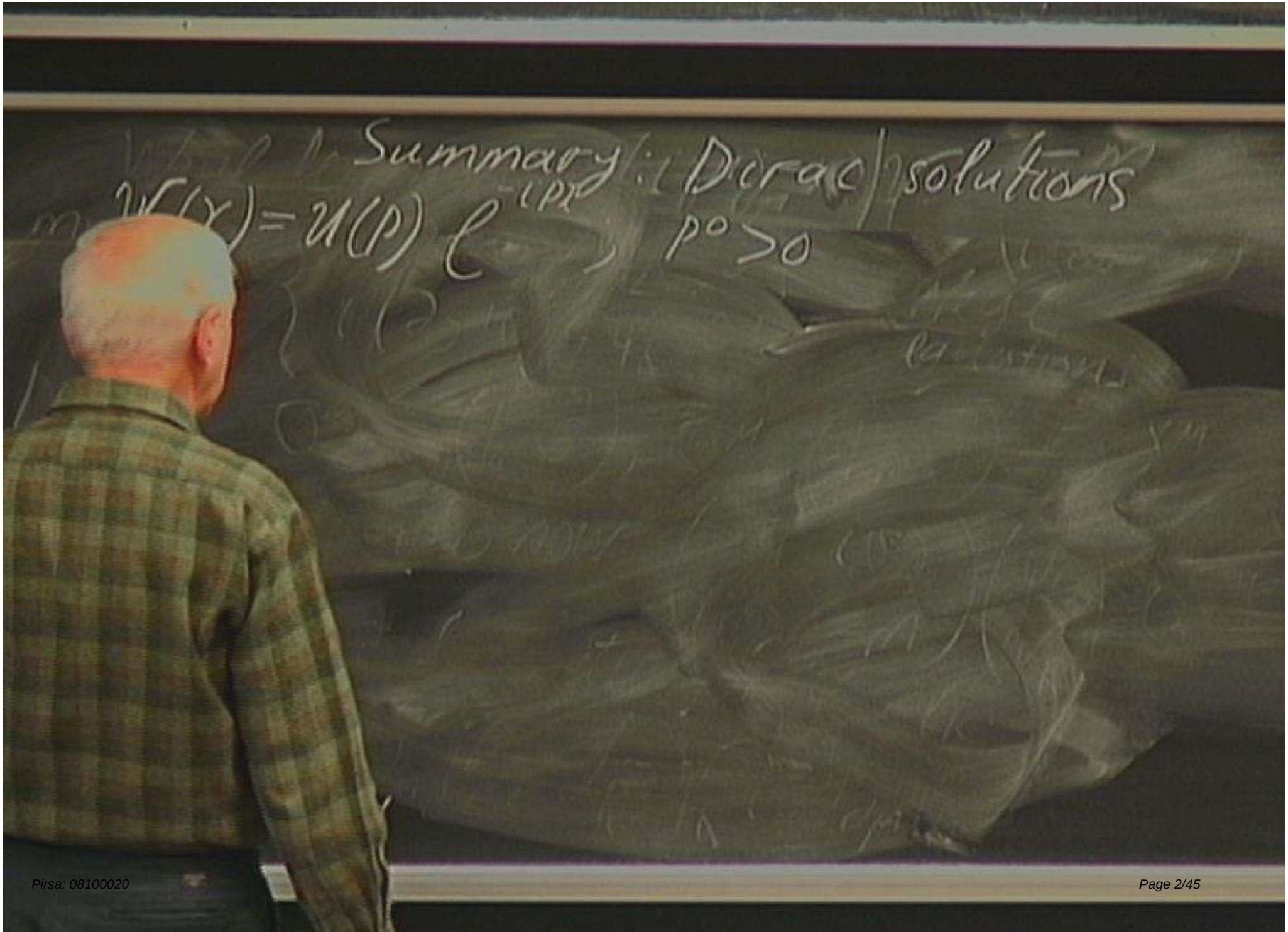
Title: Quantum Field Theory 1 - Lecture 8B

Date: Oct 29, 2008 03:30 PM

URL: http://pirsa.org/08100020

Abstract: Quantum Field Theory I course taught by Volodya Miransky of the University of Western Ontario


Summary: Dirac solutions
$\psi(x) = u(p)\, e^{-(px)}$, $p^0 > 0$

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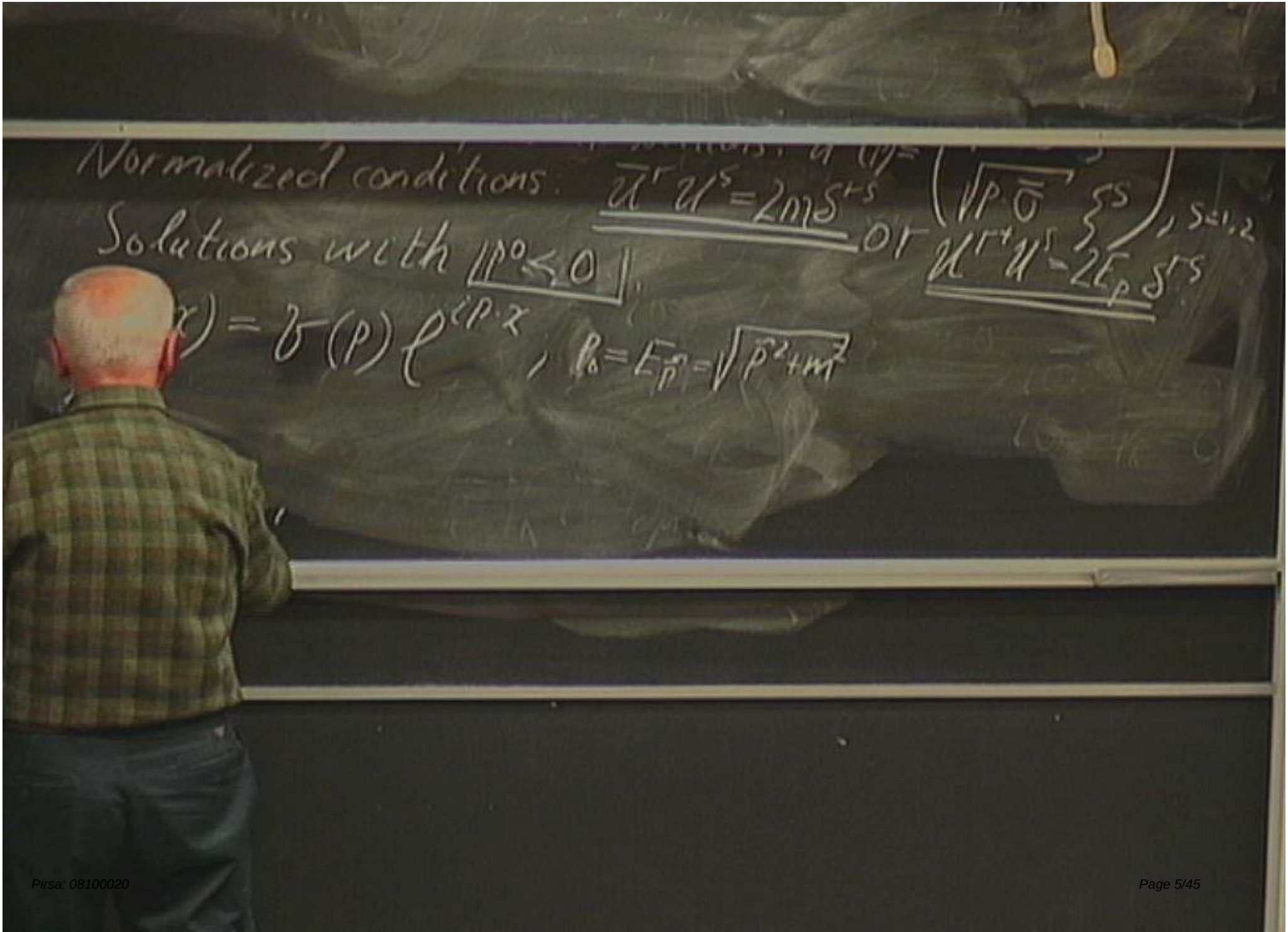
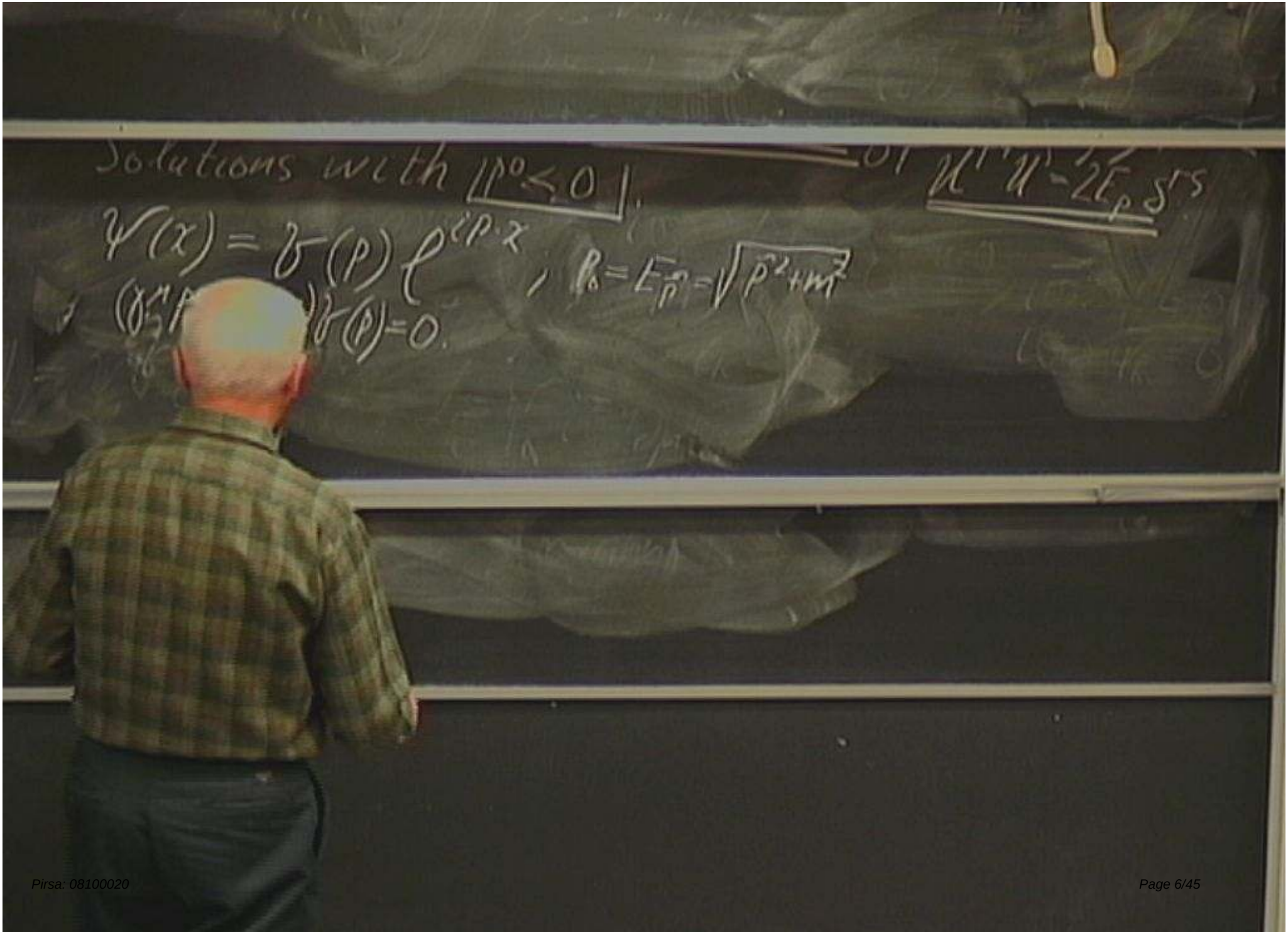
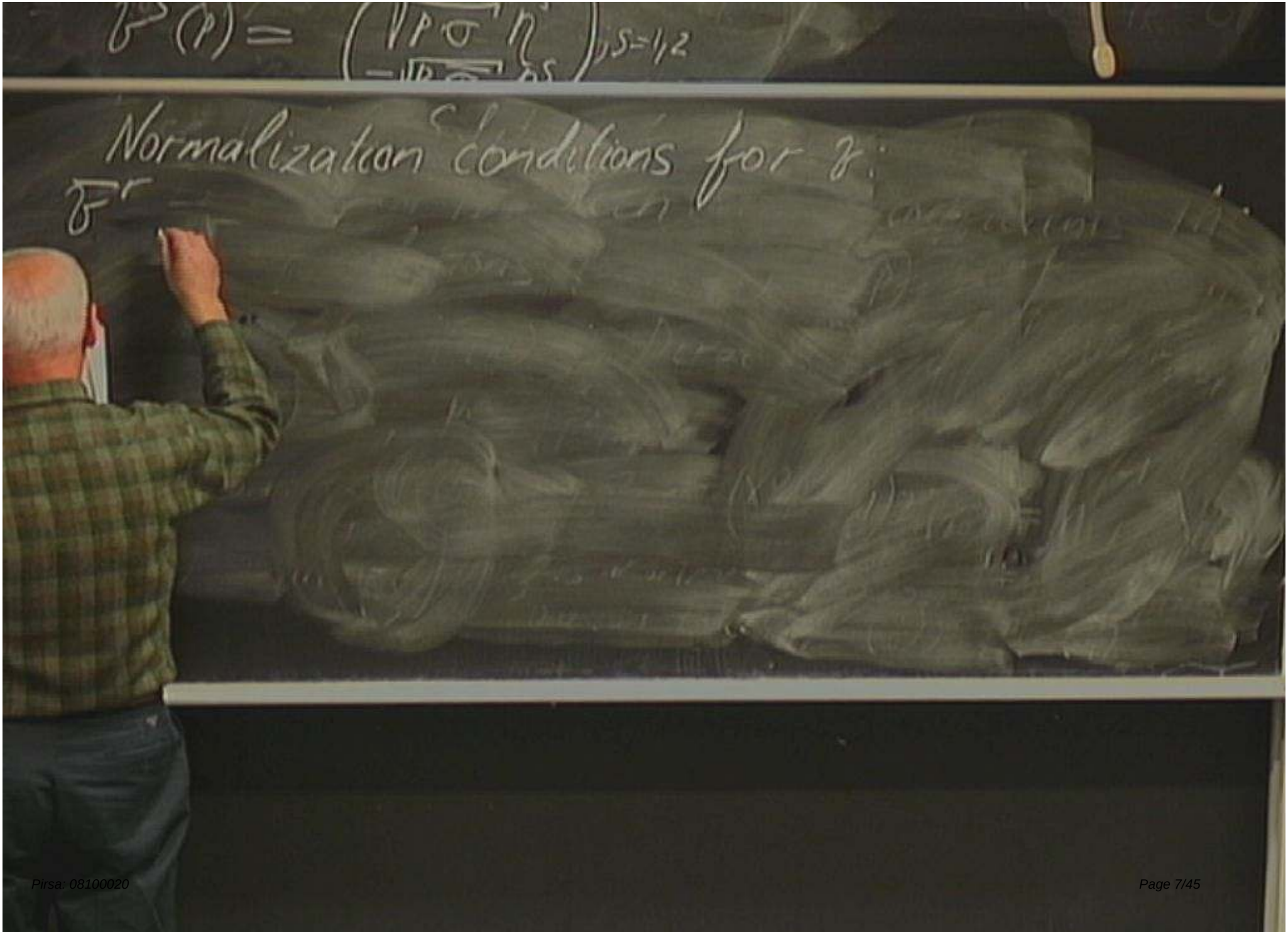
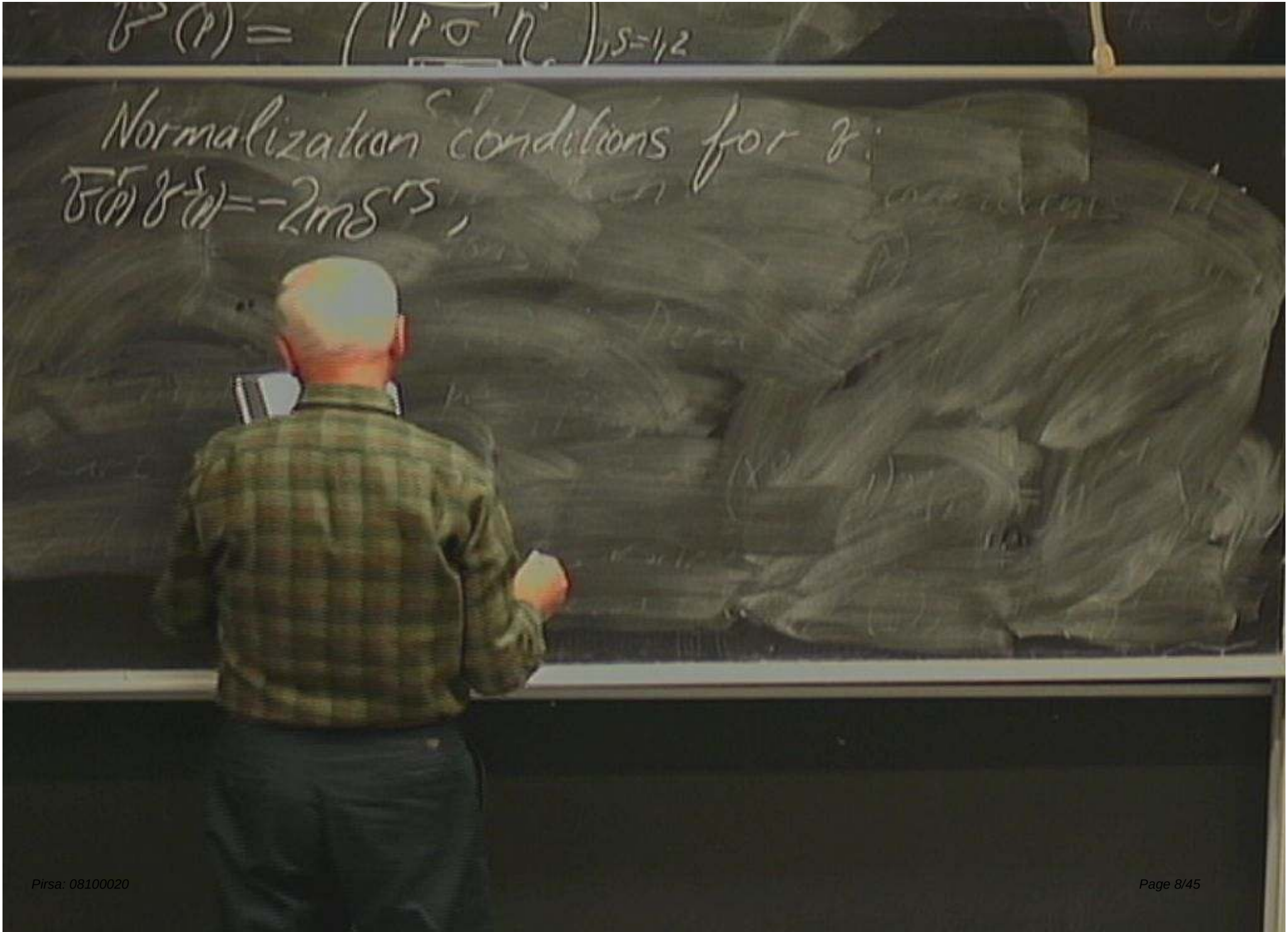
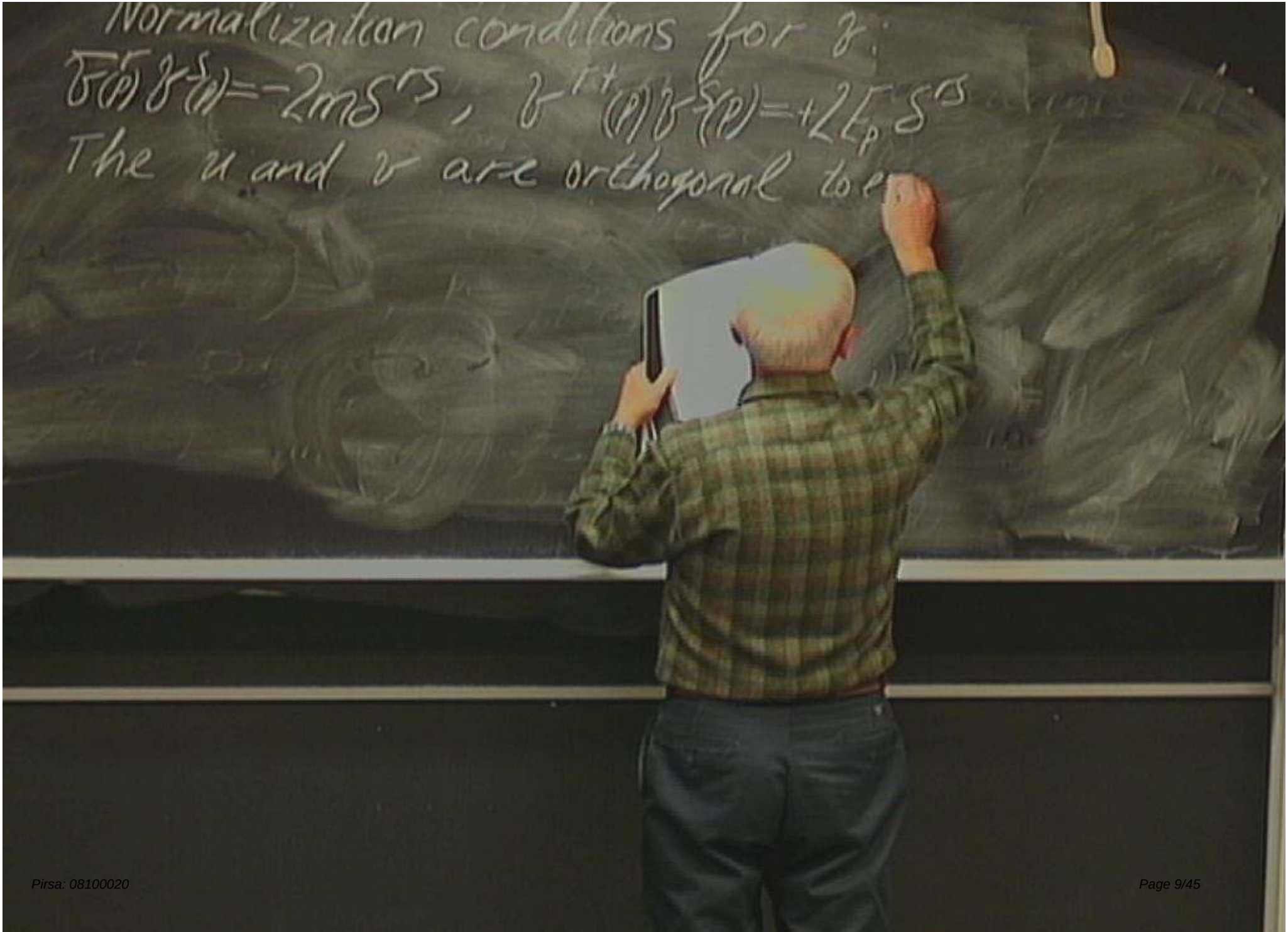
Normalized conditions: $\bar{u}^r u^s = 2m$

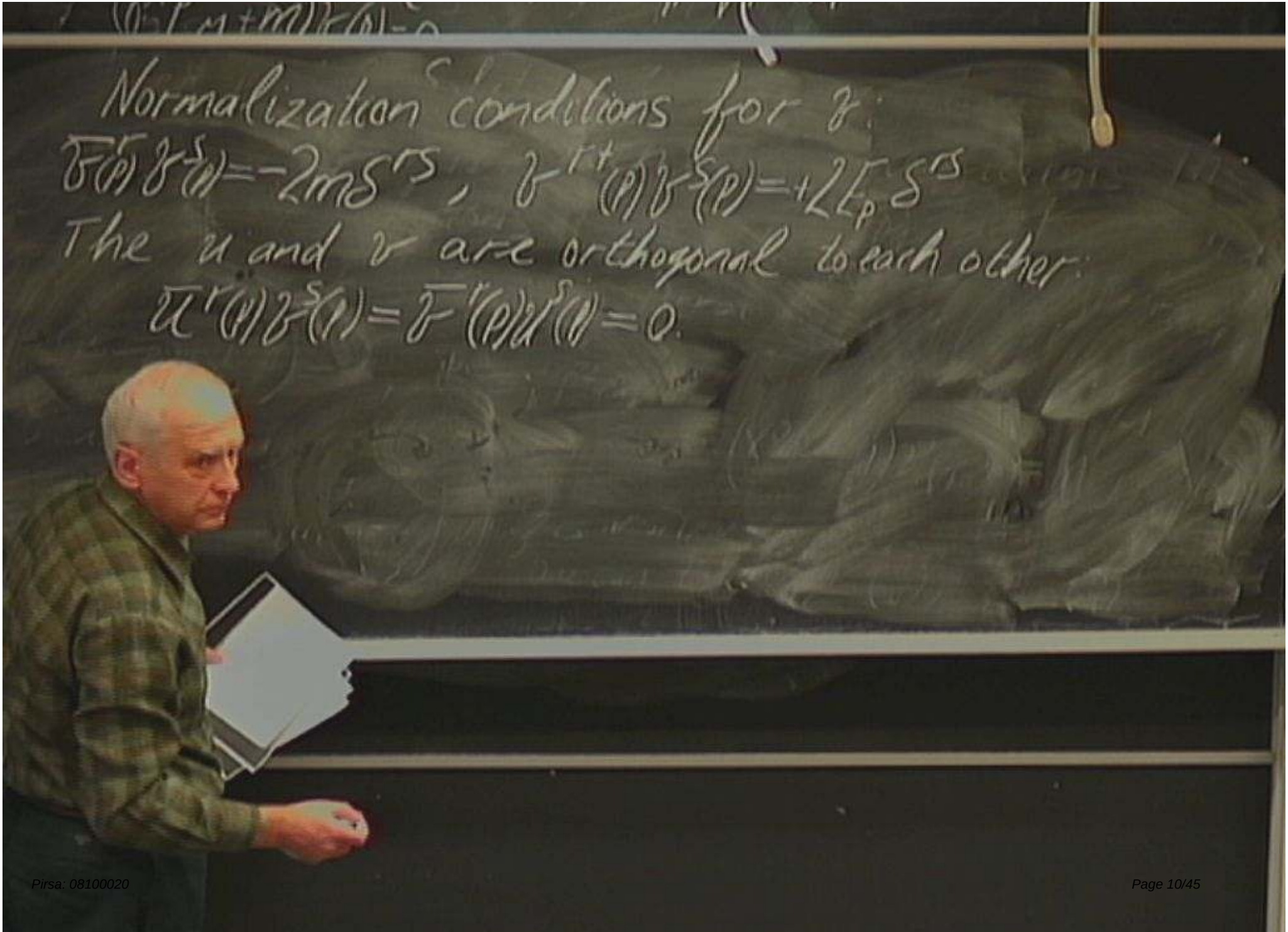

Normalized conditions
Solutions with
$\psi(x) = b(p)\, e^{ip\cdot x}$, $p_0 = E_p = \sqrt{p^2+m^2}$

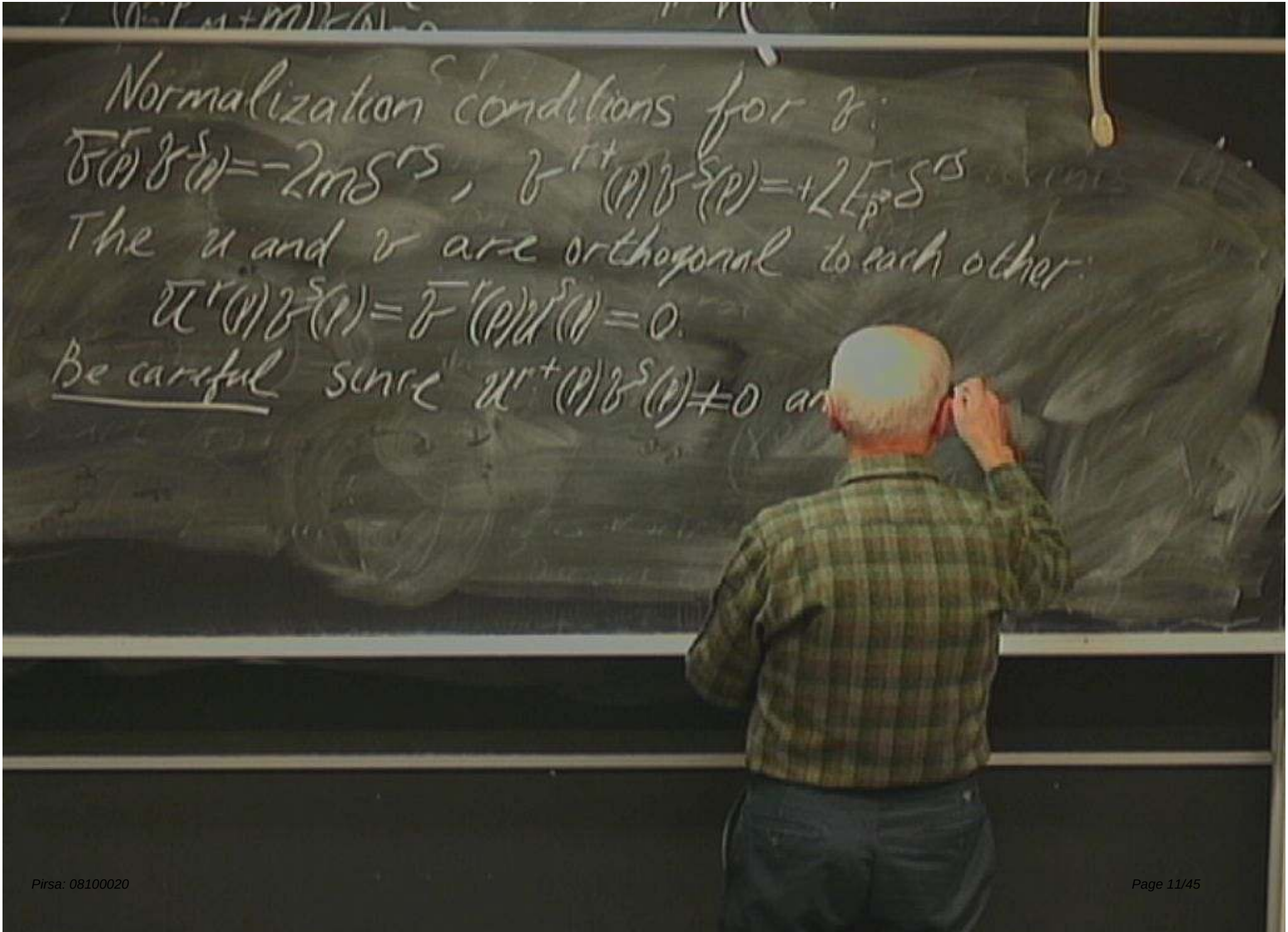

Solutions with $p^0 < 0$:
$\psi(x) = v(p)\, e^{ip\cdot x}$, $p_0 = -E_p = -\sqrt{p^2 + m^2}$
$(\gamma^\mu p_\mu + m)\, v(p) = 0$.

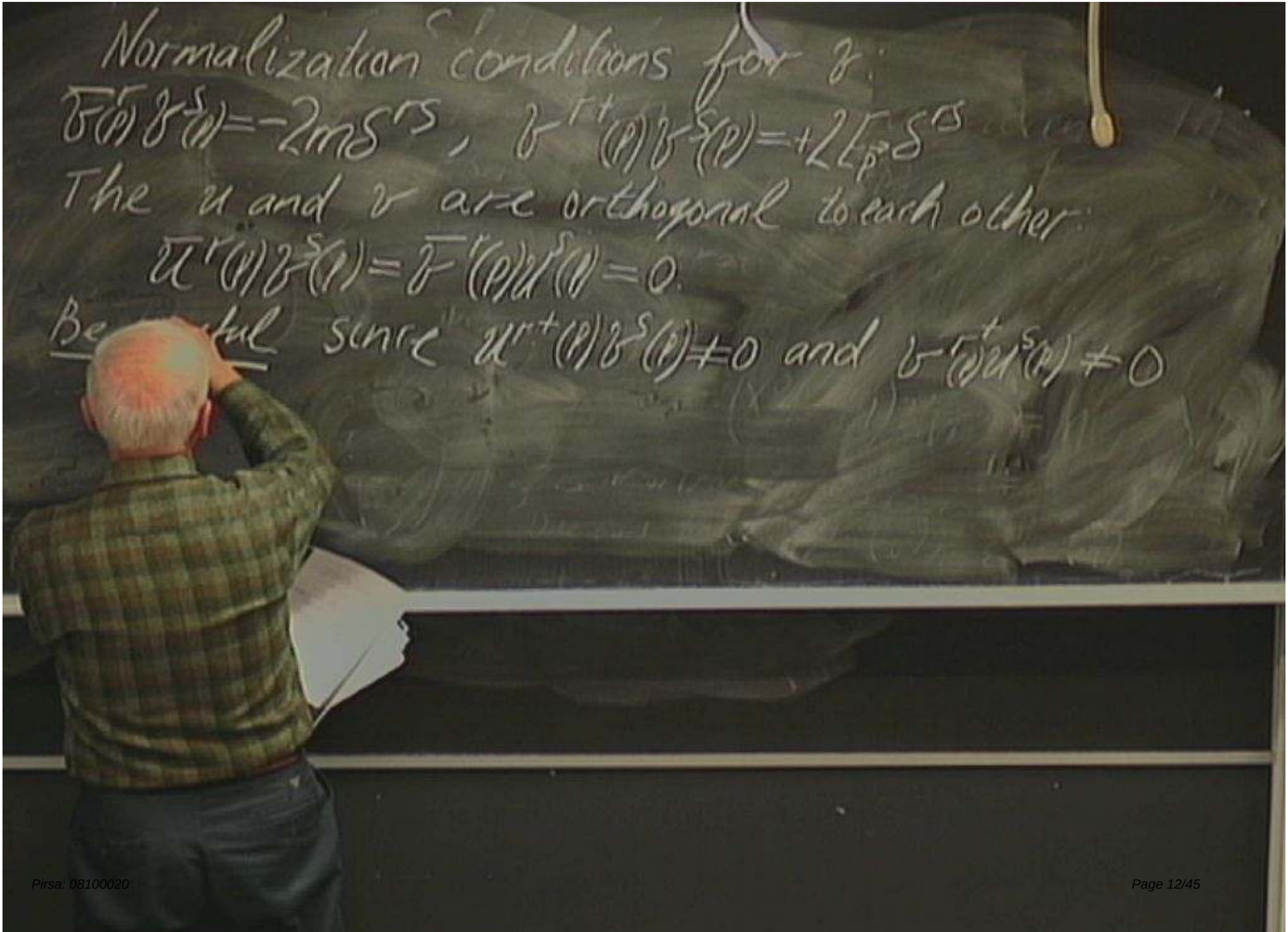

Normalization conditions for

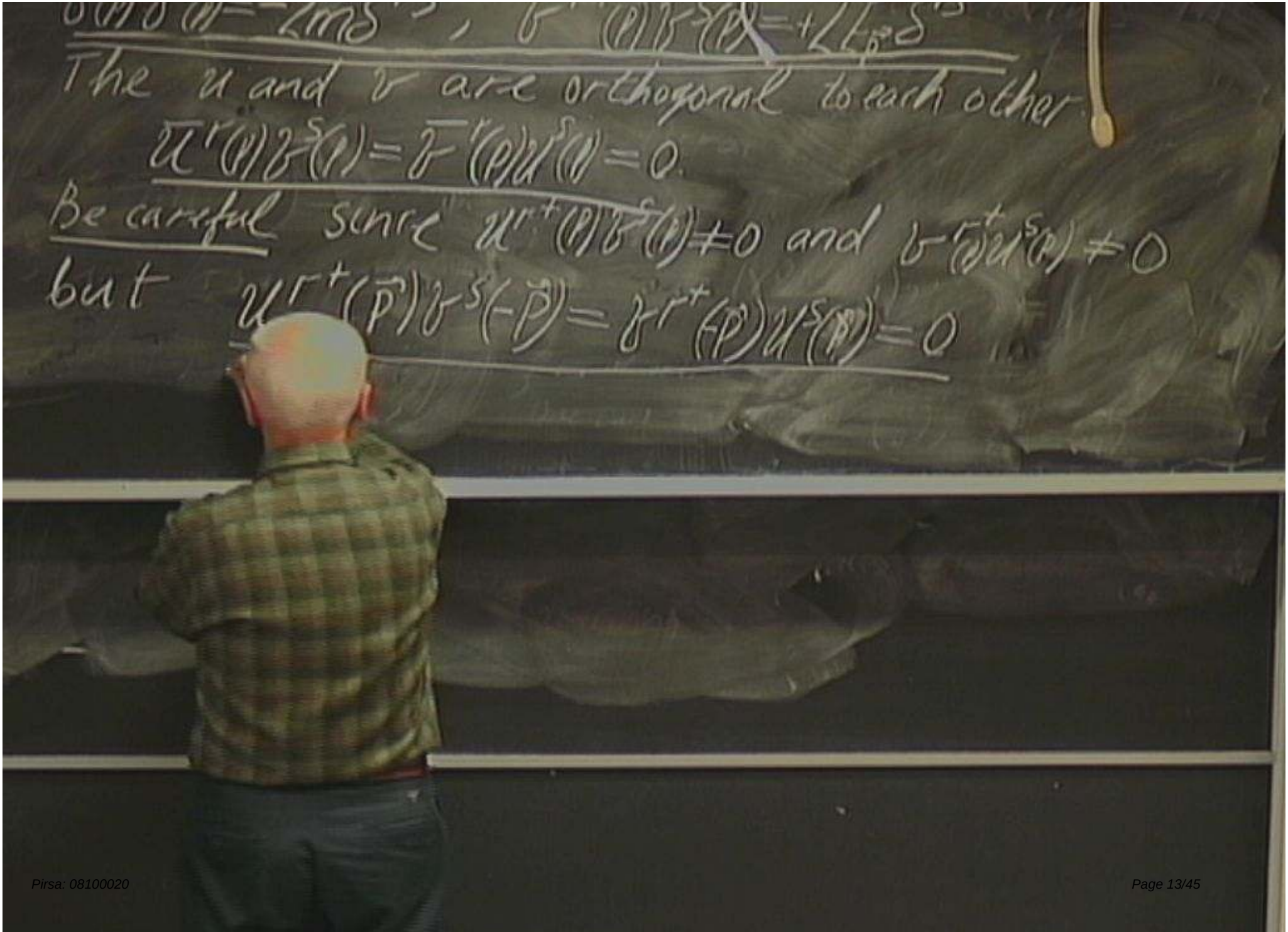

Normalization conditions for


Normalization conditions for u:
The u and v are orthogonal to e


Normalization conditions for u, v:
$\bar{u}^r(p)u^s(p) = -2m\delta^{rs}$, $v^{\dagger r}(p)v^s(p) = +2E_p\delta^{rs}$
The u and v are orthogonal to each other:
$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0$.


Normalization conditions for v:
$\bar{v}^r(p)v^s(p) = -2m\delta^{rs}$, $v^{r\dagger}(p)v^s(p) = +2E_p\delta^{rs}$
The u and v are orthogonal to each other:
$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0$.
Be careful since $u^{r\dagger}(p)v^s(p) \neq 0$ an


Normalization conditions for v:
$\bar{v}^r(p) v^s(p) = -2m\delta^{rs}$, $v^{r\dagger}(p) v^s(p) = +2E_p \delta^{rs}$
The u and v are orthogonal to each other:
$\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0$.
Because since $u^{r\dagger}(p) v^s(p) \neq 0$ and $v^{r\dagger}(p) u^s(p) \neq 0$


The u and v are orthogonal to each other
$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0$
Be careful since $u^{r+}(p)v^s(p) \neq 0$ and $v^{r+}(p)u^s(p) \neq 0$
but $u^{r+}(\vec{p})v^s(-\vec{p}) = v^{r+}(-\vec{p})u^s(\vec{p}) = 0$

$(\not{p} + m)v(p) = 0$

$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}, s = 1,2$

Normalization conditions for v:

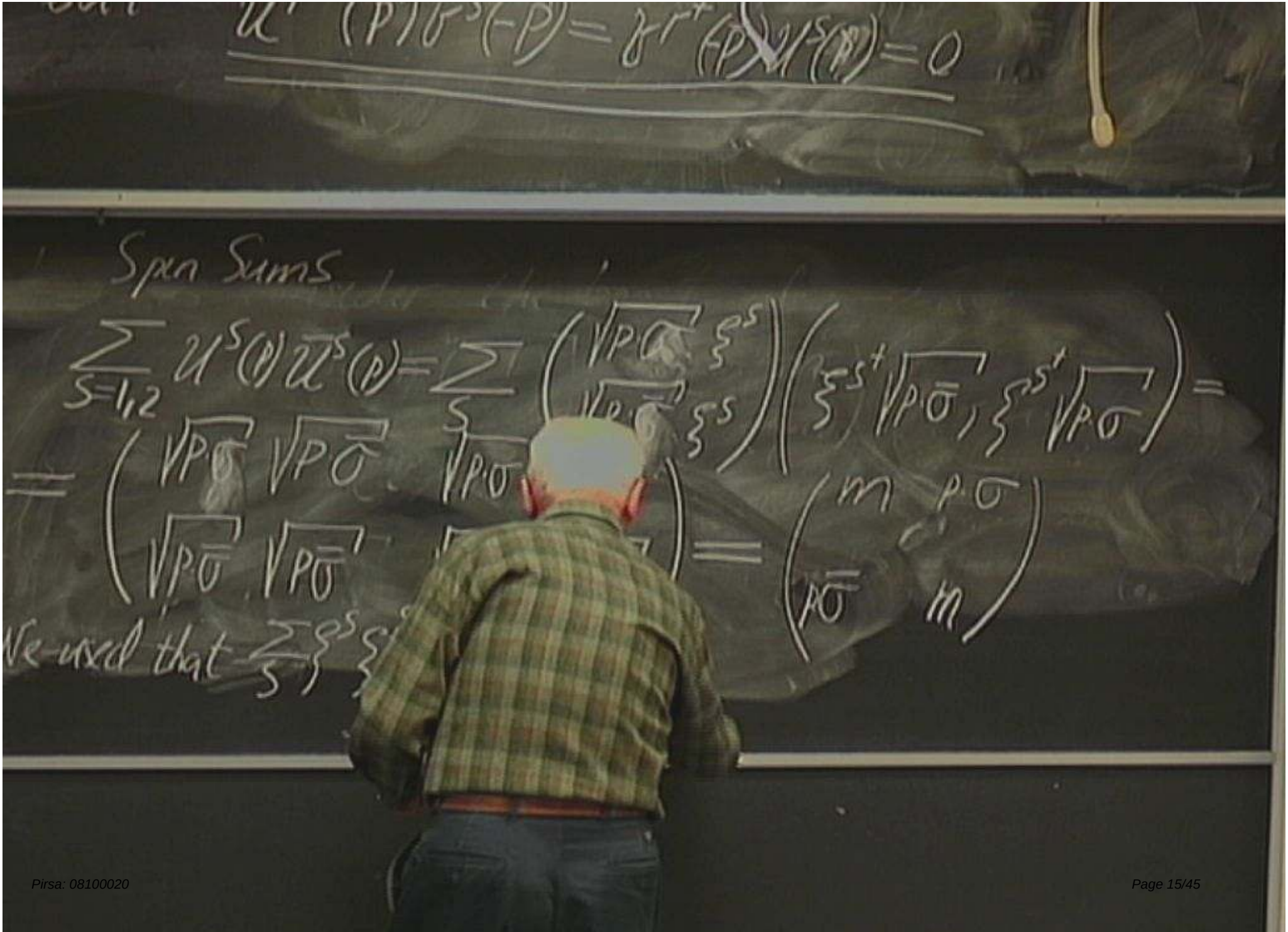
$\bar{v}^r(p) v^s(p) = -2m\delta^{rs}, \quad v^{r\dagger}(p) v^s(p) = +2E_p \delta^{rs}$

The u and v are orthogonal to each other:

$\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0$

Be careful since $u^{r\dagger}(p) v^s(p) \neq 0$ and $v^{r\dagger}(p) u^s(p) \neq 0$

but $u^{r\dagger}(\vec{p}) v^s(-\vec{p}) = v^{r\dagger}(\vec{p}) u^s(\vec{p}) = 0$


Spin Sums
We used that

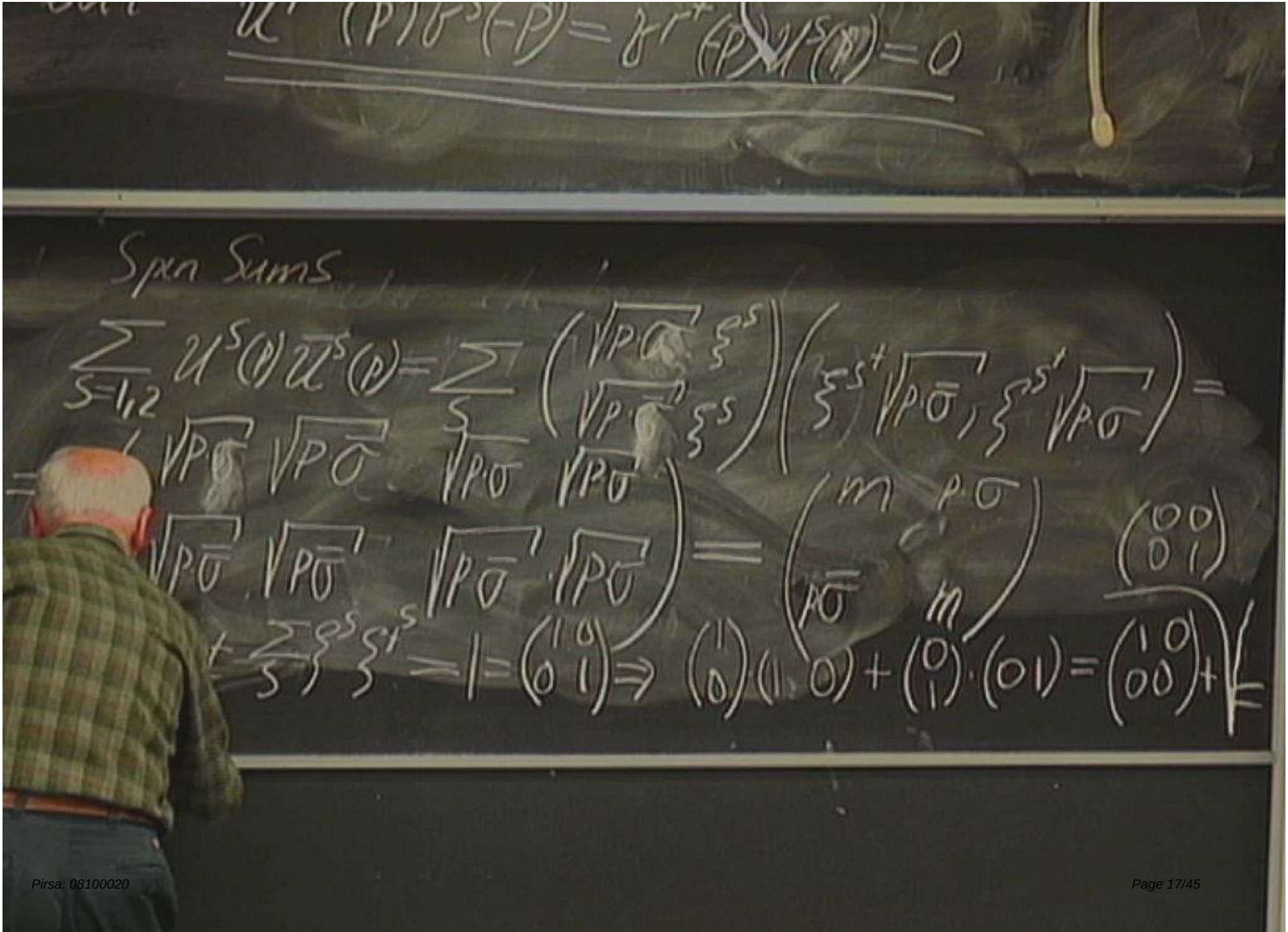
$$\bar{u}^r(p)\gamma^5(-p) = \gamma^r(-p)u^s(p) = 0$$

Spin Sums

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\sigma} \right) =$$

$$= \begin{pmatrix} \sqrt{p\sigma}\sqrt{p\bar{\sigma}} & \sqrt{p\sigma}\sqrt{p\sigma} \\ \sqrt{p\bar{\sigma}}\sqrt{p\bar{\sigma}} & \sqrt{p\bar{\sigma}}\sqrt{p\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\bar{\sigma} & m \end{pmatrix}$$

We used that $\sum_s \xi^s \xi^{s\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ (


Spin Sums

$u^{\dagger}(p)\gamma^{5}(-p) = \bar{u}^{r}(-p)\,u^{s}(p) = 0$

Spin Sums

$$\sum_{s=1,2} u^{s}(p)\bar{u}^{s}(p) = \sum_{s} \begin{pmatrix} \sqrt{p\sigma}\,\xi^{s} \\ \sqrt{p\bar{\sigma}}\,\xi^{s} \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\sigma} \right) =$$

$$= \begin{pmatrix} \sqrt{p\sigma}\sqrt{p\bar{\sigma}} & \sqrt{p\sigma}\sqrt{p\sigma} \\ \sqrt{p\bar{\sigma}}\sqrt{p\bar{\sigma}} & \sqrt{p\bar{\sigma}}\sqrt{p\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

We used that $\sum_{s} \xi^{s}\xi^{s\dagger} = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ \ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0\ \ 1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\bar{u}^r(p)\gamma^5(-p) = \gamma^r(-p)u^s(p) = 0$$

Spin Sums

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\sigma}\,\xi^s \\ \sqrt{p\bar{\sigma}}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\sigma} \right) =$$

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$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$u^{s'}(p)\,v^{s}(-p) = \bar{u}^{s'}(-p)\,v^{s}(p) = 0$

Spin Sums

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\cdot\sigma} \right) =$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \not{p} + m$$

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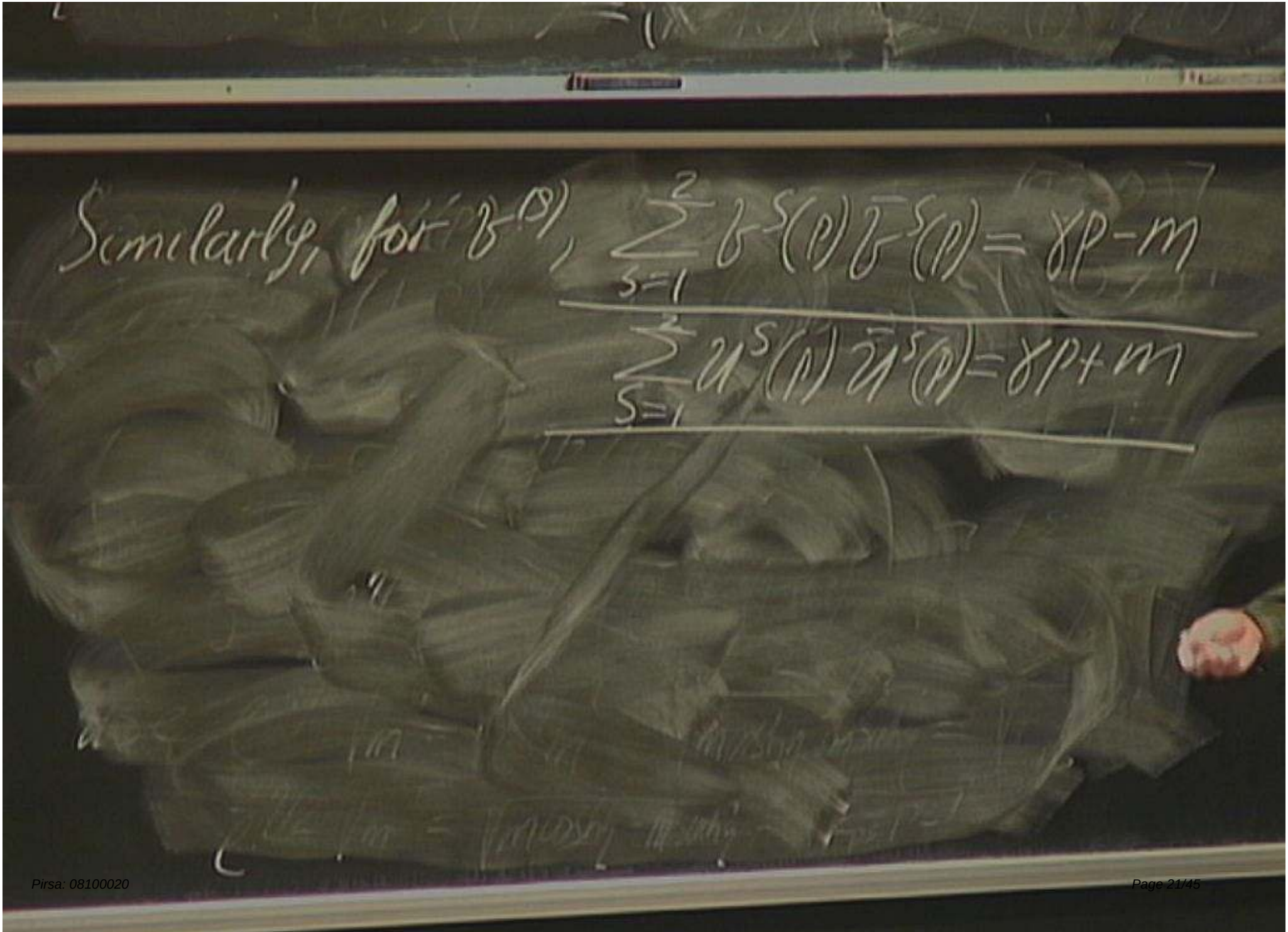
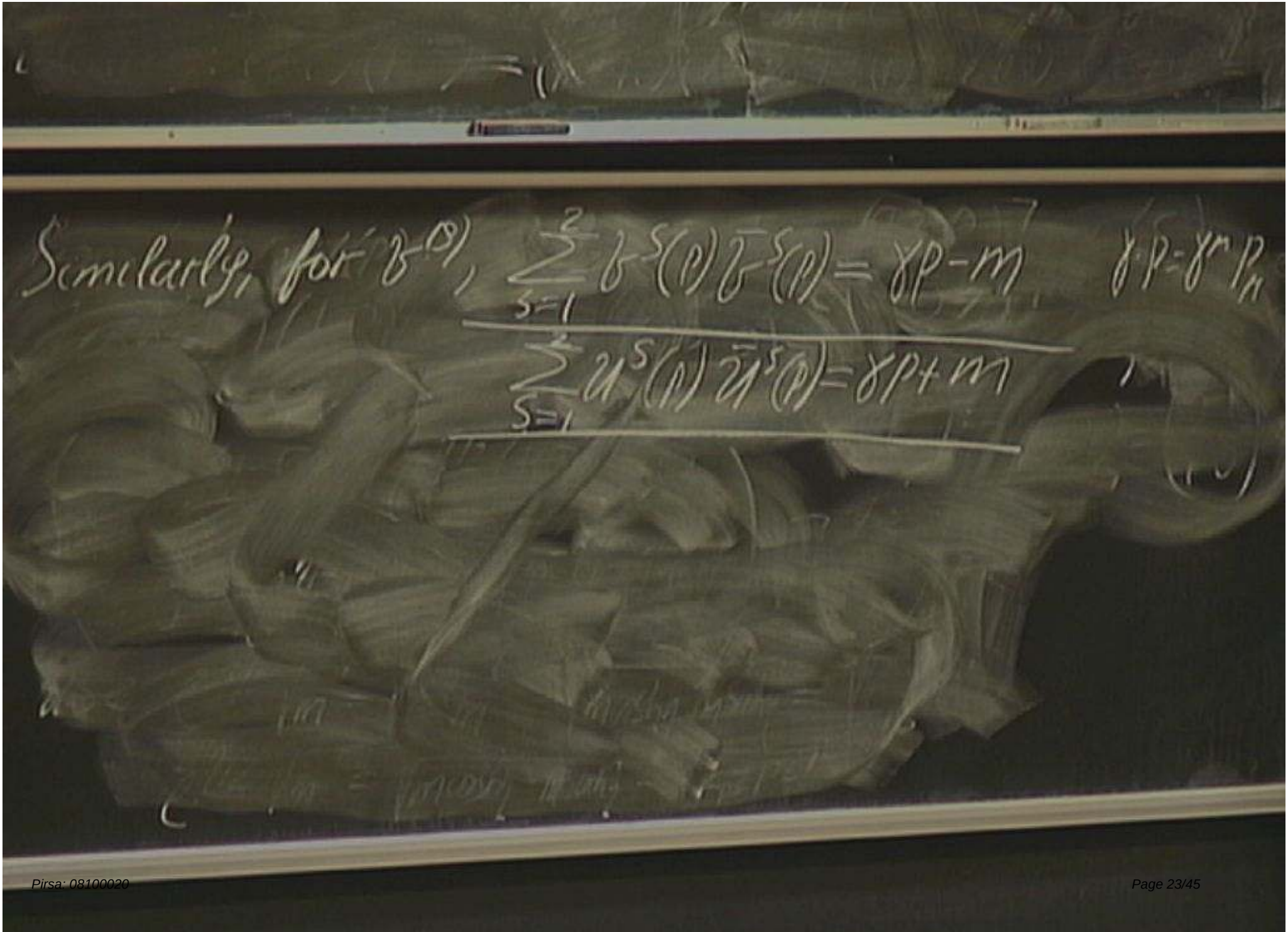
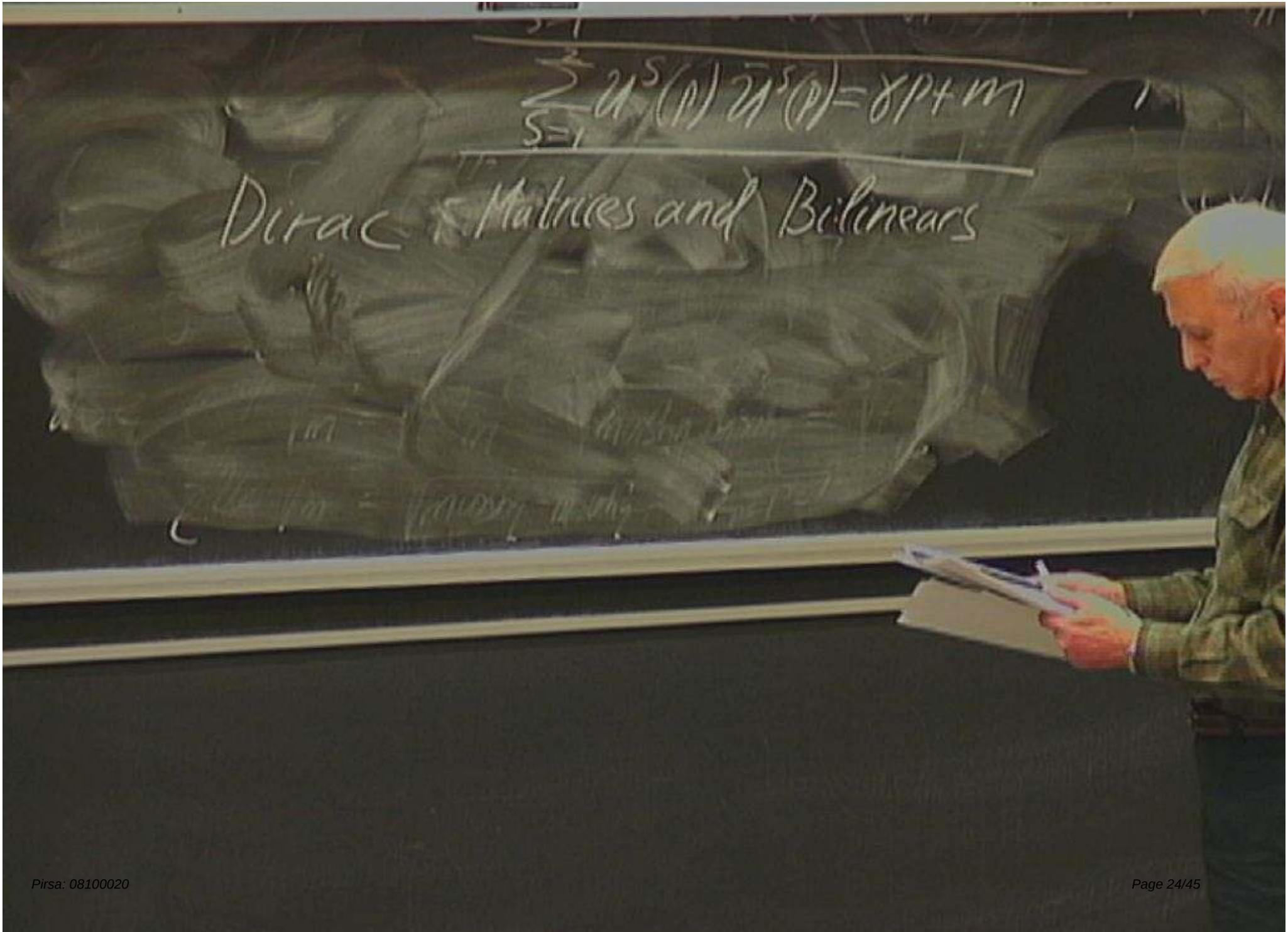
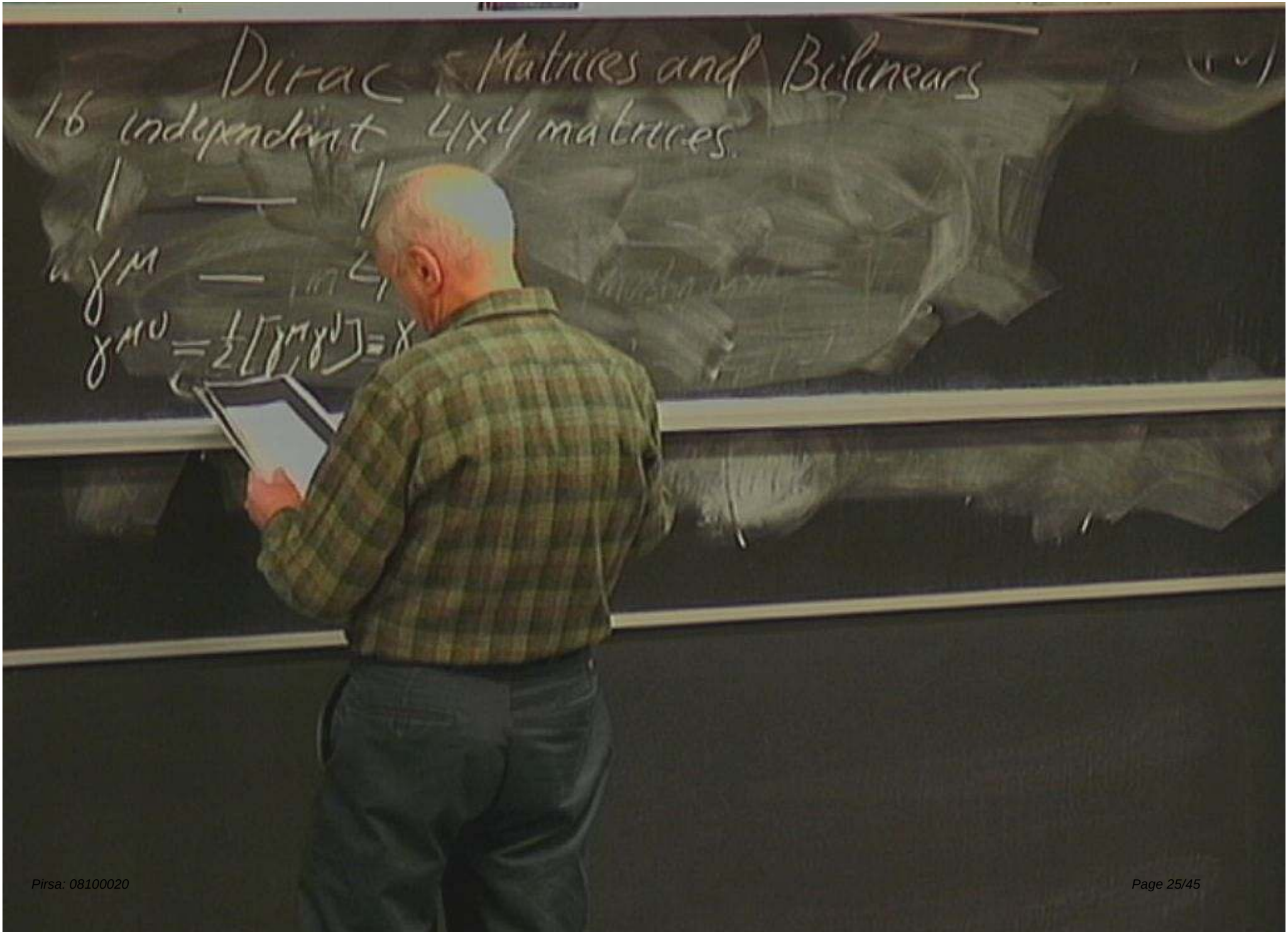

Similarly, for $v^{(0)}$, $\sum_{s=1}^{2} v^s(p)\bar{v}^s(p) = \not{p} - m$
$\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = \not{p} + m$

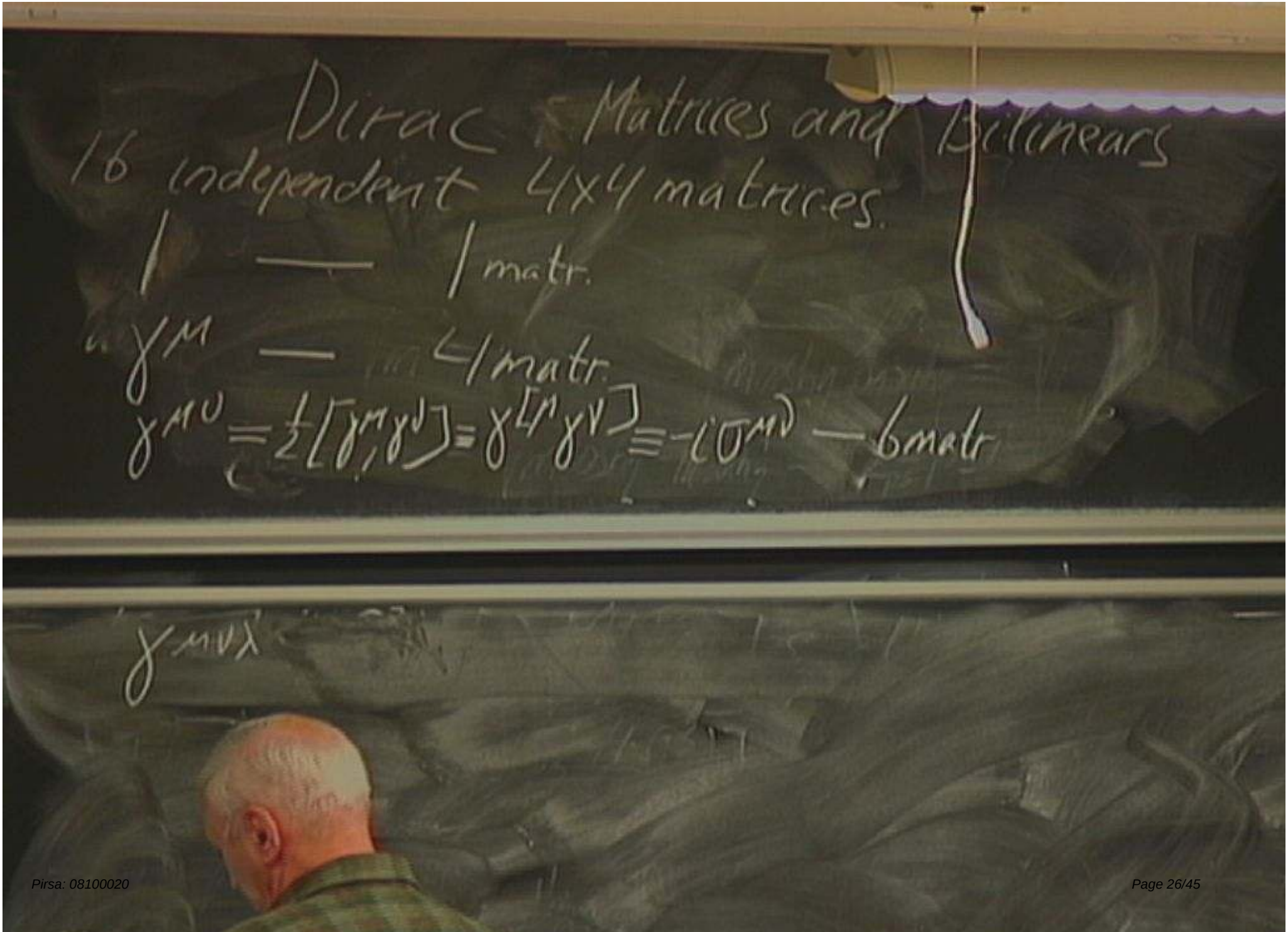
Similarly, for $v^{(s)}$, $\sum_{s=1}^{2} v^s(p)\bar{v}^s(p) = \not{p} - m$

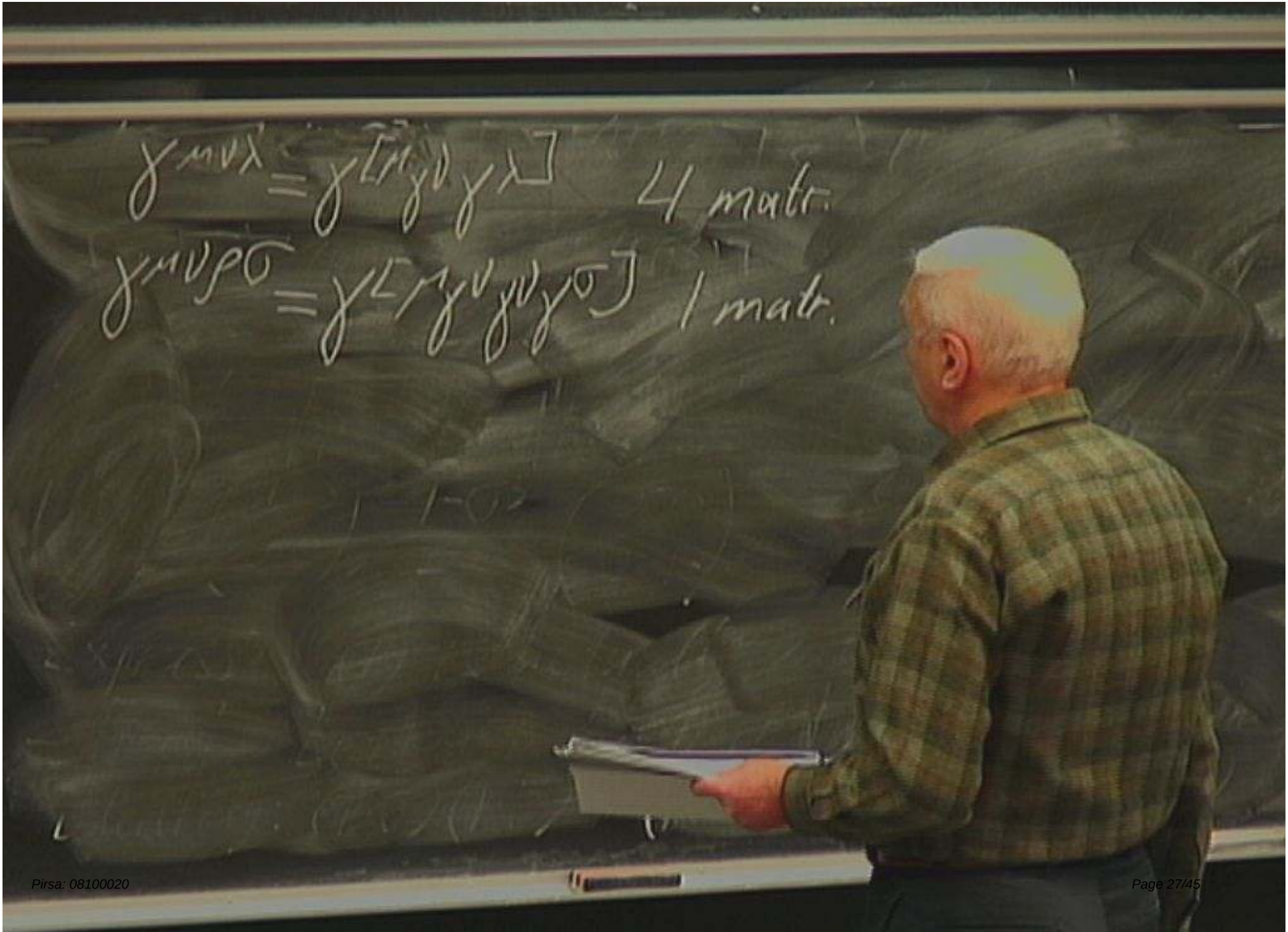
$\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = \not{p} + m$

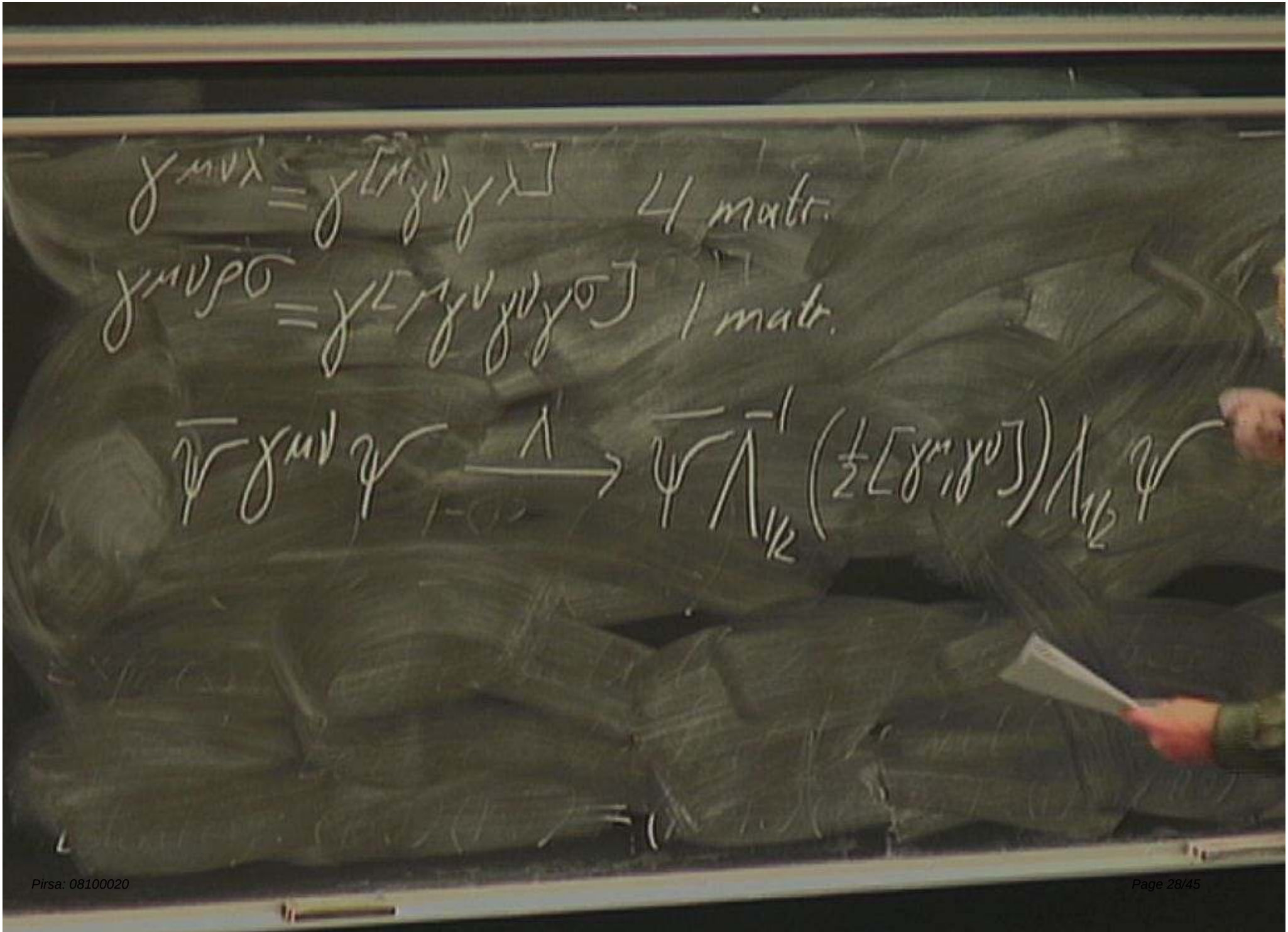

Similarly, for $\beta=0$, $\sum_{s=1}^{2} b^s(p)\bar{b}^s(p) = \not{p} - m$ $\not{p} = \gamma^\mu p_\mu$
$\sum_{s=1}^{2} u^s(p)\bar{u}^s(p) = \not{p} + m$

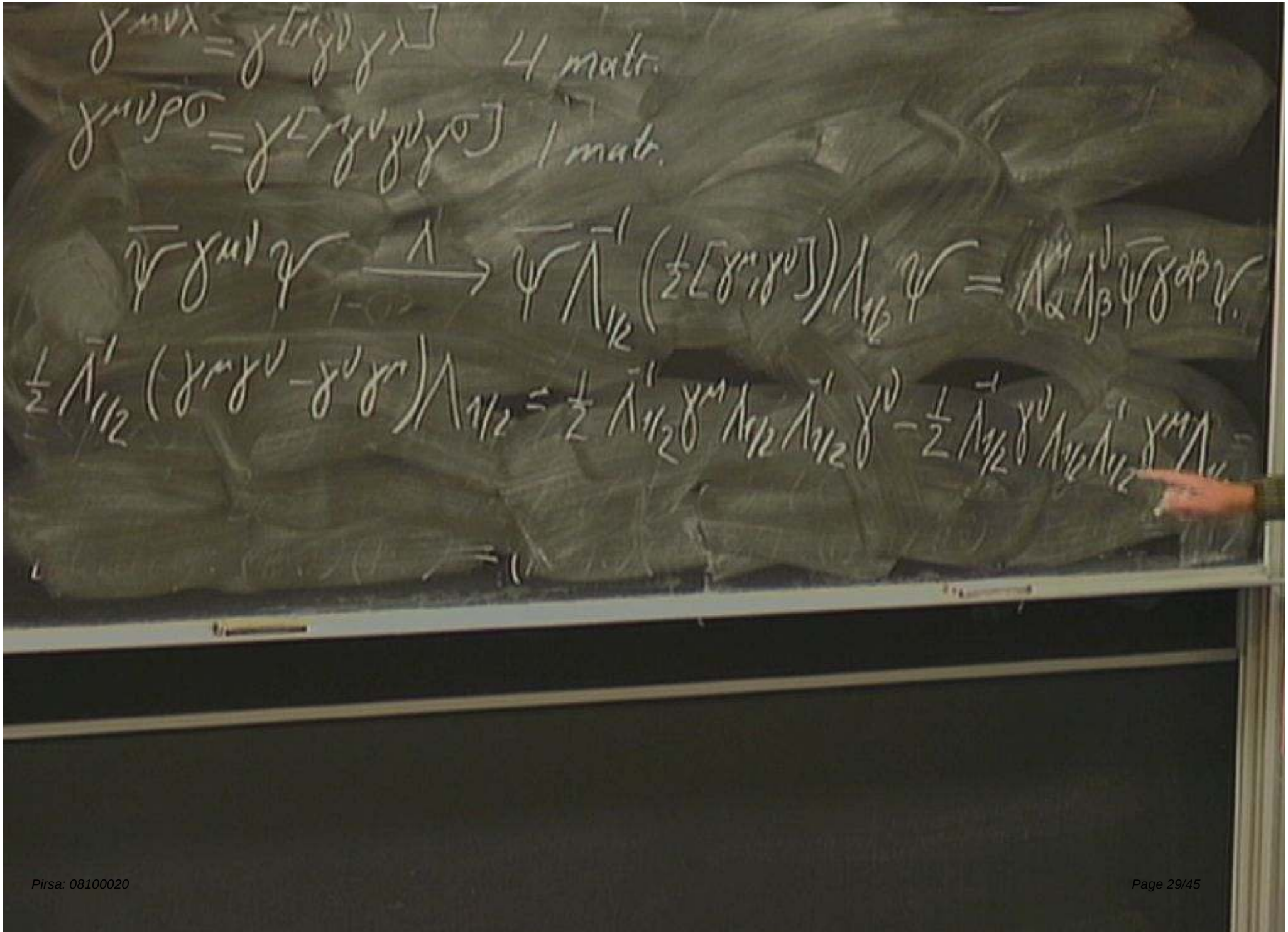

$\sum_{s=1}^{2} u^s(p)\,\bar{u}^s(p) = \not{p} + m$
Dirac Matrices and Bilinears

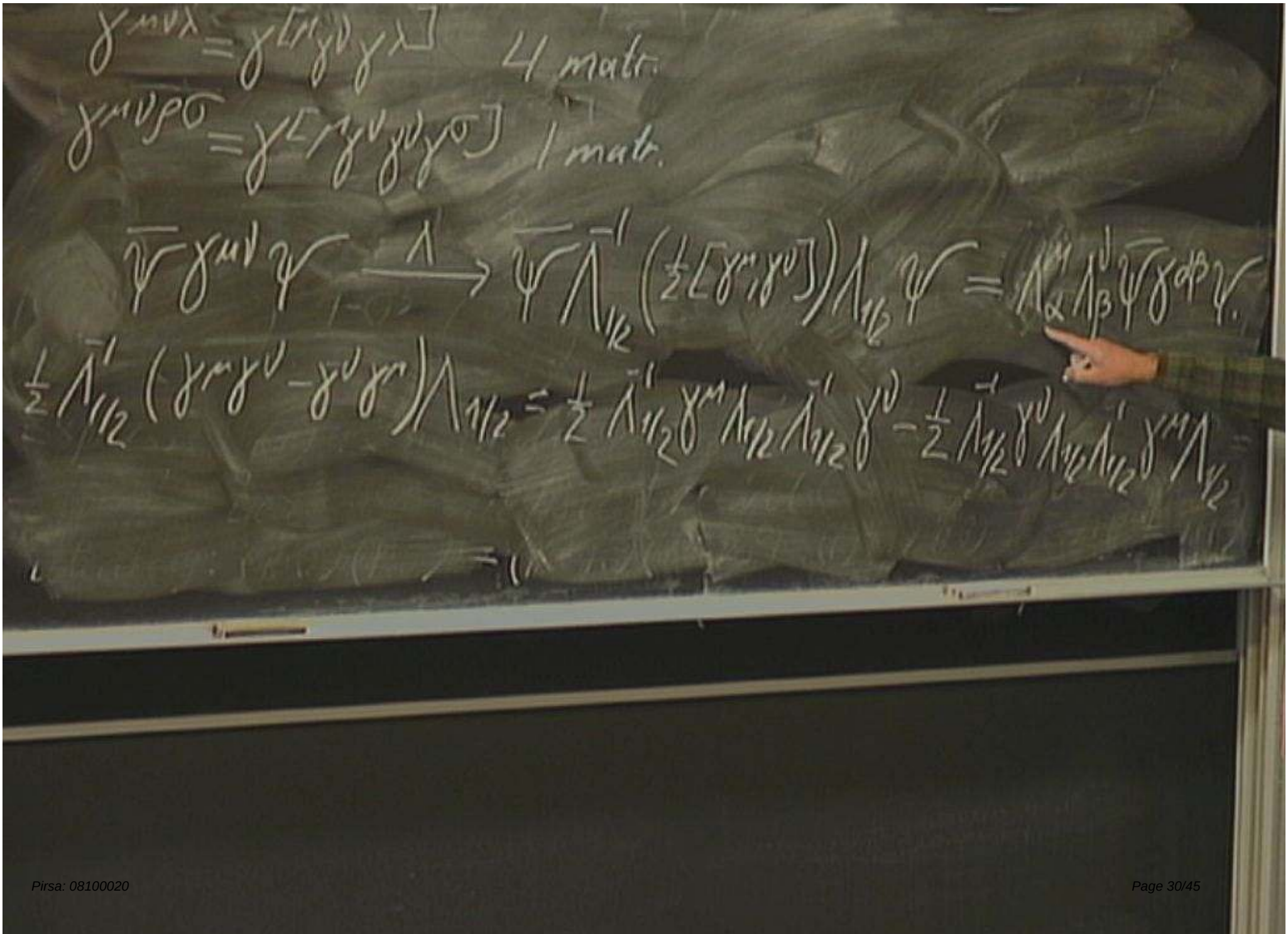

Dirac Matrices and Bilinears
16 independent 4x4 matrices

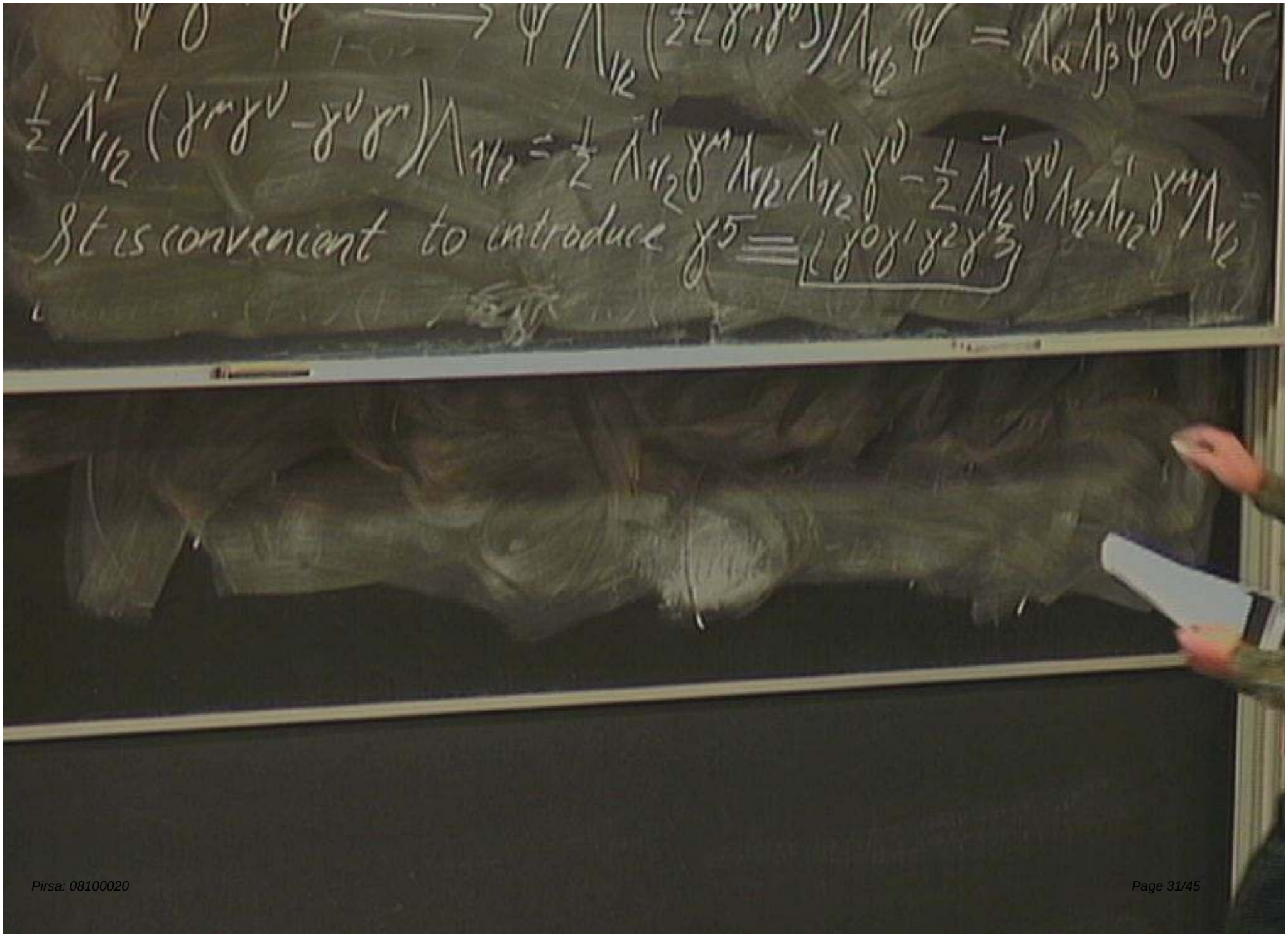

Dirac Matrices and Bilinears
16 independent 4x4 matrices.
1 — 1 matr.
γμ — 4 matr.
γμν = i/2 [γμ, γν] = γ[μ γν] ≡ −(σμν) — 6 matr
γμνλ

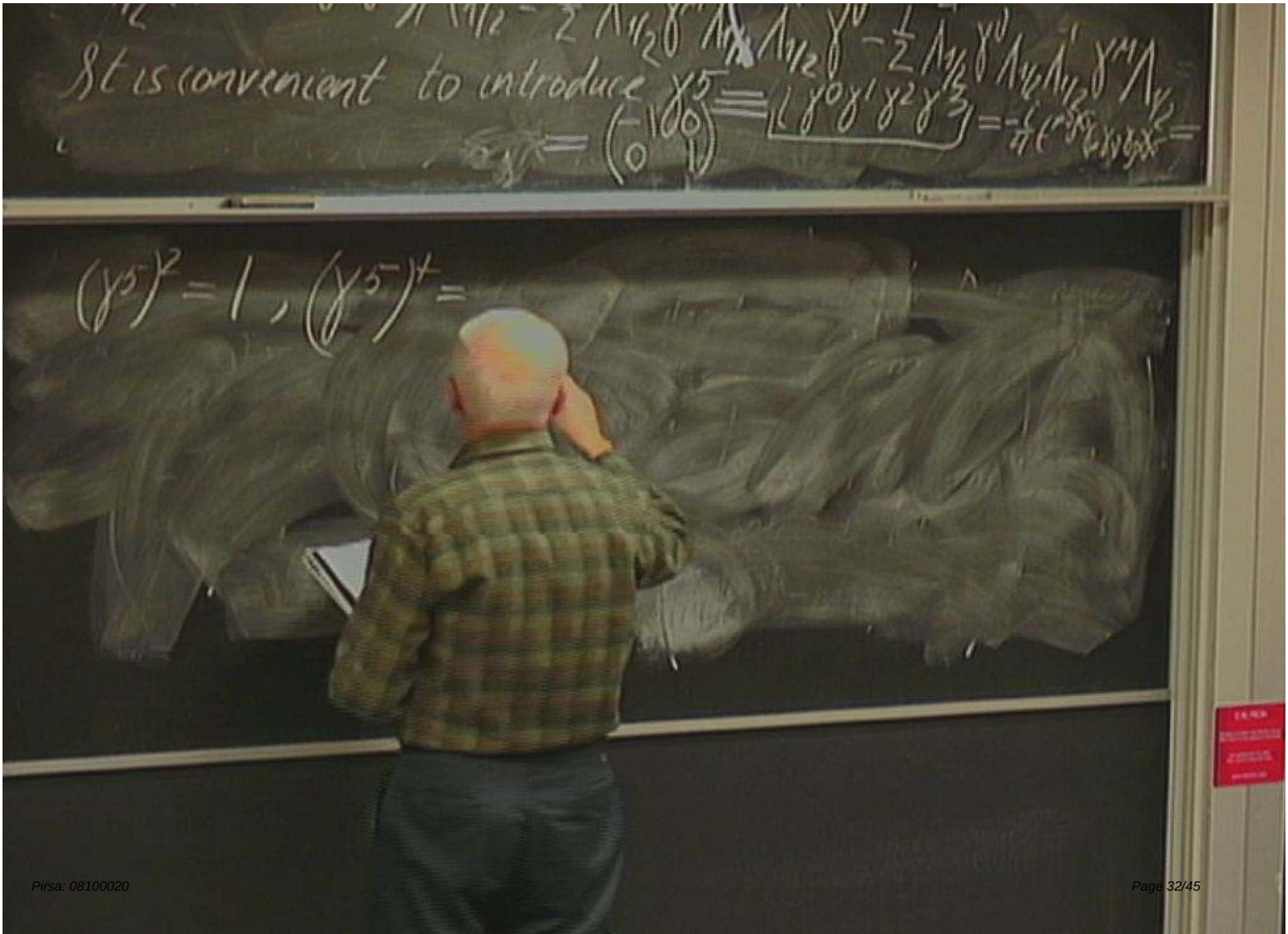

$\gamma^{\mu\nu\lambda} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda]}$ 4 matr.
$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]}$ 1 matr.

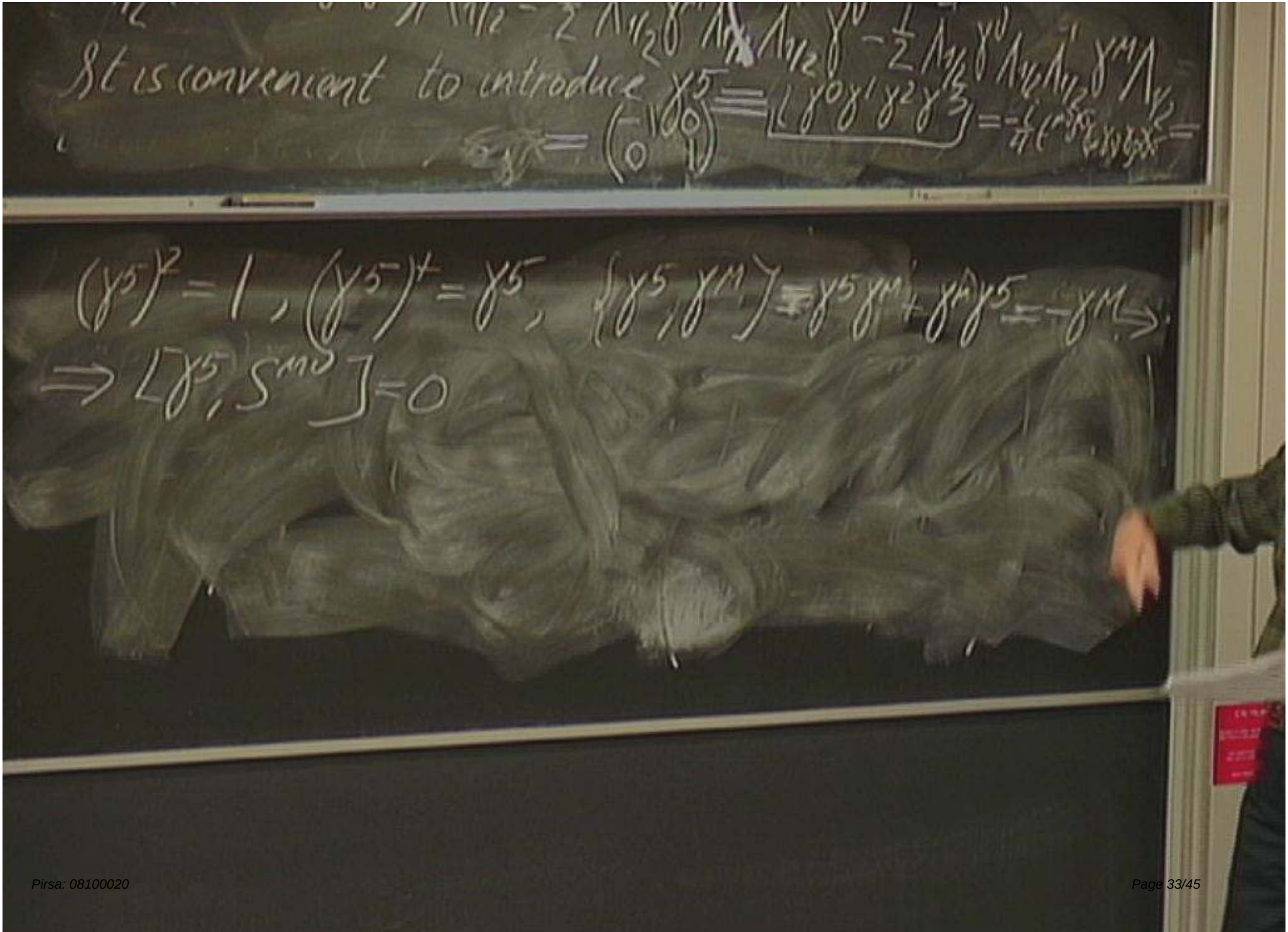

$\gamma^{\mu\nu\lambda} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\lambda]}$ 4 matr.
$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]}$ 1 matr.
$\bar{\psi}\gamma^{\mu\nu}\psi \xrightarrow{\Lambda} \bar{\psi}\Lambda^{-1}_{1/2}\left(\frac{1}{2}[\gamma^{\mu},\gamma^{\nu}]\right)\Lambda_{1/2}\psi$

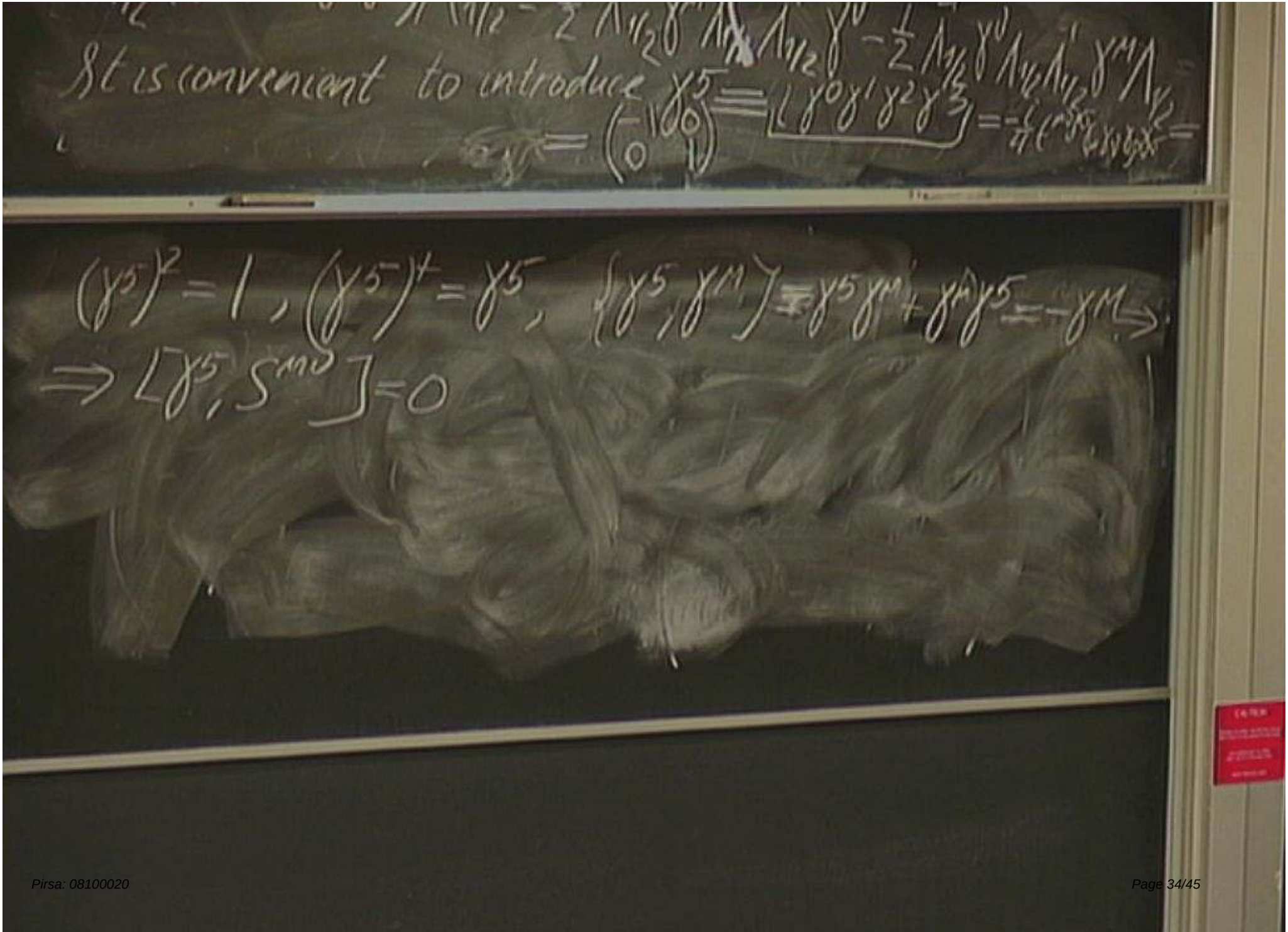

4 matr.
1 matr.

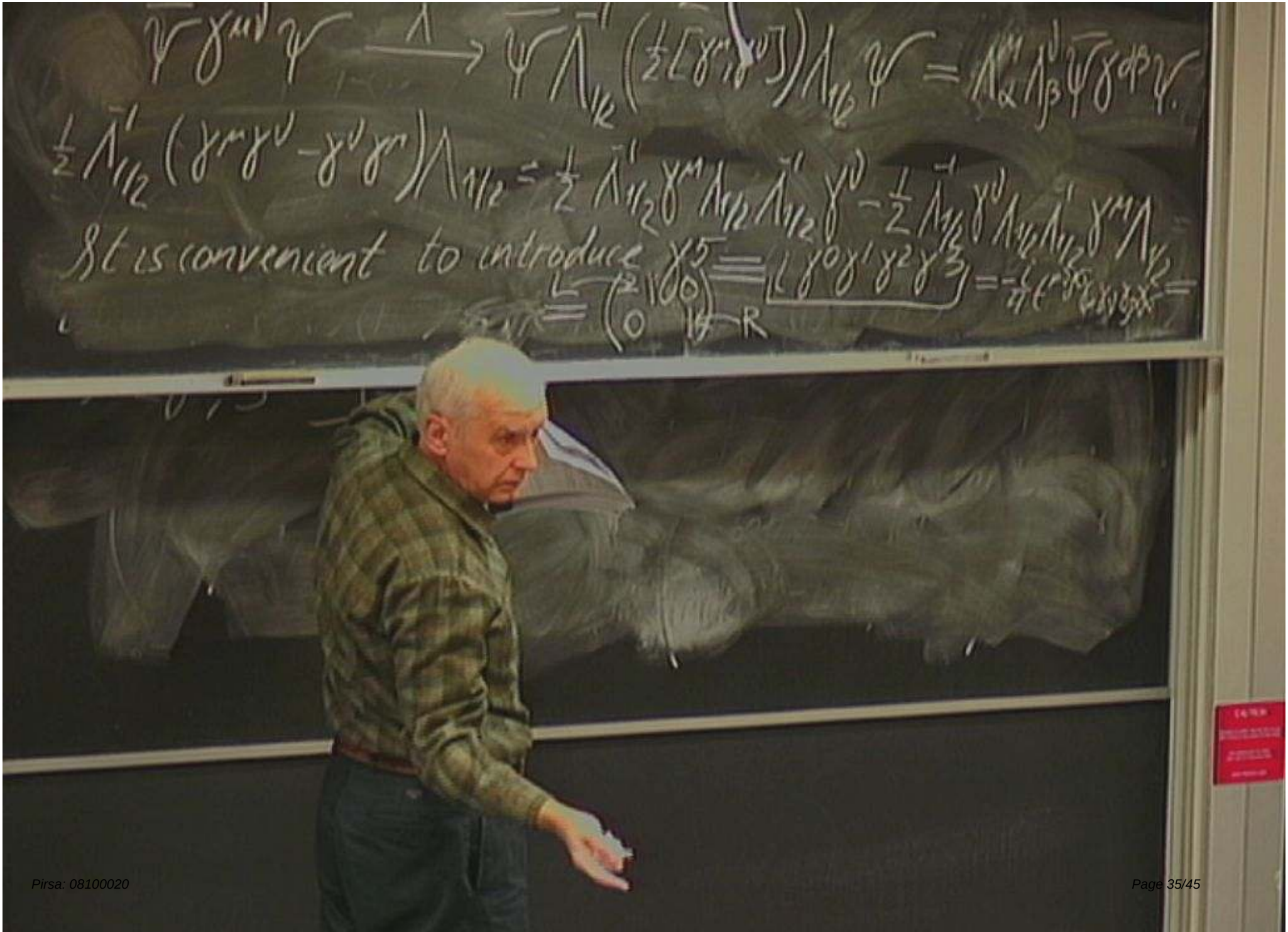

4 matr.
1 matr.

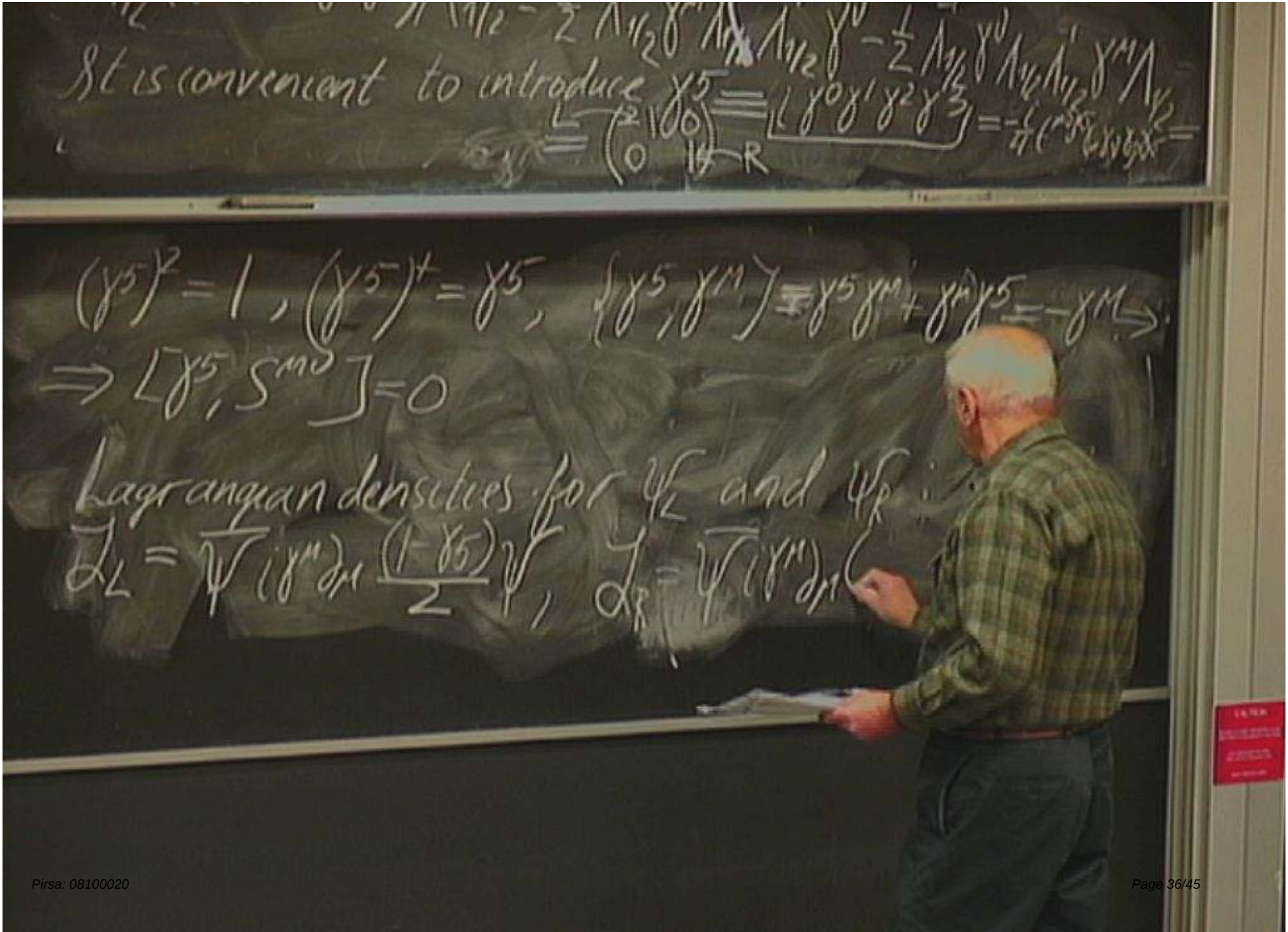

It is convenient to introduce


It is convenient to introduce
$(\gamma^5)^2 = 1, (\gamma^5)^+ =$


It is convenient to introduce
$(\gamma^5)^2 = 1$, $(\gamma^5)^+ = \gamma^5$, $\{\gamma^5, \gamma^m\} = \gamma^5\gamma^m + \gamma^m\gamma^5 = 0$
$\Rightarrow [\gamma^5, S^{mn}] = 0$


It is convenient to introduce
$(\gamma^5)^2 = 1$, $(\gamma^5)^+ = \gamma^5$, $\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = -\gamma^\mu + \gamma^\mu = 0$
$\Rightarrow [\gamma^5, S^{\mu\nu}] = 0$


$\bar{\psi}\gamma^{\mu\nu}\psi \xrightarrow{\Lambda} \bar{\psi}\Lambda^{-1}_{1/2}\left(\tfrac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\Lambda_{1/2}\psi = \Lambda^\mu_\alpha\Lambda^\nu_\beta\bar{\psi}\gamma^{\alpha\beta}\psi$
$\tfrac{1}{2}\Lambda^{-1}_{1/2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)\Lambda_{1/2} = \tfrac{1}{2}\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2}\Lambda^{-1}_{1/2}\gamma^\nu - \tfrac{1}{2}\Lambda^{-1}_{1/2}\gamma^\nu\Lambda_{1/2}\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2}$
It is convenient to introduce $\gamma^5 = [\gamma^0\gamma^1\gamma^2\gamma^3] = -\frac{i}{4!}\epsilon^{\alpha\beta\gamma\delta}\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta =$
$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ R


It is convenient to introduce γ^5
$(\gamma^5)^2 = 1$, $(\gamma^5)^+ = \gamma^5$, $\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = -\gamma^\mu\gamma^5 + \gamma^\mu\gamma^5 = 0$
$\Rightarrow [\gamma^5, S^{\mu\nu}] = 0$
Lagrangian densities for ψ_L and ψ_R:
$\mathcal{L}_L = \bar{\psi}_L i\gamma^\mu \partial_\mu \frac{(1-\gamma^5)}{2}\psi_L$, $\mathcal{L}_R = \bar{\psi}_R i\gamma^\mu \partial_\mu$

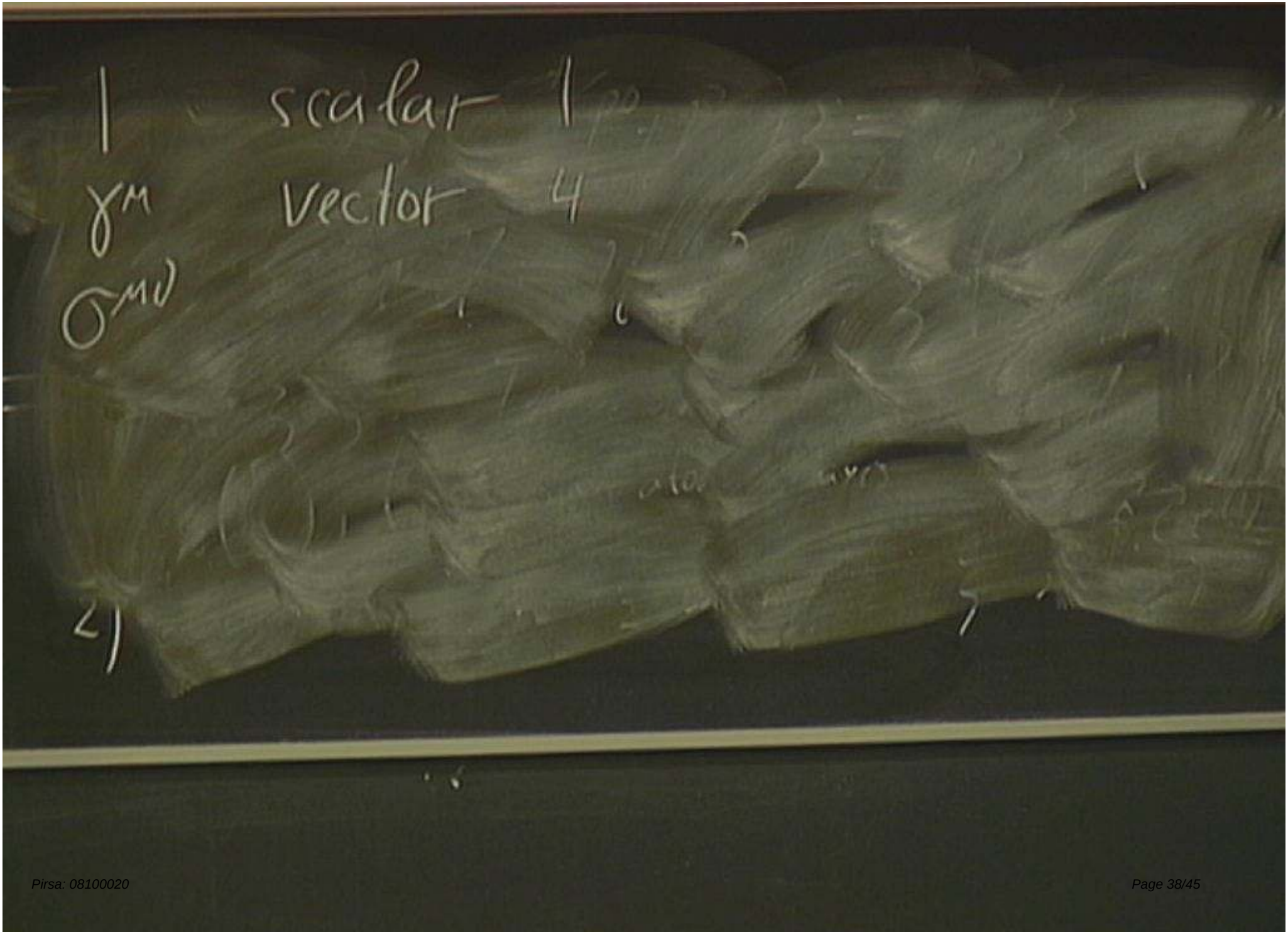
It is convenient to introduce $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = R$

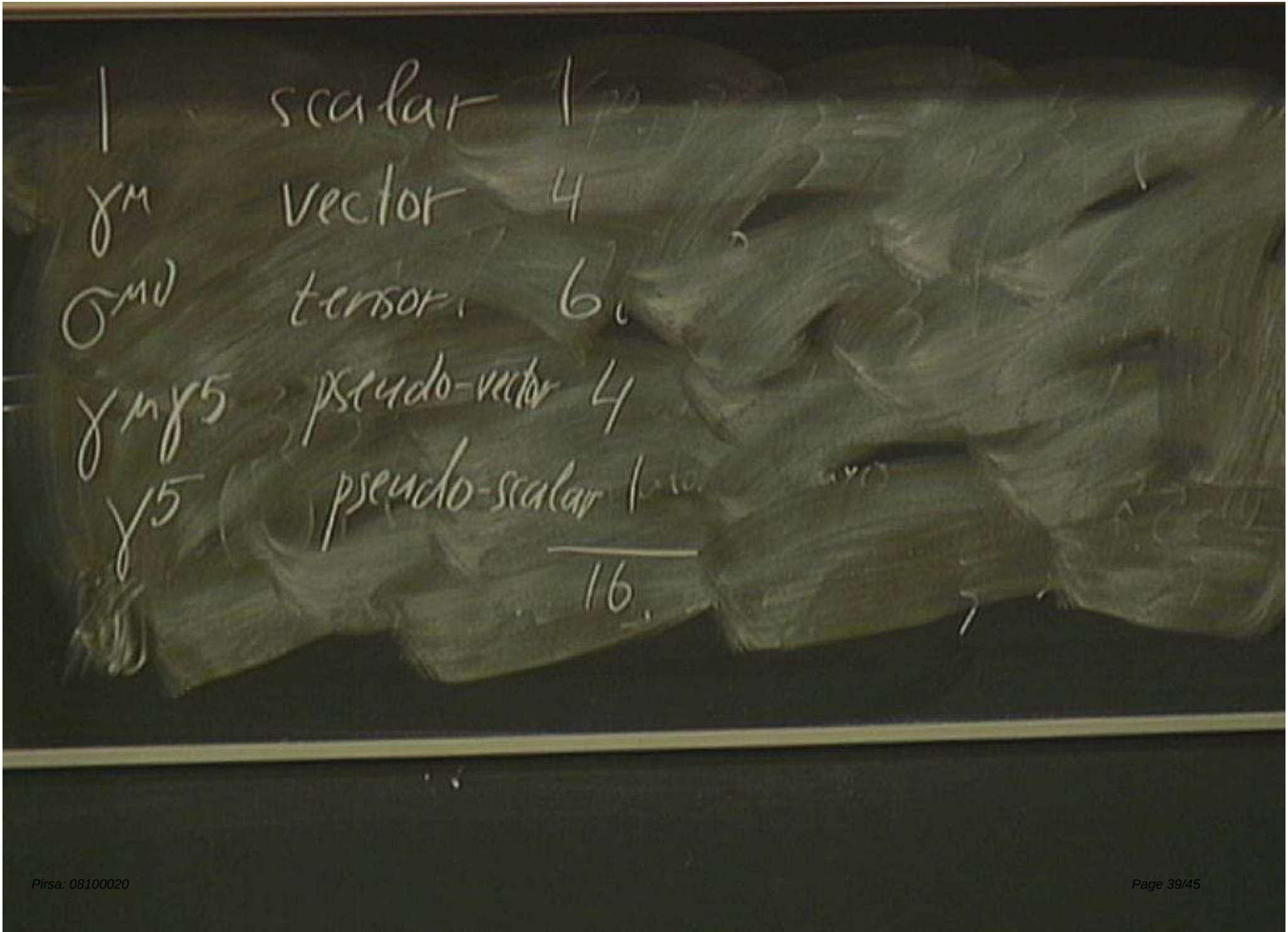
$(\gamma^5)^2 = 1$, $(\gamma^5)^\dagger = \gamma^5$, $\{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = -\gamma^\mu + \gamma^\mu = 0$

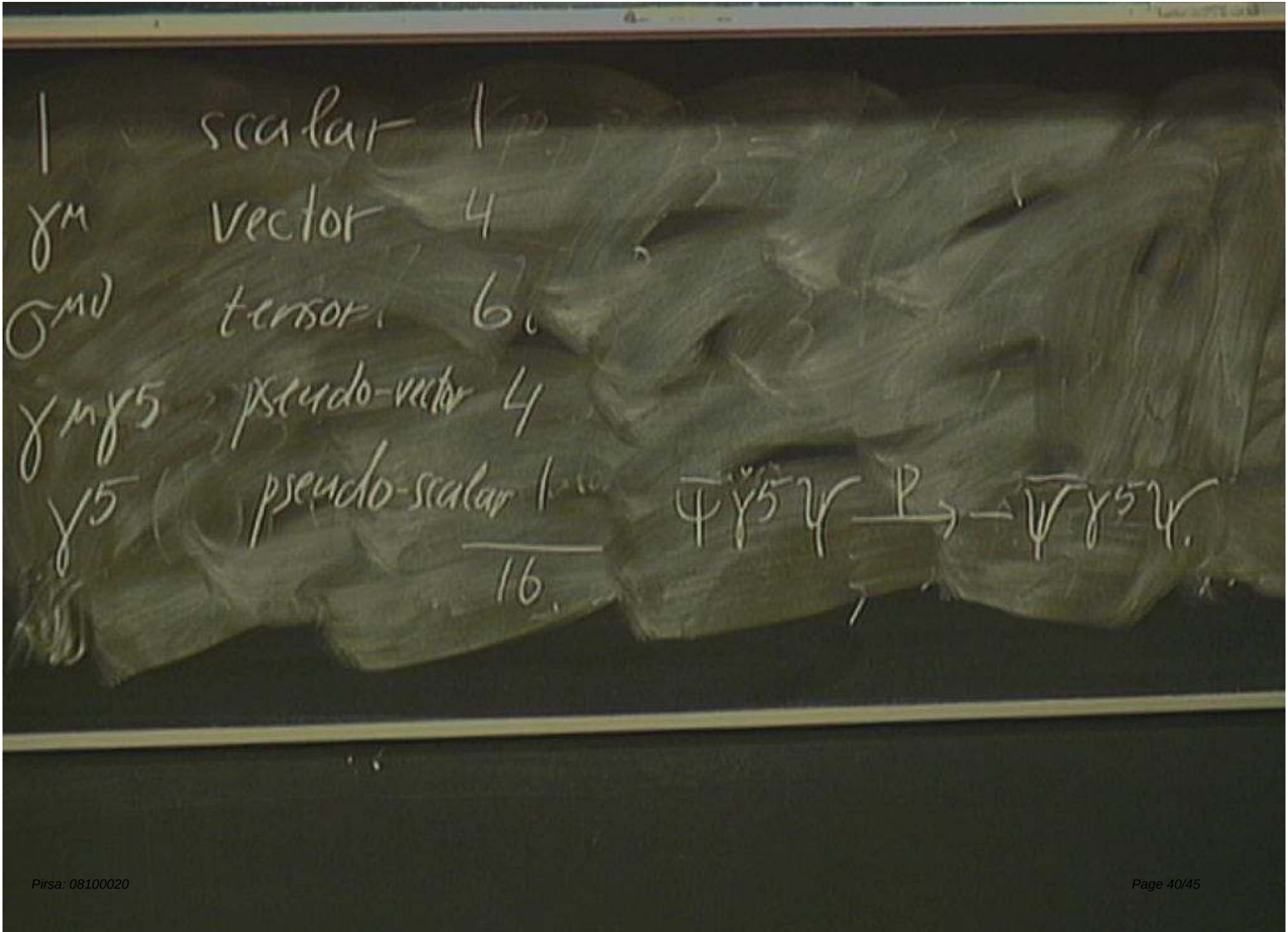
$\Rightarrow [\gamma^5, S^{\mu\nu}] = 0$

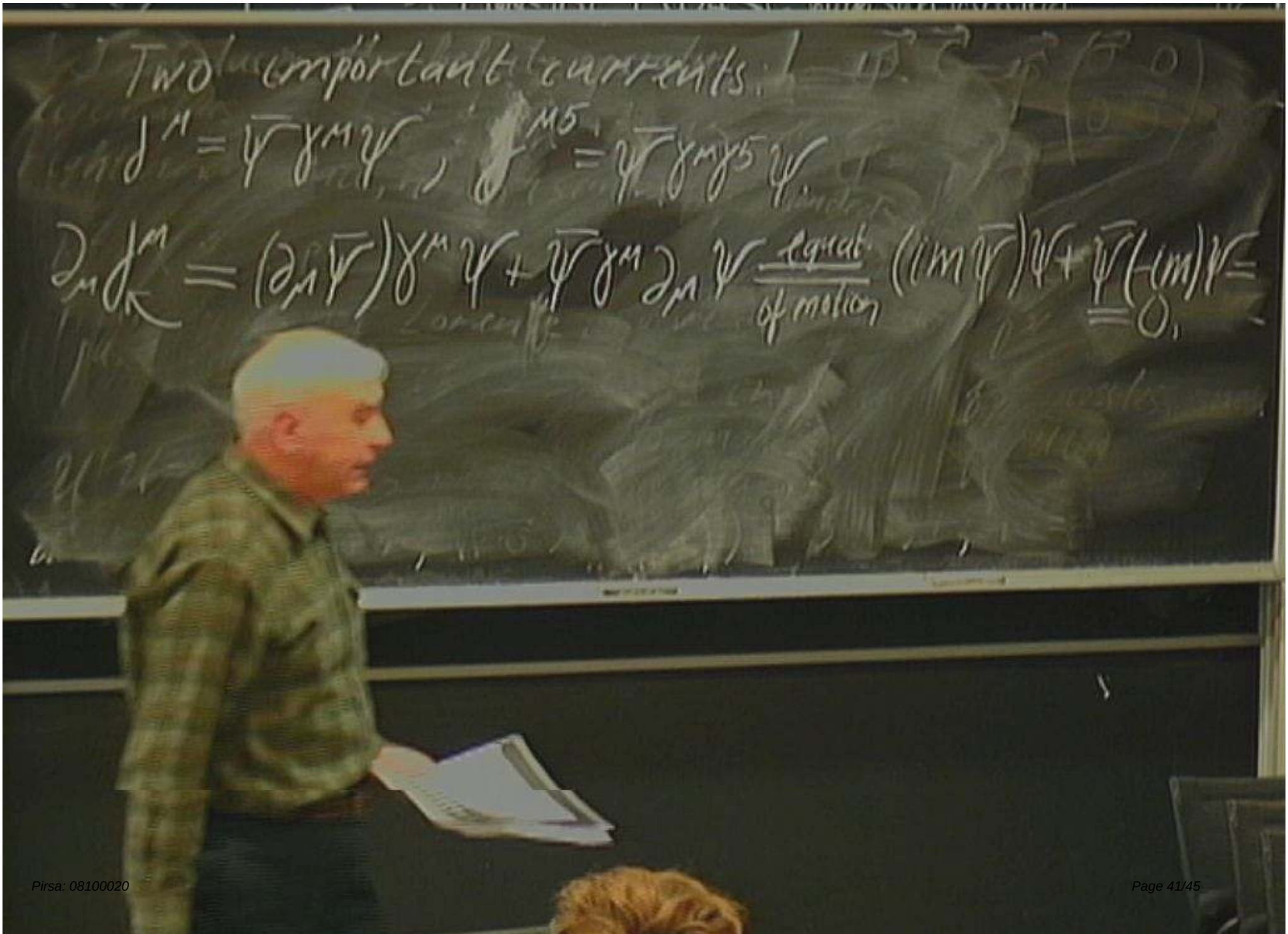
Lagrangian densities for ψ_L and ψ_R:

$\mathcal{L}_L = \bar{\psi}\, i\gamma^\mu \partial_\mu \frac{(1-\gamma^5)}{2}\psi$, $\mathcal{L}_R = \bar{\psi}\, i\gamma^\mu \partial_\mu \frac{(1+\gamma^5)}{2}\psi$


1 scalar 1
γ^μ vector 4
$\sigma^{\mu\nu}$


1 scalar 1
γ_μ vector 4
$\sigma_{\mu\nu}$ tensor 6
$\gamma_\mu\gamma_5$ pseudo-vector 4
γ_5 pseudo-scalar 1
16


1 scalar 1
γ^μ vector 4
$\sigma^{\mu\nu}$ tensor 6
$\gamma^\mu\gamma^5$ pseudo-vector 4
γ^5 pseudo-scalar 1
16
$\bar{\psi}\gamma^5\psi \xrightarrow{P} -\bar{\psi}\gamma^5\psi$


Two important currents:
$j^\mu = \bar{\psi}\gamma^\mu\psi$, $j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi$.
$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi \overset{\text{eqn of motion}}{=} (im\bar{\psi})\psi + \bar{\psi}(-im)\psi = 0$

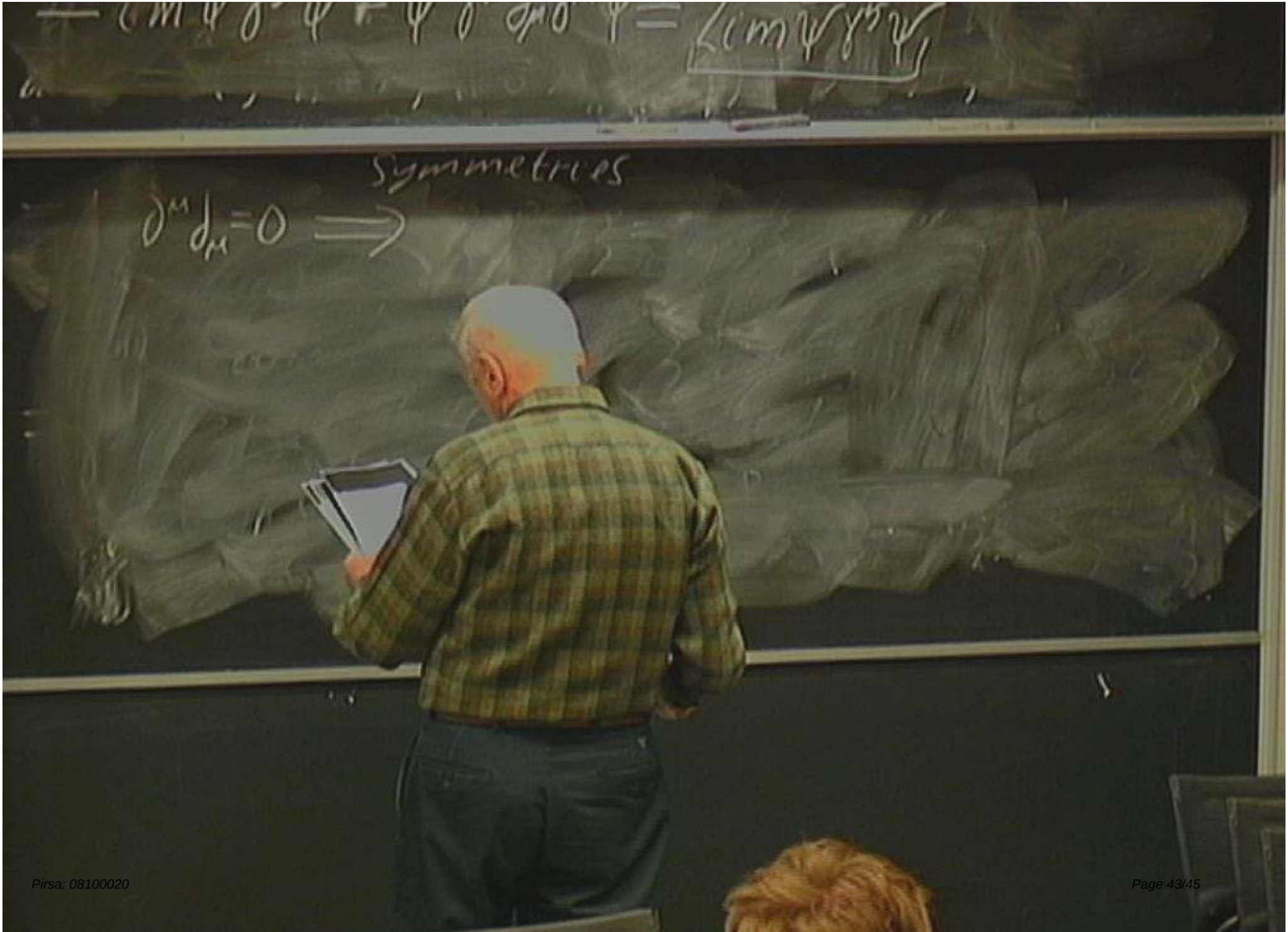
Two important currents:

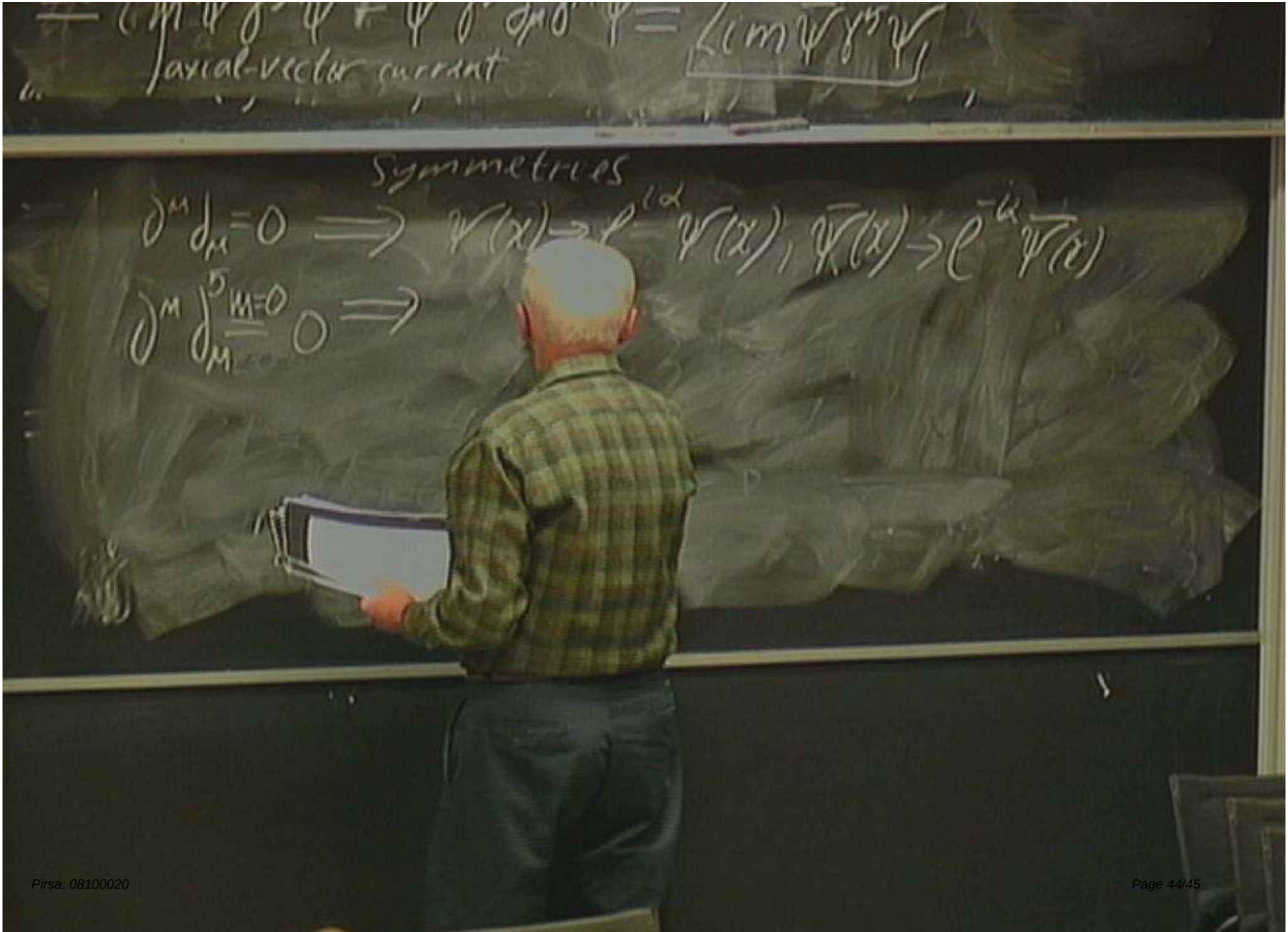
$$j^\mu = \bar\psi \gamma^\mu \psi, \quad j^{\mu 5} = \bar\psi \gamma^\mu \gamma^5 \psi$$

$$\partial_\mu j^\mu = (\partial_\mu \bar\psi)\gamma^\mu \psi + \bar\psi \gamma^\mu \partial_\mu \psi \underset{\text{of motion}}{\overset{\text{Equat.}}{=}} (im\bar\psi)\psi + \bar\psi(-im)\psi = 0,$$

$$\partial_\mu j^{\mu 5} = 2im\bar\psi\gamma^5\psi \qquad \partial_\mu j^{\mu 5} = (\partial^\mu \bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\partial_\mu\gamma^\mu\gamma^5\psi =$$

$$= im\bar\psi\gamma^5\psi + \bar\psi\gamma^5\partial_\mu\gamma^\mu\psi = \underline{2im\bar\psi\gamma^5\psi}$$


symmetries
$\partial^\mu j_\mu = 0 \Longrightarrow$


axial-vector current
Symmetries

$= i\partial_\mu \bar\psi \gamma^\mu \gamma^5 \psi + \bar\psi \gamma^5 \partial_\mu \gamma^\mu \psi = \underline{2im\bar\psi\gamma^5\psi}$

axial-vector current

$\partial^\mu j_\mu = 0 \Longrightarrow \psi(x) \to e^{i\alpha}\psi(x),\ \bar\psi(x) \to e^{-i\alpha}\bar\psi(x)$

$\partial^\mu j^5_\mu = 0 \Longrightarrow \psi(x) \to e^{i\omega\gamma^5}\psi(x),\ \bar\psi(x) \to \bar\psi(x) e^{i\omega\gamma^5}$

useful two chiral currents.

chiral transformations

$j^\mu_L = \bar\psi \gamma^\mu \frac{1-\gamma^5}{2}\psi,\quad j^\mu_R = \bar\psi\gamma^\mu\frac{1+\gamma^5}{2}\psi,\quad \psi^\dagger = \psi^\dagger\gamma^0 \Rightarrow \psi^\dagger e^{-i\omega\gamma^5}\gamma^0 =$

$= \psi^\dagger\gamma^5 e^{i\omega\gamma^5} = \bar\psi$

chiral currents