Title: Reflecting magnons from D-branes

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Abstract: The discovery of integrability in the large N limit of the prototypical realization of the AdS/CFT correspondence has raised the hope that the spectrum of scale dimensions in N=4 SYM (and strings in AdS_5 x S^5) might be known exactly, i.e. to all orders in the coupling constant. So far, most of the efforts focused on closed strings and periodic boundary conditions. In this talk I will discuss how these ideas are extended to open string and open boundary conditions. In particular how to obtain exact expressions for the boundary scattering matrices, discuss the integrability of the boundary conditions and how finite size effects may be taken into account.

Reflecting magnons from D-branes

Diego Correa - DAMTP - Cambridge

October 21, 2008

- arXiv:0712.1361 [hep-th], H-Y.Chen, D.Correa
- arXiv:0808.0452 [hep-th], D.Correa, C.Young

Motivations

- AdS/CFT motivations
- Stringest tests of AdS/CFT
  Interpolation between weak and strong 't Hooft coupling descriptions
- Ultimate goal:
  Exact spectrum (as function of the coupling) in the large N limit
- Exploring (other) integrable structures underlying AdS/CFT
  Integrable open boundary conditions

Outline

- Prototypical AdS/CFT example.
- Review on:

closed (open) strings in $AdS_5 \times S^5 \leftrightarrow$ periodic (open) spin chains

- Spectrum problem as magnons scattering problem (Bethe Ansatz)
- Symmetries of the problem fix scattering matrix (exact λ dependence)
- Similar analysis for reflection matrices, including open boundary conditions
- Sketch of finite size corrections

AdS/CFT Correspondence [Maldacena 97]

The CFT $\mathcal{N} = 4$ Super Yang-Mills with gauge group $SU(N)$ in $d = 4$

is conjectured to be **equivalent** to

Type IIB string theory in the geometric background $AdS_5 \times S^5$

Why is this still a conjecture after 10 years?

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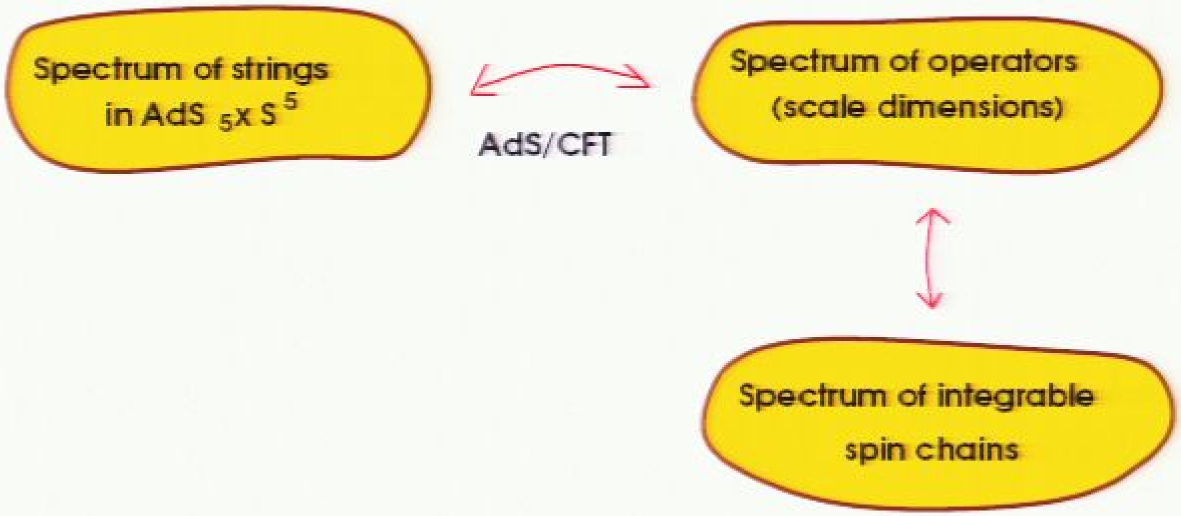
Why is this still a conjecture after 10 years?

- 't Hooft limit: taking $N \to \infty$ and keeping fixed $\lambda = g_{YM}^2 N = R^4/\alpha'^2$

strong/weak coupling duality $\Rightarrow$ hard to prove

Planar limit of $\mathcal{N} = 4$ SYM/ Free strings in $AdS_5 \times S^5$



- For closed strings the dual operators are single trace operators
- Spin chain labeling in the $SU(2)$ sector:

$$Z \equiv \uparrow \quad Y \equiv \downarrow \quad \text{tr}(ZZYZYZ) \leftrightarrow |\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\rangle$$

space of these single trace operators as periodic spin chains

⊛ At 1-loop, anomalous dim. from Heisenberg spin chain [Minahan, Zarembo 02]

$$D_1 \sim \frac{\lambda}{8\pi^2}\sum_l (1 - P_{l,l+1})$$

$P_{l,l+1}$ permutes spins of sites l^{th} and $l+1^{th}$

Dismissing boundary conditions, 1 magnon eigenfunction:

$$|p\rangle = \sum_n e^{ipn} |\overbrace{\uparrow\uparrow\ldots\uparrow}^{n-1}\downarrow\uparrow\uparrow\ldots\uparrow\rangle \qquad \text{with} \qquad E(p) = \frac{\lambda}{2\pi^2}\sin^2\left(\frac{p}{2}\right)$$

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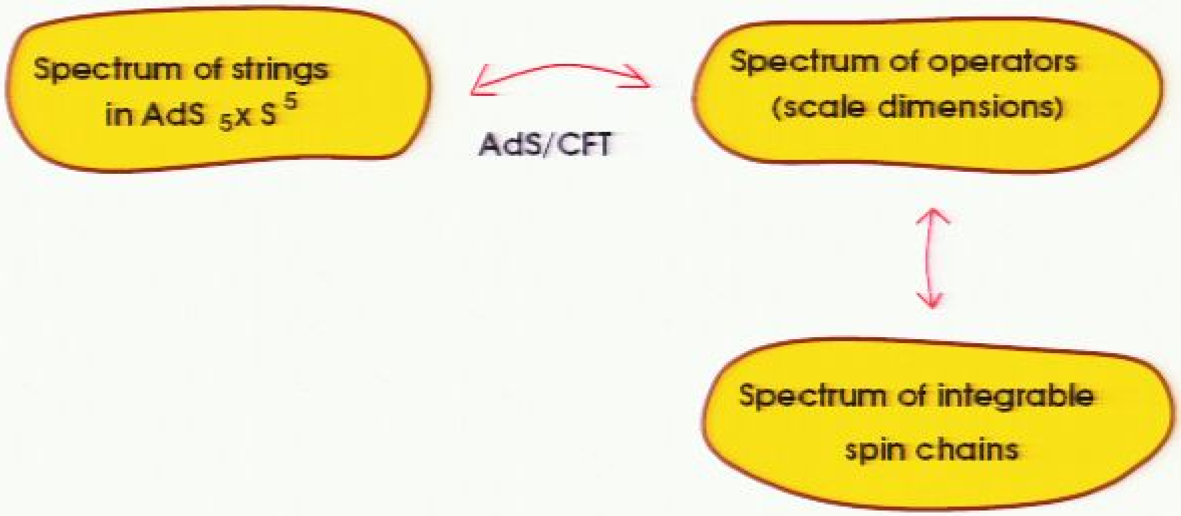

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Bethe ansatz for 2 magnons:

$$|p_1, p_2\rangle = \sum_{n_1<n_2} \left(e^{ip_1n_1+ip_2n_2} \qquad\qquad \right)|\overbrace{\uparrow\uparrow\ldots\uparrow}^{n_1-1}\downarrow\overbrace{\uparrow\uparrow\ldots\uparrow}^{n_2-n_1-1}\downarrow\uparrow\uparrow\ldots\uparrow\rangle$$

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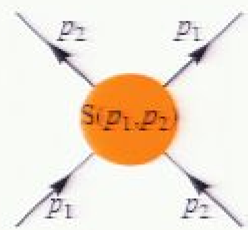
$$|p_1, p_2\rangle = \sum_{n_1 < n_2} \left(e^{ip_1 n_1 + ip_2 n_2} + S(p_1, p_2) e^{ip_2 n_1 + ip_1 n_2}\right) |\overbrace{\uparrow\uparrow \ldots \uparrow}^{n_1 - 1} \downarrow \overbrace{\uparrow\uparrow \ldots \uparrow}^{n_2 - n_1 - 1} \downarrow \uparrow\uparrow \ldots \uparrow\rangle$$

is eigenfunction with $E = E(p_1) + E(p_2)$ if

$$S(p_1, p_2) = \frac{\varphi(p_2) - \varphi(p_1) + i}{\varphi(p_2) - \varphi(p_1) - i} \qquad \text{with} \qquad \varphi(p_j) = \frac{1}{2} \cot\left(\frac{p_j}{2}\right)$$



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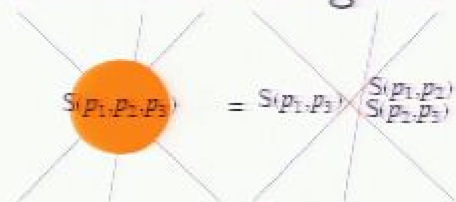

$$S(p_1, p_2, p_3) = S(p_1, p_2)S(p_1, p_3)S(p_2, p_3)$$

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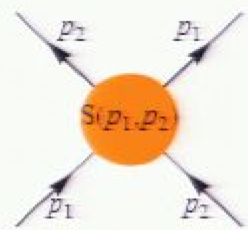

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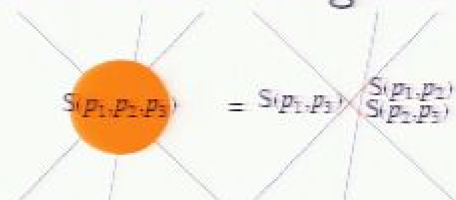

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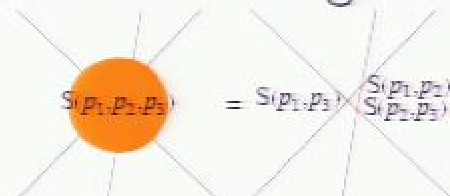

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Chain of finite size → periodic boundary conditions:

$$\Psi(n_1 + L, n_2, \ldots, n_M) = \Psi(n_1, n_2 + L, \ldots, n_M) = \cdots = \Psi(n_1, n_2, \ldots, n_M)$$

Magnons momenta are restricted to satisfy the Bethe equations. For M magnons

$$\exp(i p_j L) = \prod_{k \neq j}^{M} S(p_j, p_k) = \prod_{k \neq j}^{M} \frac{\varphi(p_k) - \varphi(p_j) + i}{\varphi(p_k) - \varphi(p_j) - i} \qquad \varphi(p_j) = \frac{1}{2} \cot\left(\frac{p_j}{2}\right)$$

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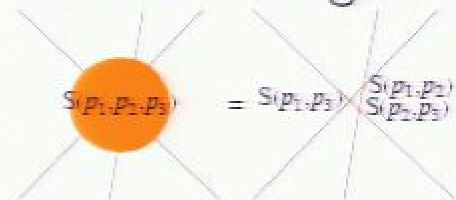

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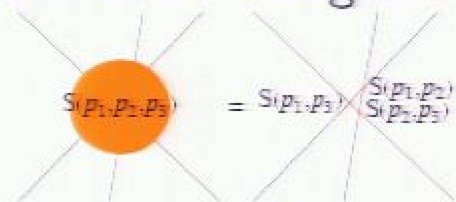

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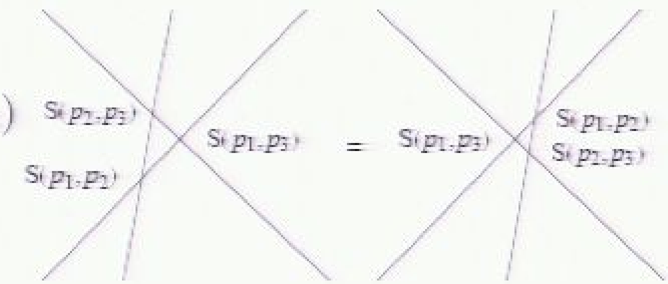
⊛ These ideas have to be generalized in two ways:

(i) More general operators than those of the $SU(2)$ sector. We can still think of magnon impurities in a vacuum of $ZZ\ldots ZZ$, but magnon impurities will carry a flavor. Thus, $S(p_j, p_k)$ are really scattering matrices. $S(p_1, p_2, p_3)$ would admit different factorizations. Integrability requires the Yang-Baxter equation

$$S(p_2,p_3)\cdot S(p_1,p_3)\cdot S(p_1,p_2) = S(p_1,p_2)\cdot S(p_1,p_3)\cdot S(p_2,p_3)$$



(ii) We want to extend the anomalous dimension calculation from 1-loop to higher-loop (eventually all-loop):

Spin chain with nearest neighbor int. $\rightarrow$ Spin chain with long range int.

Residual symmetry [Beisert 05]

However, we don't know the all-loop spin chain Hamiltonian...

How can we know about:

1. The exact dispersion relaction?
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⊛ Pick up a vacuum: tr$(ZZ\cdots Z)$ with symmetries $SU(2|2)^2 \subset PSU(2,2|4)$

- Impurities propagating in this reference state should accommodate into representations of $SU(2|2)^2$
- For instance, the 8 bosons $(X, \bar{X}, Y, \bar{Y}, D_\mu Z)$ + 8 fermions transform in the fundamental representation of $SU(2|2)^2$

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⊛ Take a single $SU(2|2)$. Even part, $SU(2)\times SU(2)$, generated by $\mathfrak{R}^{\mathfrak{a}}_{\mathfrak{b}}$ and $\mathfrak{L}^{\alpha}_{\beta}$. Odd part is generated by fermionic Q^{α}_{b} and S^{β}_{a}. The central extension is related to the anomalous dimension we are interested in, $C=\frac{1}{2}(\Delta-J)$

For example: $$\{Q^{\alpha}_{b},S^{\beta}_{a}\}=\delta^{a}_{b}\mathfrak{L}^{\alpha}_{\beta}+\delta^{\alpha}_{\beta}\mathfrak{R}^{\mathfrak{a}}_{\mathfrak{b}}+\delta^{\alpha}_{\beta}\delta^{a}_{b}C$$

$SU(2|2)$ would in principle admit two additional central charges

$$\{Q^\alpha_a, Q^\beta_b\} = \epsilon^{\alpha\beta}\epsilon_{ab}P \qquad \{S^a_\alpha, S^b_\beta\} = \epsilon_{\alpha\beta}\epsilon^{ab}K$$

On the other hand, magnons also carry momentum p (another quantum number)

Beisert: Momentum p should be related to the additional central extensions

$$P = g(e^{-ip} - 1) \qquad K = g(e^{ip} - 1)$$

Elementary magnons ⟶ fundamental representation of $SU(2|2)$ which is $2|2$ dimensional: $(\phi^a|\psi^\alpha)$

$$\mathfrak{R}^a{}_b|\phi^c\rangle = \delta^c_b|\phi^a\rangle - \frac{1}{2}\delta^a_b|\phi^c\rangle \qquad \mathfrak{L}^\alpha{}_\beta|\psi^\gamma\rangle = \delta^\gamma_\beta|\psi^\alpha\rangle - \frac{1}{2}\delta^\alpha_\beta|\psi^\gamma\rangle$$

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The coefficients (a, b, c, d) satisfy: $ad - bc = 1$

Either (a, b, c, d) or (C, P, K) label fundamental rep.

$$C = \frac{1}{2}(ad + bc) \qquad P = ab \qquad K = cd$$

$$ad - bc = 1 \quad \text{implies} \quad C^2 - PK = \tfrac{1}{2}$$

which gives the exact dispersion relation

$$\Delta - J = 2C = \pm\sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)} \qquad 16g^2 = \frac{\lambda}{\pi^2}$$

Aside: By a totally different approach, using a matrix model, the exact dispersion relation can be obtained including the factor $\frac{\lambda}{\pi^2}$ [Berenstein, Correa, Vázquez]

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$$S^a{}_\alpha|\phi^b\rangle = c\ \epsilon^{ab}\epsilon_{\alpha\beta}|\psi^\beta\rangle \qquad S^a{}_\alpha|\psi^\beta\rangle = d\ \delta^\beta_\alpha|\phi^a\rangle$$

The coefficients (a, b, c, d) satisfy: $\qquad ad - bc = 1$

Either (a, b, c, d) or (C, P, K) label fundamental rep.

$$C = \frac{1}{2}(ad + bc) \qquad P = ab \qquad K = cd$$

$$ad - bc = 1 \quad \text{implies} \quad C^2 - PK = \tfrac{1}{2}$$

which gives the exact dispersion relation

$$\Delta - J = 2C = \pm\sqrt{1 + 16g^2 \sin^2\left(\frac{p}{2}\right)} \qquad 16g^2 = \frac{\lambda}{\pi^2}$$

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⊛ The S-matrix is $16^2 \times 16^2$. Demanding it commutes with the generators of the $psu(2|2)^2 \times \mathbb{R}^3$ algebra, it is completely fixed up to a phase

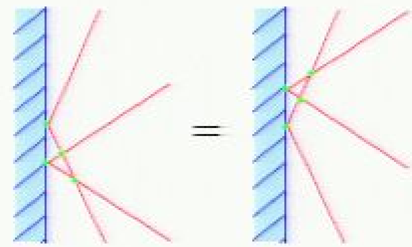
- The resulting S-matrix satisfies the Yang-Baxter equation
- Crossing symmetry constrains the phase [Janik],[Beisert, Hernandez, Lopez]

Extend the analysis for open strings/open chains

- Semi-infinite chain. To obtain eigenfunctions we also need a reflection matrix

$$|p\rangle = \sum_n \left(e^{ipn} + \mathcal{R}(p) e^{-ipn} \right) |Z^n \chi ZZZ \cdots \rangle$$

- Integrability of boundary condition requires boundary Yang-Baxter



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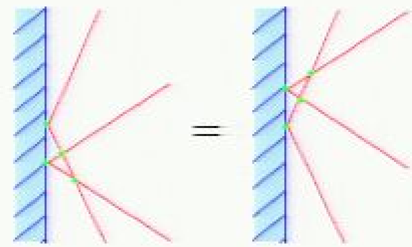

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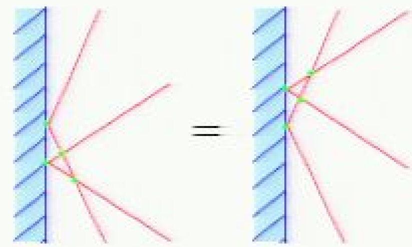
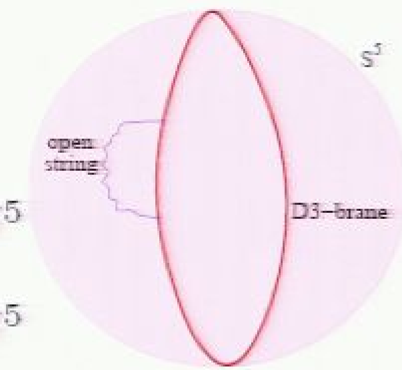


- D-branes set open boundary conditions for strings

Interesting cases in $AdS_5 \times S^5$

- D3-brane, whose wordvolume is $R \times S^3 \subset AdS_5 \times S^5$
- D5-brane, whose wordvolume is $AdS_4 \times S^2 \subset AdS_5 \times S^5$
- D7-brane, whose wordvolume is $AdS_5 \times S^3 \subset AdS_5 \times S^5$



⊛ For D3-branes, CFT is still $\mathcal{N}=4$ SYM. When the D3-brane carries N units of ang. momentum, the dual operator is

$$\det(Z) = \epsilon^{j_1 \cdots j_N}_{i_1 \cdots i_N} Z^{i_1}_{j_1} \cdots Z^{i_N}_{j_N}$$

Adding an open string to it

$$\epsilon^{i_1 \cdots i_N}_{j_1 \cdots j_N} Z^{j_1}_{i_1} \cdots Z^{j_{N-1}}_{i_{N-1}} \overbrace{(YY \cdots YY)^{j_N}_{i_N}}^{open\ string}$$

⊛ For D5 or D7 branes, the gauge theory is $\mathcal{N}=4$ SYM plus hypermultiplets in the fundamental of the gauge group

$$\bar{\phi}_j (ZZ \cdots Z \chi Z \cdots ZZ)^j_i \phi^i$$

For D5 case, the matter fields live in a 3-dim defect

⊛ Knowing the exact reflection matrices is very desirable. It can be used to put into context weak and strong coupling known results on integrability:

	$\lambda \ll 1$	$\lambda \gg 1$
	1-loop integrability	Class. open string integrability
(max.) D3	✓ [Berenstein, Vázquez]	✓ [Mann, Vázquez]
D5	✓ [DeWolfe, Mann]	✗ [Mann, Vázquez]
D7	✓ [Erler, Mann]	✓ [Mann, Vázquez]

⊛ In all these D-branes an $S^d \subset S^5$ is specified vacuum state carries angular momentum in the S^5 different relative orientations between the D-brane and vacuum ang. momentum will matter.

Residual symmetries preserved by reflections will be different:

$$\epsilon^{i_1 \cdots i_N}_{j_1 \cdots j_N} Z^{j_1}_{i_1} \cdots Z^{j_{N-1}}_{i_{N-1}} (YY \cdots YY)^{j_N}_{i_N} \qquad \text{symmetry is } SU(1|2)^2$$

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Different cases will also present different boundary degrees of freedom.

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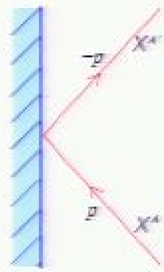

- **Reflection matrix:** For fundamental magnons

$$\mathcal{R}_L(p) : \; (\phi^a|\psi^\alpha) \;\rightarrow\; (\phi^a|\psi^\alpha)$$

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$$[\mathcal{R}_L(p), \mathfrak{J}] = [\mathcal{R}_L(p), \mathfrak{L}^\alpha_\beta] = [\mathcal{R}_L(p), \mathfrak{Q}^\alpha] = [\mathcal{R}_L(p), \mathfrak{S}_\alpha] = 0$$

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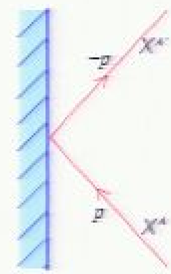

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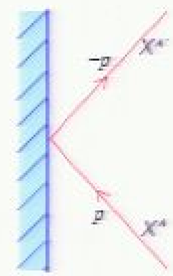

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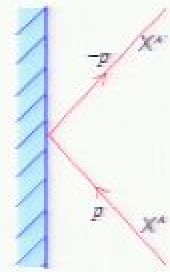

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Fixes the exact reflection matrix up to a phase (which will be also a function of λ)

$$\mathcal{R}_L(p) = \mathcal{R}_{0L}(p) \begin{pmatrix} -e^{ip} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[Hofman, Maldacena]

- Boundary Yang-Baxter is obeyed

➢ Cases D3 and D7 are expected to be integrable to all order in λ

D5 case with some detail

Matter fields live in a 3d defect: $SO(1,3) \to SO(1,2)$

D5 specifies a max. $S^2 \subset S^5$: $SO(6) \to SO(3)_{123} \times SO(3)_{456}$

Boundary fields:

	$SO(3)_{123} \times SO(3)_{456}$	$SO(1,2)$	Δ
ϕ^a	$[0, \frac{1}{2}]$	$[0]$	$\frac{1}{2}$
$\psi^{\alpha\dot{a}}$	$[\frac{1}{2}, 0]$	$[\frac{1}{2}]$	1

$\mathcal{N}=4$ SYM field ↗ bound. field ↗

$\bar{\phi}_j(ZZ\cdots Z\chi Z\cdots ZZ)^j_i\phi^i$

accommodate in rep of the residual symmetry:

$SU(2|2)_D \subset SU(2|2) \times SU(2|2)$

They come with a definite $\Delta - J_Z$

(i) if $J_Z = J_{12}$ $(Z = \Phi^1 + i\Phi^2)$ ➜ $(\phi^a, \psi^{\alpha 1})$ have the lowest $\Delta - J_Z = \frac{1}{2}$ and form a fundamental of $SU(2|2)_D$

(ii) if $J_Z = J_{56}$ $(Z = \Phi^5 + i\Phi^6)$ ➜ ϕ^1 have the lowest $\Delta - J_Z = 0$ and is a singlet of $SU(2|2)_D$

Let's go through the case (ii), where boundaries are singlets of $SU(2|2)_D$

What about bulk magnons? They are in a $(\boxslash;\widetilde{\boxslash})$ of the bulk symmetry $SU(2|2)\times\widetilde{SU(2|2)}$. But, with respect to the diagonal $SU(2|2)_D$ they are in a $\boxslash\otimes\boxslash$, with with labels (a,b,c,d) and $(-a,b,c,-d)$. This correspond to central charges:

$$C_D=\sqrt{1+16g^2\sin(\tfrac{p}{2})^2}\,,\qquad P_D=0\,,\qquad K_D=0\,,$$

The reflection matrix is therefore a map

$$\mathcal{R}:\boxslash\otimes\boxslash\to\boxslash\otimes\boxslash$$

and is fixed by the bosonic symmetries to be of the form

$$\begin{aligned}
\mathcal{R}\,|\phi^a_p\times\tilde\phi^b_p\rangle &= A_R|\phi^{\{a}_{-p}\times\tilde\phi^{b\}}_{-p}\rangle+B_R|\phi^{[a}_{-p}\times\tilde\phi^{b]}_{-p}\rangle+\tfrac{1}{2}C_R\epsilon^{ab}\epsilon_{\tilde\alpha\tilde\beta}|\psi^{\tilde\alpha}_{-p}\times\tilde\psi^{\tilde\beta}_{-p}\rangle\\
\mathcal{R}\,|\psi^{\tilde\alpha}_p\times\tilde\psi^{\tilde\beta}_p\rangle &= D_R|\psi^{\{\tilde\alpha}_{-p}\times\tilde\psi^{\tilde\beta\}}_{-p}\rangle+E_R|\psi^{[\tilde\alpha}_{-p}\times\tilde\psi^{\tilde\beta]}_{-p}\rangle+\tfrac{1}{2}F_R\epsilon_{ab}\epsilon^{\tilde\alpha\tilde\beta}|\phi^a_{-p}\times\tilde\phi^b_{-p}\rangle\\
\mathcal{R}\,|\phi^a_p\times\tilde\psi^{\tilde\beta}_p\rangle &= G_R|\psi^{\tilde\beta}_{-p}\times\tilde\phi^a_{-p}\rangle+H_R|\phi^a_{-p}\times\tilde\psi^{\tilde\beta}_{-p}\rangle\\
\mathcal{R}\,|\psi^{\tilde\alpha}_p\times\tilde\phi^b_p\rangle &= K_R|\psi^{\tilde\alpha}_{-p}\times\tilde\phi^b_{-p}\rangle+L_R|\phi^b_{-p}\times\tilde\psi^{\tilde\alpha}_{-p}\rangle
\end{aligned}$$

Fermionic symmetries fix $A_R(p), \cdots, L_R(p)$ up to an overall factor

$$A_R = -R_0(p)\frac{x^-}{x^+} \qquad F_R = R_0(p)\frac{f(x^- + x^+)(x^- - x^+)^2}{\eta^2 x^+(1 + x^- x^+)}$$

$$B_R = R_0(p)\frac{x^-(x^- + (x^+)^3)}{(x^+)^2(1 + x^- x^+)} \qquad G_R = R_0(p)\frac{x^- + x^+}{2x^+}$$

$$C_R = -R_0(p)\frac{\eta^2(x^- + x^+)}{f x^+(1 + x^- x^+)} \qquad H_R = R_0(p)\frac{x^+ - x^-}{2x^+}$$

$$D_R = R_0(p) \qquad K_R = R_0(p)\frac{x^+ - x^-}{2x^+}$$

$$E_R = -R_0(p)\frac{x^+ + (x^-)^3}{x^-(1 + x^- x^+)} \qquad L_R = R_0(p)\frac{x^- + x^+}{2x^+}$$

There is a hidden dependence on the coupling in the spectral parameters

$$e^{ip} = \frac{x^+}{x^-} \qquad \text{constrained to} \qquad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}$$

⊛ Taking $g \to 0$ we recover the 1-loop reflection matrix of [DeWolfe, Mann]

⊛ Boundary Yang-Baxter equation: a direct computation shows that many matrix elements of the bYBE are non-vanishing. For instance,

$$\langle(\phi^1 \times \tilde{\phi}^2)_{-p_1}, (\phi^1 \times \tilde{\phi}^1)_{-p_2}|(\text{bYBE})|(\phi^1 \times \tilde{\phi}^2)_{p_1}, (\phi^1 \times \tilde{\phi}^1)_{p_2}\rangle =$$
$$\frac{x_2^-(x_1^+ - x_1^-)^2(x_1^+ + x_1^-)(x_2^+ - x_2^-)(x_2^+ + x_2^-)(x_1^+ - x_2^-)(x_1^- - x_2^-)(x_1^- + x_2^+)}{4x_1^+(x_2^+)^2(x_1^- - x_2^+)^2(x_1^+ + x_2^+)^2(1 + x_1^+ x_1^-)(1 - x_1^+ x_2^+)}$$

How is this consistent with the 1-loop integrability? All non-vanishing matrix element of the bYBE for scalar bulk excitations vanish in the weak coupling limit

$$\langle(\phi^1 \times \tilde{\phi}^2)_{-p_1}, (\phi^1 \times \tilde{\phi}^1)_{-p_2}|(\text{bYBE})|(\phi^1 \times \tilde{\phi}^2)_{p_1}, (\phi^1 \times \tilde{\phi}^1)_{p_2}\rangle =$$
$$-\frac{256 g^2 e^{-\frac{i}{2}(3p_1+p_2)}(1 - 2e^{ip_1} + e^{i(p_1+p_2)})\cos(\frac{p_1}{2})\sin^4(\frac{p_1}{2})\cos(\frac{p_2}{2})\sin^2(\frac{p_2}{2})\sin(\frac{p_1+p_2}{2})\sin(\frac{p_1-p_2}{2})}{(1 + e^{2ip_2}(5 - 4\cos(p_1)) + 2e^{ip_2}(\cos(p_1) - 2))^2} + \mathcal{O}(g^4)$$

Integrability in the D5 case is just an accident of the 1-loop order.

⊛ Summary of cases and results

	Residual symmetry	Bound. d.o.f	Bound. YBE	
D3 vacuum Z	$SU(2\|2)^2$	$(\boxslash;\boxslash)$	✓	[Hofman, Maldacena]
D3 vacuum Y	$SU(1\|2)^2$	$(\mathbf{1};\mathbf{1})$	✓	
D5 vacuum Z	$SU(2\|2)_D$	$\boxslash$	✗	[Correa, Young]
D5 vacuum Y	$SU(2\|2)_D$	$\mathbf{1}$	✗	
D7 vacuum Z	$SU(2)^2 \times SU(2\|2)$	$(\mathbf{1};\boxslash)$	✓	[Correa, Young]
D7 vacuum Y	$SU(1\|2)^2$	$(\mathbf{1};\mathbf{1})$	✓	

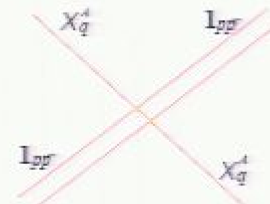
- In all cases, all-loop for the reflection matrices of magnons are obtained from symmetry arguments up to an overall factor.
- The overall factor is constrained by demanding crossing symmetry.

Crossing transformation: particle $\leftrightarrow$ anti-particle

$$(E, p) \quad \leftrightarrow \quad (-E, -p)$$

- **Singlet state:** Particle/anti-particle pair singlet of $SU(2|2)$

$$|\mathbf{1}_{p,\bar{p}}\rangle = -\frac{b(p)}{a(\bar{p})}\epsilon_{ab}|\phi^a_p \phi^b_{\bar{p}}\rangle + \epsilon_{\alpha\beta}|\psi^\alpha_p \psi^\beta_{\bar{p}}\rangle$$



It must scatter trivially with a third magnon

$$\mathcal{S}(\bar{p}, q) \cdot \mathcal{S}(p, q)|\mathbf{1}_{p,\bar{p}}\rangle \otimes |X_q\rangle = |X_q\rangle \otimes |\mathbf{1}_{p,\bar{p}}\rangle$$

This imposes a condition on the scattering phase σ [Janik]

$$\sigma(\bar{p}, q)\sigma(p, q) = \frac{y^-}{y^+}\frac{(x^- - y^+)(1 - 1/x^+y^+)}{(x^- - y^-)(1 - 1/x^+y^-)} \qquad e^{ip} = \frac{x^+}{x^-} \quad e^{iq} = \frac{y^+}{y^-}$$

Difficult to solve: double-crossing non-trivial $\quad \rightarrow \quad \sigma(\bar{\bar{p}}, q) \neq \sigma(p, q)$

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D3 vacuum Y	$SU(1\|2)^2$	$(\mathbf{1};\mathbf{1})$	✓	
D5 vacuum Z	$SU(2\|2)_D$	$\square$	✗	[Correa, Young]
D5 vacuum Y	$SU(2\|2)_D$	$\mathbf{1}$	✗	
D7 vacuum Z	$SU(2)^2 \times SU(2\|2)$	$(\mathbf{1};\square)$	✓	[Correa, Young]
D7 vacuum Y	$SU(1\|2)^2$	$(\mathbf{1};\mathbf{1})$	✓	

This imposes a condition on the scattering phase σ [Janik]
In all cases all-loop for the reflection matrices of magnons are obtained from

$$\sigma(\bar{p},q)\sigma(p,q) = \frac{y^-(x^- - y^+)(1 - 1/x^+y^+)}{y^+(x^- - y^-)(1 - 1/x^+y^-)} \qquad e^{ip} = \frac{x^+}{x^-} \quad e^{iq} = \frac{y^+}{y^-}$$

Difficult to solve: double-crossing non-trivial $\rightarrow$ $\sigma(\bar{\bar{p}},q) \neq \sigma(p,q)$

Crossing transformation: particle $\leftrightarrow$ anti-particle

$$(E, p) \quad \leftrightarrow \quad (-E, -p)$$

- **Singlet state:** Particle/anti-particle pair singlet of $SU(2|2)$

$$|\mathbf{1}_{p,\bar{p}}\rangle = -\frac{b(p)}{a(\bar{p})}\epsilon_{ab}|\phi_p^a \phi_{\bar{p}}^b\rangle + \epsilon_{\alpha\beta}|\psi_p^\alpha \psi_{\bar{p}}^\beta\rangle$$

It must scatter trivially with a third magnon

$$\mathcal{S}(\bar{p}, q) \cdot \mathcal{S}(p, q)|\mathbf{1}_{p,\bar{p}}\rangle \otimes |X_q\rangle = |X_q\rangle \otimes |\mathbf{1}_{p,\bar{p}}\rangle$$

This imposes a condition on the scattering phase σ [Janik]

$$\sigma(\bar{p}, q)\sigma(p, q) = \frac{y^-}{y^+}\frac{(x^- - y^+)(1 - 1/x^+y^+)}{(x^- - y^-)(1 - 1/x^+y^-)} \qquad e^{ip} = \frac{x^+}{x^-} \quad e^{iq} = \frac{y^+}{y^-}$$

Difficult to solve: double-crossing non-trivial $\rightarrow$ $\sigma(\bar{\bar{p}}, q) \neq \sigma(p, q)$

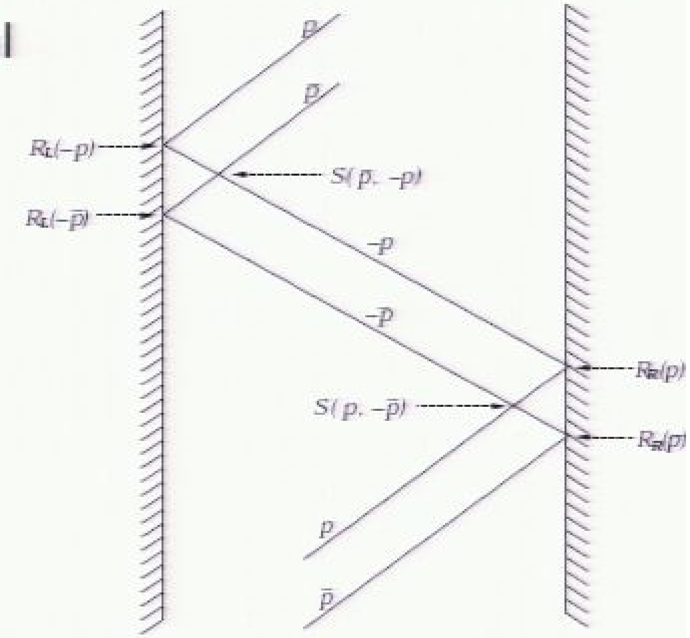
- Reflection of a singlet $|\mathbf{1}_{p,\bar{p}}\rangle$ must also be trivial

$$\mathcal{R}_R(p)\cdot\mathcal{S}(p,-\bar{p})\cdot\mathcal{R}_R(\bar{p})|\mathbf{1}_{p,\bar{p}}\rangle = |\mathbf{1}_{-\bar{p},-p}\rangle$$

This imposes a condition on the phase $\mathcal{R}_{0R}(p)$
(in the cases with $SU(1|2)^2$ symmetry)

$$\mathcal{R}^2_{0R}(p)\mathcal{R}^2_{0R}(\bar{p}) = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}}\sigma^2(p,-\bar{p})$$

$\mathcal{R}_{0R}(p)\mathcal{R}_{0R}(-p) = 1$ Unitarity constraint



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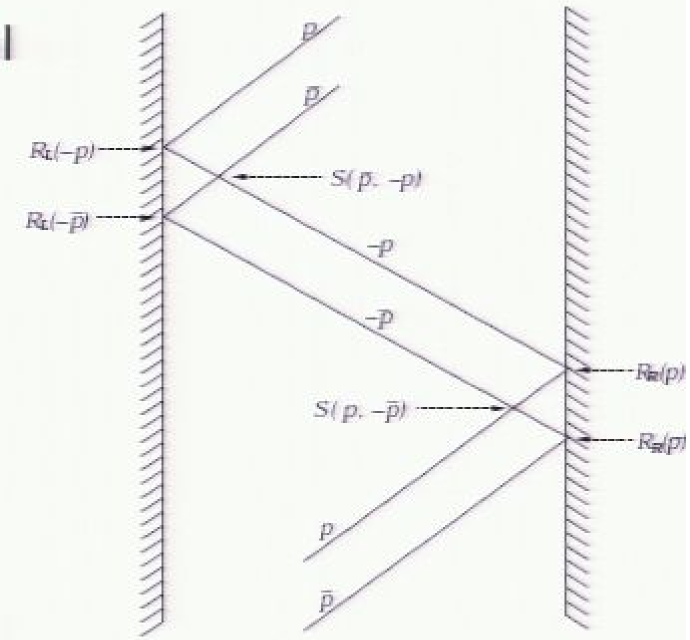

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⊛ In terms of the known bulk dressing factor $\sigma(p,q)$ it is easy to find a solution to this system, also matching the weak coupling factor [Chen,Correa]

$$\mathcal{R}_{0R}^2(p) = e^{2ip}\sigma(-p,p) \qquad \text{exact dressing phase}$$

Summary and conclusions so far

- Spectral problem in the planar limit of $\mathcal{N} = 4$ SYM, as the scattering of magnons in spin chains.
  - Eventually, all-loop spectrum from asymptotic Bethe ansatz for spin chain with long-range interactions.

- The all-loop scattering matrices for the Bethe ansatz known up to a phase by the symmetries of the problem.

- The overall phase is constrained by crossing symmetry

- Analogous asymptotic Bethe ansatz for open boundary conditions (different realizations of open strings ending on D-branes)

- The all-loop reflection matrices are also obtained. In some cases (D3 and D7) boundary Yang-Baxter equation is obeyed.

Is that it?

- The actual spectrum will come out from the Bethe equations; for chains of finite size.
- For periodic chains, wrapping effects are ignored by the asymptotic S-matrix

That is, when the range of interaction exceeds the length of the chain. In a chain of length L, wrapping effects turn up at order λ^L.

In the $SU(2)$ sector, the shortest non-BPS trace has $L = 4$:

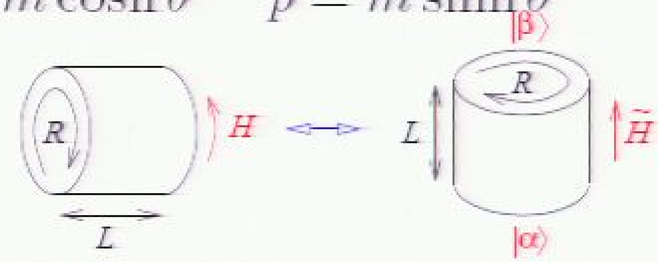
$$\text{tr}(ZZYY) + \#\text{tr}(ZYZY)$$

⇨ A four-loop computation is needed to calculate the failure of the asymptotic Bethe Ansatz. A really tough computation: [Fiamberti, Santambrogio, Sieg, Zanon]

- Finite size correction for open boundaries:
  - ❒ Even vacuum state would receive them
  - ❒ They might be relevant earlier in the weak coupling expansion (They would turn up at two-loop order in the simplest case)

⊛ Thermodynamic Bethe Ansatz (TBA) in relativistic integrable field theories

$E^2 = m^2 + p^2$ param. by rapidity θ : $E = m\cosh\theta$ $p = m\sinh\theta$



Double-wick rotation: $\theta \to \tilde{\theta} = \theta + i\frac{\pi}{2}$

In the "mirror" theory: $\tilde{E} = ip$ $\tilde{p} = iE$

The partition function of the theory in the cylinder can be seen in two ways:

$$Z(R, L) = \mathrm{Tr}\left(e^{-RH(L)}\right) = \langle\alpha|e^{-L\tilde{H}(R)}|\beta\rangle$$

For $R \to \infty$ but finite L: $e^{-RE_0(L)} \approx \langle\alpha|e^{-L\tilde{H}(R)}|\beta\rangle$

where the boundary states in the mirror theory (now in an infinity line) can be written as a superposition of asymptotic states

$$|\beta\rangle = \exp\left(\int_{\infty}^{\infty} d\theta \mathcal{R}(i\tfrac{\pi}{2} - \theta)A^\dagger(-\theta)A^\dagger(\theta)\right)|0\rangle$$ $A^\dagger$ creates an asymp. particle

The probability amplitude in the mirror and for $R \to \infty$ can be computed with saddle point approx. As a result,

$$E_0(L) = -m \int_{-\infty}^{\infty} \frac{d\theta}{4\pi} \cosh\theta \log\left(1 + \mathcal{R}(i\tfrac{\pi}{2} + \theta)\mathcal{R}(i\tfrac{\pi}{2} - \theta)e^{-\epsilon(\theta)}\right)$$

which has to be feeded with $\epsilon(\theta)$, solution of $\Phi(\theta) \equiv \frac{d \log \mathcal{S}(\theta)}{i d\theta}$

$$\epsilon(\theta) = 2mL\cosh\theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Phi(\theta - \theta') \log\left(1 + \mathcal{R}(i\tfrac{\pi}{2} + \theta')\mathcal{R}(i\tfrac{\pi}{2} - \theta')e^{-\epsilon(\theta')}\right)$$

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⊛ TBA can be extended to our non-relativistic scattering model. Done in [Janik,Bajnok] for periodic boundary conditions

⊛ For open boundary conditions already ground state receive finite size correction and this should be simpler to observe. In the simpler case the TBA result can be compared with a two-loop explicit calculation.

Notice:

- Dispersion relation non-relativistic, then it is different in the mirror theory:

$$\tilde{E} = 2\text{arcsinh}\left(\frac{\sqrt{1+\tilde{p}^2}}{4g}\right)$$

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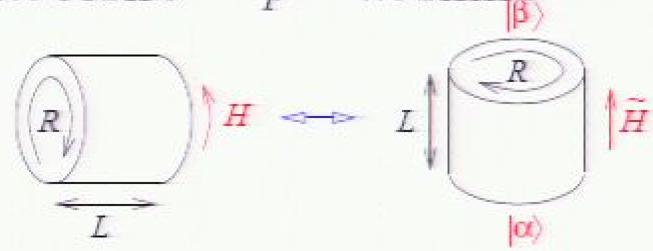

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