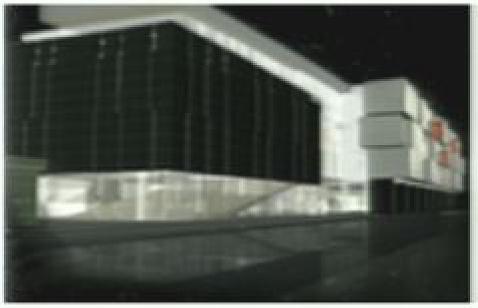
Title: Dyons with potentials: duality and black hole thermodynamics

Date: Oct 16, 2008 02:00 PM

URL: http://pirsa.org/08100021

Abstract: A modified version of the double potential formalism for the electrodynamics of dyons is constructed. Besides the two vector potentials, this manifestly duality invariant formulation involves four additional potentials, scalar potentials which appear as Lagrange multipliers for the electric and magnetic Gauss constraints and potentials for the longitudinal electric and magnetic fields. In this framework, a static dyon appears as a Coulomb-like solution without string singularities. Dirac strings are needed only for the Lorentz force law, not for Maxwell\'s equations. The magnetic charge no longer appears as a topological conservation law but as a surface integral on a par with electric charge. The theory is generalized to curved space. As in flat space, the string singularities of dyonic black holes are resolved. As a consequence all singularities are protected by the horizon and the thermodynamics is shown to follow from standard arguments in the grand canonical ensemble.

Perimeter Institute. October 16, 2008



Dyons with potentials

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Electric-magnetic duality   Motivation

Special type of symmetries in modern theoretical physics

Simplest context : electromagnetism with electric and magnetic sources

Application to thermodynamics of black holes dyons

New formalism making EM duality manifest

with A. Gomberoff **arXiv:0705.0632,** [hep-th], Phys. Rev. D 78 (2008) 025025

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Black hole dyons

First law for Reissner-Nordstrøm BH

RN dyon

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

$$N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}},$$

$$A = -\frac{Q}{r} dt + P(1 - \cos\theta) d\phi.$$

infer thermodynamics for parameter variations from purely electric case using duality

$$\delta M = \frac{\kappa}{8\pi} \delta \mathcal{A} + \phi_H \delta Q + \psi_H \delta P$$

$$\Delta = M^2 - (Q^2 + P^2), \quad r_\pm = M \pm \sqrt{\Delta}$$

$$\kappa = \frac{r_+ - r_-}{2r_+^2}, \quad \mathcal{A} = 8\pi\left[M^2 - \frac{Q^2 + P^2}{2} + M\sqrt{\Delta}\right].$$

$$\phi_H = \frac{Q}{r_+}, \quad \psi_H = \frac{P}{r_+}$$

Problem: excluded in action based derivations of 1st law and Euclidean approaches because of string singularity and absence of magnetic potential

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EM duality

No sources

equations of motion

$$\begin{cases} dF = 0 \\ d^*F = 0 \end{cases}$$

field strength

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$$

$$F_{0i} = -E_i$$

$$F_{ij} = \epsilon_{ijk} B^i$$

invariance under

$$\begin{pmatrix} F \\ {}^*F \end{pmatrix} \to M \begin{pmatrix} F \\ {}^*F \end{pmatrix}$$

$$\det M \neq 0$$

action principle

$$dF = 0 \Longrightarrow F = dA$$

$$S[A] = \frac{1}{2} \int F \wedge {}^*F$$

gauge invariance

$$\delta_\Lambda A = d\Lambda, \quad \delta_\Lambda S = 0$$

EM duality — No sources

Hamiltonian action principle
$$S_H[A_\mu, E^i] = \int d^4x(-\dot{E}^i A_i - \mathcal{H} - A_0 \partial_i E^i)$$

canonical structure
$$\{A_i(t,x), -E^j(t,y)\} = \delta_i^j \delta^3(x-y)$$

Hamiltonian density
$$\mathcal{H} = \frac{1}{2}(E^i E_i + B^i B_i) \qquad B^i = \epsilon^{ijk}\partial_j A_k$$

solve Gauss constraint
$$\partial_i E^i = 0 \Longrightarrow E^i = \epsilon^{ijk}\partial_j Z_k$$

reduced action principle
$$S_R[Z_T^i, A_i^T] = \int d^4x(-\epsilon^{ijk}\partial_j Z_k \dot{A}_i - \mathcal{H})$$

invariant under duality rotations
$$\begin{cases} \delta_D A_i = Z_i, \\ \delta_D Z_i = -A_i \end{cases}, \quad \delta_D S_R = 0$$

Deser & Teitelboim Phys. Rev. D 13 (1976) 1592

EM duality

No sources

doublet notation

$$A_i^a = \begin{pmatrix} A_i \\ Z_i \end{pmatrix}$$

$$B^{ai} = \epsilon^{ijk}\partial_j A_k^a = \begin{pmatrix} B^i \\ E^i \end{pmatrix}$$

$$\delta_D A_{ia} = \epsilon_{ab} A_i^b$$

manifestly invariant action

$$S_{SS}[A_\mu^a] = \frac{1}{2}\int d^4x \left[\epsilon_{ab} B^{ai}(\partial_0 A_i^b - \partial_i A_0^b) - \delta_{ab} B^{ai} B_i^b\right]$$

A_0^a spurious

Schwarz & Sen Nucl. Phys. B411 (1994) 35

EM duality

Point-particle sources

duality requires both electric and magnetic sources

magnetic pole $q_n^a = (g_n, 0)$ electron $q_n^a = (0, e_n)$ dyon $q_n^a = (g_n, e_n)$

current $$j^{a\mu}(x) = \sum_n q_n^a \int_{\Gamma_n} \delta^{(4)}(x - z_n) dz_n^\mu$$

pole at origin $$j_M^\mu(x) = g\delta_0^\mu \delta^{(3)}(x) \qquad B^i = \frac{g}{4\pi}\frac{x^i}{r^3}$$

singular potential $$A = -\frac{g}{4\pi}(\cos\theta + 1)d\varphi \qquad A_i = \frac{g}{4\pi}\begin{pmatrix} \frac{y}{r(r-z)} \\ -\frac{x}{r(r-z)} \\ 0 \end{pmatrix}$$

regularize $$A_i \to A_i^\epsilon = \frac{g}{4\pi}\begin{pmatrix} \frac{y}{R(R-z)} \\ -\frac{x}{R(R-z)} \\ 0 \end{pmatrix}$$

$$R = \sqrt{r^2 + \epsilon^2}$$

magnetic field: semi-infinite solenoid

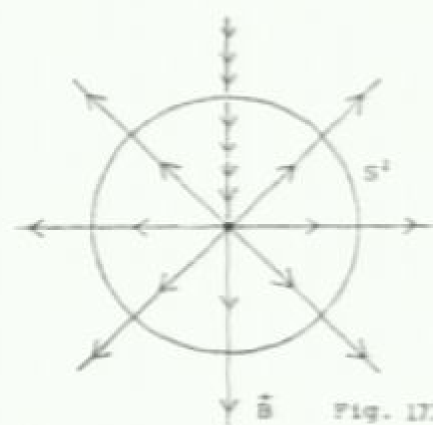
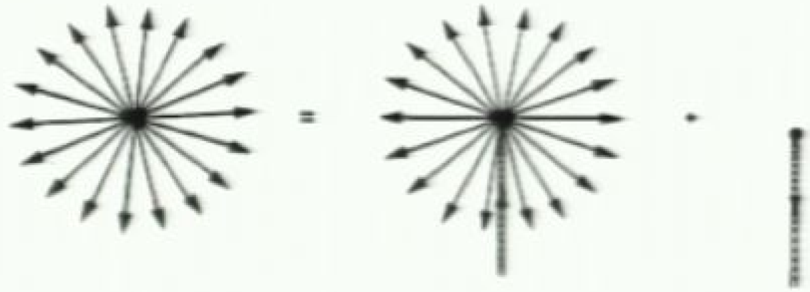


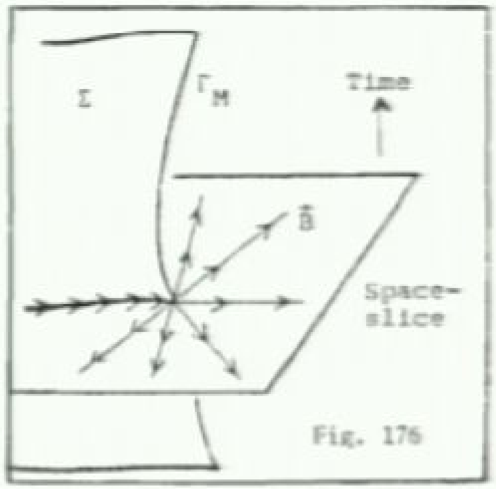
Fig. 173

EM duality

Dirac strings

add dynamical string





$$G^{\mu\nu}_{\Sigma}(x) = g \int_{\Sigma} \delta^{(4)}(x-y) dy^{\mu} \wedge dy^{\nu}$$

$\partial\Sigma = \Gamma_M$ $\qquad y^{\mu}(\tau,\sigma),\ y^{\mu}(\tau,0) = z^{\mu}_{M}(\tau)$

property $\qquad d^{*}G_{\Sigma} = {}^{*}G_{\partial\Sigma} = {}^{*}j_M$

modified field strength $\qquad F^{D} = dA + {}^{*}G_{\Sigma} \qquad dF^{D} = {}^{*}j_M$

action principle

$$S^{D}[A_{\mu}(x), y^{\mu}(\tau,\sigma), z_e(\tau)] = \frac{1}{2}\int F^{D} \wedge {}^{*}F^{D} + e\int_{\Gamma_e} A$$
$$-m_e \int_{\Gamma_e} \sqrt{-dz^{\mu}_{e} dz_{\mu e}} - m_M \int_{\Gamma_M} \sqrt{-dz^{\mu}_{M} dz_{\mu M}}$$

Dirac veto: electron cannot touch string

$$eg = 2\pi N \hbar$$

quantization condition

Dirac Phys. Rev. 74 (1948) 817

EM duality

Remarks on Dirac strings

(i) asymmetric treatment of both types of sources, manifest Poincaré invariance but no duality

(ii) non trivial U(1) fiber bundles, no strings

Wu & Yang Phys. Rev. D12 (1975) 3845, D14 (1976) 437

(iii) Dirac strings related to de Rham currents and Poincaré duality

De Rham, Variétés différentiables, Hermann, 1960

BH dyons

Duality

"complicated" proof of duality between electric and magnetic BH

partition function through semi-classical evaluation of Euclidean path integral

$Z(\beta, P)$ vs $Z(\beta, \phi_H)$

Hawking & Ross, Phys. Rev. D52 (1995) 5685

additional Legendre transformation needed to compare

Dyons and duality   Extended formulation in flat space

New construction   dynamical longitudinal fields and non spurious scalar potentials

$$A^a_\mu \equiv (A_\mu, Z_\mu) \qquad C^a \equiv (C, Y) \qquad \vec{B}^a \equiv (\vec{B}, \vec{E}) \qquad \vec{B}^a = \vec{\nabla} \times \vec{A}^a + \vec{\nabla} C^a$$

external sources $\partial_\mu j^{a\mu} = 0$

action principle $I[A^a_\mu, C^a] = I_M[A^a_\mu, C^a] + I_I[A^a_\mu; j^{a\mu}],$ $I_I[A^a_\mu; j^{a\mu}] = \int d^4x\, \epsilon_{ab} A^a_\mu j^{b\mu}.$

$$I_M[A^a_\mu, C^a] = \frac{1}{2} \int d^4x \left[\epsilon_{ab}(\vec{B}^a + \vec{\nabla} C^a) \cdot (\partial_0 \vec{A}^b - \vec{\nabla} A^b_0) - \vec{B}^a \cdot \vec{B}_a\right].$$

Maxwell's equations

$$\left\{ \begin{array}{l} A^a_0 : \ \ \vec{\nabla} \cdot \vec{B}^a \equiv \nabla^2 C^a = j^{0a} \qquad C^a : \ \nabla^2 C_a = \epsilon_{ab}(\vec{\nabla} \cdot \partial_0 \vec{A}^b - \nabla^2 A^b_0) \\ \vec{A}^a : \ -\epsilon_{ab} \partial_0 \vec{B}^b + \vec{\nabla} \times \vec{B}_a = \epsilon_{ab} \vec{j}^b. \end{array} \right.$$

no strings, duality manifest

G.B. & Gomberoff, Phys. Rev. D78 (2008) 025025

Dyons and duality Extended formulation in flat space

fixed point particle dyon at origin $j^{a\mu}(x) = 4\pi Q^a \delta_0^\mu \delta^3(x)$

Coulomb-type solution $A^a = -\frac{\epsilon^{ab} Q_b}{r} dt, \; C^a = -\frac{Q^a}{r}$

resolution of string singularity

gauge invariance $\delta_\epsilon A^a_\mu = \partial_\mu \epsilon^a, \qquad \delta_\epsilon C^a = 0$

magnetic charge is surface integral, no longer a topologically charge

canonical pairs $(\vec{A}^T, -\vec{\nabla} \times \vec{Z}^T), (\vec{A}^L, -\vec{\nabla} Y) \; (\vec{Z}^L, \vec{\nabla} C)$

gauge fixing the magnetic Gauss constraint gives back standard EM

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Dyons and duality   Extended formulation with dynamical dyons

dynamical point particle dyons need strings for Lorentz force law

$$G^{a\mu\nu} = \sum_n q_n^a \int_{\Sigma_n} \delta^{(4)}(x - y_n) dy_n^\mu \wedge dy_n^\nu \qquad y_n^\mu(\sigma_n, \tau_n), \quad y_n^\mu(0, \tau_n) = z_n^\mu(\tau_n)$$

total action $\quad I_M + I_I + I_k + \frac{1}{2}\int d^4x\, \epsilon_{ab}\left[2\partial^i C^a \alpha_i^b - \beta^{ai}\alpha_i^b + \beta_T^{ai}\partial_0\gamma_i^b)\right].$

$$I_k[z_n^\mu] = -\sum_n m_n \int_{\Gamma_n} \sqrt{-dz_n^\mu dz_{n\mu}} \qquad \beta^{ai} = G^{a\ 0i} \qquad \alpha_i^a = \frac{1}{2}\epsilon_{ijk}\, G^{a\ jk} \qquad \beta_T^{ai} = \epsilon^{ijk}\partial_j \gamma_k^a$$

variation with respect to z_n^μ gives Lorentz force law

veto: string attached to dyon n cannot cross any other dyon

leads to standard quantization condition for dyons $\quad \epsilon_{ab} q_n^a q_m^b = 2\pi N \hbar$

Dyons and duality: Extended formulation and Lorentz invariance

action with string terms: related to Dirac's action through elimination of auxiliary fields

(indirect) proof of Lorentz invariance

Poincaré generators without sources:

$$\vec{P} = -\frac{1}{2}\int d^3x\, \epsilon_{ab}\vec{B}^a \times \vec{B}^b, \qquad \vec{J} = -\int d^3x\, \epsilon_{ab}\vec{B}^a(\vec{x}\cdot\vec{B}^b),$$

$$\vec{K} = -\int d^3x\, \vec{x}\left(\frac{1}{2}\vec{B}^a\cdot\vec{B}^a\right).$$

transformations:

$$\begin{aligned}
\delta_Q C^a &= 0,\\
\delta_Q A^a_i &= \partial_i\lambda^a_Q - \epsilon^{ab}B_{bi}\xi^0 - \epsilon_{ijk}\xi^j B^{ak},\\
\delta_Q B^a_i &= -\epsilon^{ijk}\partial_j(\epsilon^{ab}B_{bk}\xi^0) - \partial_j(B^{aj}\xi^i) + \partial_j(B^{ai}\xi^j),\\
\delta_Q A^a_0 &= \partial_0\lambda^a_Q + \epsilon^{ab}B_{bi}\xi^i,
\end{aligned}$$

$$\xi(\omega,a)^\mu = -(\omega^\mu{}_i x^i + a^\mu), \qquad \lambda^a_Q = -\epsilon^{ab}\nabla^{-2}\partial_i(B^i_b\xi^0) + \nabla^{-2}\partial_i(\epsilon^{ijk}B^a_j\xi_k).$$

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Dyons and duality   Extended formulation in curved space

need to generalize first order ADM formulation of Einstein-Maxwell

no non singular longitudinal magnetic field $\mathcal{B}^i_{ADM} = \epsilon^{ijk}\partial_j A_k$

curved space $\mathcal{B}^{ai} = \epsilon^{ijk}\partial_j A^a_k + \sqrt{g}\partial^i C^a$

matter action
$$I_M[A^a_\mu, C^a, g_{ij}, N, N^i] = \frac{1}{8\pi}\int d^4x \Big[(\mathcal{B}^{ai} + \sqrt{g}\partial^i C^a)\epsilon_{ab}(\partial_0 A^b_i - \partial_i A^b_0) - \frac{N}{\sqrt{g}}\mathcal{B}^i_a\mathcal{B}^a_i - \epsilon_{ab}\epsilon_{ijk}N^i\mathcal{B}^{aj}\mathcal{B}^{bk}\Big]$$

total action
$$I[z,u] = \int d^4x\,[a_A(z)\partial_0 z^A - u^\alpha\gamma_\alpha]$$

kinetic term
$$a_A(z)\partial_0 z^A = \frac{\pi^{ij}}{16\pi}\partial_0 g_{ij} - \frac{\mathcal{E}^i}{4\pi}\partial_0 A_i + \frac{\sqrt{g}\partial^i C}{4\pi}\partial_0 Z_i$$

Lagrange multipliers and constraints $u^\alpha \equiv (N, N^i, A^a_0)$ $\gamma_\alpha \equiv (\mathcal{H}_\perp, \mathcal{H}_i, \mathcal{G}_a)$

$$\mathcal{H}_\perp = \frac{1}{16\pi}(\mathcal{H}^{ADM}_\perp + \mathcal{H}^{mat}_\perp), \quad \mathcal{H}_i = \frac{1}{16\pi}(\mathcal{H}^{ADM}_i + \mathcal{H}^{mat}_i), \quad \mathcal{G}_a = \frac{1}{4\pi}\epsilon_{ab}\partial_i\mathcal{B}^{bi}$$

gauge parameters $\epsilon^\alpha \equiv (\xi^\perp, \xi^i, \lambda^a)$ smeared constraints $\Gamma[\epsilon] = \int d^3x\,\gamma_\alpha\epsilon^\alpha$

Dyons and duality — Extended formulation in curved space

first class gauge algebra

$$\{\Gamma[\epsilon_1],\Gamma[\epsilon_2]\} = \Gamma[[\epsilon_1,\epsilon_2]] \qquad \{H[\xi],H[\eta]\} = H[[\xi,\eta]_{SD}] + G[[\xi,\eta]_B]$$

surface deformation algebra

$$[\xi,\eta]^{\perp}_{SD} = \xi^i\partial_i\eta^{\perp} - \eta^i\partial_i\xi^{\perp}, \qquad [\xi,\eta]^a_B = \mathcal{B}^{ai}\epsilon_{ijk}\xi^j\eta^k - \frac{\epsilon^{ac}\mathcal{B}_{ci}}{\sqrt{g}}(\xi^{\perp}\eta^i - \eta^{\perp}\xi^i)$$

$$[\xi,\eta]^i_{SD} = g^{ij}(\xi^{\perp}\partial_j\eta^{\perp} - \eta^{\perp}\partial_j\xi^{\perp}) + \xi^j\partial_j\eta^i - \eta^j\partial_j\xi^i$$

diffeomorphisms $\xi^{\perp} = N\eta^0 \quad \xi_i = g_{i\mu}\eta^{\mu} \qquad \mathcal{L}_{\eta}g_{\mu\nu} \approx \delta_{\xi}g_{\mu\nu}$

sources $I_{ADM} + I_M + I_J$

$$I_J[A^a_{\mu},C^a,y^{\mu}] = \frac{1}{4\pi}\int d^4x\,\epsilon_{ab}\left[A^a_{\mu}j^{b\mu} + \sqrt{g}\partial^i C^a\alpha^b_i - \frac{1}{2}\beta^{ai}\alpha^b_i - \frac{1}{2}\beta^{ai}_T\partial_0\gamma^b_i\right]$$

$$\beta^{aTi} = \epsilon^{ijk}\partial_j\gamma^a_k$$

1-1 correspondence of solutions to those of covariant equations

resolved RN dyon

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad A^a = -\epsilon^{ab}Q_b\left(\frac{1}{r} - \frac{1}{r_+}\right)dt$$

$$N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}}. \qquad C^a = -Q^a\int_r^{\infty}\frac{dr'}{r'^2N(r')}$$

Dyons and duality   1st law of black hole mechanics

surface integrals $\delta_z(\gamma_\alpha\epsilon^\alpha) = \delta z^A \frac{\delta(\gamma_\alpha\epsilon^\alpha)}{\delta z^A} - \partial_i k^i_\epsilon$ Regge & Teitelboim Ann. Phys. 88 (1974) 286

solution of constraints z^A_s solution of linearized constraints δz^A_s

time independent solution z^a_s, u^α_s $\partial_0 z^A_s = 0$

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for RN dyon $$\oint_{S} d^{n-1}x_i\ k^{mat.i}_u = -\frac{1}{4\pi}\int_0^\pi d\theta\int_0^{2\pi} d\phi\ \epsilon_{ab}A^a_0\delta\mathcal{B}^{bLi}$$

Dyons and duality   1st law of black hole mechanics

first law
$$\oint_{S^\infty} d^3x_i\, k^i_u[z,\delta z] = \oint_{S_{r_+}} d^3x_i\, k^i_u[z,\delta z]$$

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Dyons and duality Extended formulation in curved space

first class gauge algebra $\{\Gamma[\epsilon_1],\Gamma[\epsilon_2]\}=\Gamma[[\epsilon_1,\epsilon_2]]$ $\{H[\xi],H[\eta]\}=H[[\xi,\eta]_{SD}]+G[[\xi,\eta]_B]$

surface deformation algebra

$$[\xi,\eta]^{\perp}_{SD}=\xi^i\partial_i\eta^{\perp}-\eta^i\partial_i\xi^{\perp}, \qquad [\xi,\eta]^a_B=\mathcal{B}^{ai}\epsilon_{ijk}\xi^j\eta^k-\frac{\epsilon^{ac}\mathcal{B}_{ci}}{\sqrt{g}}(\xi^{\perp}\eta^i-\eta^{\perp}\xi^i)$$

$$[\xi,\eta]^i_{SD}=g^{ij}(\xi^{\perp}\partial_j\eta^{\perp}-\eta^{\perp}\partial_j\xi^{\perp})+\xi^j\partial_j\eta^i-\eta^j\partial_j\xi^i$$

diffeomorphisms $\xi^{\perp}=N\eta^0 \quad \xi_i=g_{i\mu}\eta^{\mu}$ $\mathcal{L}_{\eta}g_{\mu\nu}\approx\delta_{\xi}g_{\mu\nu}$

sources $I_{ADM}+I_M+I_J$

$$I_J[A^a_\mu,C^a,y^\mu]=\frac{1}{4\pi}\int d^4x\,\epsilon_{ab}\left[A^a_\mu j^{b\mu}+\sqrt{g}\partial^i C^a\alpha^b_i-\frac{1}{2}\beta^{ai}\alpha^b_i-\frac{1}{2}\beta^{ai}_T\partial_0\gamma^b_i\right]$$

$$\beta^{aTi}=\epsilon^{ijk}\partial_j\gamma^a_k$$

1-1 correspondence of solutions to those of covariant equations

resolved RN dyon

$$ds^2 = -N^2dt^2+N^{-2}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \qquad N=\sqrt{1-\frac{2M}{r}+\frac{Q^2+P^2}{r^2}}.$$

$$A^a = -\epsilon^{ab}Q_b\left(\frac{1}{r}-\frac{1}{r_+}\right)dt \qquad C^a = -Q^a\int_r^\infty\frac{dr'}{r'^2N(r')}$$

Dyons and duality   1st law of black hole mechanics

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Black hole thermodynamics

path integral

$$Z[\beta,\phi,\psi] = \int \mathcal{D}\Phi e^{I_e^T}$$

Euclidean action

$$I_e^T = \int_0^\beta d\tau \Big(i \int d^3x\, a_A(z)\partial_0 z^A - (\boldsymbol{H} - \phi^c \boldsymbol{Q} - \psi^c \boldsymbol{P}) \Big) + \text{ghost terms}$$

leading contribution: action evaluated at
Euclidean Reissner-Nordstrom dyon

$$I_e^{mat}(RND) = \beta\phi_H Q + \beta\psi_H P$$

$$I_e^{grav}(RND) = -\beta M + \frac{1}{4}\mathcal{A}$$

total result

$$\Psi_G = -\beta M + \frac{1}{4}\mathcal{A} + \beta\phi_H Q + \beta\psi_H P$$

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Conclusion

construction of an explicitly duality invariant version of electromagnetism

enhanced gauge invariance

static dyon described by Coulomb fields without string singularities

electric and magnetic charges are surface integrals

applications in the context of thermodynamics of BH dyons

Central extensions: Surface charges & generalized Killing vectors

eneralized conserved charges:
$$\begin{cases} d\omega^{n-p} \approx 0 \\ \omega^{n-p} \sim \omega^{n-p} + d\eta^{n-p-1} + t^{n-p}, \quad t^{n-p} \approx 0 \end{cases}$$

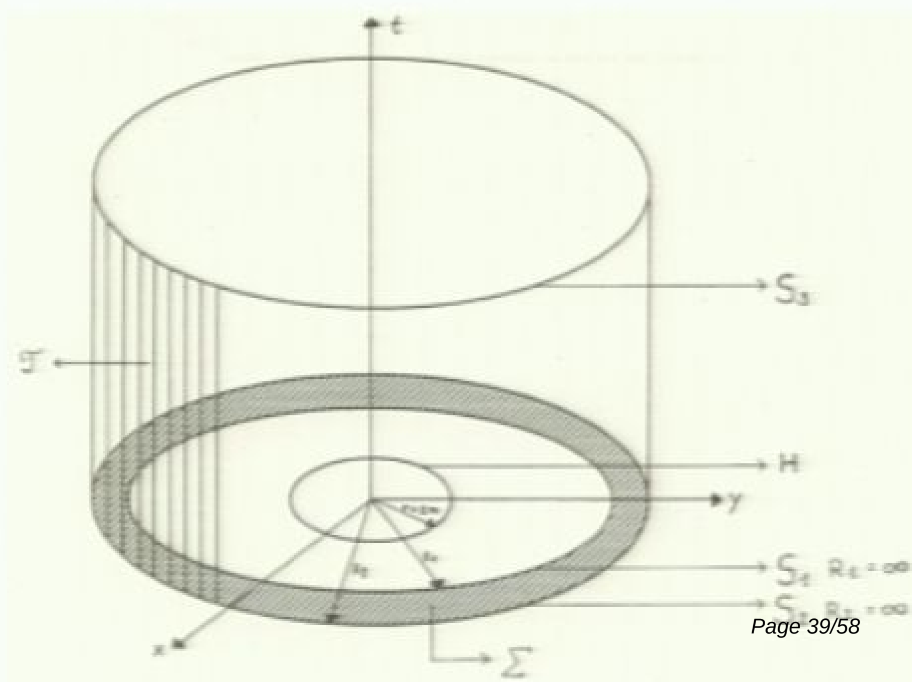
$p = 1:$ conserved currents associated with global symmetries

irreducible gauge theories (no 2,3-forms):

$$H^{n-p}_{char}(d) = 0 \ for \ p \geqslant 3$$

$$H^{n-2}_{char}(d) \longleftrightarrow f^\alpha \ such \ that \ R^i_\alpha(f^\alpha) = 0$$

charges: $$Q_f[\phi^s] = \oint_{S^{n-2}} k^{n-2}_f[\phi^s]$$



Central extensions: Surface charges & generalized Killing vectors

Examples

semi-simple YM theory: $\delta_\epsilon A^a_\mu = D_\mu \epsilon^a = 0 \Longrightarrow \epsilon^a = 0$

EM: $\delta_\epsilon A_\mu = \partial_\mu \epsilon = 0 \Longrightarrow \epsilon = cte \quad \longleftrightarrow \quad k^{n-2} = {}^*F$

electric charge $Q = \oint_{S^{n-2}} {}^*F$

GR: $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 0 \Longrightarrow \xi^\mu = 0$

linearized gravity: $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} = 0 \Longrightarrow \xi^\mu$ Killing vector of $\bar{g}_{\mu\nu}$

$$k_\xi[h;\bar{g}] = \frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}\sqrt{-\bar{g}}\Big(\xi^\nu \bar{D}^\mu h + \xi^\mu \bar{D}_\sigma h^{\sigma\nu} + \xi_\sigma \bar{D}^\nu h^{\sigma\mu} + \frac{1}{2} h \bar{D}^\nu \xi^\mu + \frac{1}{2} h^{\mu\sigma} \bar{D}_\sigma \xi^\nu + \frac{1}{2} h^{\nu\sigma} \bar{D}^\mu \xi_\sigma - (\mu \longleftrightarrow \nu)\Big),$$

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Central extensions: Algebra & asymptotics

global symmetry $\mathcal{L}_\xi \bar{g}_{\mu\nu} = 0 \Longrightarrow \quad \delta^1_\xi h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu}$

$\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ Poincaré invariance of Pauli-Fierz theory

Algebra: Surface charges form a representation of the algebra of Killing vectors

$$\{Q_{\xi_1}, Q_{\xi_2}\} := \delta^1_{\xi_1} Q_{\xi_2} = Q_{[\xi_1,\xi_2]}$$

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$g_{\mu\nu} = \bar{g}_{\mu\nu} + O(\frac{1}{r^{\chi_{\mu\nu}}})$ at boundary $r \longrightarrow \infty$

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Central extensions: Algebra & asymptotics

new feature: asymptotic Killing vectors $\mathcal{L}_\xi \bar{g}_{\mu\nu} \to 0$ to leading order

that preserve the fall-off conditions $\mathcal{L}_\xi g_{\mu\nu} = O(\frac{1}{r^{\chi_{\mu\nu}}})$

suitable tuning of fall-off conditions on metrics and asymptotic Killing vectors:

centrally extended charge representation of algebra of asymptotic Killing vectors

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NB: central extension vanishes for exact symmetries of the background

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Central extensions: Asymptotically ADS spacetimes

non trivial asymptotic Kvf= conformal Kvf of flat boundary metric

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = -(1+\frac{r^2}{l^2})d\tau^2 + +(1+\frac{r^2}{l^2})^{-1}dr^2 + r^2 \sum_A f_A (dy^A)^2,$$

n>3: $\mathfrak{so}(n-1,2)$ only exact Killing vectors of AdS, no central extension

n=3: pseudo-conformal algebra in 2 dimensions, 2 copies of Wit algebra

charge algebra: 2 copies of Virasoro

$$i\{\mathcal{L}^\pm_m, \mathcal{L}^\pm_n\} = (m-n)\mathcal{L}^\pm_{m+n} + \frac{c}{12}m(m^2-1)\delta_{n+m},$$

$$\{\mathcal{L}^\pm_m, \mathcal{L}^\mp_n\} = 0,$$

Brown & Henneaux CMP 104 (1986) 207

where $c = \frac{3l}{2G}$ is the central charge for the anti-de Sitter case.

similar results in de Sitter spacetimes at timelike infinity

Central extensions: Algebra & asymptotics

new feature: asymptotic Killing vectors $\mathcal{L}_\xi \bar{g}_{\mu\nu} \to 0$ to leading order

that preserve the fall-off conditions $\mathcal{L}_\xi g_{\mu\nu} = O(\frac{1}{r^{\chi_{\mu\nu}}})$

suitable tuning of fall-off conditions on metrics and asymptotic Killing vectors:

centrally extended charge representation of algebra of asymptotic Killing vectors

$$\{Q_{\xi_1}, Q_{\xi_2}\} := \delta_{\xi_1} Q_{\xi_2} = Q_{[\xi_1,\xi_2]} + K_{\xi_1,\xi_2} \qquad K_{\xi_1,\xi_2} = \oint_{S^\infty} k_{\xi_2}[\mathcal{L}_{\xi_1}\bar{g}, \bar{g}]$$

NB: central extension vanishes for exact symmetries of the background

G.B. & F. Brandt Nucl. Phys. B633 (2002) 3-82

G.B. & G. Compère JMP 49 (2008) 042901

Central extensions: Asymptotically ADS spacetimes

non trivial asymptotic Kvf= conformal Kvf of flat boundary metric

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu}dx^\mu dx^\nu = -(1+\frac{r^2}{l^2})d\tau^2 + +(1+\frac{r^2}{l^2})^{-1}dr^2 + r^2\sum_A f_A(dy^A)^2,$$

n>3: $\mathfrak{so}(n-1,2)$ only exact Killing vectors of AdS, no central extension

n=3: pseudo-conformal algebra in 2 dimensions, 2 copies of Wit algebra

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Central extensions: Asymptotically flat spacetimes

conformal boundary in asymptotically flat spacetimes: null infinity

Introducing the retarded time $u = t - r$, the luminosity distance r and angles θ^A on the $(n-2)$-sphere by $x^1 = r\cos\theta^1$, $x^A = r\sin\theta^1 \cdots \sin\theta^{A-1}\cos\theta^A$, for $A = 2, \ldots, n-2$, and $x^{n-1} = r\sin\theta^1 \cdots \sin\theta^{n-2}$, the Minkowski metric is given by

$$d\bar{s}^2 = -du^2 - 2\,du\,dr + r^2 \sum_{A=1}^{n-2} s_A (d\theta^A)^2, \tag{3.1}$$

where $s_1 = 1$, $s_A = \sin^2\theta^1 \cdots \sin^2\theta^{A-1}$ for $2 \leqslant A \leqslant n-2$. The (future) null boundary is defined by $r = \text{constant} \to \infty$ with u, θ^A held fixed.

$$\mathfrak{bms}_n$$

$$\begin{aligned}
\xi^u &= T(\theta^A) + u\partial_1 Y^1(\theta^A) + o(r^0), \\
\xi^r &= -r\partial_1 Y^1(\theta^A) + o(r), \\
\xi^A &= Y^A(\theta^B) + o(r^0), \qquad A = 1, \ldots, n-2,
\end{aligned}$$

$Y^A(\theta^A)$ conformal Kvf of n-2 sphere

$T(\theta^A)$ "supertranslations", arbitrary function on n-2 sphere

Central extensions: Asymptotically flat spacetimes

$$\widehat{\xi} = [\xi, \xi']$$

$$\hat{T} = Y^A \partial_A T' + T \partial_1 Y'^1 - Y'^A \partial_A T - T' \partial_1 Y^1,$$
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algebra: semi-direct product with abelian ideal $\mathfrak{i}_{n-2}$

n>4: $\mathfrak{so}(n-1,1) \ltimes \mathfrak{i}_{n-2}$

n=4: conformal algebra in 2d $\ltimes \; \mathfrak{i}_2$

$$\cup$$
$$\mathfrak{so}(3,1)$$

Bondi-Metzner-Sachs (1962)

Central extensions: Asymptotically flat spacetimes

=3: no restriction on $Y(\theta)$ $J_n = \xi(T=0, Y=\exp(in\theta))$ $P_n \equiv \xi(T=\exp(in\theta), Y=0)$

$$\mathrm{i}[J_m, J_n] = (m-n)J_{m+n},$$
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copy of Wit algebra acting on $\mathfrak{i}_1$

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Ashtekar et al. Phys. Rev. D55 (1997) 669

charge algebra:

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relation to AdS_3 similar to contraction between $\mathfrak{so}(2,2) \to \mathfrak{iso}(2,1)$

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G.B. & G. Compère CQG 24 (2007) F15

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Central extensions: Extremal Kerr in 4d

Strominger et al: use formalism to compute central extension for extremal Kerr in 4d

combine with Cardy formula to argue for microscopic explanation of Bekenstein-Hawking entropy

Guica, Hartman, Song & Strominger,
The Kerr/CFT correspondence, arXiv:0809.4266

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