Title: Explorations of Covariant Canonical Gravity

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Abstract: The standard Hamiltonian formulation of (first order) gravity breaks manifest covariance both in its retention of the Lorentz group as a local gauge group and in its discrepant treatment of spacelike and timelike diffeomorphisms. Here we promote more covariant alternatives for canonical quantum gravity that address each of these problems, and discuss the implications for both the classical and the quantum theory of gravity. By retaining the full local Lorentz group, one gains significant insight into the geometric and algebraic properties of the Hamiltonian dynamics. As an example, we discuss the possibility of computing the internal spin angular momentum of asymptotically flat spacetimes, which may lead to insight into the nature of spin in quantum gravity. By treating the spacelike and timelike diffeomorphisms on equal footing, using techniques from geometric quantization we find a new representation of the quantum constraints where the total Hamiltonian is kinematical in the same vein as the Gauss and diffeomorphism constraints. Finally, we discuss the possibility of a manifestly 4-dimensional symplectic form on the Lagrangian phase space.

Covariance and General Relativity

- The 3+1, canonical, Hamiltonian analysis of general relativity breaks explicit covariance in two ways
  - Global Covariance: General covariance broken by splitting spacetime into space and time and treating two parts on unequal footing
  - Local Covariance: Local Lorentz group (in Einstein-Cartan gravity) is broken to subgroup of rotations

- Classically this is *usually* not a problem
  - Global Covariance not manifest but regained when E-C equations solved
  - Local covariance doesn't usually play a significant role in classical theory

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Quantum Problems

- Local gauge group plays major role in quantum theory
  - In LQG local Spin(3,1) broken to SU(2)
  - Discrete areas and volumes are representations of SU(2)
- Some indication that discreteness may not hold in covariant treatment
  - Discreteness related to compactness of gauge group
  - Not true in covariant treatment of 2+1 gravity
  - Does not appear to hold in covariant 3+1 LQG
  - Known only at kinematical level, dynamics not understood
- Splitting of spacetime manifest in problem of time
  - In QM constraint and gauge orbit factoring occur in one step: "frozen time"
  - Hamiltonian constraint implements dynamics and invariance under timelike diffeomorphisms

The Approach

- Faced with problems of time and local covariance one can take different routes
    - Take problem at face value and explore consequences
    - Guess we are doing something wrong or missing something
- We want to adopt radically conservative view
    - Retain as much of canonical theory as possible (symplectic form, constraints, commutator)
    - Re-work theory to address these issues
- Do canonical analysis but focus on Lagrangian phase space with Hamiltonian phase space as submanifold
- Problem of covariance has consequences for both classical and quantum gravity

A Simple Example

- Start with simple non-relativistic single particle action:

$$S = \int_{-\infty}^{t} \left(\tfrac{1}{2} m\, \dot{x} \cdot \dot{x} - V(x)\right) dt$$

- Arbitrary variation splits into bulk and boundary piece

$$\underbrace{\boldsymbol{\delta} S = m\dot{x} \cdot \boldsymbol{\delta} x\Big|_{t}}_{\boldsymbol{J}} - \underbrace{\int_{-\infty}^{t} (m\ddot{x} + \nabla V) \cdot \boldsymbol{\delta} x\, dt}_{\boldsymbol{\theta}}\,.$$

- Think of $\boldsymbol{\delta}$ as exterior derivative on phase space

$\boldsymbol{J}$:Symplectic one-form $\boldsymbol{\theta}$:Lagrangian one-form

$$\boldsymbol{\Omega} = -\boldsymbol{\delta} \boldsymbol{J} = \boldsymbol{\delta} x \wedge \boldsymbol{\delta}(m\dot{x})\Big|_{t}$$:Symplectic form

…A Simple Example

- Focus on the Lagrangian one-form:

$$\boldsymbol{\theta} = 0 \quad \leftrightarrow \quad m\ddot{x} = -\nabla V$$:Bulk EOM

- Exterior derivative also gives symplectic form:

$$\boldsymbol{\delta\delta S} = \boldsymbol{\delta J} + \boldsymbol{\delta\theta} = 0 \longrightarrow \boxed{\boldsymbol{\delta\theta} = -\boldsymbol{\delta J} = \boldsymbol{\Omega}}$$

- Consider the time translation vector field:

$$\bar{\boldsymbol{t}} = \dot{x} \cdot \frac{\delta}{\boldsymbol{\delta} x} + \ddot{x} \cdot \frac{\delta}{\boldsymbol{\delta}\dot{x}} \qquad \boldsymbol{\theta}(\bar{\boldsymbol{t}}) = \int -(m\ddot{x} + \nabla V) \cdot \dot{x}\, dt$$

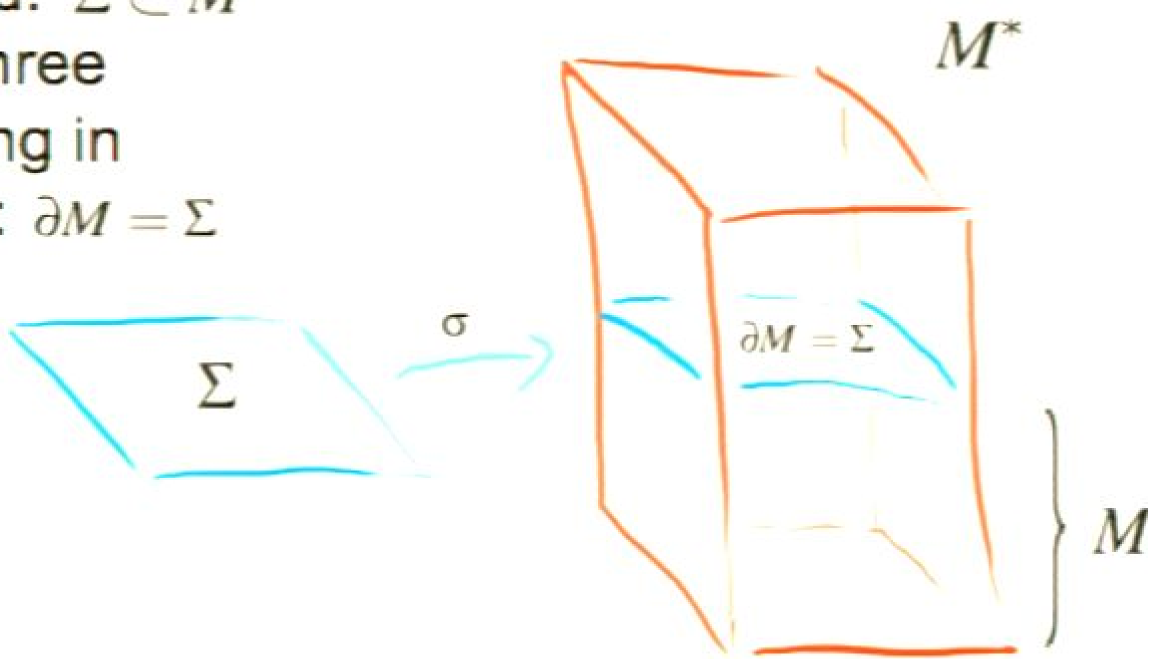
$$= -\left(\tfrac{1}{2}mv^2 + V(x)\right) = -H$$

- Hamilton's equations impose that the Lagrangian one-form is time independent:

$$\boxed{\mathcal{L}_{\bar{t}}\boldsymbol{\theta} = (\boldsymbol{\delta\theta})(\bar{\boldsymbol{t}}) + \boldsymbol{\delta}(\boldsymbol{\theta}(\bar{\boldsymbol{t}})) = 0 \quad \longleftrightarrow \quad \boldsymbol{\Omega}(\bar{\boldsymbol{t}},\ ) = \boldsymbol{\delta} H}$$

Apply to Diff-invariant field theory

- Wish to apply method to generally covariant field theories
  - Ideas apply readily
  - Apply to Einstein-Cartan gravity
- Full manifold is M^*
  - Embed three manifold: $\Sigma \subset M^*$
  - Dynamical arena is three manifold and everything in past of three manifold: $\partial M = \Sigma$



General Relativity (Einstein-Cartan)

- Dynamical variables:
  - Spin(3,1) connection coefficients: $\varpi = \frac{1}{4}\gamma_{[I}\gamma_{J]}\,\varpi^{IJ}$
  - Tetrad, Frame Field, veirbein: $\varepsilon = \frac{i}{2}\gamma_I\,\varepsilon^I$
- Einstein-Cartan Action:

$$\boxed{S = \int_M \star\,\varepsilon\,\varepsilon\,R_\varpi}$$

$\star = -i\gamma_5$:internal dual

- Bulk equations of motion:

$$\delta\varpi : \quad D_\varpi(\star\varepsilon\,\varepsilon) = 0$$ (Vanishing torsion)

$$\delta\varepsilon : \quad \varepsilon\star R_\varpi - \star R_\varpi\,\varepsilon = 0$$ (Einstein equations)

- Boundary variables

$$e \equiv \sigma^*\varepsilon \qquad \omega \equiv \sigma^*\varpi$$

Lagrangian and Symplectic forms

- Symplectic two-form:

$$\boxed{\Omega = \int_\Sigma \star\delta\omega \wedge \delta(e\, e)}$$

Degenerate!

- Lagrangian one-form

$$\boxed{\theta = \int_M -D_\varpi(\star\varepsilon\,\varepsilon)\delta\varpi + (\star R_\varpi\,\varepsilon - \varepsilon\star R_\varpi)\,\delta\varepsilon}$$

- Define Noether vector: $\theta(\bar{W})$ boundary functional
- Three relevant Noether vectors:

$$\begin{cases} \bar{\lambda} = \int_M -D_\varpi\lambda\,\frac{\delta}{\delta\varpi} + [\lambda,\varepsilon]\,\frac{\delta}{\delta\varepsilon} & \lambda \in spin(3,1) \\ \bar{N} = \int_M \mathcal{L}_{\bar{N}}\varpi\,\frac{\delta}{\delta\varpi} + \mathcal{L}_{\bar{N}}\varepsilon\,\frac{\delta}{\delta\varepsilon} & \bar{N} \text{ :spacelike (tangent to boundary)} \\ \bar{t} = \int_M \mathcal{L}_{\bar{t}}\varpi\,\frac{\delta}{\delta\varpi} + \mathcal{L}_{\bar{t}}\varepsilon\,\frac{\delta}{\delta\varepsilon} & \bar{t} \text{ :timelike 4-vector} \end{cases}$$

Constraints

- Noether vectors define constraints if symmetry is to hold
- Gauge (Gauss) Constraint:

$$\boxed{C_G(\bar{\lambda}) = -\boldsymbol{\theta}(\bar{\boldsymbol{\lambda}}) = \int_\Sigma -D_\omega \lambda \star e\, e}$$

- 3-diffeomorphism constraint

$$\boxed{C_D(\bar{N}) = -\boldsymbol{\theta}(\bar{\boldsymbol{N}}) = \int_\Sigma \mathcal{L}_{\bar{N}} \omega \star e\, e}$$

- Time translation, Hamiltonian constraint

$$C_{tot}(t,\lambda) = -\boldsymbol{\theta}(\bar{\boldsymbol{t}}) = C_H(t) + C_G(\lambda)$$

$$\lambda \equiv -\varpi(\bar{t})$$

$$\boxed{C_H(t) = \int_\Sigma - \star [t, e] R_\omega}$$

$$t \equiv \varepsilon(\bar{t}) = \tfrac{i}{2}\gamma_I\, t^I$$

Hamilton's equations for EC-gravity

- As before guess the form of Hamilton's equations to be:

$$\mathcal{L}_{\bar{t}}\theta = 0 \quad \longrightarrow \quad \Omega(\bar{t},\ ) = \delta C_{tot}(t, \lambda)$$

- Exterior derivative is on full Lagrangian phase space:

$$\boldsymbol{\Omega}(\bar{t},\ ) = \boldsymbol{\delta} C_{tot}(\lambda, t)\,|_{\lambda, t} + C_H(\boldsymbol{\delta} t) + C_G(\boldsymbol{\delta}\lambda)$$

- Identifying Components:

HAM1: $C_{tot}(\lambda, t) = C_H(t) + C_G(\lambda) = 0$

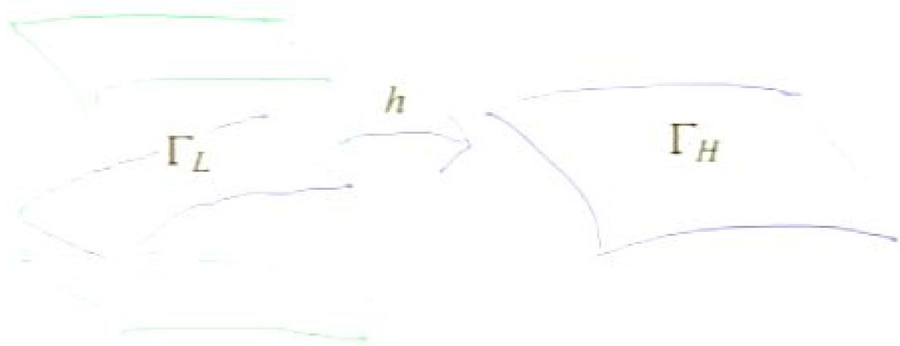
HAM2: $\Omega(\bar{t},\ ) = \delta C_{tot}(t, \lambda)\big|_{t,\lambda}$

- Since symplectic form is degenerate HAM2 also constrains phase space:

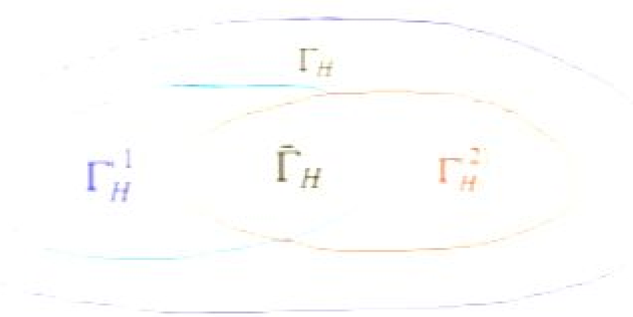
$$\exists \bar{X} \mid \boldsymbol{\Omega}(\bar{X},\ ) = \boldsymbol{\delta} X \quad \rightarrow \mathcal{L}_{\bar{Z}} X = 0 \quad \forall \bar{Z} \mid \boldsymbol{\Omega}(\bar{Z},\ ) = 0$$

Constrained phase space

- Spacelike embedding induces projection of Lagrangian to Hamiltonian phase space



- Physical submanifold is intersection of two submanifolds
  - Submanifold where HAM1 holds: $\Gamma_H^{(1)}$
  - Submanifold where HAM2 holds: $\Gamma_H^{(2)}$
  - Physical submanifold $\bar{\Gamma}_H = \Gamma_H^{(1)} \cap \Gamma_H^{(2)}$



Equivalence with Einstein equations

•HAM1 gives spatial components of Einstein-equations:

$$C_H(\boldsymbol{\delta} t) = C_G(\boldsymbol{\delta}\lambda) = 0$$

$$\sigma^*\left(D_\varpi \star \varepsilon\,\varepsilon\right) = 0$$

$$\sigma^*\left(\star R_\varpi \varepsilon - \varepsilon \star R_\varpi\right) = 0$$

•HAM2 gives time components of Einstein-equations:

$$\boldsymbol{\Omega}(\bar{t},\ ) = \boldsymbol{\delta} C_{tot}(\lambda, t)\,|_{\lambda, t}$$

$$\sigma^*\left(i_{\bar{t}}(D_\varpi \star \varepsilon\,\varepsilon)\right) = 0$$

$$\sigma^*\left(i_{\bar{t}}(\star R_\varpi \varepsilon - \varepsilon \star R_\varpi)\right) = 0$$

•Together this is the full set of Einstein equations on spacelike hypersurface

Constraint Algebra

- Let $\bar{t}_1$ and $\bar{t}_2$ be two different choices for time evolution vector field:

$$\bar{\boldsymbol{t}}_{\mathbf{1,2}} = \int_M \mathcal{L}_{\bar{t}_{1,2}}\varepsilon \,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\varepsilon} + \mathcal{L}_{\bar{t}_{1,2}}\varpi \,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\varpi} \qquad C_{tot}(t_{1,2},\lambda_{1,2}) = -\boldsymbol{\theta}(\bar{\boldsymbol{t}}_{\mathbf{1,2}})$$

- Consider the commutator:

$$[\bar{\boldsymbol{t}}_{\mathbf{1}}, \bar{\boldsymbol{t}}_{\mathbf{2}}] = \int_M \mathcal{L}_{[\bar{t}_1,\bar{t}_2]}\varepsilon \,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\varepsilon} + \mathcal{L}_{[\bar{t}_1,\bar{t}_2]}\varpi \,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\varpi}$$

- Define a theta-bracket:

$$\boxed{\{C_{tot}(t_1,\lambda_1), C_{tot}(t_2,\lambda_2)\}_{\boldsymbol{\theta}} \equiv -\boldsymbol{\theta}([\bar{\boldsymbol{t}}_{\mathbf{1}}, \bar{\boldsymbol{t}}_{\mathbf{2}}])}$$

- Relation with Poisson bracket:

$$\boxed{\{C^{(1)}_{tot}, C^{(2)}_{tot}\}_{\boldsymbol{\theta}} = \{C^{(1)}_{tot}, C^{(2)}_{tot}\}_{Poisson} + \iota_{\bar{\boldsymbol{t}}_{\mathbf{2}}}\mathcal{L}_{\bar{\boldsymbol{t}}_{\mathbf{1}}}\boldsymbol{\theta} - \iota_{\bar{\boldsymbol{t}}_{\mathbf{1}}}\mathcal{L}_{\bar{\boldsymbol{t}}_{\mathbf{2}}}\boldsymbol{\theta}}$$

Local Covariance

(Retention of local Lorentz Group)

Relation with (A)dS algebra

- The theta bracket is closed, for GR it gives:

$$\boldsymbol{\theta([\bar{t}_1,\bar{t}_2])} = C_H(-\varepsilon([\bar{t}_1,\bar{t}_2])) + C_G(\varpi([\bar{t}_1,\bar{t}_2]))$$

- Only Hamiltonian constraint is changed
- Explicit computation of constraint algebra yields:

$$\{C_G(\lambda_1)\,,C_G(\lambda_2)\} = C_G([\lambda_1,\lambda_2])$$

$$\{C_G(\lambda)\,,C_H(t)\} = C_H([\lambda,t])$$

$$\{C_H(t_1)\,,C_H(t_2)\} \overset{\bar{\Gamma}_H}{\approx} -\tfrac{\Lambda}{3}C_G([t_1,t_2]) - C_G(\underbrace{C}_{\text{Weyl tensor}}([\bar{t}_1,\bar{t}_2]))$$

- Constraint algebra is deformation of de Sitter, anti-de Sitter, or Poincare algebra depending on sign of c.c.

Asymptotically flat spacetimes

- Restrict configurations to asymptotically flat or (A)dS spacetimes
  - Boundary term must be added to action

$$S = \frac{1}{k}\int_M \star\,\varepsilon\,\varepsilon\,(R_\varpi - \tfrac{\Lambda}{6}\,\varepsilon\,\varepsilon) - \frac{1}{k}\int_{\partial M} \star\,\varepsilon\,\varepsilon\,\varpi$$

- Each constraint picks up boundary term
  - Total Hamiltonian becomes true Hamiltonian

$$H(t) = \frac{1}{k}\int_\Sigma - \star\,[t.e]\,(R_\omega - \tfrac{\Lambda}{3} e\,e) + \frac{1}{k}\int_{\partial\Sigma} \star[t.e]\,\omega$$

$$G(\lambda) = \frac{1}{k}\int_\Sigma \star\lambda\,D_\omega(e\,e) - \frac{1}{k}\int_{\partial\Sigma} \star\,\lambda\,e\,e$$

- On constraint manifold, boundary term is ADM energy-momentum

$$P^I = \frac{1}{4k}\int_{\partial\Sigma} \epsilon^I{}_{JKL}\,e^J\,\omega^{KL}$$

Spin angular momentum operator

- What happens when a spinor falls into a blackhole? Where does the (internal) spin go?
  - Can the spin be encoded in the gravitational field alone?

- Boundary term of Gauss constraint, interpreted as spin generator

$$\sigma^{IJ} \equiv \frac{1}{4k} \int_{\partial\Sigma} \epsilon^{IJ}{}_{KL}\, e^K\, e^L$$

- On constraint subspace, algebra of boundary terms is precisely (A)dS/Poincare algebra

$$\{\sigma, \sigma\} \sim \sigma$$
$$\{\sigma, P\} \sim P$$
$$\{P, P\} \sim \tfrac{\Lambda}{3}\,\sigma$$

Spin Invariant

- Poincare algebra has two Casimirs:

ADM Mass

$$C_1 = P_I P^I \sim M^2$$

Quartic Casimir (spin invariant)

$$C_2 = W_I W^I \sim M^2\, s(s+1)$$

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Spin in Quantum Theory

- Spin operator generates large gauge transformations:

$$e^{\{\lambda_{IJ}\sigma^{IJ},\bullet\}} \in \mathcal{G}/\mathcal{G}_0$$

- Projection of gauge group induced by (timelike) momentum:

$$Spin(3,1) \xrightarrow{P} SU_{(P)}(2)$$
$$\{g \,|\, g(P) = P\}$$

- Compactify hypersurface by including point at infinity:

$$g : \Sigma + \{\infty\} \rightarrow SU_{(P)}(2)$$
$$S^3 \quad \rightarrow \quad S^3$$

- Configurations characterized by homotopy class of map:

$$\pi_3(S^3) = \mathbb{Z}$$

- In quantum theory states are eigenvalues of map suggesting that spin is quantized

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(Spacelike and Timelike Diffeomorphisms)

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Global Covariance

(Spacelike and Timelike Diffeomorphisms)

Geometric Quantization

- Adopt basic idea of geometric quantization
- Focus on, Lagrangian as opposed to symplectic one-form
- Wave function will be functional of Lagrangian variables
- Recall basic idea of geometric quantization:
  - Introduce complex line bundle over Hamiltonian phase space
  - Pre-quantum wave-function is section of bundle
  - Symplectic one-form becomes U(1) connection
- Following along these lines
  - Introduce complex line bundle over Lagrangian phase space
  - Pre-quantum wave function is section of bundle
  - Lagrangian one-form becomes U(1) connection

The pre-quantum theory

- Consider a symplectomorphism generated by X:

$$\Omega \to \Omega' + \mathcal{L}_{\bar{X}}\Omega$$

$$\mathcal{L}_{\bar{X}}\Omega = \delta\Omega(\bar{X},\ ) = 0 \quad \to \quad \Omega(\bar{X},\ ) = \delta X$$

- Under same transformation one-forms change by:

$$J \to J + \mathcal{L}_{\bar{X}}J = J - \delta\phi \qquad \text{or} \qquad \theta \to \theta + \mathcal{L}_{\bar{Y}}\theta = \theta + \delta\phi'$$

$$\phi = (J(\bar{X}) - X) \qquad \qquad \phi' = (\theta(\bar{Y}) - Y)$$

- Interpret this as U(1) transformation of connection defined as follows:

$$\mathcal{D}_J = \delta - iJ \qquad \text{or} \qquad \mathcal{D}_\theta = \delta + i\theta$$

$$\mathcal{D}_J\mathcal{D}_J = i\,\Omega \qquad \qquad \mathcal{D}_\theta\mathcal{D}_\theta = i\,\Omega$$

Transformation of wavefunction

- Pre-quantum wavefunction is functional of all phase variables:

$$\Psi = \Psi[\omega, e] \qquad \text{or} \qquad \Psi = \Psi[\varpi, \varepsilon]$$

- Under the same transformation, wavefunction is "covariantly Lie-dragged":

$$\begin{aligned} \Psi[\omega, e] \to e^{i\phi}\Psi[\omega', e'] \\ = \Psi - i\phi\,\Psi + \mathcal{L}_{\bar{X}}\Psi \\ \equiv \Psi + i\widehat{\mathcal{O}}(\bar{\boldsymbol{X}}, X)\,\Psi \end{aligned} \qquad \begin{aligned} \Psi[\varpi, \varepsilon] \to e^{i\phi'}\Psi[\varpi', \varepsilon'] \\ = \Psi - i\phi'\,\Psi + \mathcal{L}_{\bar{Y}}\Psi \\ \equiv \Psi + i\widehat{\mathcal{O}}(\bar{\boldsymbol{Y}}, Y)\,\Psi \end{aligned}$$

- Pre-quantum operators are Hermitian generators of U(1) transformation of pre-quantum wavefunction

$$\boxed{\hat{O}_{\boldsymbol{J}}(\bar{\boldsymbol{Y}}, Y) = i\, i_{\bar{\boldsymbol{Y}}}\mathcal{D}_{\boldsymbol{J}} - Y} \qquad \boxed{\hat{O}_{\boldsymbol{\theta}}(\bar{\boldsymbol{X}}, X) = i\, i_{\bar{\boldsymbol{X}}}\mathcal{D}_{\boldsymbol{\theta}} - X}$$

Pre-quantum operators

- Pre-quantum operators reflect associated Lie and Poisson algebras:

$$[\hat{O}_{\boldsymbol{J}}(\bar{\boldsymbol{Y}},Y)\,,\,\hat{O}_{\boldsymbol{J}}(\bar{\boldsymbol{Y}}',Y')] = i\,\hat{O}_{\boldsymbol{J}}([\bar{\boldsymbol{Y}},\bar{\boldsymbol{Y}}'],\{Y,Y'\})$$

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- Example: Gauss constraint

$$\bar{\boldsymbol{\lambda}}_{\boldsymbol{J}} = \int_{\Sigma} -D_{\omega}\lambda\,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\boldsymbol{\omega}} + [\lambda,e]\,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\boldsymbol{e}} \qquad \bar{\boldsymbol{\lambda}}_{\boldsymbol{\theta}} = \int_{M} -D_{\varpi}\lambda\,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\boldsymbol{\varpi}} + [\lambda,\varepsilon]\,\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}\boldsymbol{\varepsilon}}$$

$$C_G(\lambda) = \int_{\Sigma} -D_{\omega}\lambda \star e\,e \qquad C_G(\lambda) = \int_{M} -d\,(D_{\varpi}\lambda \star \varepsilon\,\varepsilon)$$

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Transformation of wavefunction

- Pre-quantum wavefunction is functional of all phase variables:

$$\Psi = \Psi[\omega, e] \qquad \text{or} \qquad \Psi = \Psi[\varpi, \varepsilon]$$

- Under the same transformation, wavefunction is "covariantly Lie-dragged":

$$\begin{aligned}\Psi[\omega, e] &\rightarrow e^{i\phi}\Psi[\omega', e'] \\ &= \Psi - i\phi\,\Psi + \mathcal{L}_{\bar{\boldsymbol{X}}}\Psi \\ &\equiv \Psi + i\widehat{\mathcal{O}}(\bar{\boldsymbol{X}}, X)\,\Psi\end{aligned} \qquad \begin{aligned}\Psi[\varpi, \varepsilon] &\rightarrow e^{i\phi'}\Psi[\varpi', \varepsilon'] \\ &= \Psi - i\phi'\,\Psi + \mathcal{L}_{\bar{\boldsymbol{Y}}}\Psi \\ &\equiv \Psi + i\widehat{\mathcal{O}}(\bar{\boldsymbol{Y}}, Y)\,\Psi\end{aligned}$$

- Pre-quantum operators are Hermitian generators of U(1) transformation of pre-quantum wavefunction

$$\boxed{\hat{O}_{\boldsymbol{J}}(\bar{\boldsymbol{Y}}, Y) = i\, i_{\bar{\boldsymbol{Y}}}\mathcal{D}_{\boldsymbol{J}} - Y} \qquad \boxed{\hat{O}_{\boldsymbol{\theta}}(\bar{\boldsymbol{X}}, X) = i\, i_{\bar{\boldsymbol{X}}}\mathcal{D}_{\boldsymbol{\theta}} - X}$$

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Kinematical Operators

- This is example of kinematical constraint (in both reps):
  - Pre-quantum constraint implements kinematical symmetry

$$\hat{O}(\bar{\boldsymbol{\lambda}}, C_G)\,\Psi = i\,\mathcal{L}_{\bar{\boldsymbol{\lambda}}}\Psi = 0 \quad \rightarrow \quad \Psi$$ is gauge invariant

- Same is true of diffeomorphism constraint
- Kinematical operator before polarizations are kinematical after quantization
- Kinematical Hilbert space gives generic features of quantum geometry apart from complicated dynamics
- Example: Loop Quantum Gravity
  - Kinematical Hilbert space is rigorous and unique
  - Discreteness of geometry is new quantum feature

Total Hamiltonian constraint

- Total Hamiltonian traditionally both implements dynamics and invariance under timelike diffeomorphisms
- Pre-quantum operator:

$$\bar{\boldsymbol{t}}_{\boldsymbol{J}} = \int_{\Sigma} \mathcal{L}_{\bar{t}}\omega \frac{\delta}{\delta\omega} + \mathcal{L}_{\bar{t}}e \frac{\delta}{\delta e}$$

$$\Omega(\bar{\boldsymbol{t}}_{\boldsymbol{J}}, \ ) = \delta C_{tot}$$

$$\boxed{\hat{O}_J(\bar{\boldsymbol{t}}_{\boldsymbol{J}}, C_{tot}) = i\, i_{\bar{\boldsymbol{t}}_{\boldsymbol{J}}}\boldsymbol{\delta} - \tfrac{1}{2} C_H(t)}$$

$\downarrow$

Not kinematical
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$$\bar{t}_\theta = \int_{\partial M} \mathcal{L}_{\bar{t}}\varpi \frac{\delta}{\delta\varpi} + \mathcal{L}_{\bar{t}}\varepsilon \frac{\delta}{\delta\varepsilon}$$

$$\Omega(\bar{t}_\theta, \ ) = \delta C_{tot}$$

$$\boxed{\hat{O}_\theta(\bar{t}_\theta, C_{tot}) = i\, i_{\bar{t}_\theta}\delta}$$

Kinematical!
$(because\ \ C_{tot} = -\theta(\bar{t}_\theta))$

Problem of time

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Intrinsically 4D Symplectic Formalism

- Symplectic form previously defined on full Lagrangian phase space, but has support only on Hamiltonian phase space
- Can symplectic form be smoothly extended to Lagrangian phase space? Possibly.
- Consider the split:

$$\boldsymbol{\theta} = \boldsymbol{\theta}_{\varpi} + \boldsymbol{\theta}_{\varepsilon}$$

$$\boldsymbol{\theta}_{\varpi} = \int_M \star\boldsymbol{\delta}\varpi\, D_{\varpi}(\varepsilon\,\varepsilon)$$

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4D (pre)-Symplectic form

- Define the 4D pre-symplectic form (ignoring boundary):

$$\widetilde{\boldsymbol{\Omega}} = \int_M \star\boldsymbol{\delta}(\varepsilon\,\varepsilon) \wedge \boldsymbol{\delta} R_{\varpi}$$

- Pull-back of (pre)-symplectic form to solution submanifold is identically zero

$$\phi^* \widetilde{\boldsymbol{\Omega}} = 0 \qquad \widetilde{\boldsymbol{\Omega}}(\bar{\boldsymbol{W}},\ ) = 0 \quad \forall \bar{\boldsymbol{W}} \in T\Gamma_L^{(0)}$$

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Gauge algebra

- The constraints vanish on solution to Einstein-equations:

$$C_D(\bar{V}) = \int_M \mathcal{L}_{\bar{V}}\varepsilon\,(\varepsilon \star R_{\varpi} - \star R_{\varpi}\,\varepsilon) = \int_M -\mathcal{L}_{\bar{V}}\varpi(D \star \varepsilon\,\varepsilon) \overset{\Gamma_L^{(0)}}{\approx} 0$$

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$$\boxed{\begin{array}{c} \{C_G(\lambda_1), C_G(\lambda_2)\} = C_G([\lambda_1, \lambda_2]) \\ \{C_D(\bar{V}), C_G(\lambda)\} = C_G(\mathcal{L}_{\bar{V}}\lambda) \\ \{C_D(\bar{V}_1), C_D(\bar{V}_2)\} = C_D([\bar{V}_1, \bar{V}_2]) \end{array}} \longrightarrow Lie\,(SO(3,1) \rtimes Diff_4)$$

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Conclusions

- •Alot can be done at a fundamental level to confront problems of global and local covariance
- •Addressing these problems in a satisfactory way will yield insight into both classical and quantum theory
- •One does not have to give up canonical theory to address these issues
- •One does not have to throw away time

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