Title: No-Go Theorems for Self-Correcting Quantum Memory

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Abstract: We study the possibility of a self-correcting quantum memory based on stabilizer codes with geometrically-local stabilizer generators. We prove that the distance of such stabilizer codes in D dimensions is bounded by O(L^{D-1}) where L is the linear size of the D-dimensional lattice. In addition, we prove that in D=1 and D=2, the energy barrier separating different logical states is upper-bounded by a constant independent of L. This shows that in such systems there is no natural energy dissipation mechanism which prevents errors from accumulating. Our results are in contrast with the existence of a classical 2D self-correcting memory, the 2D Ising ferromagnet.

No-Go Results for a 2D Quantum Memory Based on Stabilizer Codes

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Bravyi & Terhal, arXiv:0810.1983

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Passive or Active Error Correction

- Active Error-Correction:
  - Local Interactions plus Non-Local Reliable Classical Processing.
  - System Dynamics are Non-Equilibrium

- Passive EC or Self-Correction:

  - Built from Geometrically Local Interactions
  - System Dynamics Towards Thermal Equilibrium
  - Hardware Solution For Fault-Tolerance

- Advantages of Passive EC or Self-Correction:

  - ➢ Fault-tolerant *hardware*, lesser need for classical control, overhead, concatenation...
  - ➢ No noise thresholds to beat... `Just' go to low enough temperature.
  - ➢ Question of fundamental interest.

- Disadvantages of Passive EC:
  - ➢ Need dynamical operations (gates), hence control for computation anyway...
  - ➢ Does not use the advantage of bootstrapping QC from reliable classical computing! (but measurements are slow)

Consider only Self-Correcting Quantum Memory:

- ➢ Need to write/read unknown states to the quantum memory fault-tolerantly.
- ➢ Is it even possible in 3D or fewer space dimensions?

Classical Self-Correction in 1D

Consider 1D Ising ferromagnet. Let ↑=1 and ↓=0.
Ground-state is twice-degenerate: 11…1 or 00..0.
Ground-states are code words for the repetition code.

Energy of single spin flip excitation 000100..00 is the same as that of multi-spin flip like 011111..10.

⟹1D Ising ferromagnet is not stable against thermal fluctuations. It has a phase-transition at zero temperature to a disordered phase. Entropy dominates over energy.

⟹ There is no self-correcting mechanism.

Classical Self-Correction in 2D

For the 2D Ising ferromagnet the energy of a domain of flipped spins scales with the boundary of the domain which depends on the size of the domain. Hence, high-weight errors are energetically disfavored.

Note: Ising model not stable against local magnetic fields.

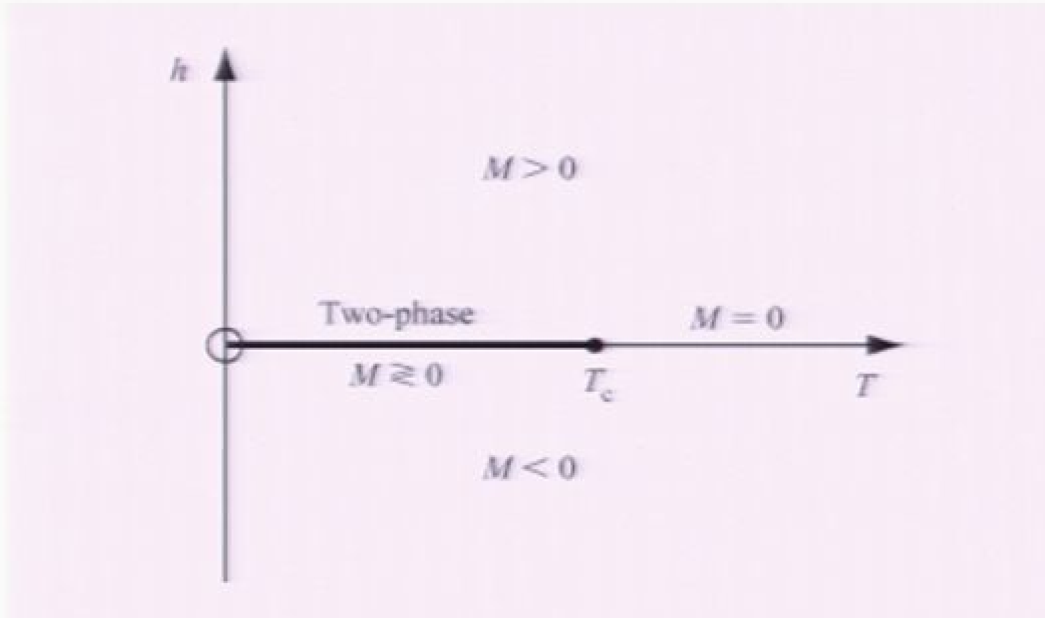
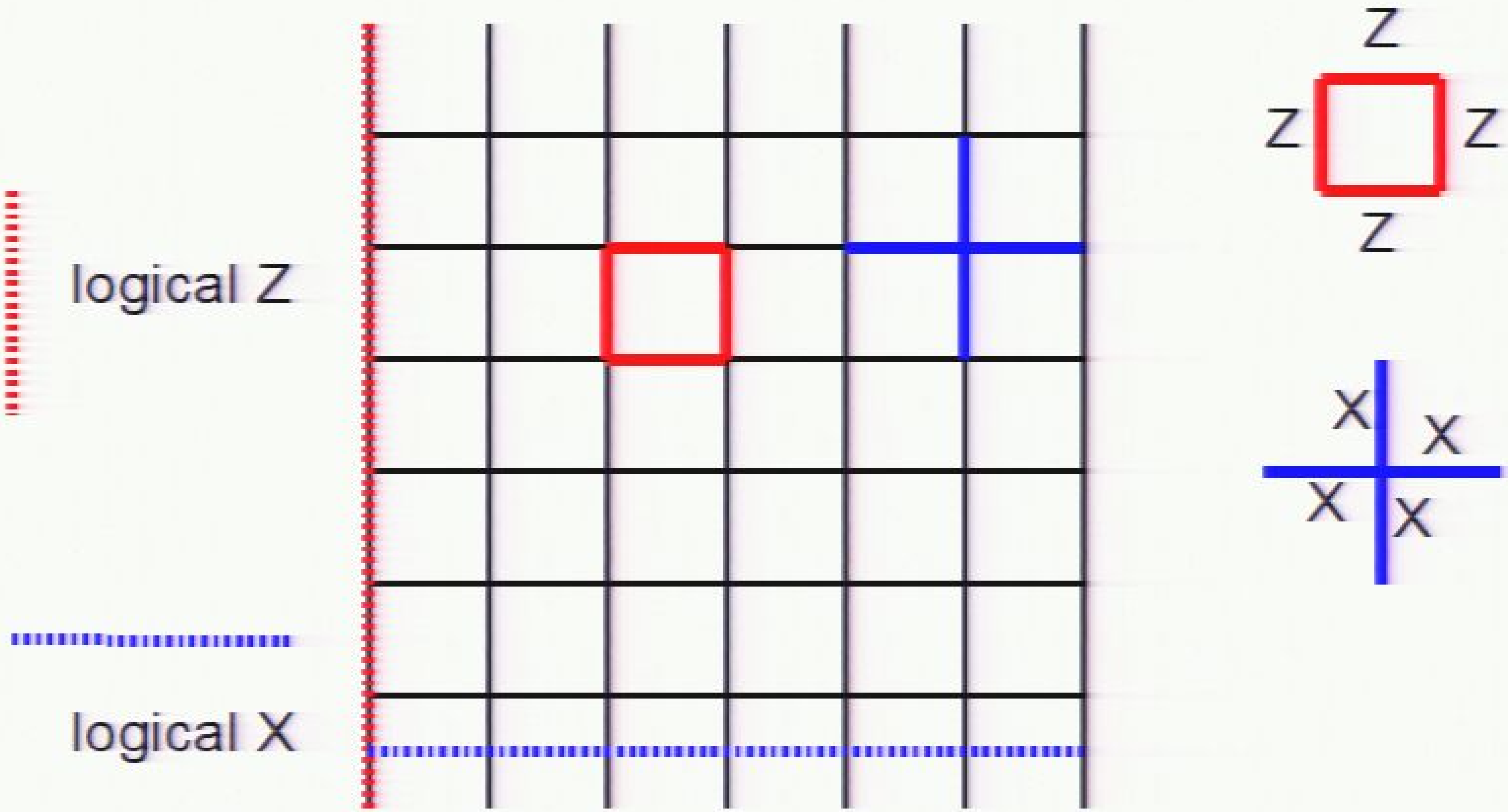


Figure 2

Phase diagram for the Ising model as a function of temperature (T) and magnetic field (h); dark portion of $h = 0$ axis represents two-phase region; T_c denotes critical point.

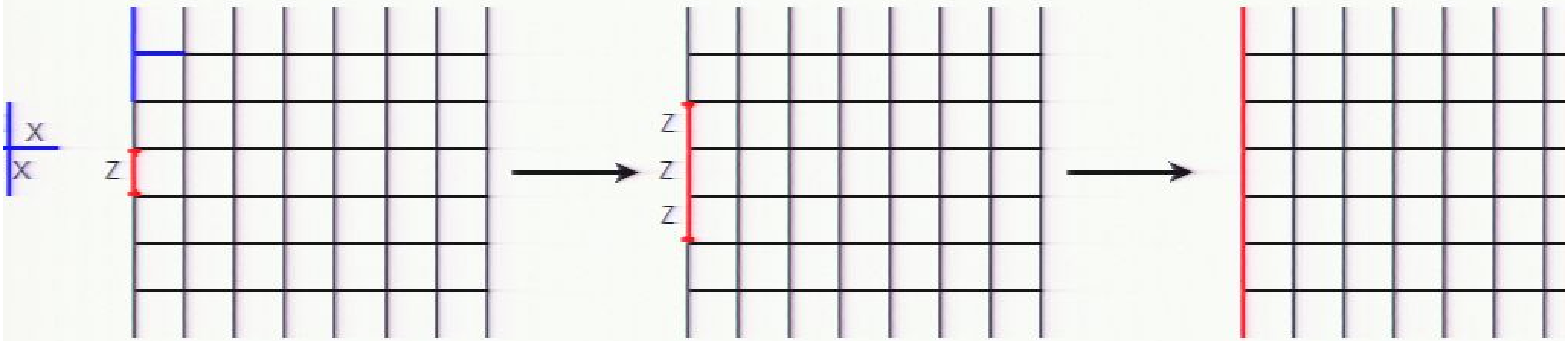
Example I: surface codes

n qubits on edges. Here n=85, L=7.



- $L \times L$ lattice. Number of qubits is $L^2 + (L-1)^2$.
- $\mathcal{N}(\mathcal{S})\backslash\mathcal{S} =< Z_{\rm vline}, X_{\rm hline} >$, one encoded qubit
- Distance is L.

Constant energy barrier for 2D surface codes



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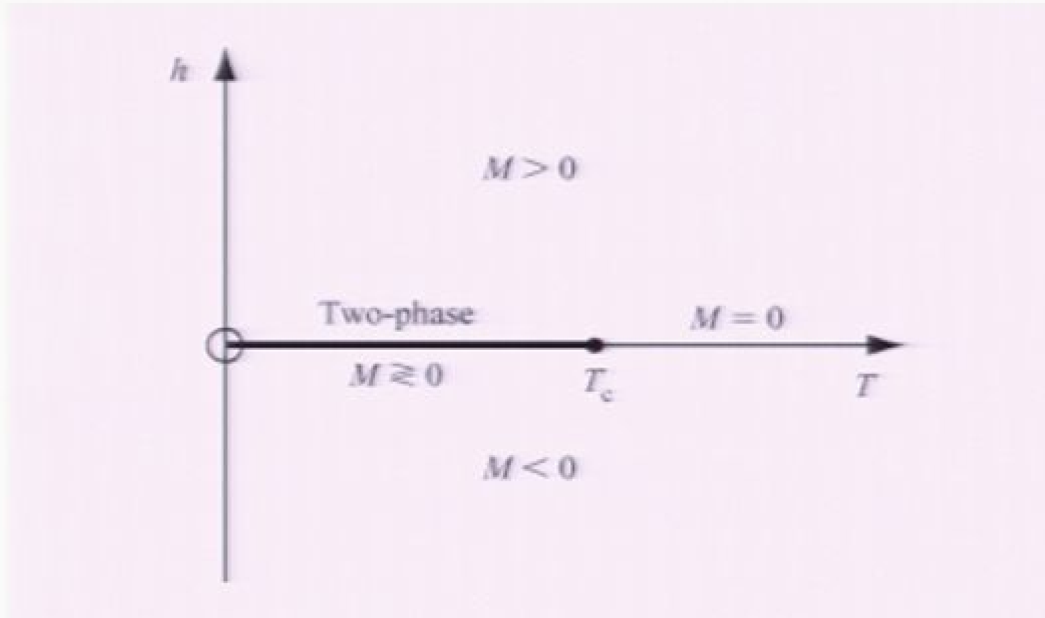

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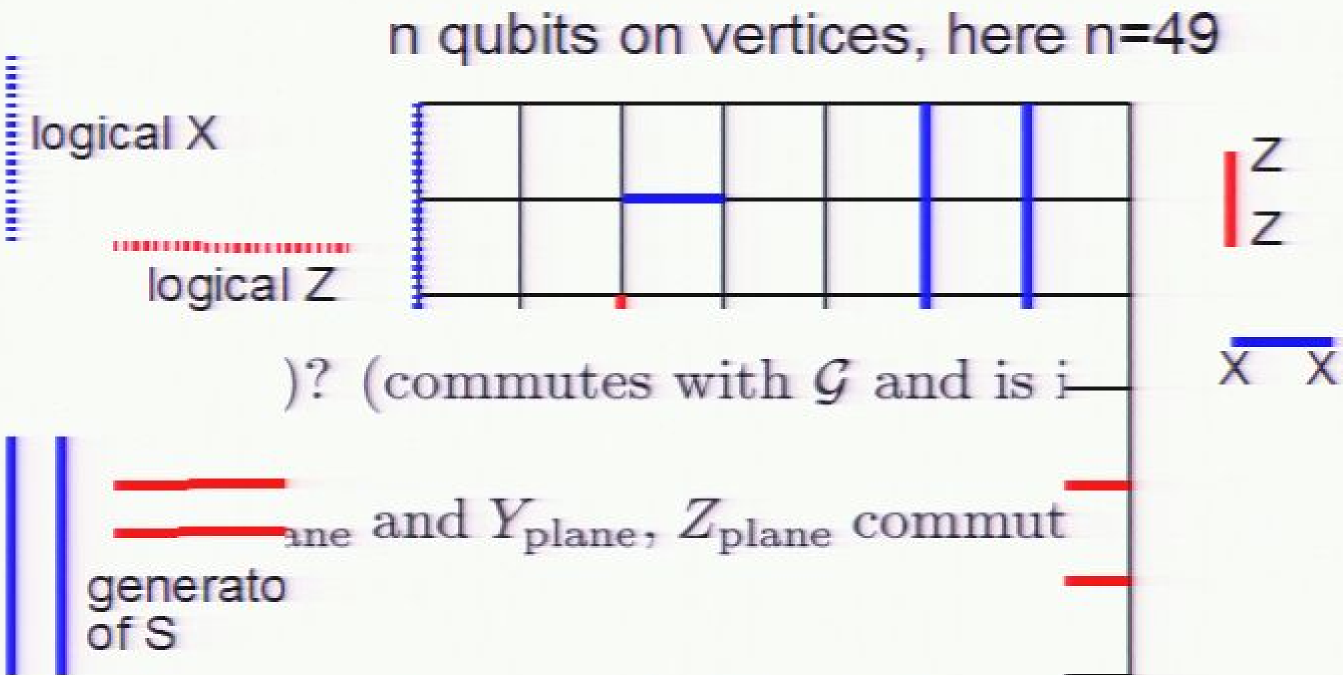

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- $\mathcal{N}(\mathcal{G})\backslash\mathcal{G} =< X_{\text{line}}, Z_{\text{line}} >$. Distance of code is L.
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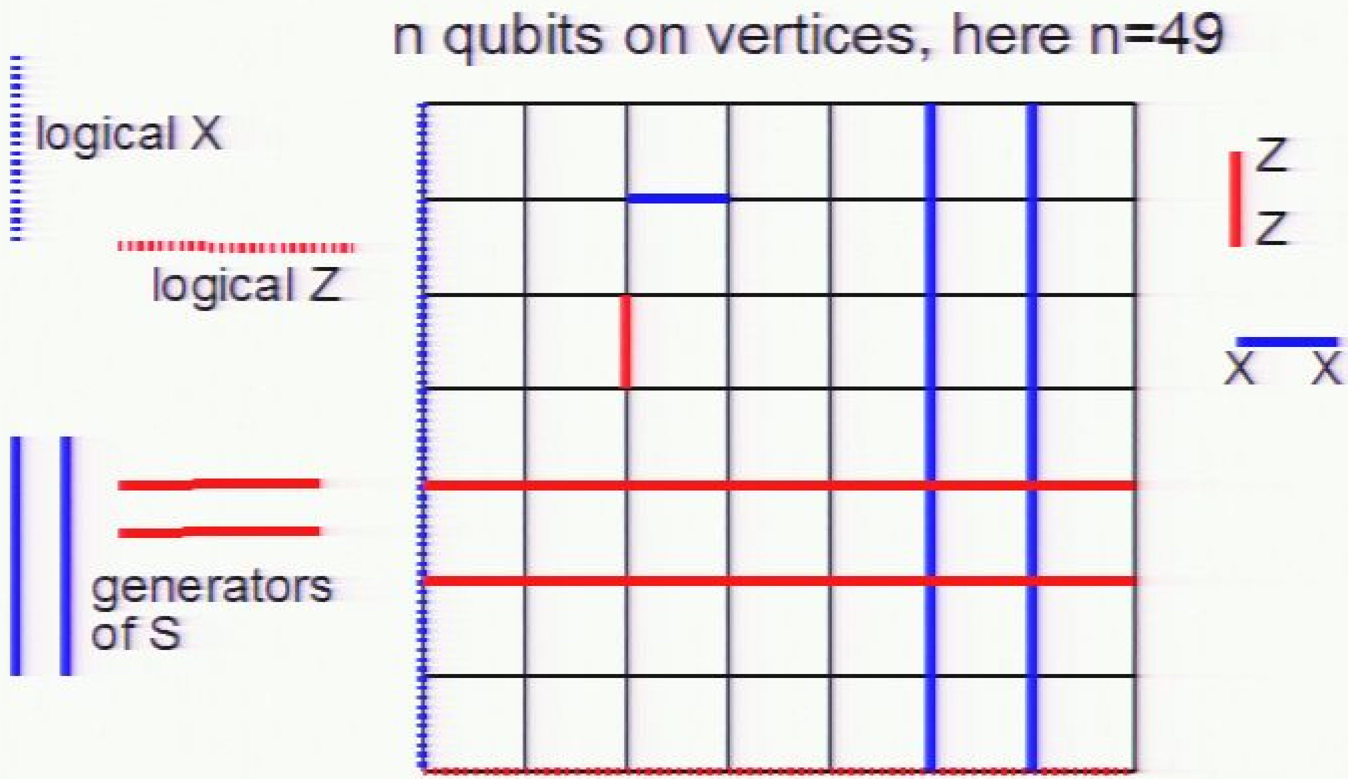

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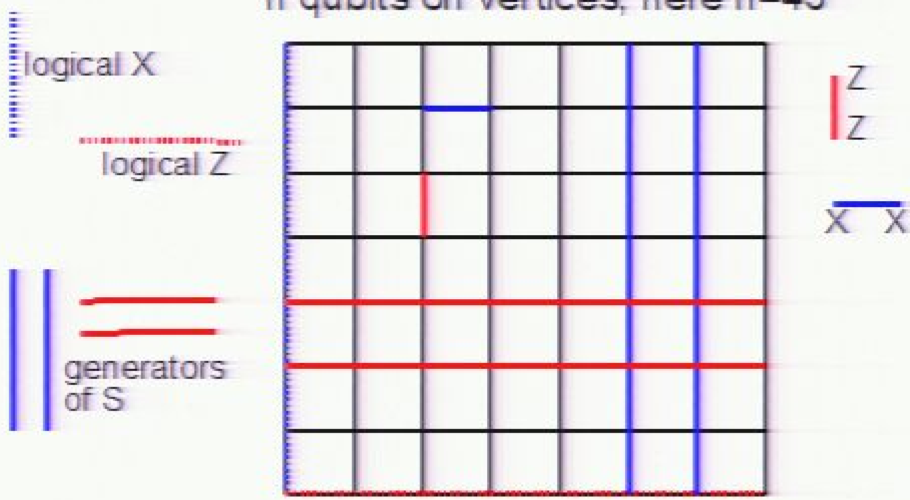
Example: 2D Bacon-Shor Codes

n qubits on vertices, here n=49



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Emerging Picture and Its Extension to Quantum Memory

- Ground-space of a Hamiltonian is a code-space of a quantum error-correcting code.
- Code has macroscopic distance, scaling with system size. (Topological Order)

Example: Ising ferromagnet as quantum stabilizer code.
$H=-\Sigma_{(i,j)\in L} Z_i Z_j$, Stabilizer $=< Z_i Z_j >$.
Logical X has distance n, the number of spins. Logical Z has distance 1 $\Longrightarrow$ distance is 1.

- In order to do a logical operator by a sequence of local errors one needs to traverse a macroscopic energy barrier (self-correction)

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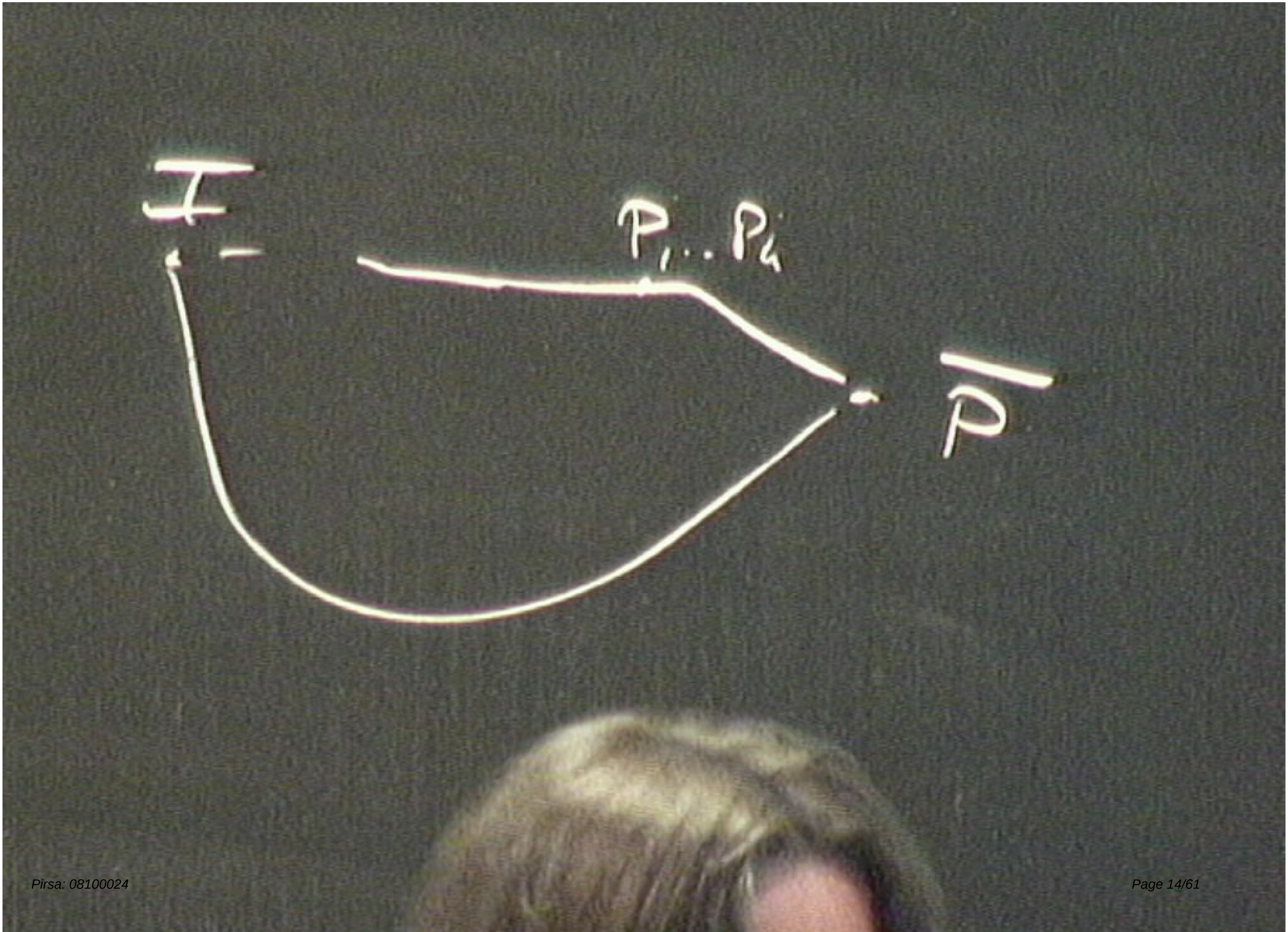
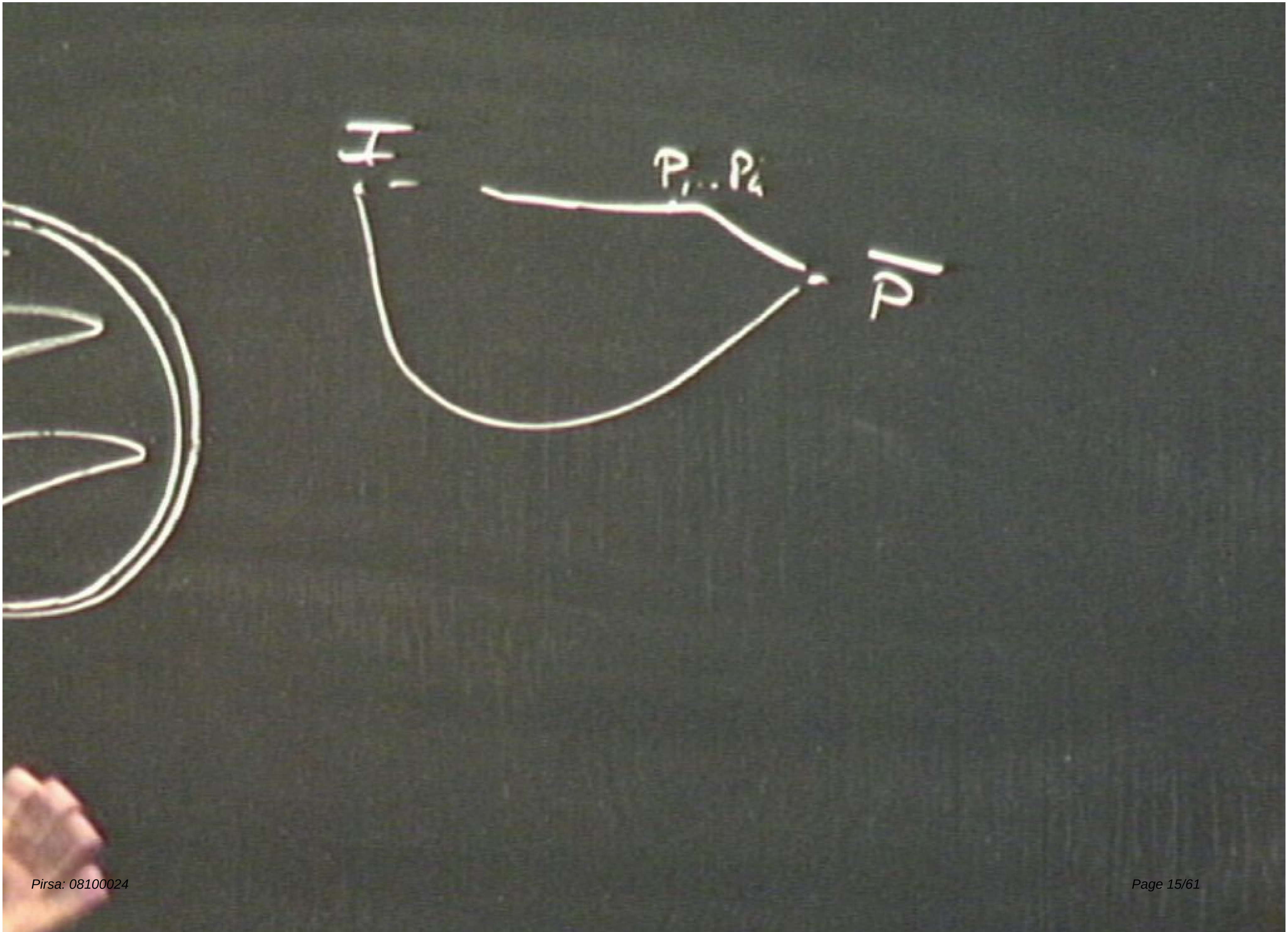
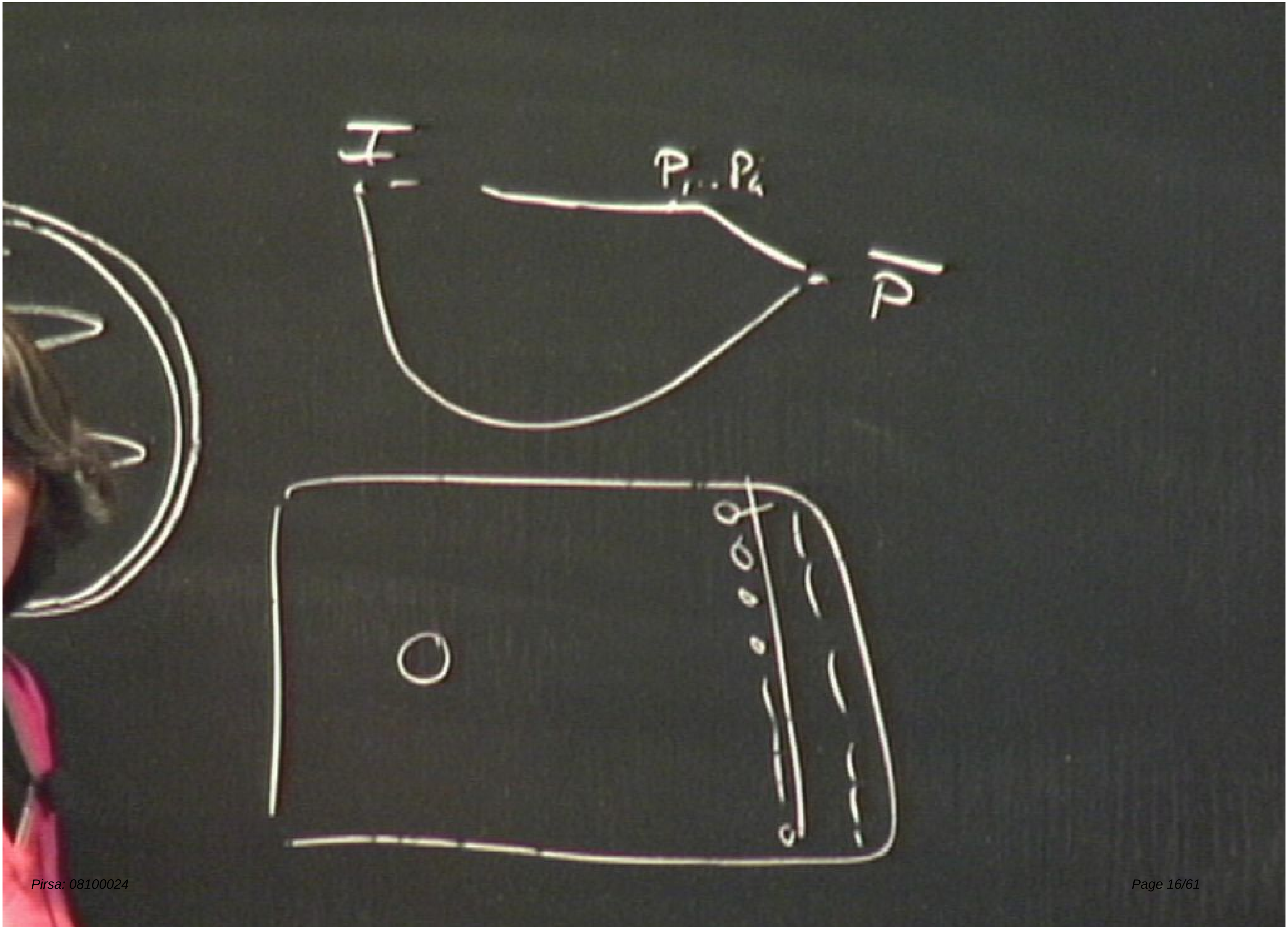
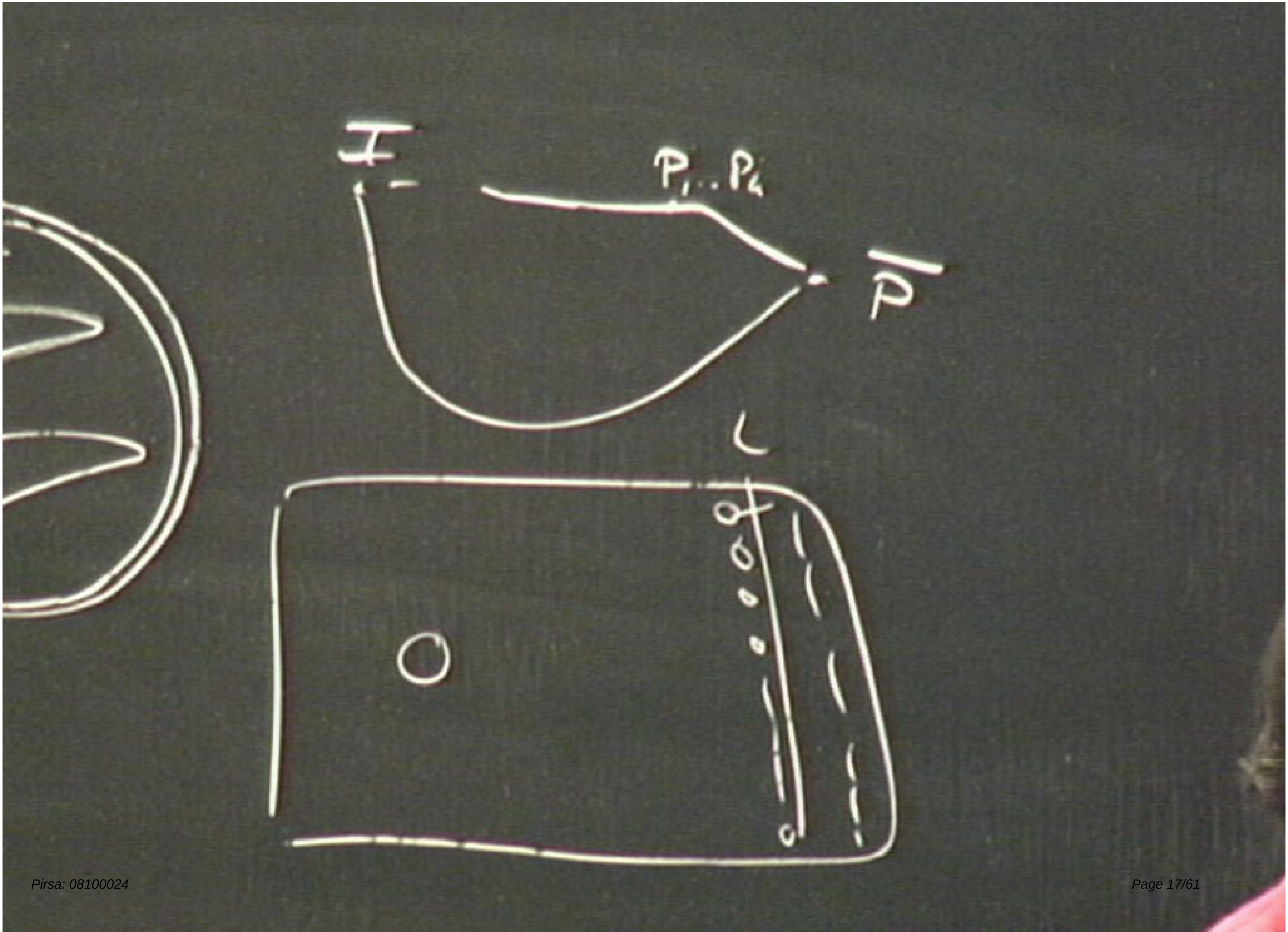
Example: domain wall in 2D Ising ferromagnet.


I_
P₁…Pₙ
P̄





$\mathcal{I}_-$
$P_1 \ldots P_n$
$\overline{P}$

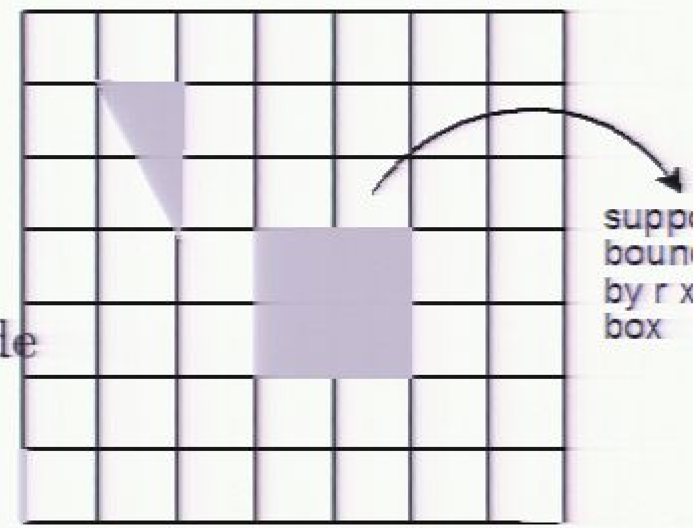


I
P1 .. P4
P
L

Quantum Memory Hamiltonians

Take stabilizer codes with geometrically-local generators.

Let $\mathcal{P}$ be the Pauli group on n qubits. Stabilizer $\mathcal{S}$ is an Abelian subgroup of $\mathcal{P}$, generated by geometrically local $\{S_i\}_{i=1}^{m}$.



Codespace is $\{|\psi\rangle \,|\, \forall\, i, S_i|\psi\rangle = |\psi\rangle\}$

Take $H = -\sum_i S_i$.

- Codespace is ground-space of H. H can encode one or several logical qubits.
- H is gapped. Ground-state energy is $-m$.
- The logical operators of the code are elements which commute with $\mathcal{S}$ but which are not in $\mathcal{S}$: $\mathcal{N}(\mathcal{S})\backslash\mathcal{S}$.

Necessary conditions for H to be a self-correcting memory:

- The distance of the code $\mathcal{S}$ scales with system size. The distance $d = \min_{P \in \mathcal{N}(\mathcal{S})\backslash\mathcal{S}} |P|$ where $|P|$ is the weight of Pauli operator P.

- the "energy barrier" scales with system size

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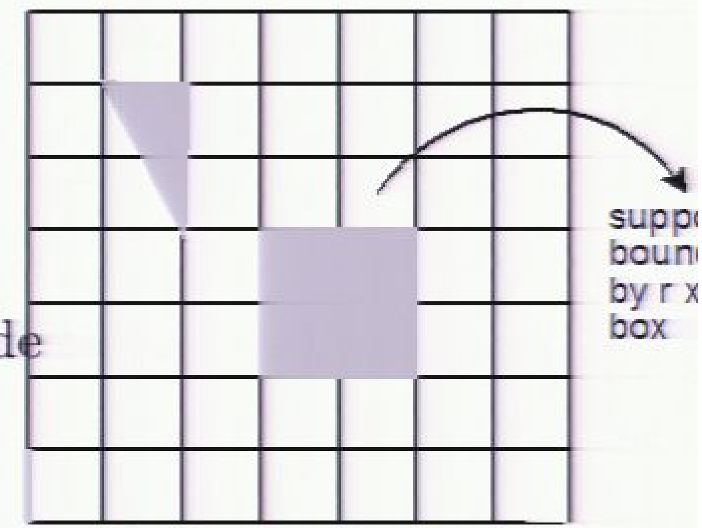
L x L lattice, qubits on vertices



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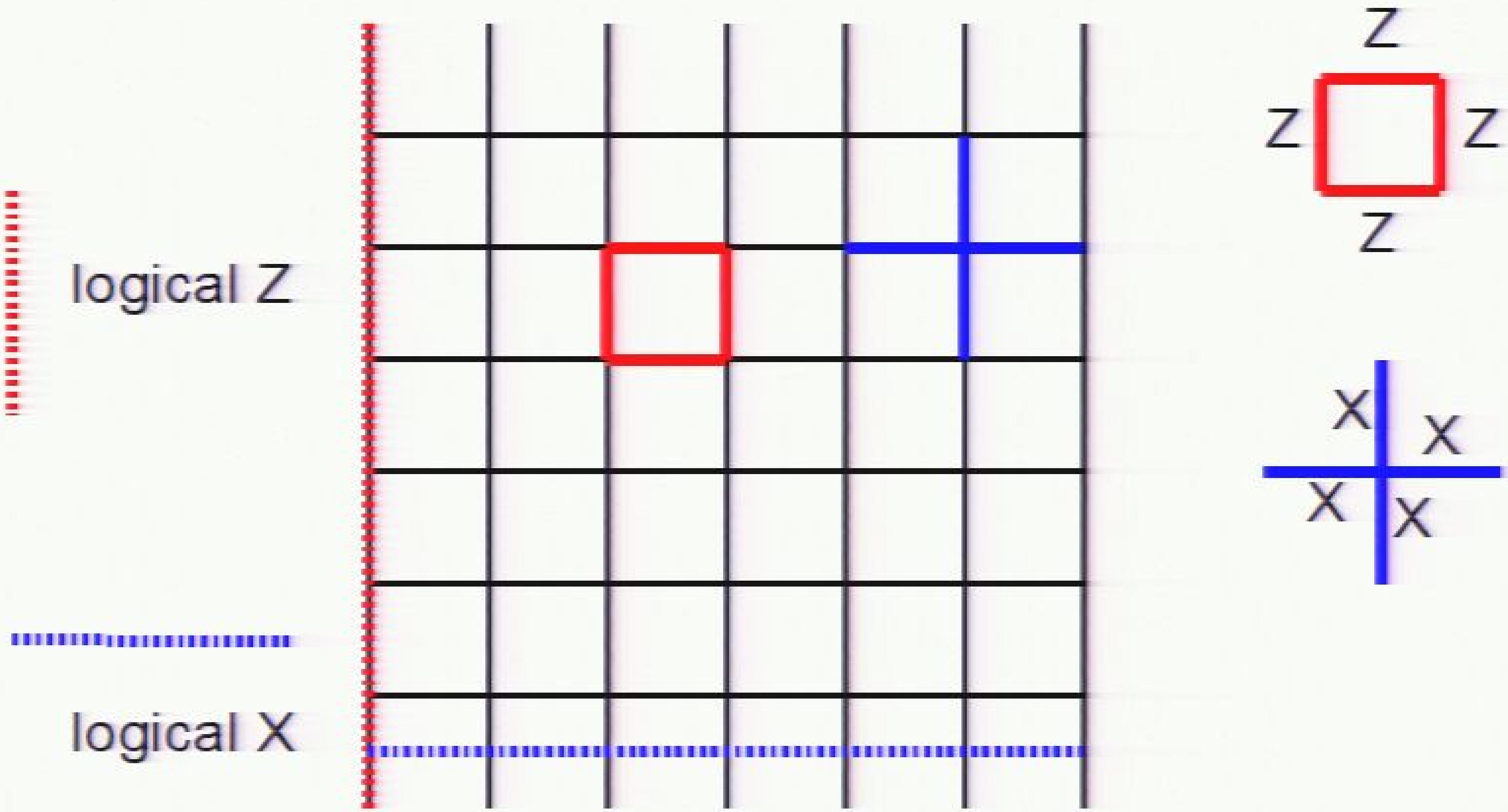

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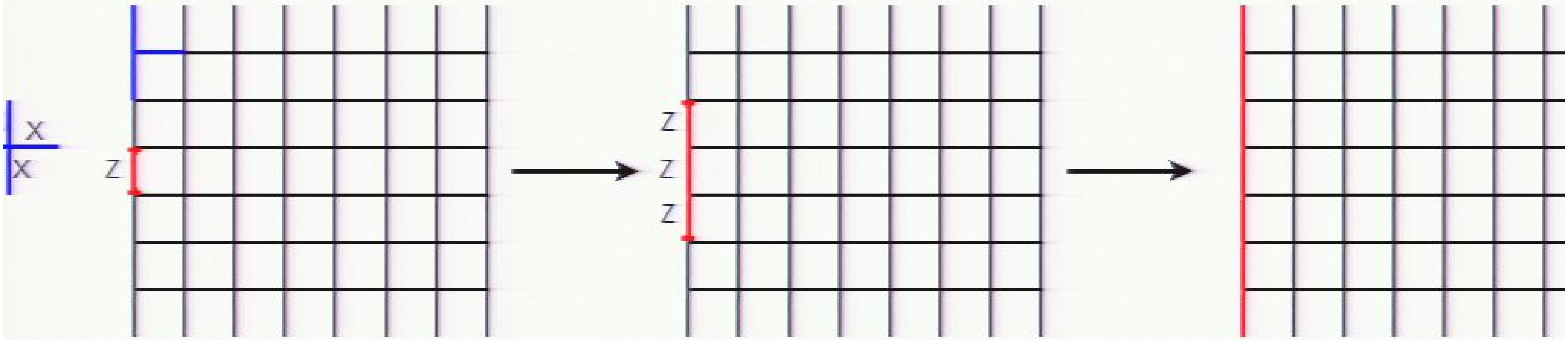

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Stabilizer Codes

Dennis *et al.* observed that 2D surface codes do not have the property of self-correction. They proposed a 4D surface code ($O(L^4)$ qubits) which has distance scaling with L^2 and $d^\ddagger$ scaling with L.

So...can we prove some general no-go results on the distance d and $d^\ddagger$? YES

Theorem I: Stabilizer codes on a D-dimensional lattice have distance $O(L^{D-1})$.

Theorem I$^+$: Pick a set of logical "system" qubits from a stabilizer code on a D-dimensional lattice and treat the other logical qubits as irrelevant gauge qubits. The distance for these system qubits is $O(L^{D-1})$.

Theorem II: The energy barrier for any stabilizer code defined on a 1 or 2-dimensional lattice is $O(1)$. No self-correction in 2D!

Conjecture: distance for stabilizer codes on a D-dimensional lattice is $O(L^{\lfloor D/2 \rfloor})$. Even 3D self-correcting stabilizer-based quantum memories may not exist.

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Energy Barrier, more formally

Consider system described by a stabilizer Hamiltonian H interacting with a bath. A complete set of energy eigenstates of the system is given by $\{P|\psi_0\rangle\}$ where $|\psi_0\rangle$ is any state in the code-space and P is a Pauli operator.

The energy cost of an operator P, $\epsilon(P)$ is determined by $\langle\psi_0|P^\dagger H P|\psi_0\rangle = -m + \epsilon(P)$.

Energy cost $\epsilon(P)$ equals (two times) the number of stabilizer elements that P anticommutes with.

Assume: environment makes single qubit errors. Creates a path from I to logical error $\overline{P}$.

Energy barrier $d^\ddagger$: least energy that environment needs to expend to implement any logical $\overline{P}$ by a sequence of single Pauli error steps.

In formula: $d^\ddagger = \min_{E\in\mathcal{N}(\mathcal{S})\backslash\mathcal{S}} \min_{\gamma\in\mathcal{W}(I,E)} \epsilon_{\max}(\gamma)$ where $\epsilon_{\max}(\gamma)$ is max energy cost along the path.

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Starting slide show...

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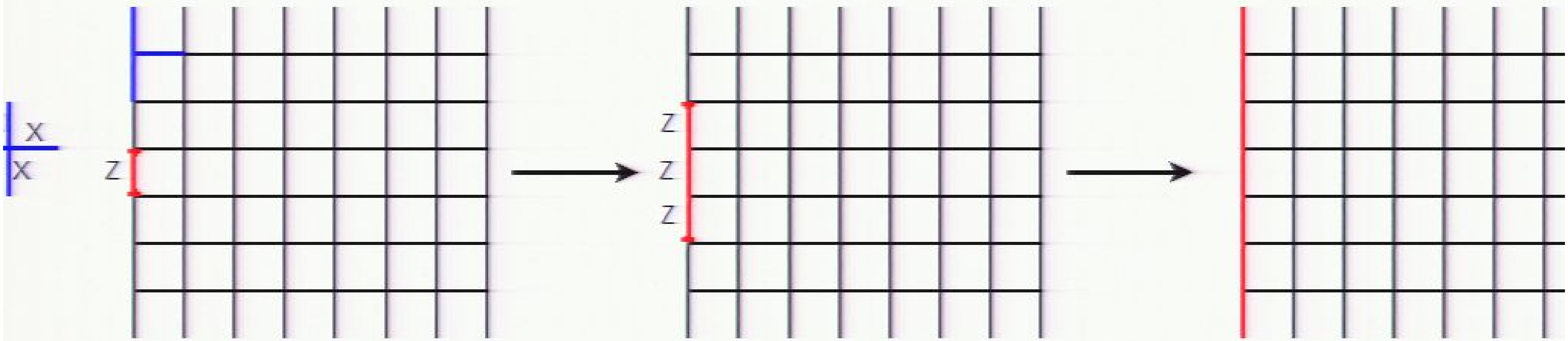

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Hamiltonians and Subsystem Codes

Hamiltonians which are sums of commuting operators seem a bit restrictive (advantage: there is a gap). Take $H = \sum_i J_i G_i$ with geometrically-local Pauli operators G_i and $O(1)$ couplings J_i.

Subsystem Codes:

- Let $\mathcal{G} =< G_1, \ldots, G_m >$ the 'gauge' group.
- Consider the center of $\mathcal{G}$: $\mathcal{S} = \mathcal{G} \cap \mathcal{N}(\mathcal{G})$. $\mathcal{S}$ is an Abelian subgroup of the Pauli group, a stabilizer!
- So $\mathcal{N}(\mathcal{S}) \backslash \mathcal{S}$ has logical operators of qubits encoded by $\mathcal{S}$. Some of these are the true 'logical' qubits and some other gauge qubits. What is the relation with $\mathcal{G}$?
- Now $\mathcal{N}(\mathcal{G}) \backslash \mathcal{G}$ has the logical operations on the logical qubits and $\mathcal{N}(\mathcal{S}) = \mathcal{N}(\mathcal{G}) \cdot \mathcal{G}$. $\mathcal{G}$ acts only on the gauge qubits. Distance $d = \min_{P \in \mathcal{N}(\mathcal{S}) \backslash \mathcal{G}} |P|$.
- Subsystem code formalism expresses symmetries of the Hamiltonian: k logical qubits $\leftrightarrow$ 2^k-degenerate eigenlevels. H block-diagonal with sectors labeled by syndromes of $\mathcal{S}$. In each sector we have a code-space for k-logical qubits and a spectrum of gauge-qubit excitations.

Example: Heisenberg Model

et $H = \sum_{(i,j)\in L} J_x X_i X_j + J_y Y_i Y_j + J_z Z_i Z_j$ where L is a 2-dimensional ttice of qubits, n qubits in total.

auge group $\mathcal{G} =< X_i X_j, Y_i Y_j, Z_i Z_j >$.

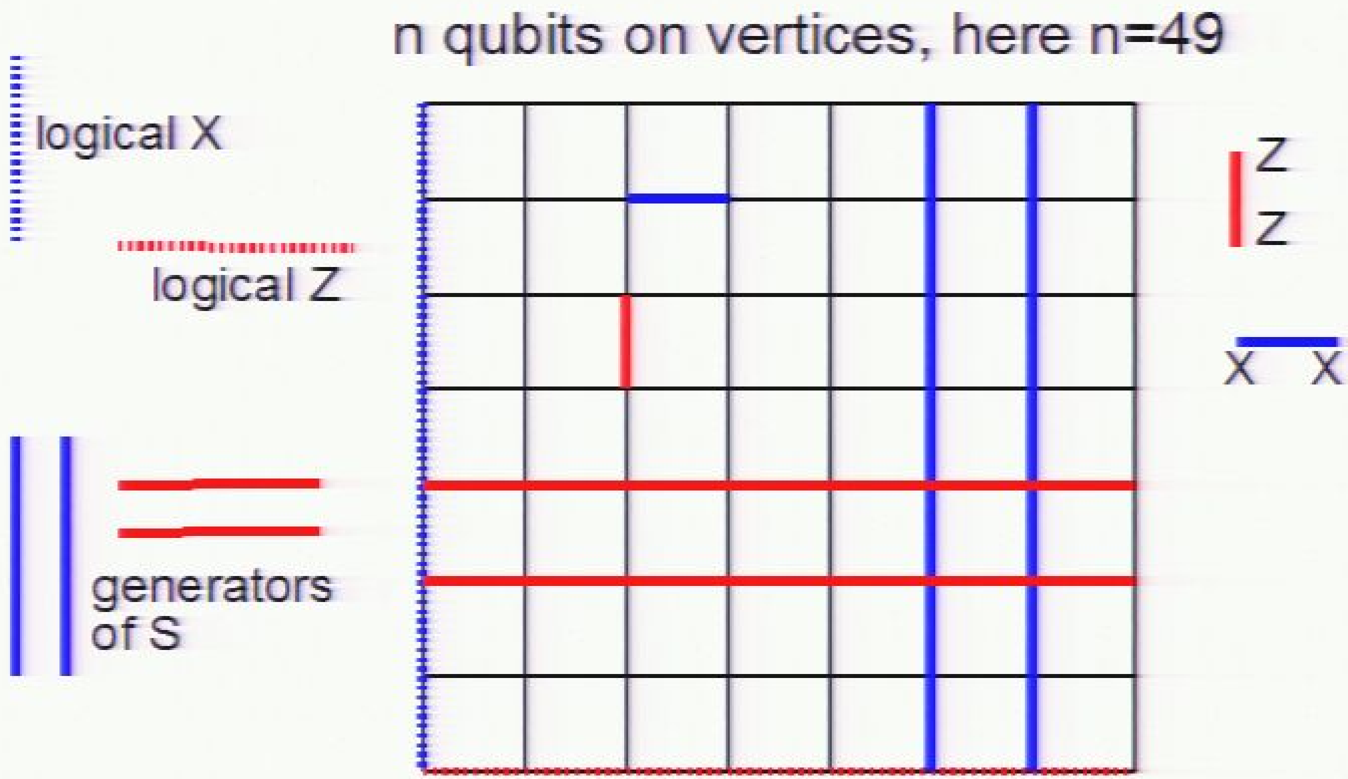
'hat is in $\mathcal{S} = G \cap \mathcal{N}(\mathcal{G})$? (commutes with $\mathcal{G}$ and is in $\mathcal{G}$)

nly plane operators X_{plane} and Y_{plane}, Z_{plane} commute with $\mathcal{G}$, are in $\mathcal{N}(\mathcal{G})$

n is even: the plane operators are in $\mathcal{G}$ and thus in $\mathcal{S}$. $\mathcal{N}(\mathcal{S}) = \mathcal{G}$. No logical qubits: $\mathcal{N}(\mathcal{G})\backslash\mathcal{G} = I$.

n is odd. Plane operators anti-commute $\Rightarrow$ cannot be in $\mathcal{S}$. $\mathcal{S} = I$ and so $\mathcal{N}(\mathcal{S})$ is the Pauli group. $\mathcal{N}(\mathcal{G})\backslash\mathcal{G} =< X_{\text{plane}}, Z_{\text{plane}} >$, one logical qubit. What is the distance of this code? $d = \min_{P \in \mathcal{N}(\mathcal{S})/\mathcal{G}} |P|$. Multiply X_{plane} by $X_i X_j$ to get a weight 1 operator, thus distance is 1 $\Rightarrow$ not good for a quantum memory.

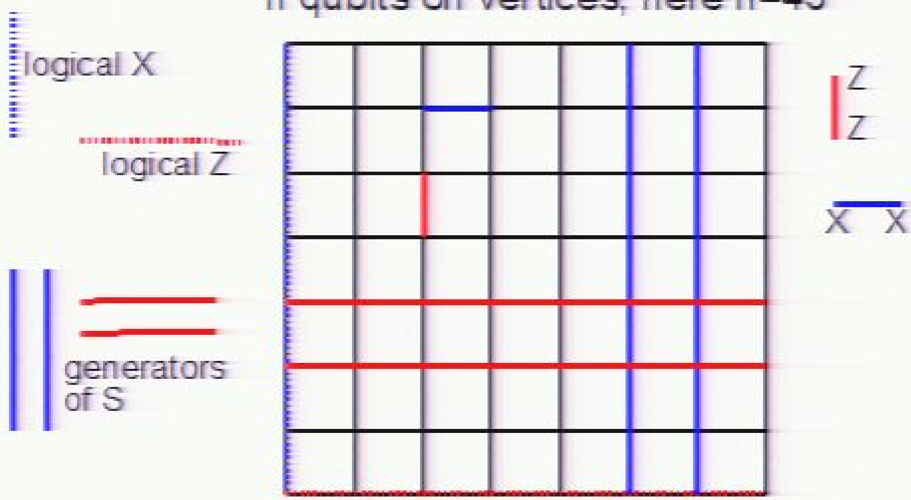
Example: 2D Bacon-Shor Codes



- $L \times L$ lattice. The gauge group is $\mathcal{G} =< X_{i,j}X_{i,j+1}, Z_{i,j}Z_{i+1,j} >$.
- $\mathcal{S}$ is generated by vertical and horizontal double line operators. Generators of $\mathcal{S}$ are nonlocal.
- $\mathcal{N}(\mathcal{G})\backslash\mathcal{G} =< X_{\text{line}}, Z_{\text{line}} >$. Distance of code is L.
- 3D model has XX, YY, ZZ gauge operators in x, y, z directions

Example: 2D Bacon-Shor Codes

n qubits on vertices, here n=49



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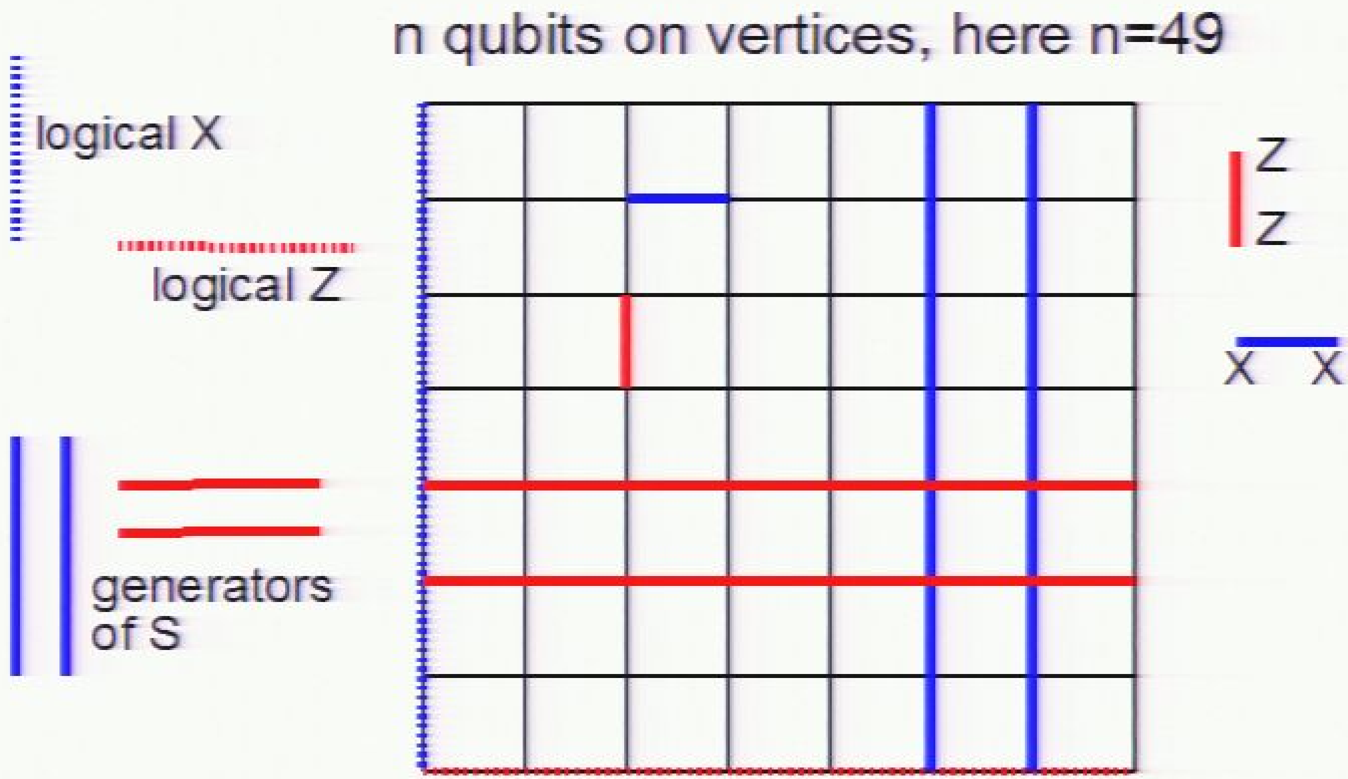

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Bounding the distance: stabilizer codes

Cleaning Lemma for Reducing Support of Logical Operators

Region M: $\mathcal{S}_M$ is restriction of $\mathcal{S}$ on M. $\mathcal{S}(M)$ contains only stabilizers wi support in M. Thus $\mathcal{S}(M) \subseteq \mathcal{S}_M$.

Consider what a logical operator $P \in \mathcal{N}(\mathcal{S})$ does on a region M, wi $|M| < d$. Call this P_M.

Imagine $P_M \in \mathcal{N}(\mathcal{S}_M)$, it commutes inside region M already. Then P_M $\mathcal{N}(\mathcal{S})$ but P_M cannot be in $\mathcal{N}(\mathcal{S})\backslash\mathcal{S}$ since $|M| < d$. Thus $P_M \in S(M)$. multiply with elements in $S(M)$ to clean out P inside M.

We argued that $\mathcal{N}(\mathcal{S}_M) \cap \mathcal{P}(M) \propto S(M)$ for regions $|M| < d$.
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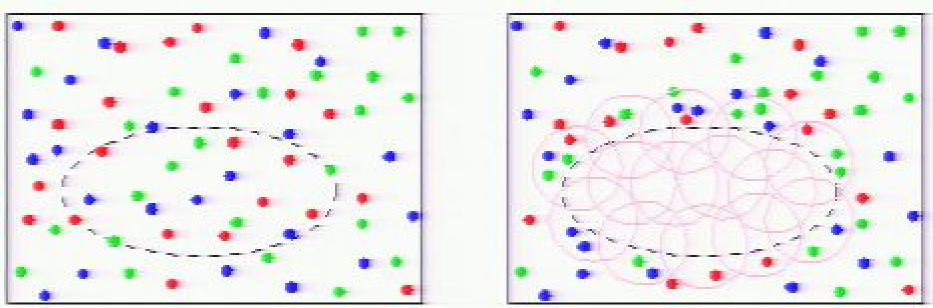

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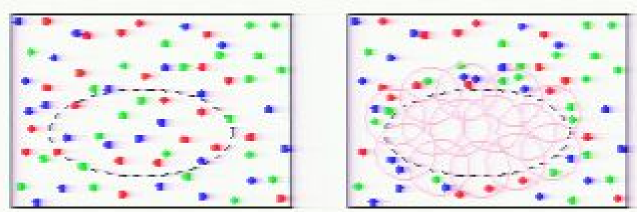

Hamiltonians and Subsystem Codes

Not all properties of H are determined by subsystem code formalism, e.g. are there gaps between different stabilizer-sectors?
Question of energy barrier scaling is more involved, depends on more details of spectrum of H.

However, it is possible to define the energy cost of Pauli error E and upper-bound it by twice the number of terms in $H = -\sum_i G_i$ with which E anti-commutes.

We can prove that 2D Bacon-Shor code is not a good quantum memory and that $d^{\ddagger} = O(d)$. First bound the distance...

Theorem III: Subsystem codes on a D-dimensional lattice have distance $O(L^{D-1})$.

10/22/2008

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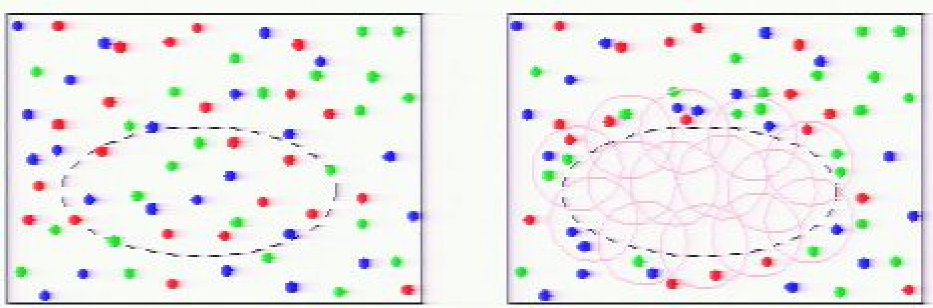

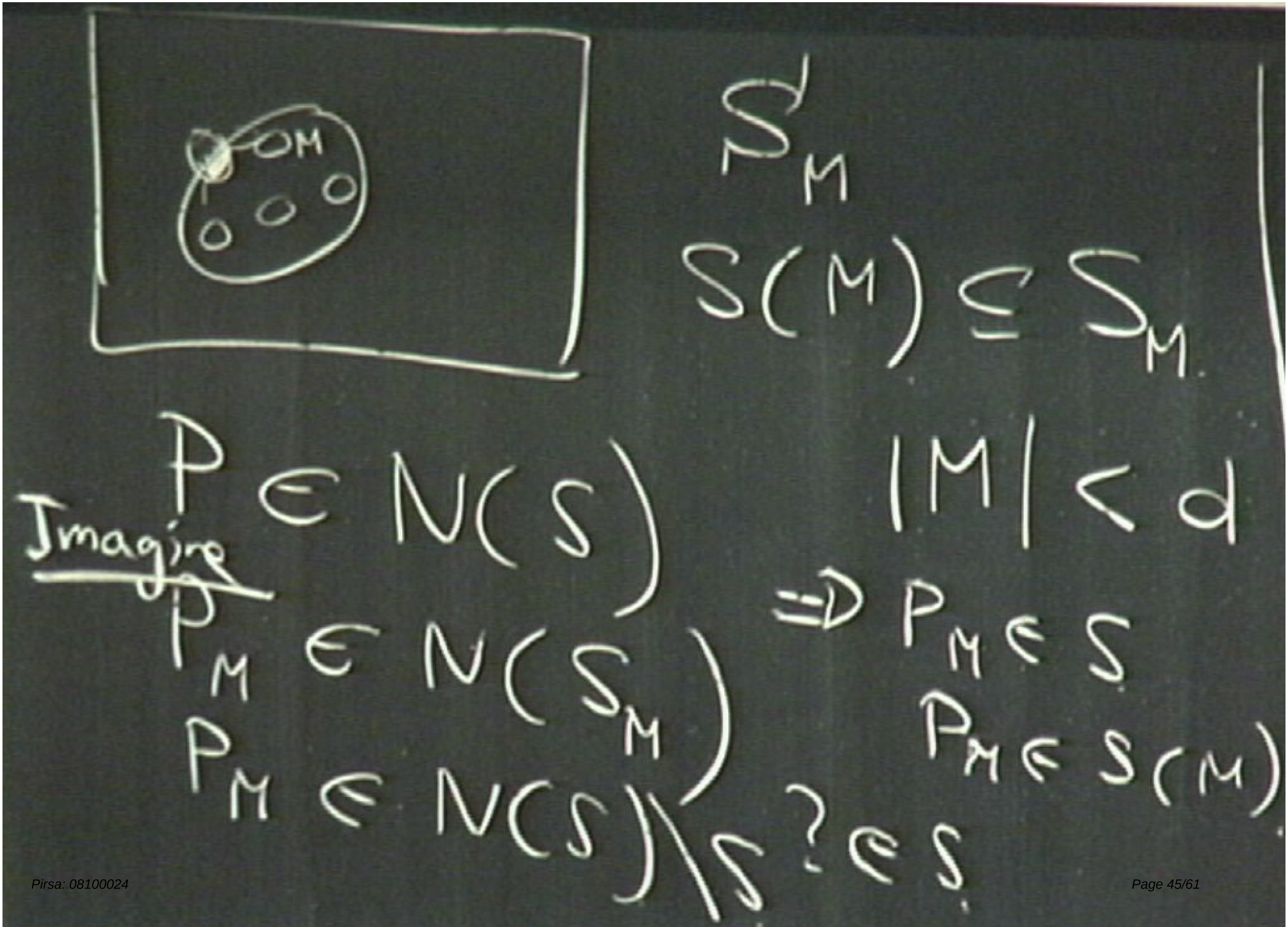
M
S^d_M
$S(M) \subseteq S_M$
$|M| < d$
Imagine
$P \in N(S)$
$P_M \in N(S_M)$
$P_M \in N(S) \backslash S$?∈ S
$\Rightarrow P_M \in S$
$P_M \in S(M)$

$N(S_M) \cap P(M)$

$= S(M)$

S_M

$< d$

S

$S(M)$

$$N(S_M) \cap P(M)$$

$$- \qquad = S(M)$$

$$S_M = N(S(M))$$

$$P_M \in N(S(M))$$

$$N(S_M) \cap P(M)$$

$$\overline{} = S(M)$$

$$S_M = N(S(M))$$

$$P_M \in N(S(M))$$

$$P_M \in S'_M$$

Cleaning Lemma

Given a logical operator P for a stabilizer code with distance d. One ca
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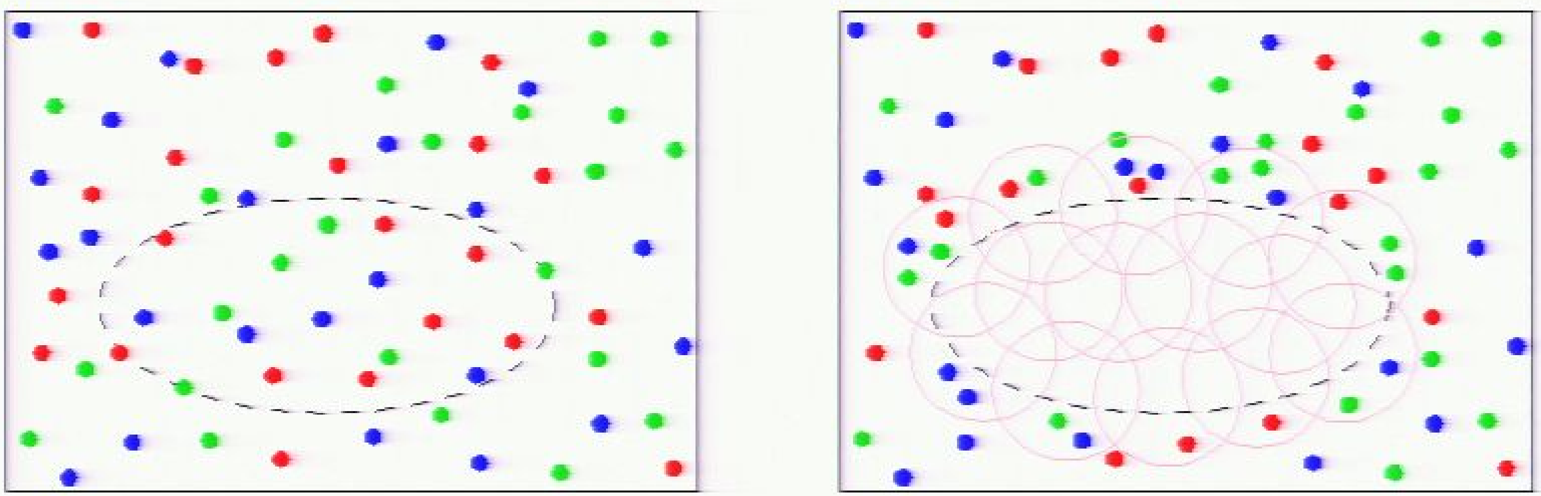
The cleaning procedure gives $P' = PS$ where $S \in \mathcal{S}$ and S acts only insi
I and the boundary of M and P' is clean (is I) on M.

We can clean many times! Make holes and see what is left...



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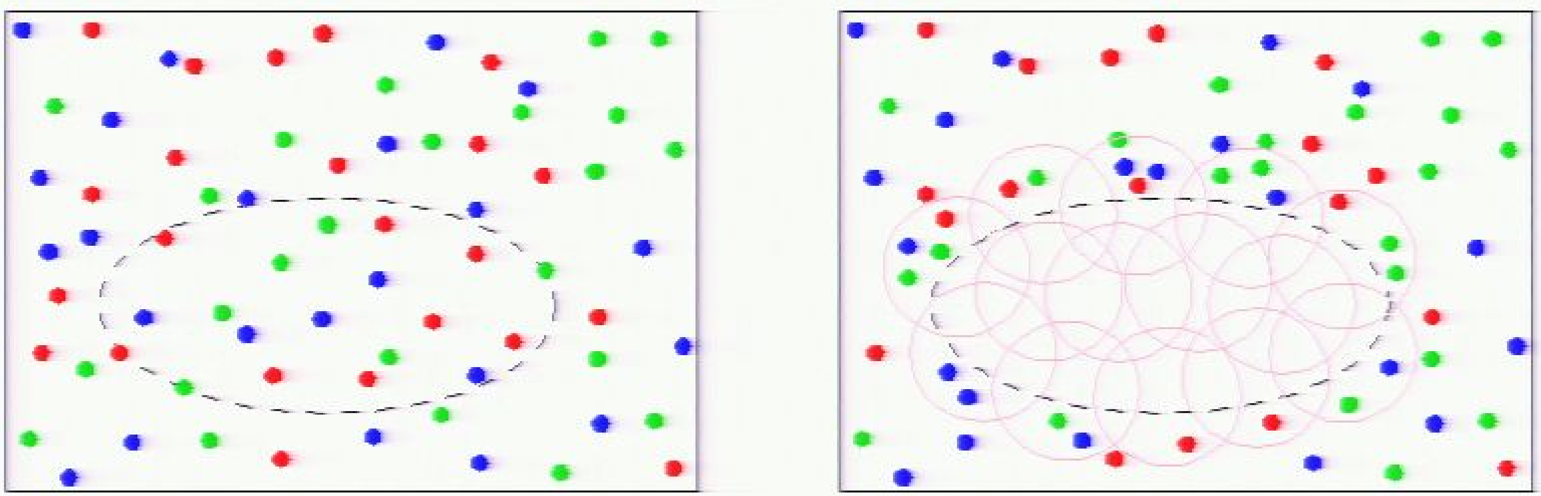
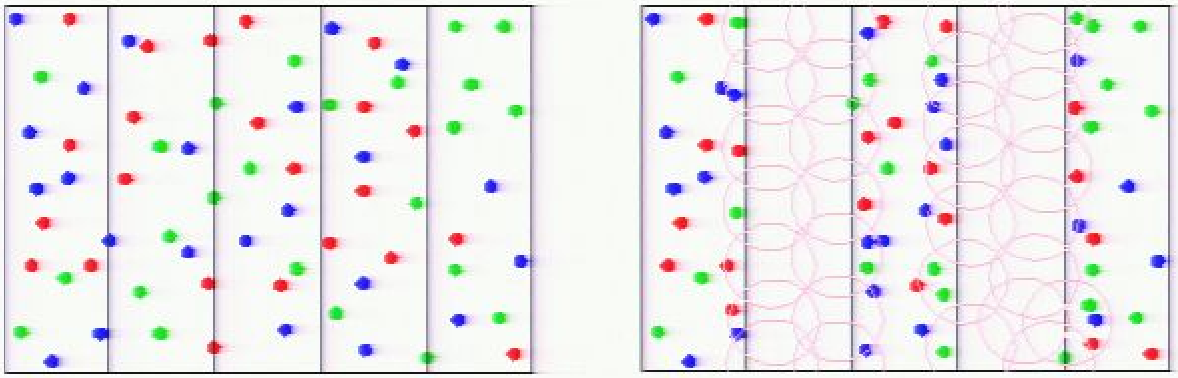


Application of cleaning lemma: bounding the distance



1. Assume for 2D stabilizer codes, the distance $d > cL$.

2. Take logical P and clean out non-adjacent strips of size cL. We get $P' = P_1 P_3 .. P_k$ on remaining strips. If strips are thick enough, cleaning one strip does not interfere with cleaning another one.

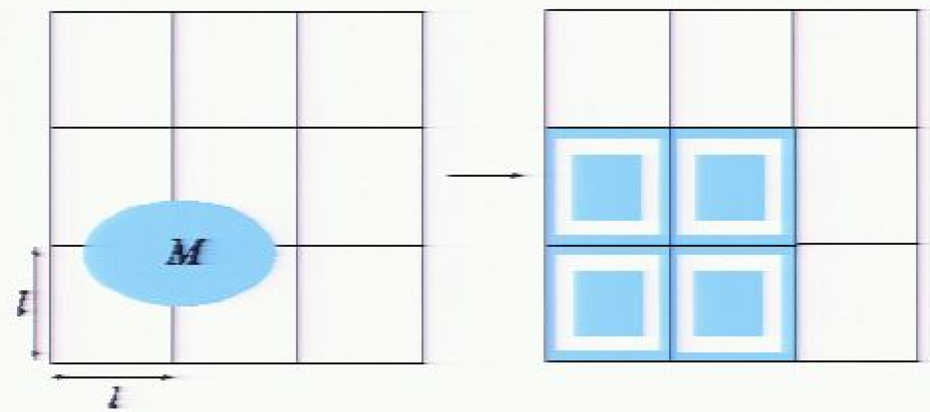
3. Strips are thick enough, so that no stabilizer generator has support on 2 strips. Thus $P_i \in \mathcal{N}(\mathcal{S})$, but there must exist $P_i \in \mathcal{N}(\mathcal{S}) \backslash \mathcal{S}$ (otherwise all $P' \in \mathcal{S}$).

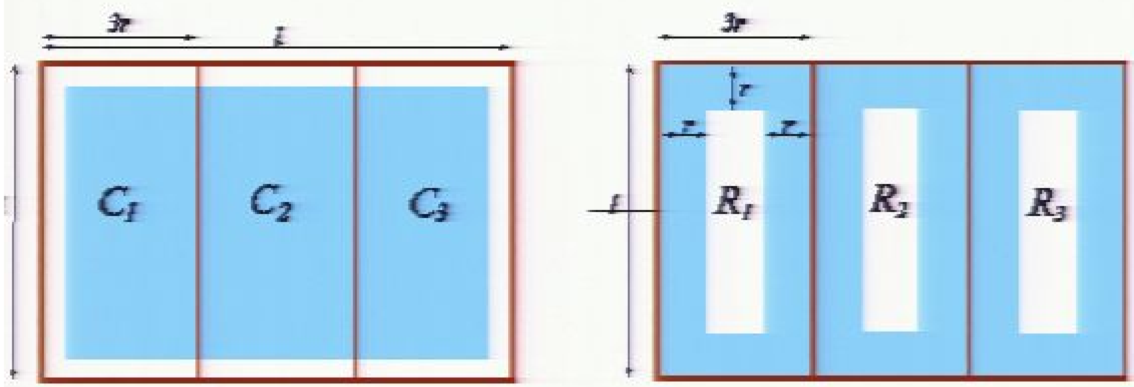
4. Hey, such P_i is a logical operator with weight at most the size of a strip, that is cL. A contradiction!

5. $\Rightarrow d \leq cL$. General D-dimensional bound is $O(L^{D-1})$

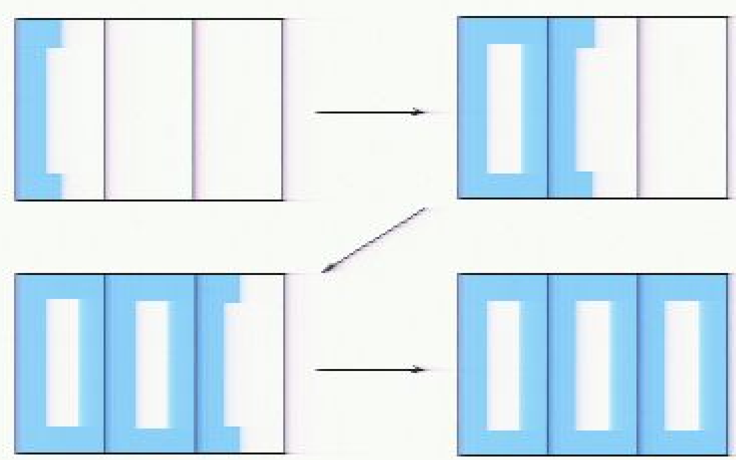
Sketch: Bounding The Energy Barrier

- Take logical op. P of min weight d. Has connected support.
- Let l=c d, boxes of size l x l and P has support on O(1) boxes
- Separate out interior of boxes. Remaining skeleton can be covered with O(1) energy effort.
- Hollow out interior of boxes (see below)





(a) (b)



Bounding The Distance: Subsystem Codes

Trouble: stabilizer group does not have local generators!
Cleaning Lemma for Subsystem Codes:

Let d be the distance of a subsystem code with locally-generated gauge group $\mathcal{G}$ and let M be a region with $|M| < d$.
For any logical operator $P \in \mathcal{N}(\mathcal{G})\backslash\mathcal{G}$ one can choose an element $S \in \mathcal{S}$ such that PS is clean (is I) on M.

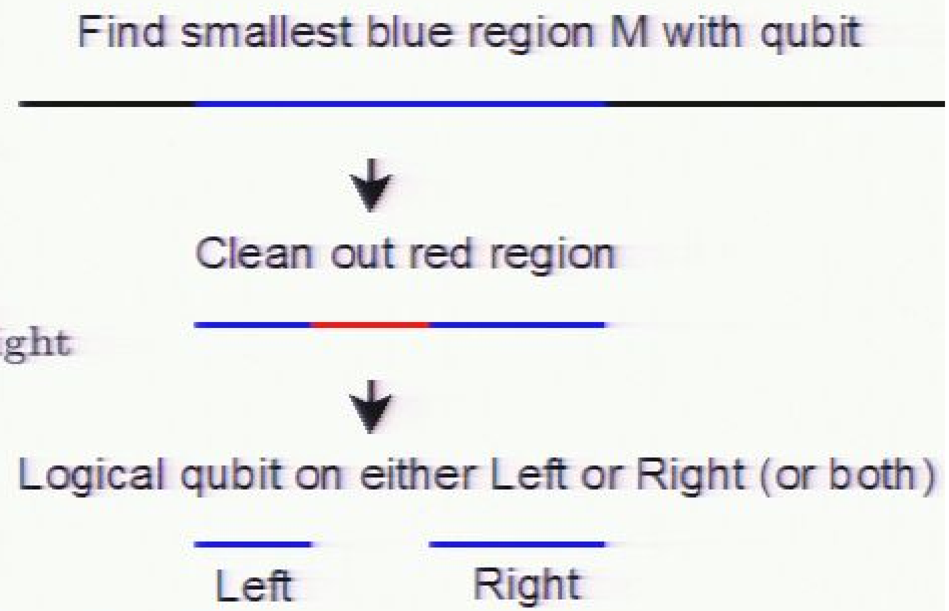
Note that we cannot repeat cleaning since S may be very nonlocal!
Another lemma is needed:

Restriction Lemma:
Take subsystem code with gauge group $\mathcal{G}$ and distance d.
Consider any subset M. Consider code $\mathcal{G}_M$. Either
(1) $\mathcal{G}_M$ has no logical qubits or
(2) $d_{\mathcal{G}_M} \geq d - |\partial M|$.

Application

1D subsystem code with gauge group $\mathcal{G}$ and distance d. Prove $d = O(1)$.

1. Let M be the smallest contiguous region such that $\mathcal{G}_M$ has a qubit.
2. Restriction Lemma implies that $d' \equiv d_{\mathcal{G}_M} \geq d - O(1)$.
3. Assume that $d' \geq c$ for some constant c and get a contradiction $\Rightarrow$
4. Thus $d' \leq c$ and hence $d \leq O(1)$.



1. If $d' \geq c$, then $|M| \geq c$

   and so if c is large enough, we can

   clean out an inner-region of M.

   We get logical operator $P' = P_{\text{left}} P_{\text{right}}$

2. $P_i \in \mathcal{N}(\mathcal{G})$ individually

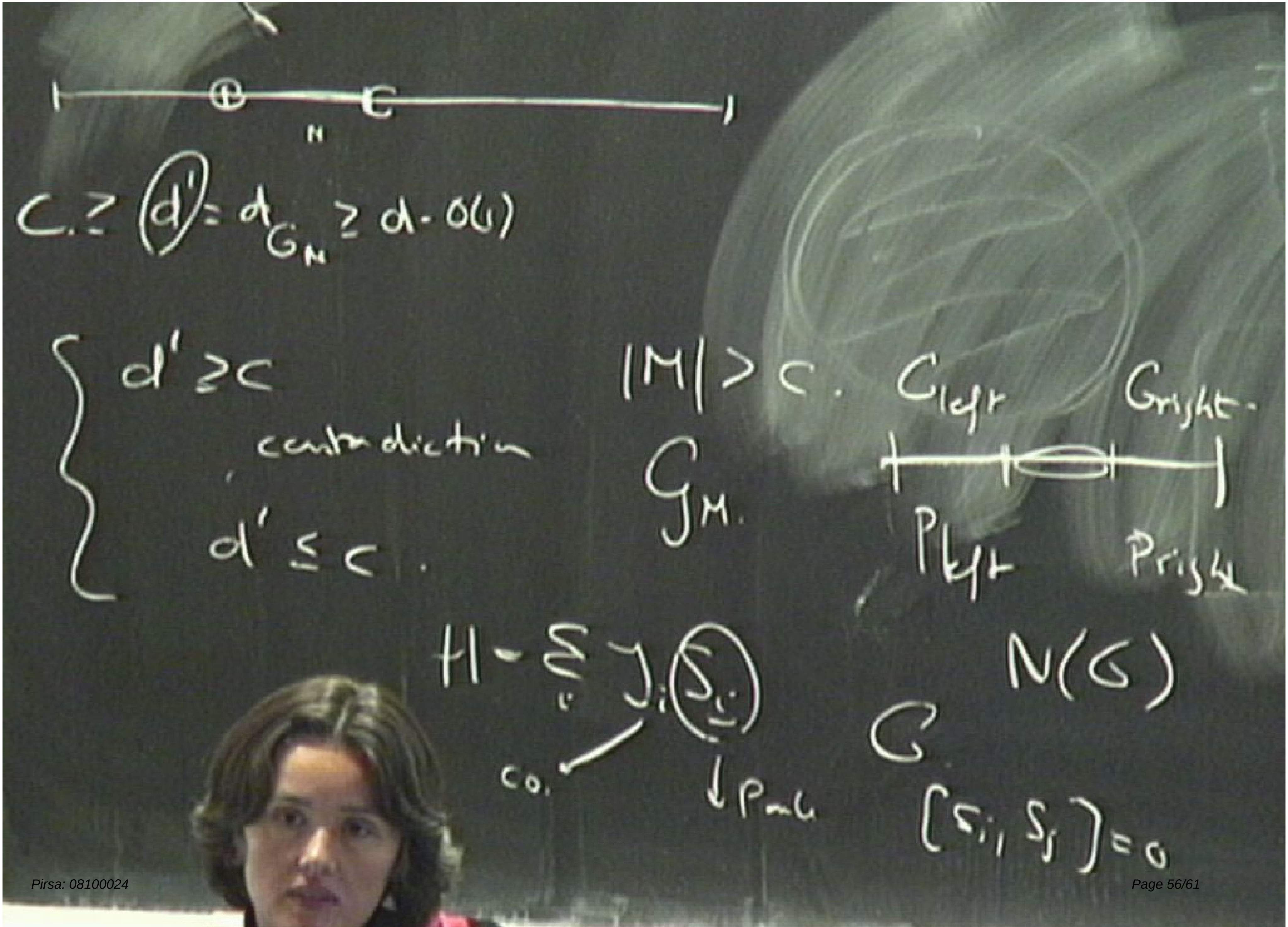
   and it cannot be that both $P_i \in \mathcal{G}$

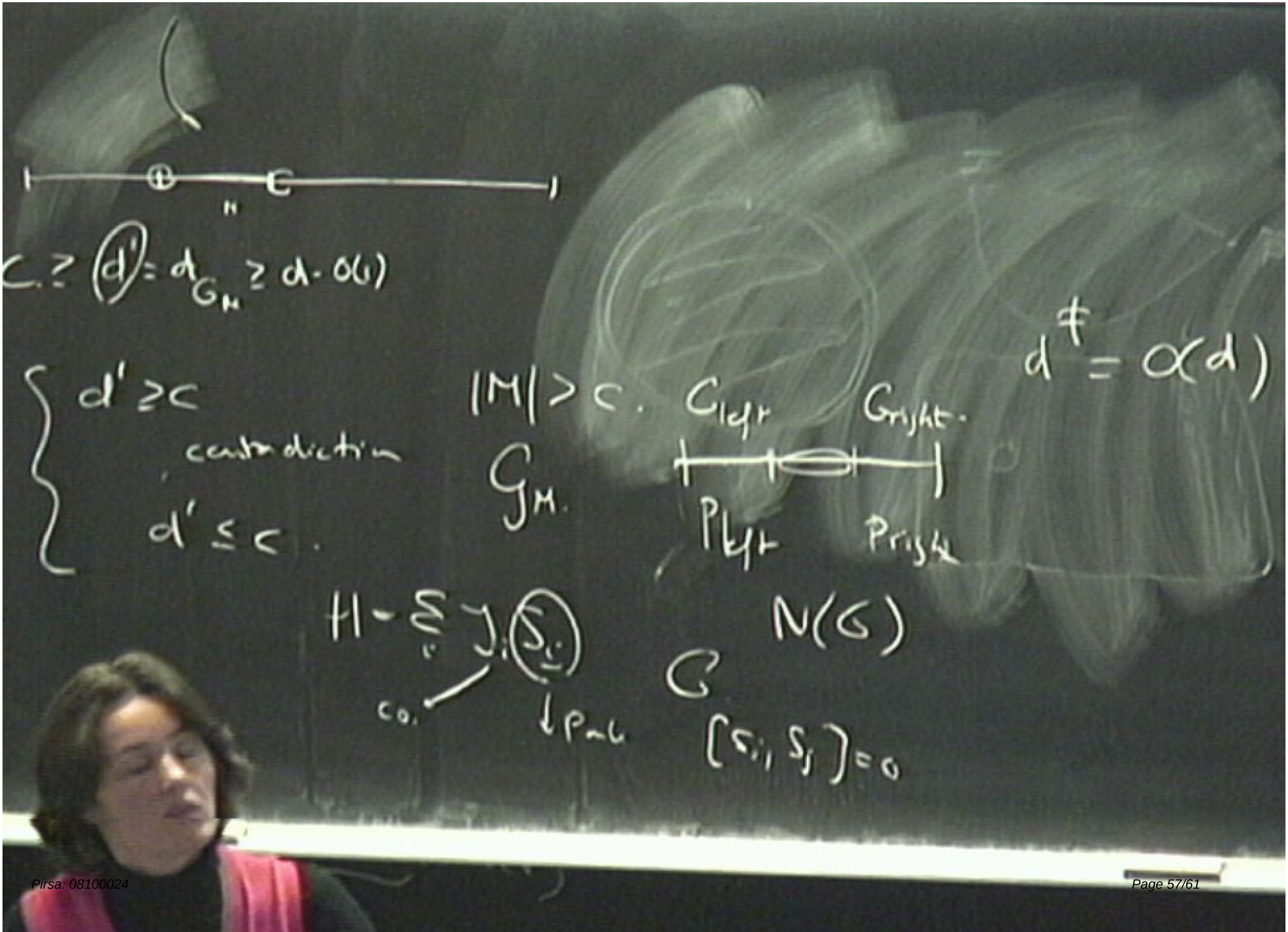
3. $\Rightarrow$ region smaller than M with a logical qubit.

   Contradiction! $\Rightarrow d' \leq c$.

Lots of *Fun* Open Problems

- Prove that distance scales as O(L) for 3D stabilizer codes or show counterexamples.
- Prove that the energy barrier is O(1) for 2D subsystem codes (or show counterexamples)
- Bound energy barrier for *best* logical qubit for 2D stabilizer codes.
- Give evidence/proof for self-correction properties of 3D subsystem codes. What are sufficient conditions for self-correction?
- Can we compute with such subsystem codes…? (like doing Clifford gates on surface codes by moving holes)


C ≥ d' = d_{G_M} ≥ d - O(1)
d' ≥ c
contradiction
d' ≤ c
|M| > c.
G_M
C_left
C_right
P_left
P_right
H = -Σ_i J_i S_i
co.
Pauli
N(G)
G
[S_i, S_j] = 0


c ≥ d' = d_{G_M} ≥ d - O(1)
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d' ≤ c
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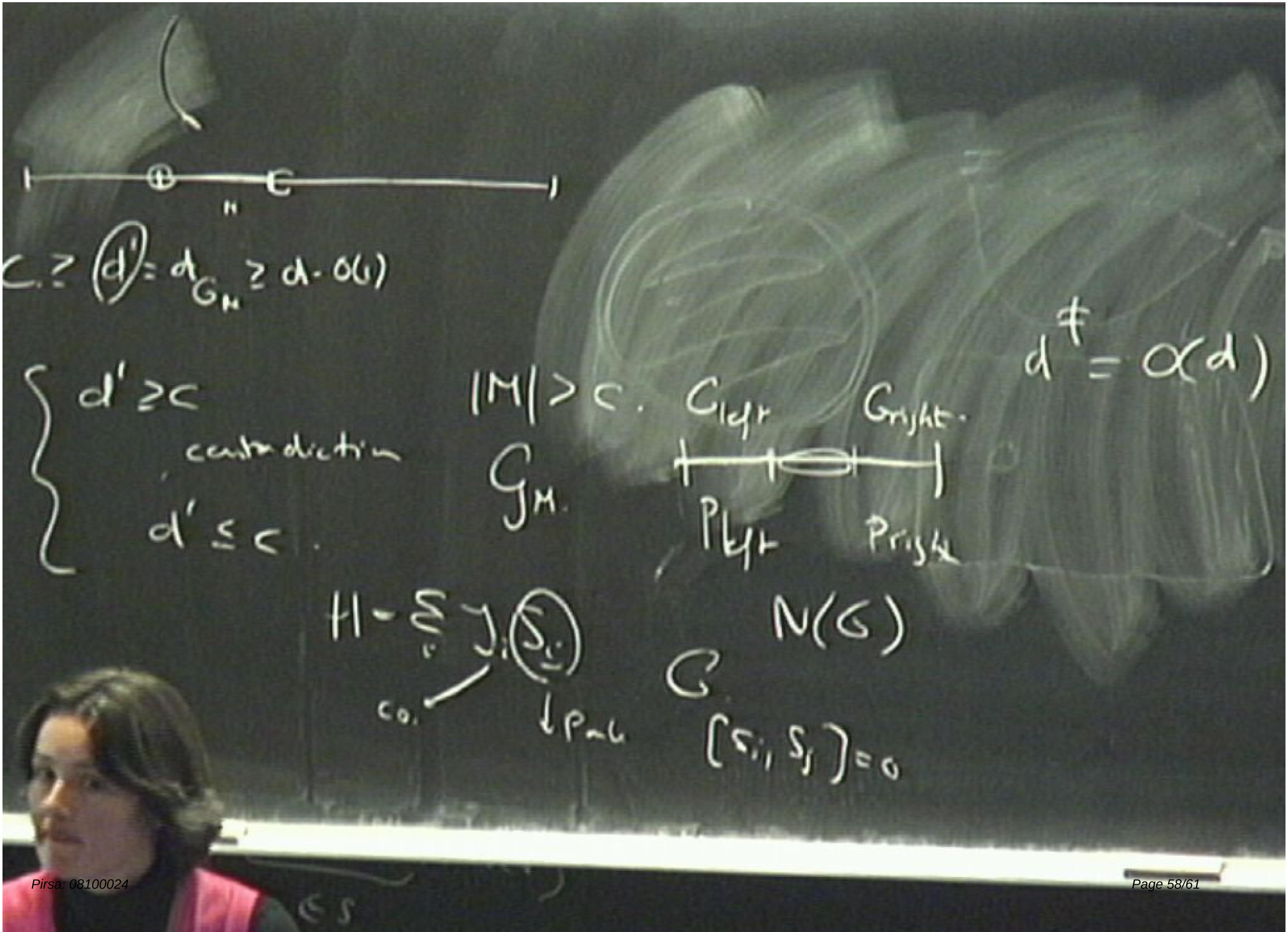
G_left
G_right
G_M
P_left
P_right
= α(d)
H = Σ J_i S_i
N(G)
G
[S_i, S_j] = 0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G_M
C_left
G_right
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P_right
d† = α(d)
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N(G)
G
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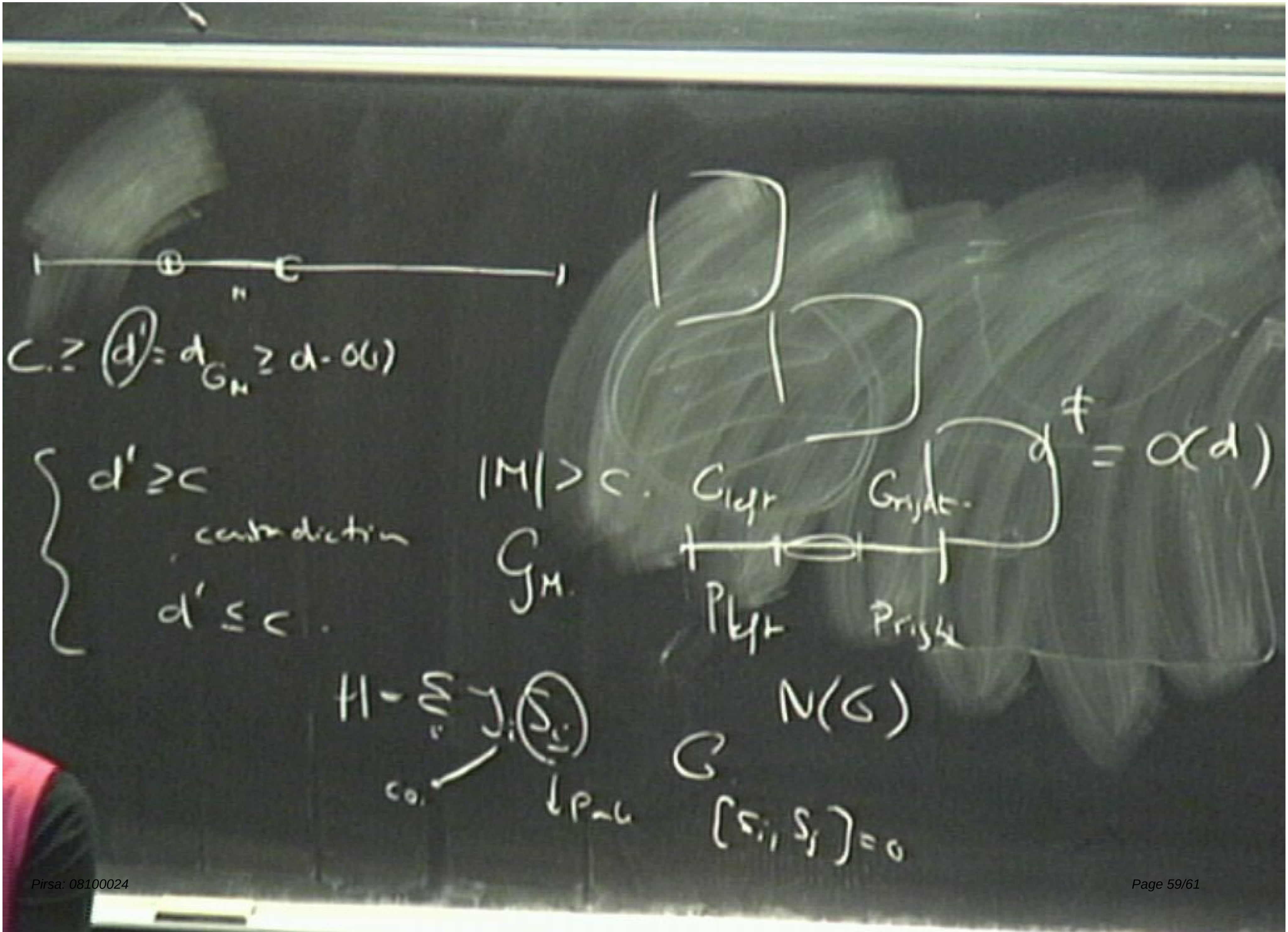
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d' ≥ c
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G_left
G_right
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P_right
d^# = α(d)
H = Σ J_i S_i
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G
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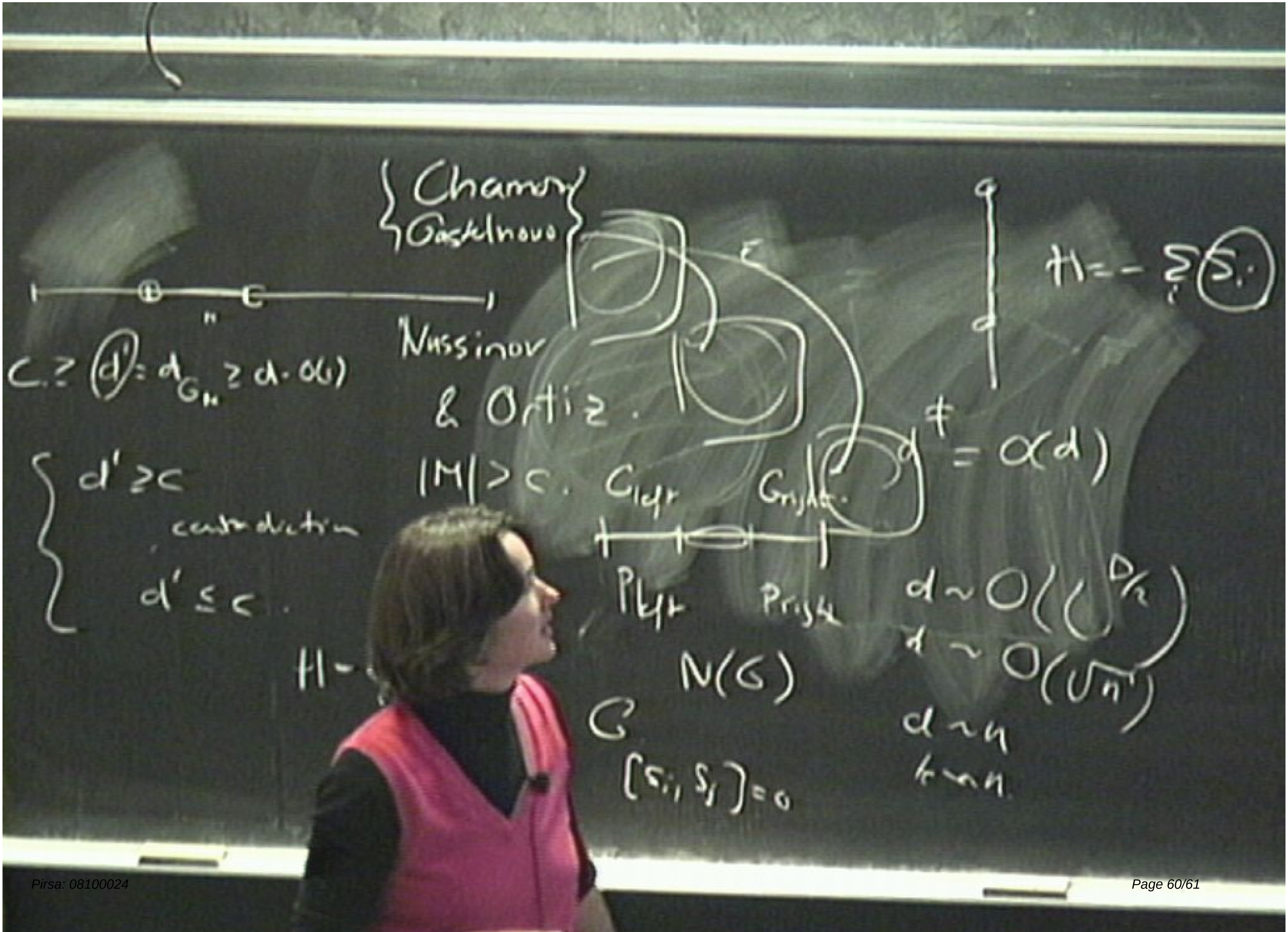
Chamon
Castelnovo
Nussinov
& Ortiz
contradiction
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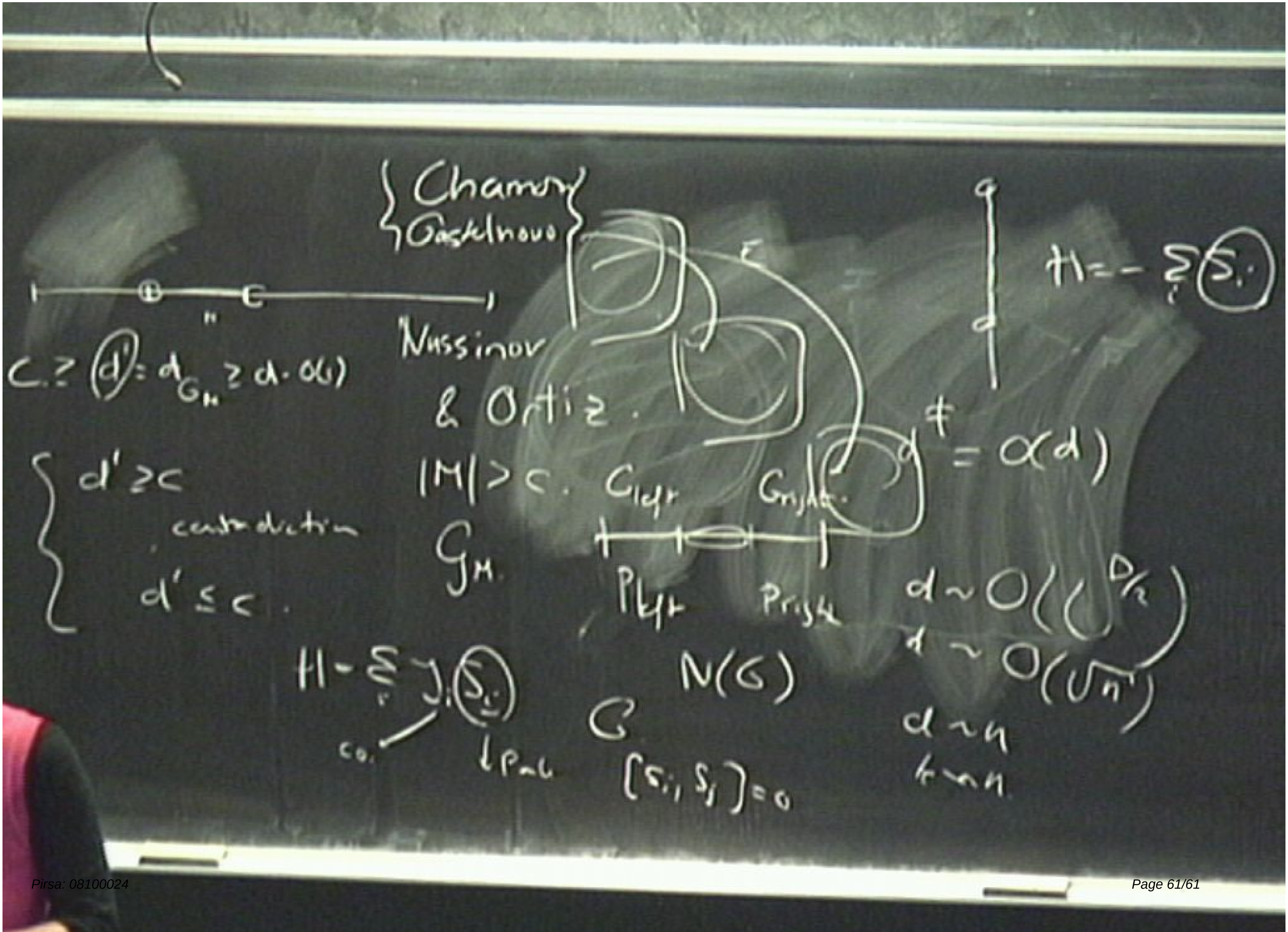
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d' ≥ c
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G_M
G_left
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P_left
P_right
N(G)
[S_i, S_j] = 0
d ~ n
d ~ O(√n)