqrt{26} - 3\sqrt{2})\theta_1 - 4 \\
v_9 &= -4\sqrt{2}\theta_1 - 4\sqrt{13} - 4 \\
v_{10} &= (((4\sqrt{2} - 4)\theta_1 - 4\sqrt{2} + 8)\theta_2 + (-8\sqrt{2} + 8))i + (-4\theta_1 - 4\sqrt{2})\theta_2 - 8 \\
v_{11} &= ((2\sqrt{26} + 6\sqrt{2} - 4\sqrt{13} - 12)\theta_1 - 8\sqrt{2} + 8)i \\
v_{12} &= \Big(((\sqrt{26} - \sqrt{2} - \sqrt{13} + 1)\theta_1 + (2\sqrt{2} - 4))\theta_2 \\
&\qquad + ((2\sqrt{2} - 4)\theta_1 - 2\sqrt{26} - 2\sqrt{2} + 2\sqrt{13} + 2)\Big)i \\
&\quad + ((-\sqrt{13} + 1)\theta_1 + 2\sqrt{2})\theta_2 - 2\sqrt{2}\theta_1 - 2\sqrt{13} - 2
\end{aligned}
$$

General Approach

1. Use a suitable symmetry group $G \subset U(d)$ (abstract error group[a]).
2. Find $T \in U(d) \setminus G$ with $G^T = G$ and T not proportional to identity.
3. Use a generic vector in an eigenspace of T as fiducial vector

$$|\phi_0\rangle = \sum_{j=0}^{\ell-1} (x_{2j} + i x_{2j+1}) \, |b_j\rangle \, .$$

4. The SIC-POVM is the projective orbit under G, i.e.

$$\{g \, |\phi_0\rangle \, \langle\phi_0| \, g^{-1} : g \in G\}.$$

5. Try to solve the resulting system of polynomial equations over $\mathbb{R}$.

[a] [Klappenecker & Rötteler, quant-ph/0010082]

Modular Computations

Solving polynomial equations using Gröbner bases

- "standard" Buchberger algorithm over cyclotomic/number field
- additional variable for field extension
  $\Longrightarrow$ computations over $\mathbb{Q}$
- use linear algebra for reduction (F4 in Magma) $\Longrightarrow$ modular techniques

Reconstructing Gröbner bases [E. Arnold 03]

- compute Gröbner bases $\mod p_i$
- if all primes p_i are "lucky", reconstruction is possible

successfully applied for $d = 10$ with 65 primes, modulus 504 digits, solutions in number field of degree 192

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Some Group Covariant SIC-POVMS

group	d	unitary automorphism	number of SIC-POVMs
H_6	6	order 3	$48 + 48$
H_8	8	order 6	64
H_9	9	order 3	$216 + 216$
H_{10}	10	order 3	$240 + 240$
H_{11}	11	order 3	$440 + 440$
H_{12}	12	order 3	384
H_{13}	13	order ≥ 3	not yet computed
H_{15}	15	order ≥ 3	not yet computed
SmallGroup(36, 11)	6	order 3	$24 + 24$
SmallGroup(64, 78)	8	order 1	16

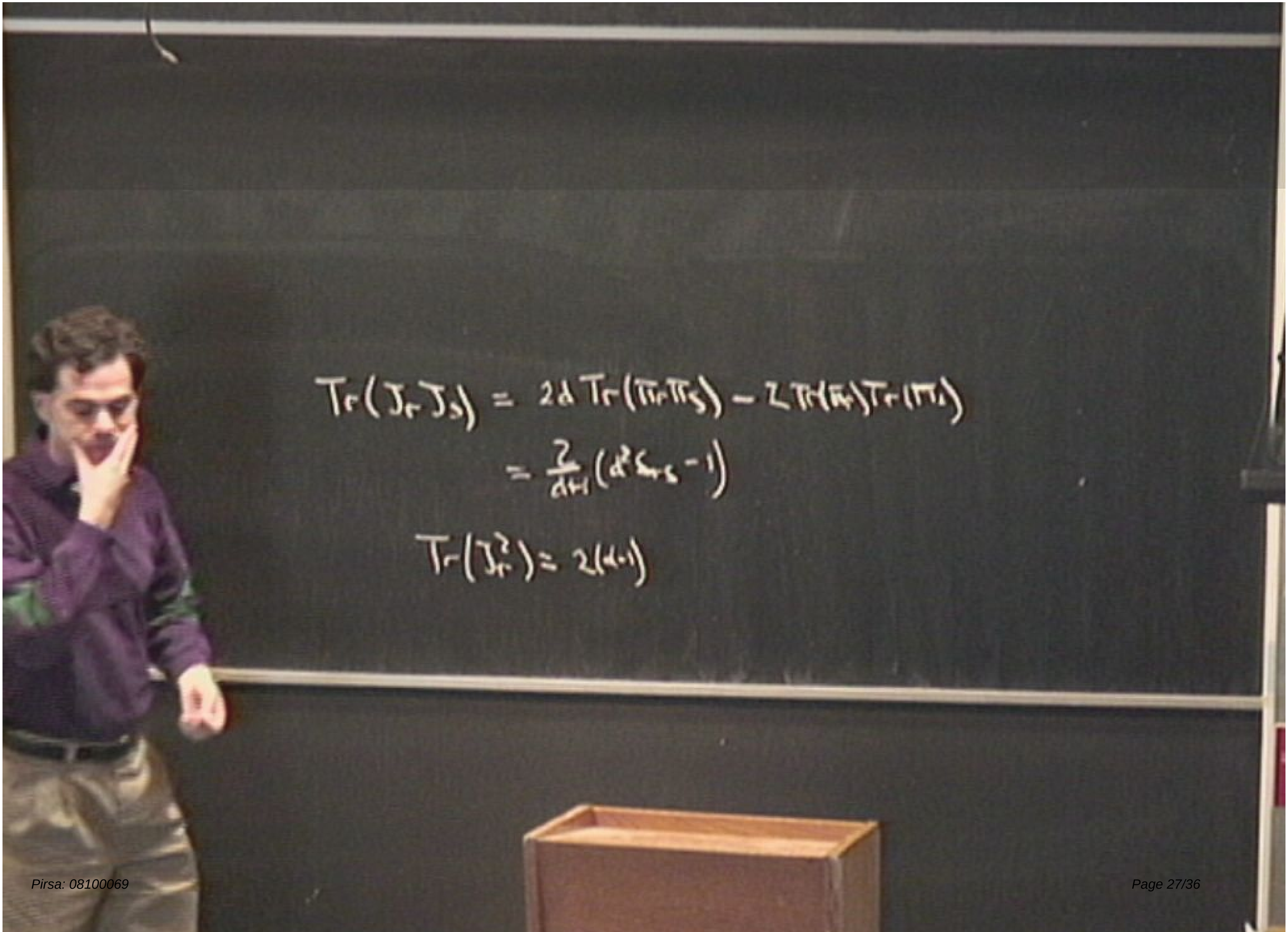
for $d \geq 10$, the classification is incomplete

Reusing SIC-POVMs

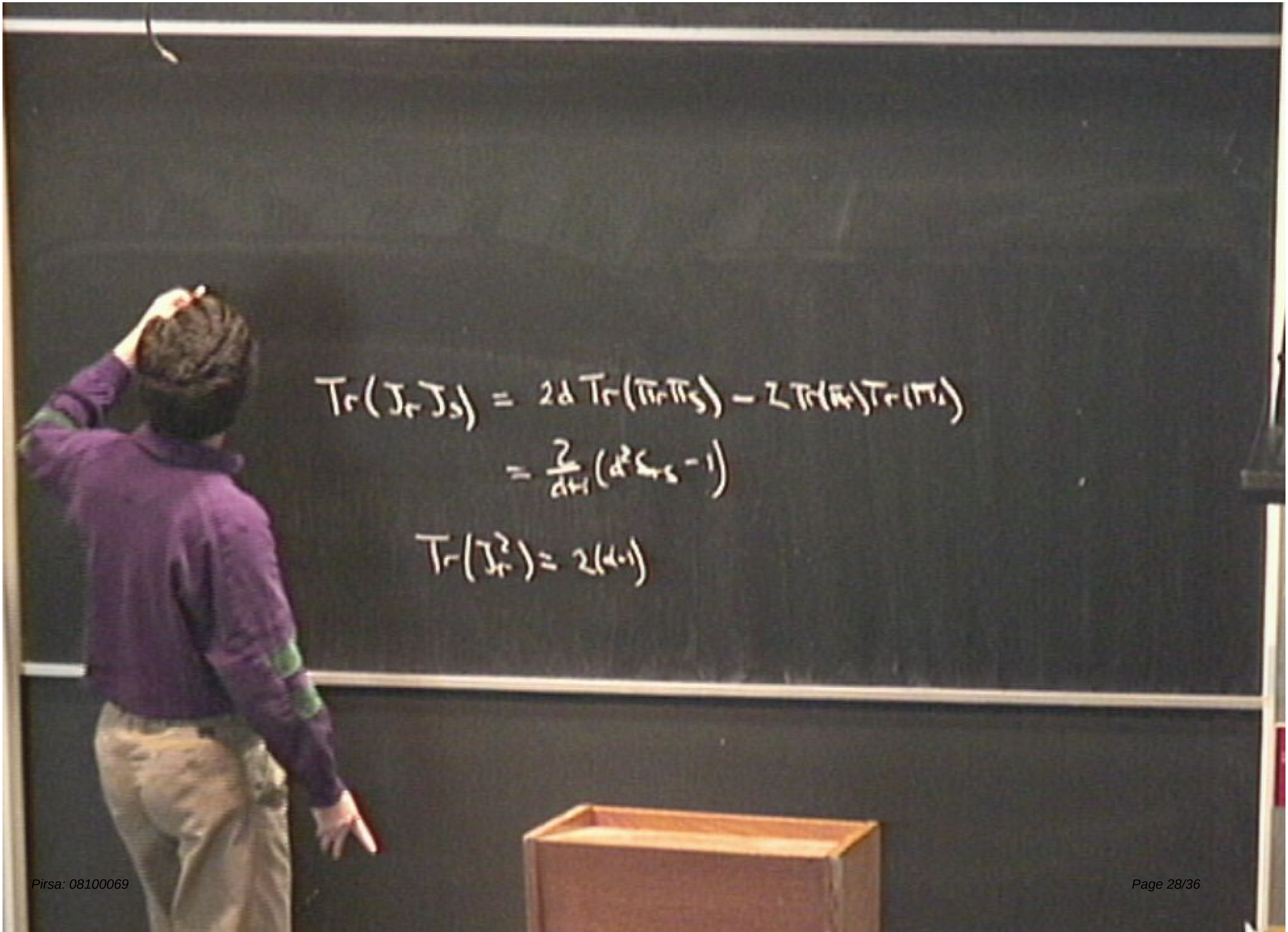
- for $d = 4$, we get 16 Weyl-Heisenberg SIC-POVMs with 16 elements.
- 23 different values for the absolute value of the mutual inner products between these 256 vectors
- define graph on these 256 vectors, two vectors are connected iff the absolute value of the inner product is $1/5$
- we get 32 maximal cliques, i.e. 32 SIC-POVMs
- each of the new SIC-POVMs is formed by 4 vectors from 4 of the original SIC-PVOMs
- the symmetry group of the new SIC-POVMs is again $\mathbb{Z}_4 \times Z_4$, related by a global base change

More Recent Results

- Up to dimension six, all Weyl-Heisenberg SIC-POVMs have an additional symmetry of order three.
- In dimension 12, we have SIC-POVMs with different symmetries of order three.
- In dimension eight, there exists SIC-POVMs for many of the abstract error groups.
- In particular, the SIC-POVM with resepect to the Pauli group (*Hoggar lines*) can be partitioned with respect to many different abstract error groups.
- Exact relation between the various SIC-POVMs is not yet clear.
- So far, no SIC-POVM without additional symmetry was found.



$\mathrm{Tr}(J_r J_s) = 2d\,\mathrm{Tr}(\Pi_r \Pi_s) - 2\,\mathrm{Tr}(\Pi_r)\mathrm{Tr}(\Pi_s)$
$= \frac{2}{d+1}(d^2 \delta_{rs} - 1)$
$\mathrm{Tr}(J_r^2) = 2(d-1)$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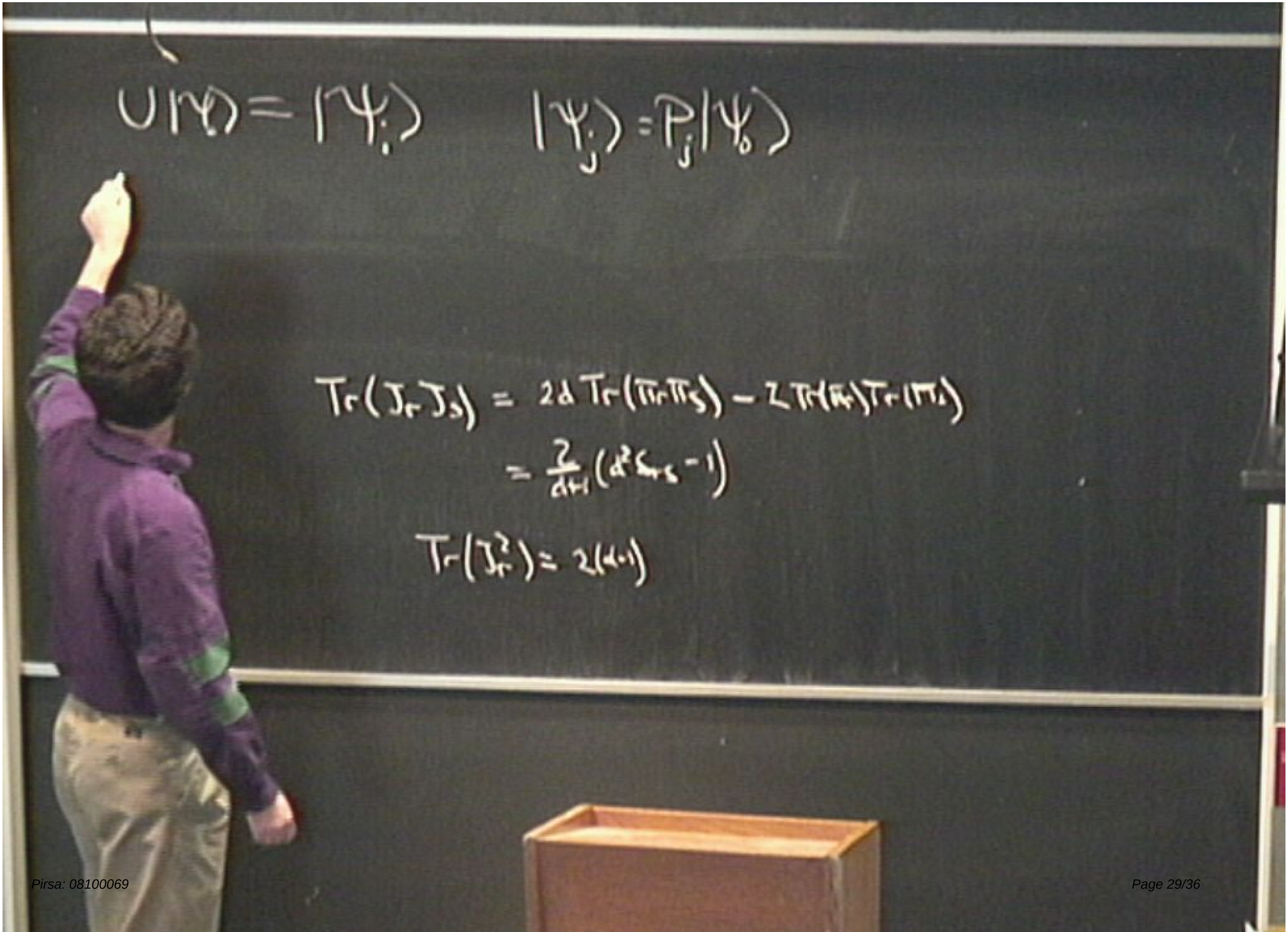
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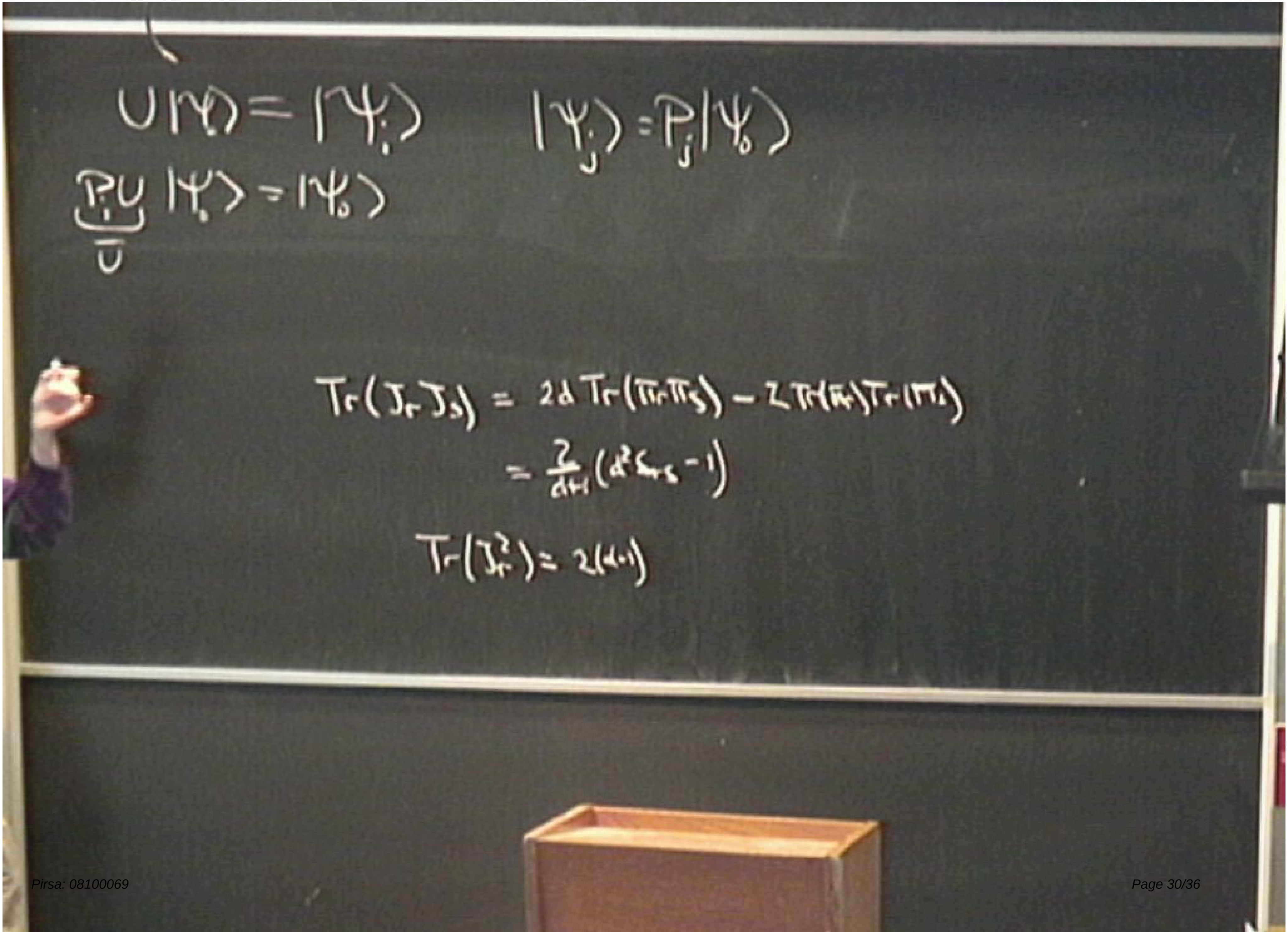
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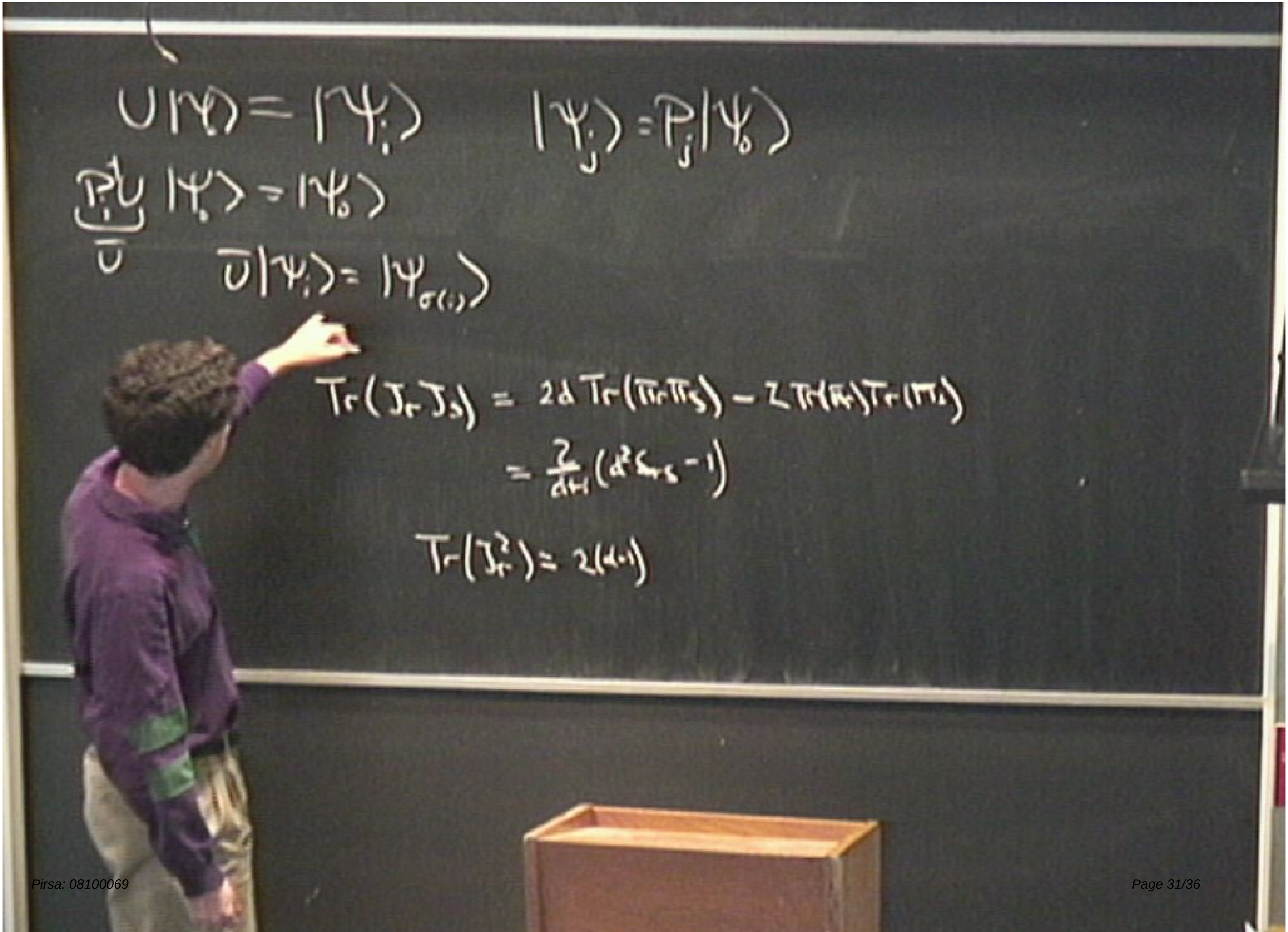
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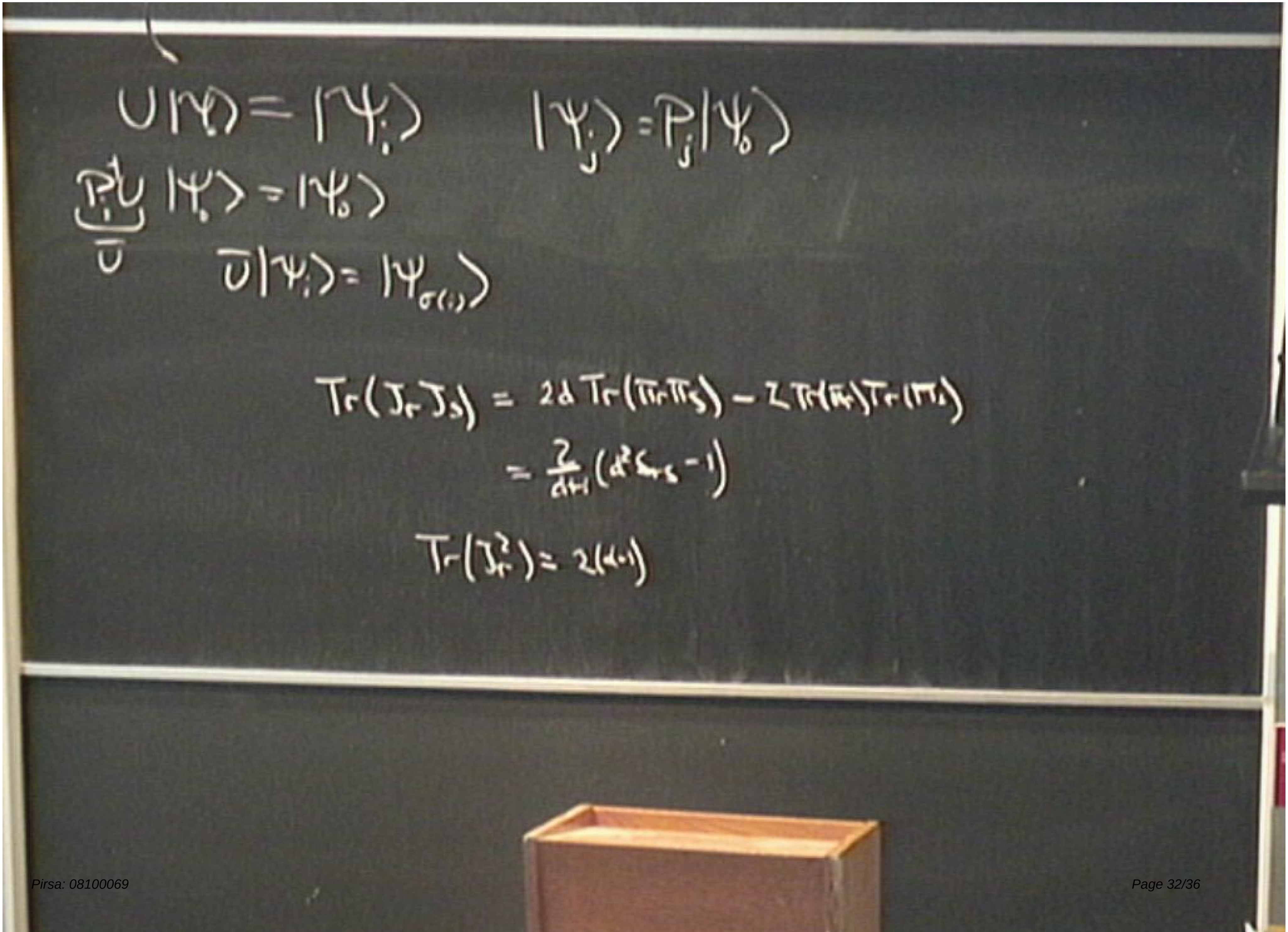
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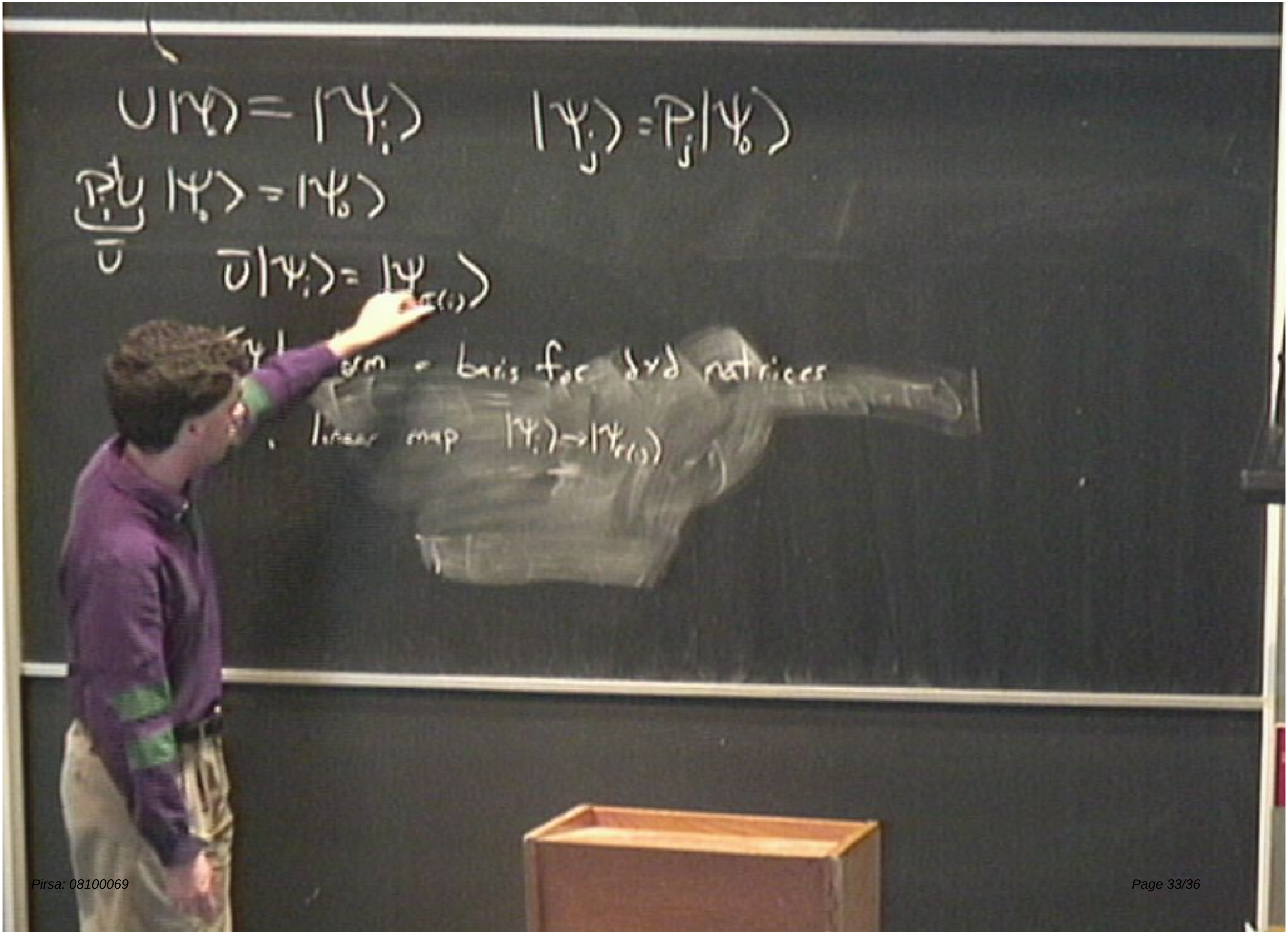
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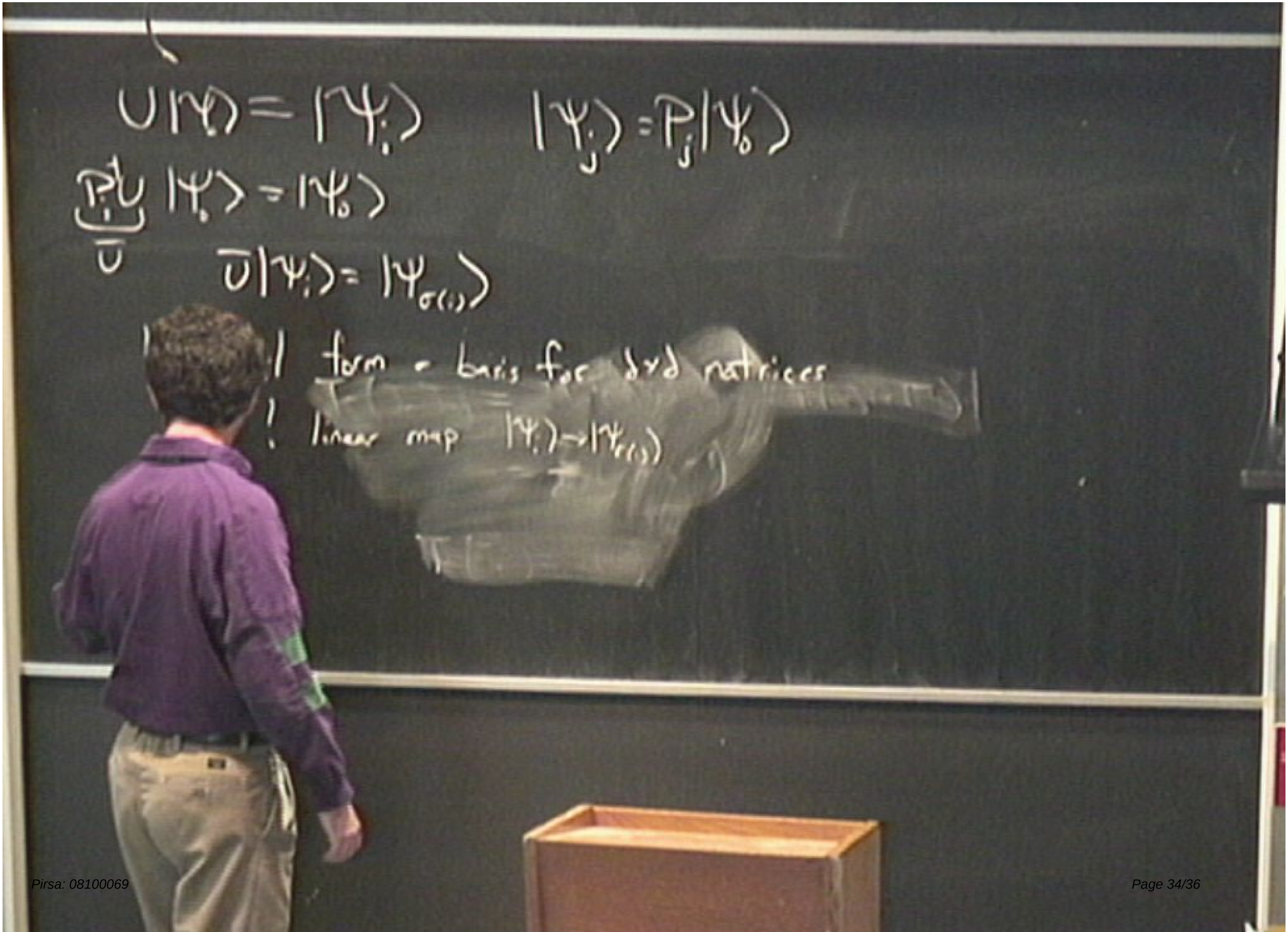
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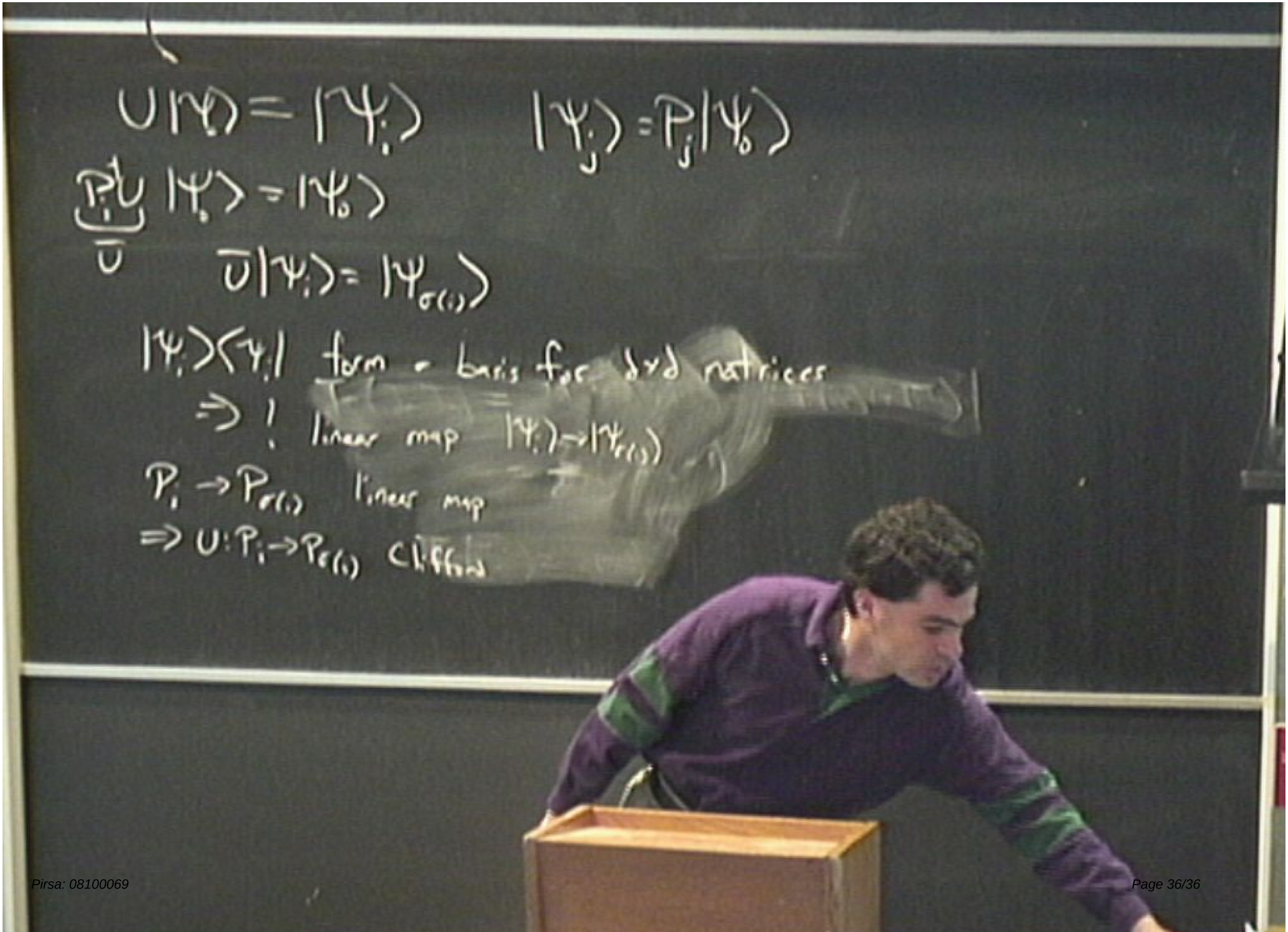
$|\psi_i\rangle\langle\psi_i|$ form a basis for $d\times d$ matrices

$\Rightarrow$! linear map $|\psi_i\rangle \to |\psi_{\sigma(i)}\rangle$

$P_i \to P_{\sigma(i)}$ linear map

$\Rightarrow U: P_i \to P_{\sigma(i)}$ Clifford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Pᵢ → Pσ(i) linear map
⇒ U: Pᵢ → Pσ(i) Clifford