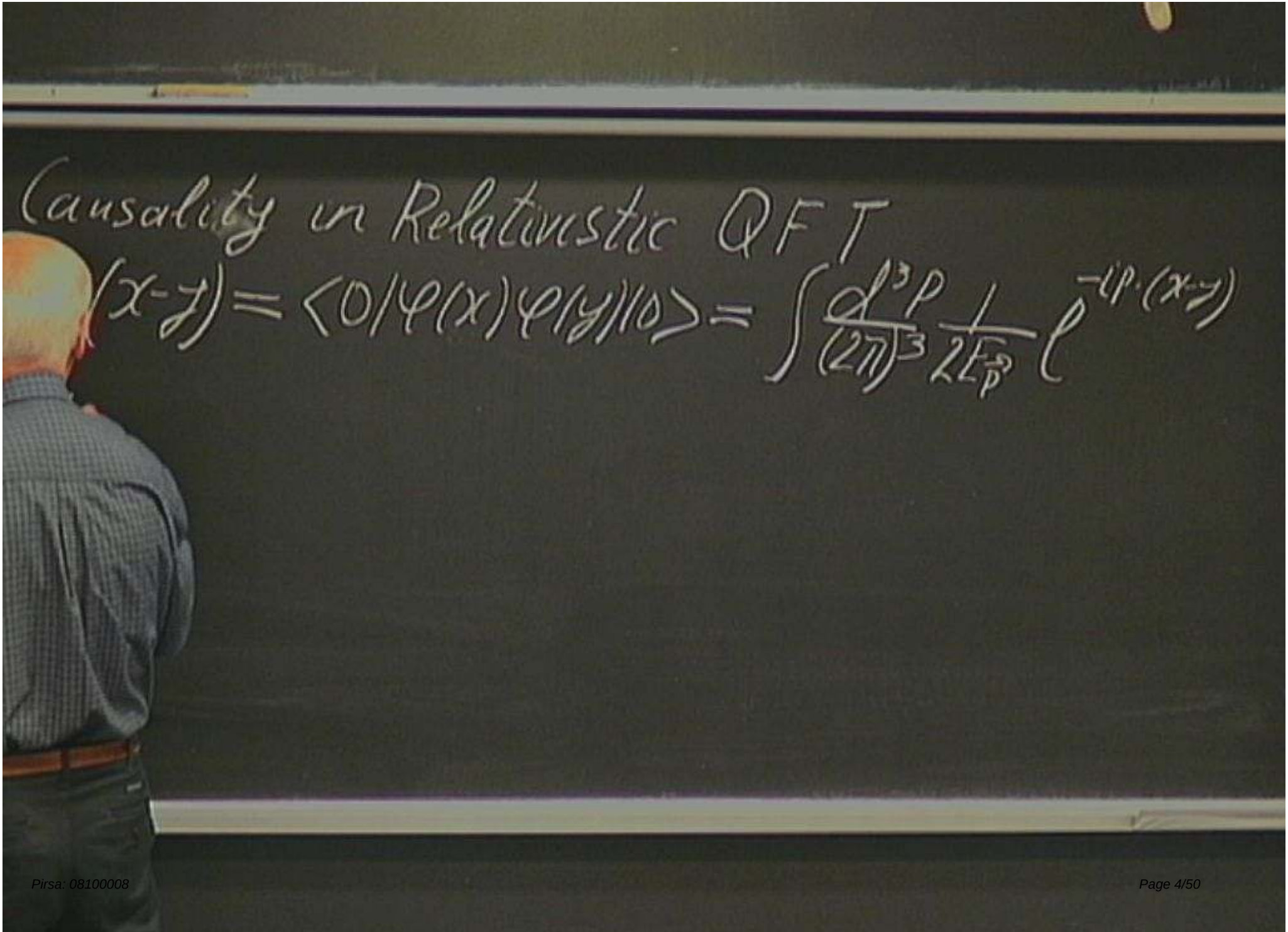
Title: Quantum Field Theory 1 - Lecture 6A

Date: Oct 15, 2008 10:30 AM

URL: http://pirsa.org/08100008

Abstract: Quantum Field Theory I course taught by Volodya Miransky of the University of Western Ontario


Causality in Relativistic QFT
$(x-y) = \langle 0|\varphi(x)\varphi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$

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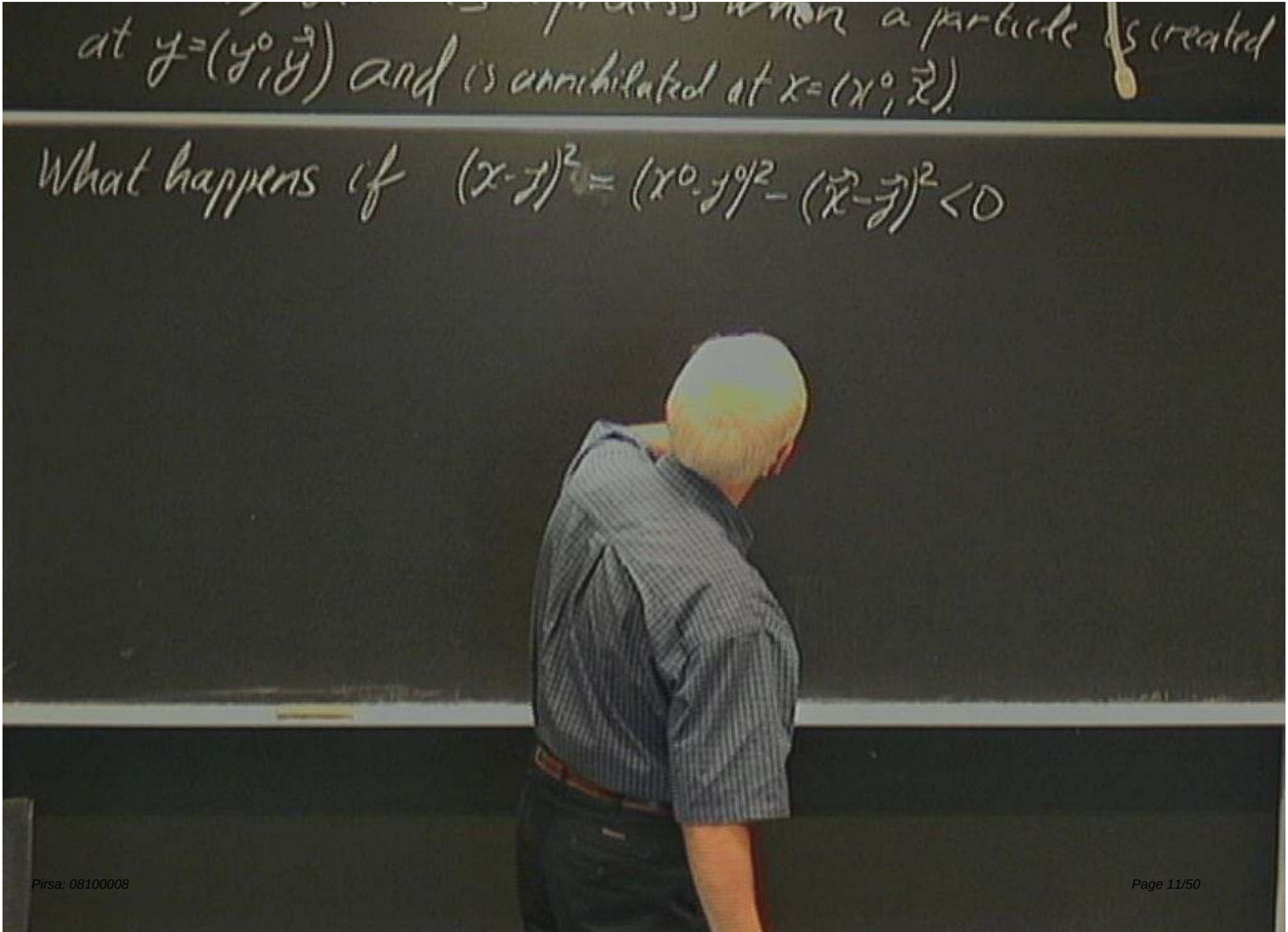
$D(x-y)$ describes a process when a particle is created at $y=(y^0,\vec{y})$ and is annihilated at $x=(x^0,\vec{x})$.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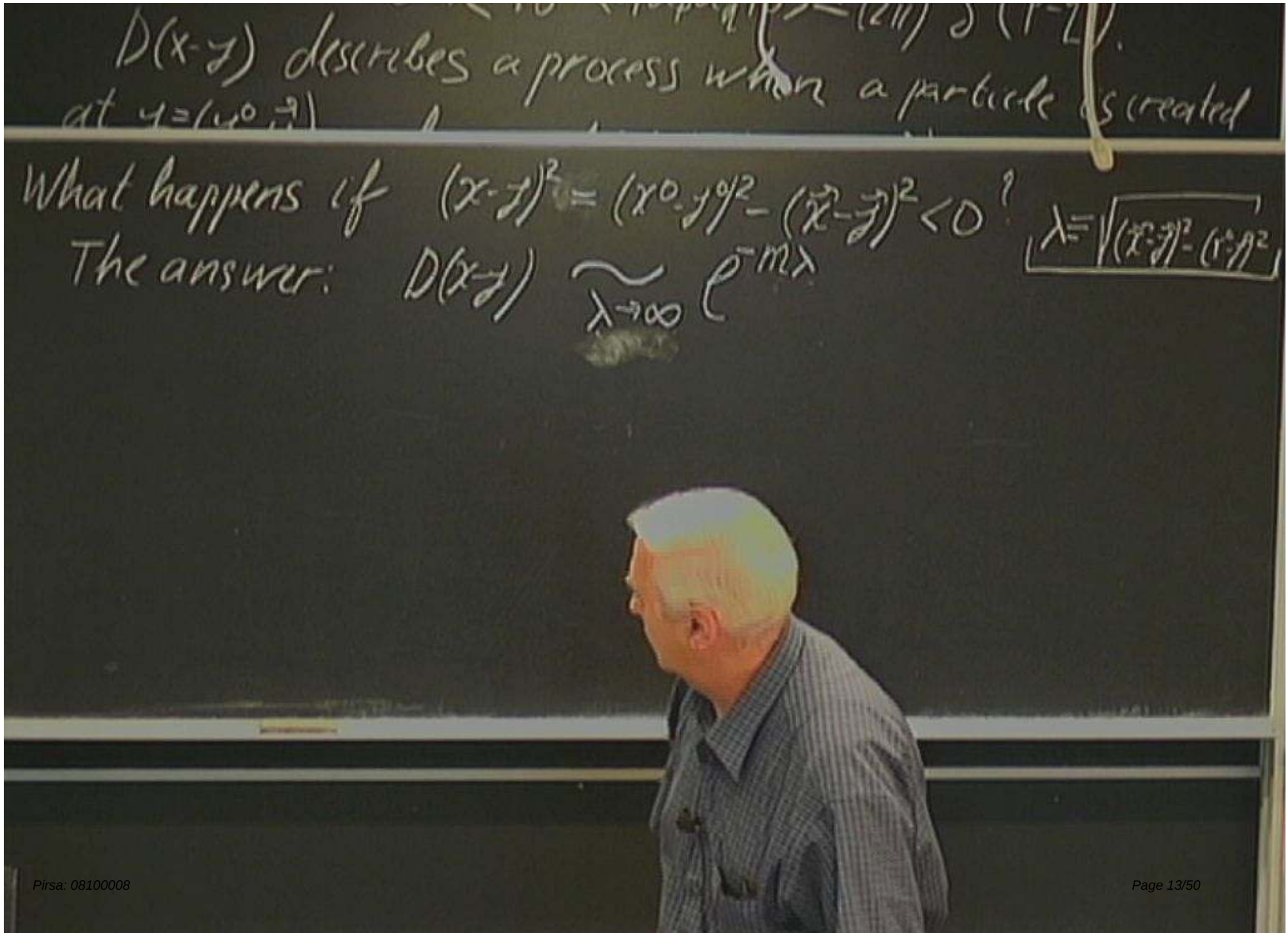

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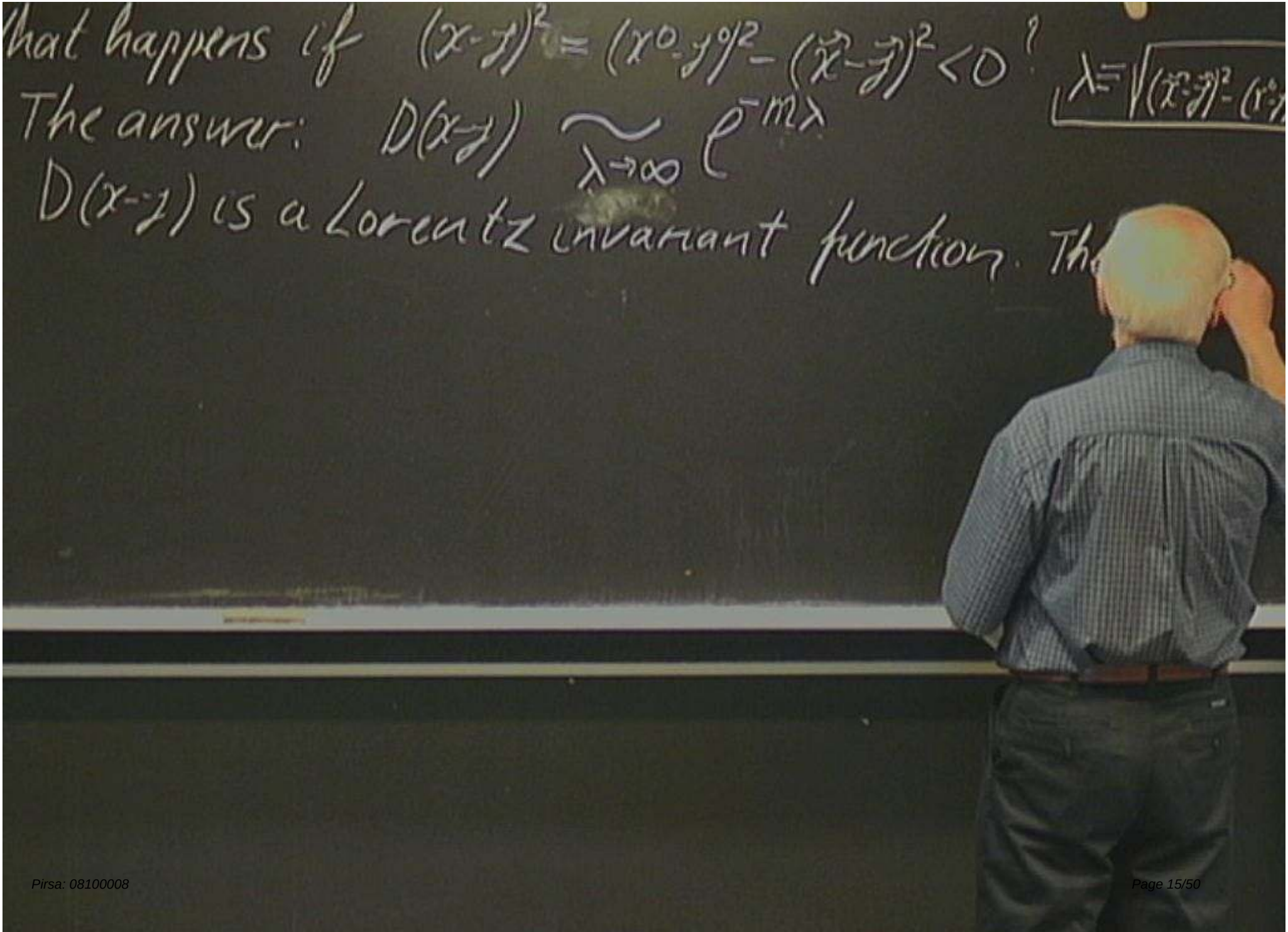
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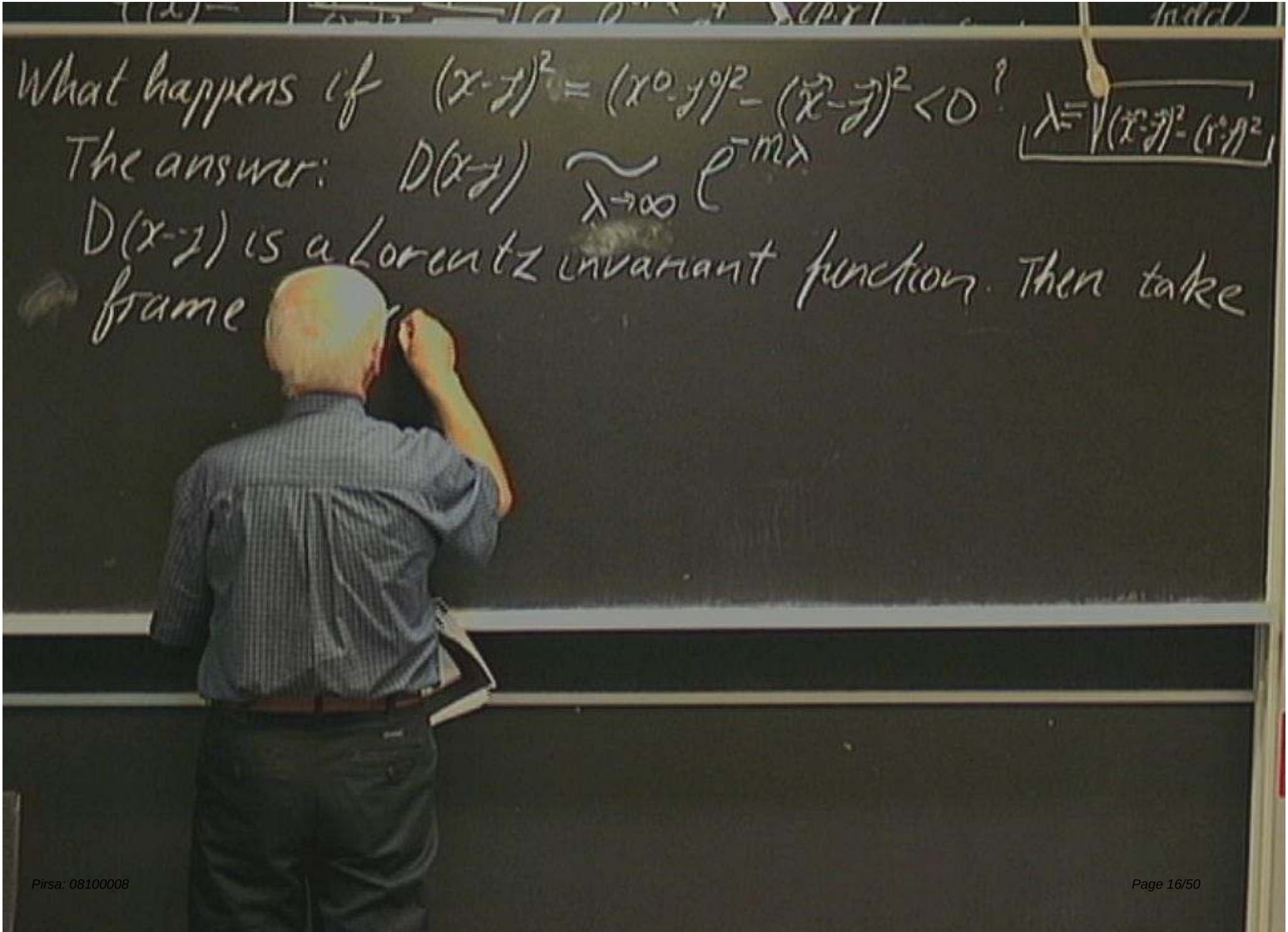
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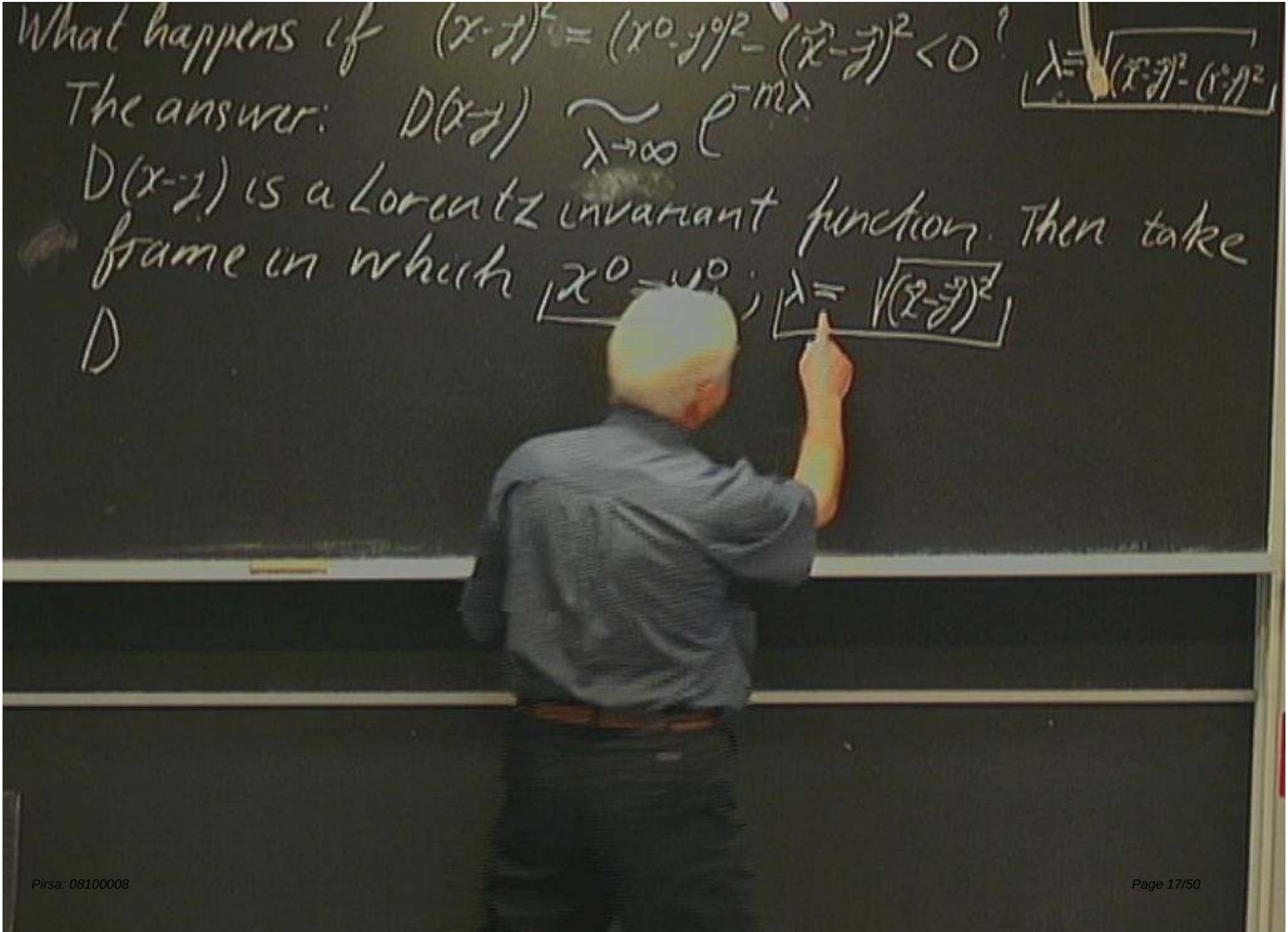
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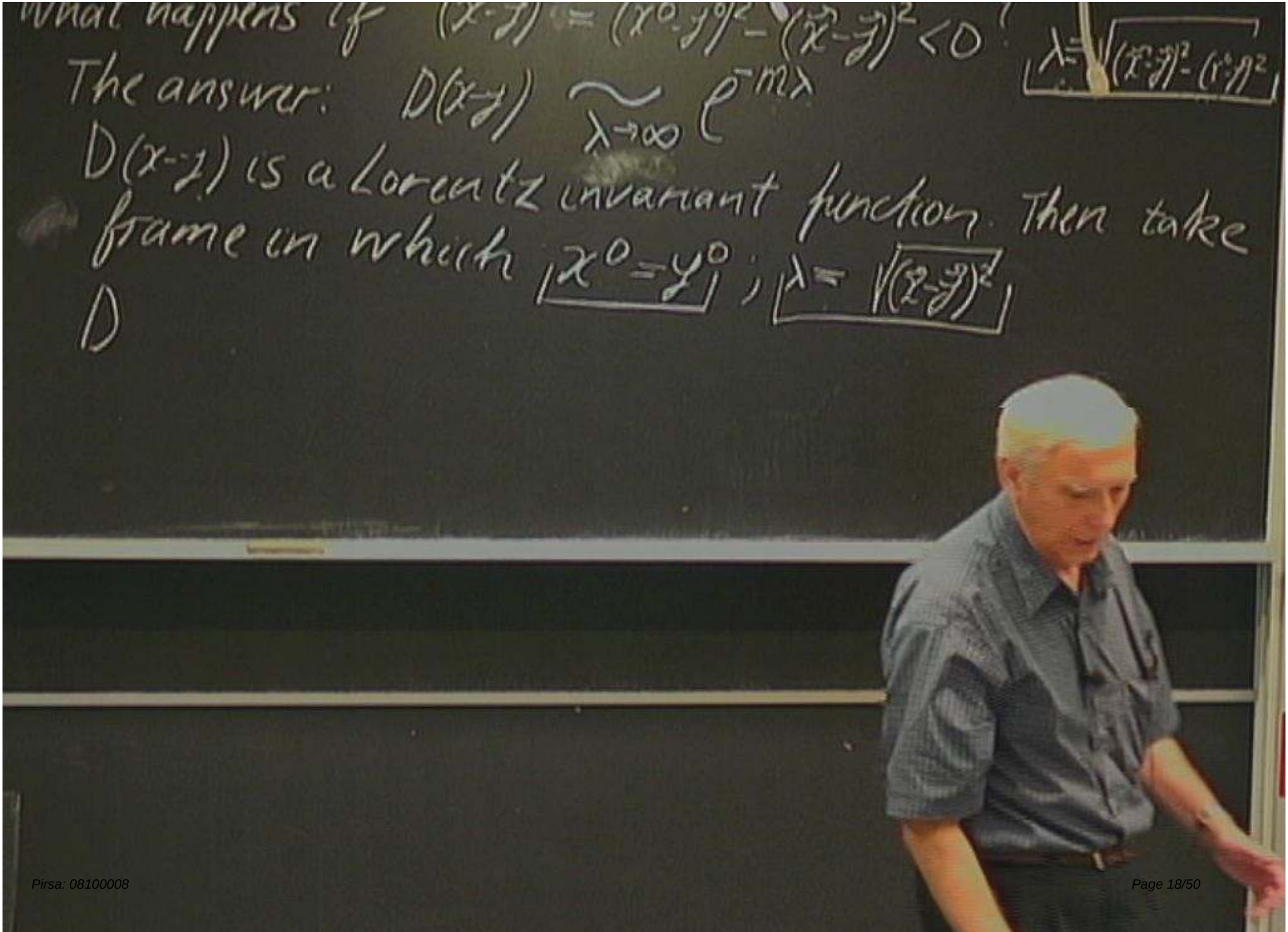
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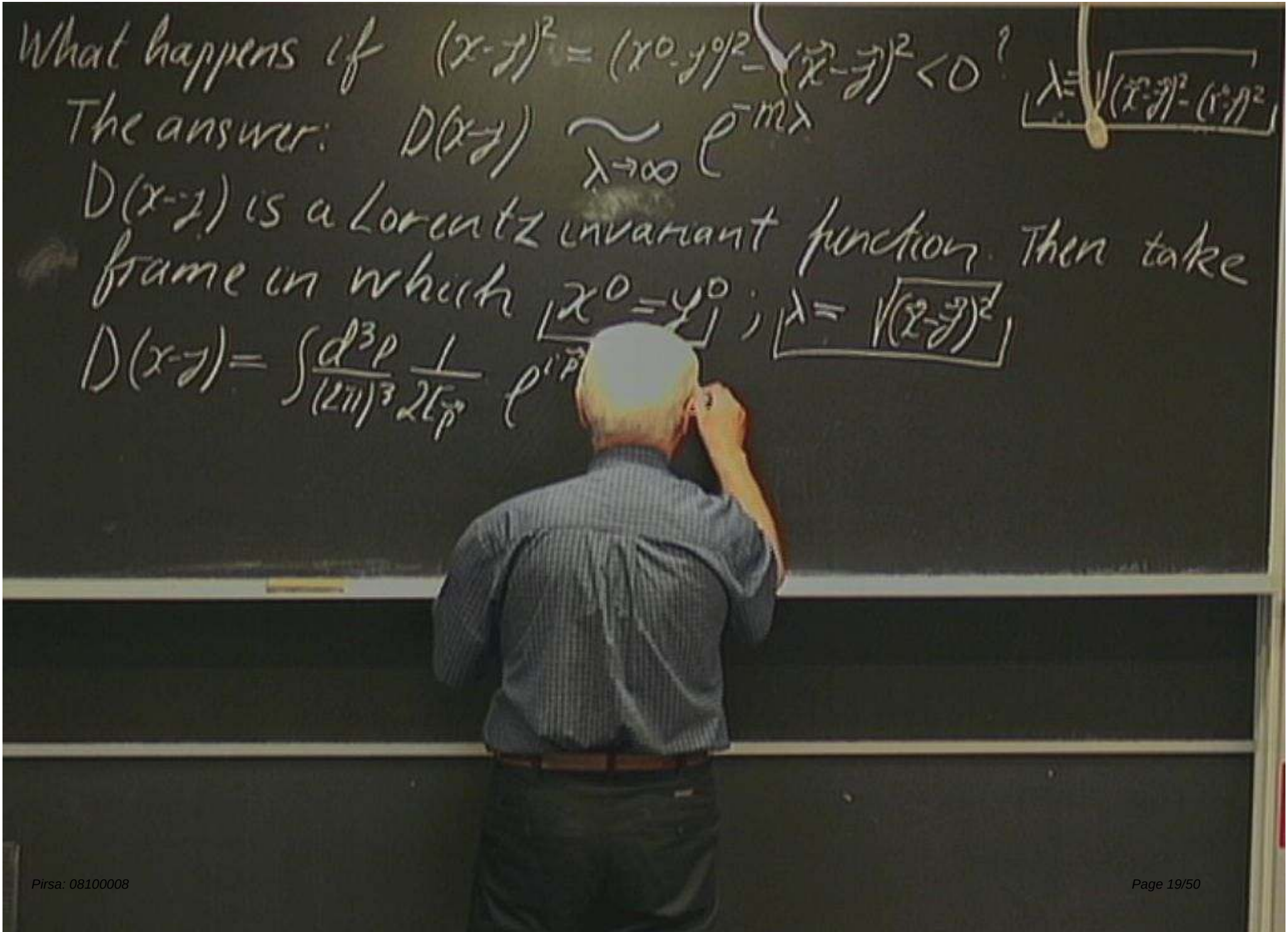
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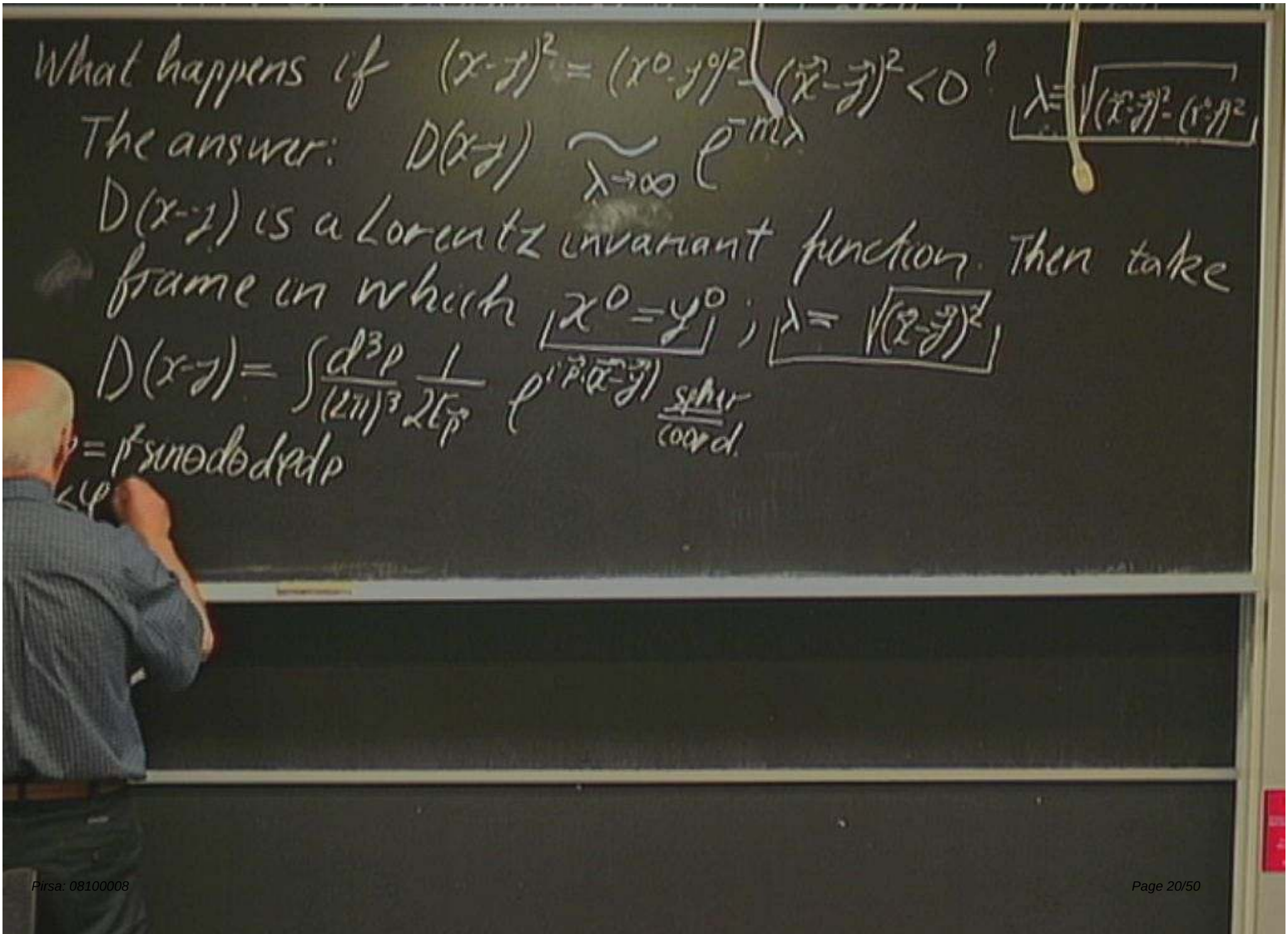
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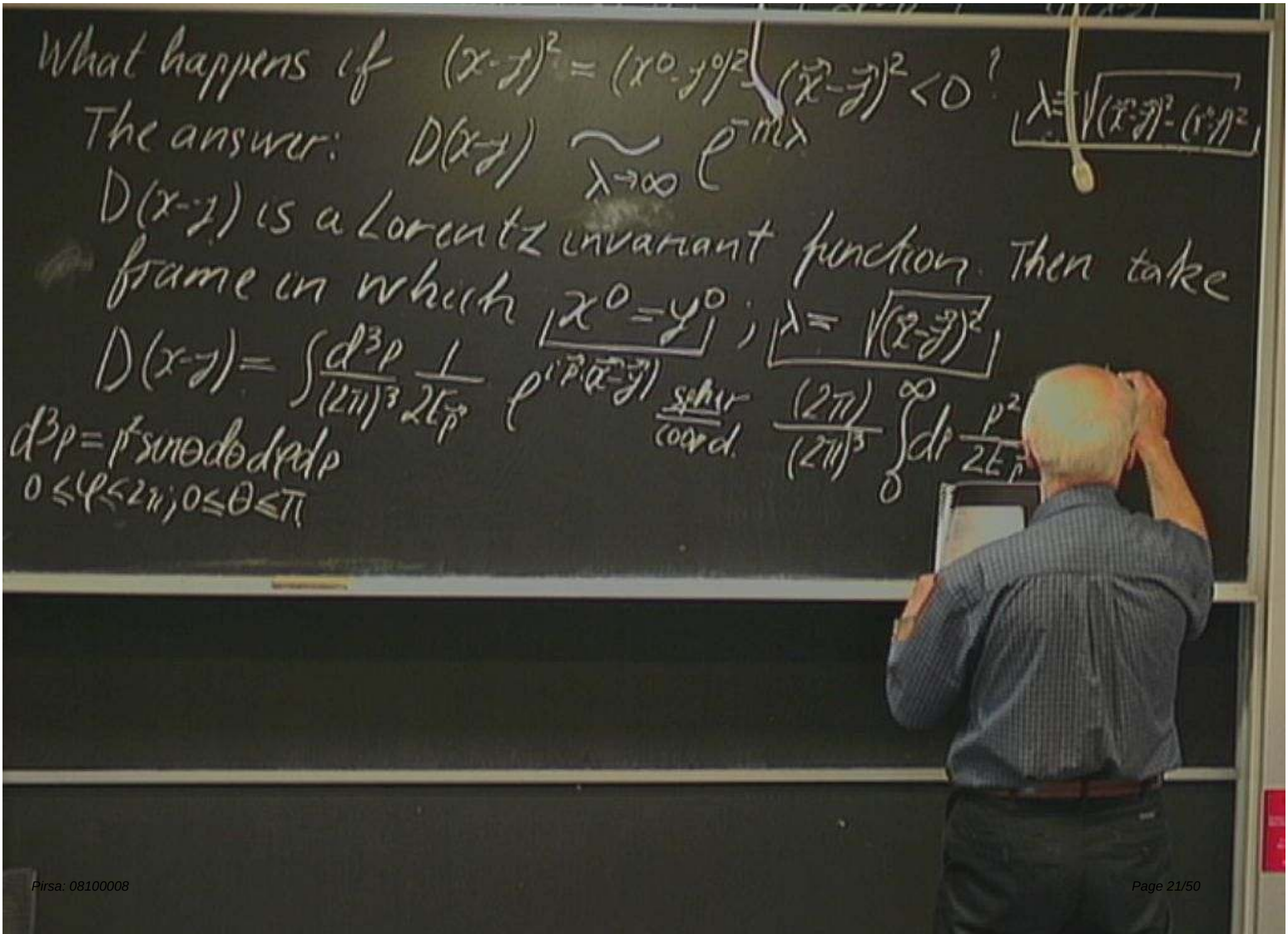
$= p^2 \sin\theta \, d\theta \, d\varphi \, dp$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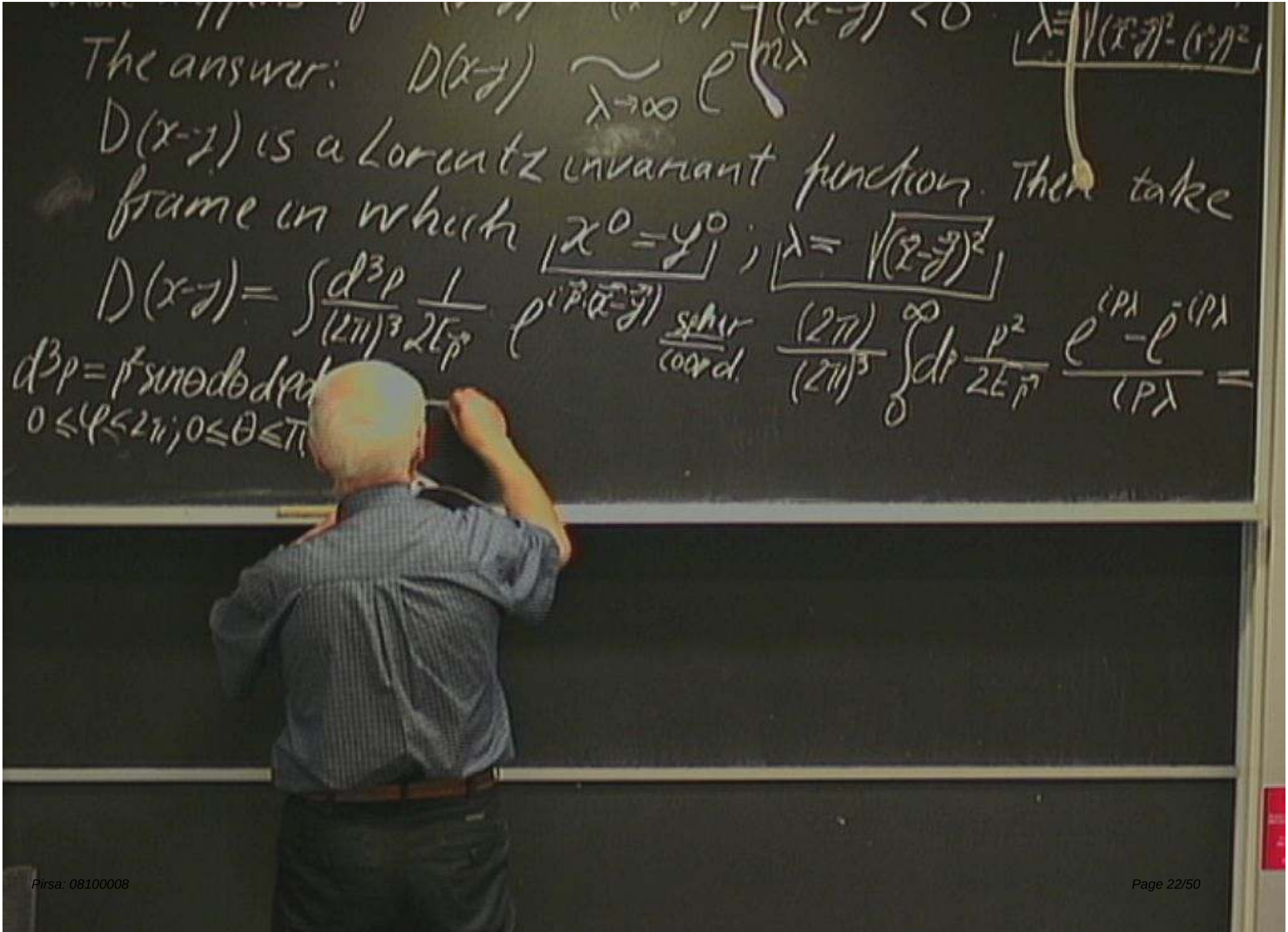
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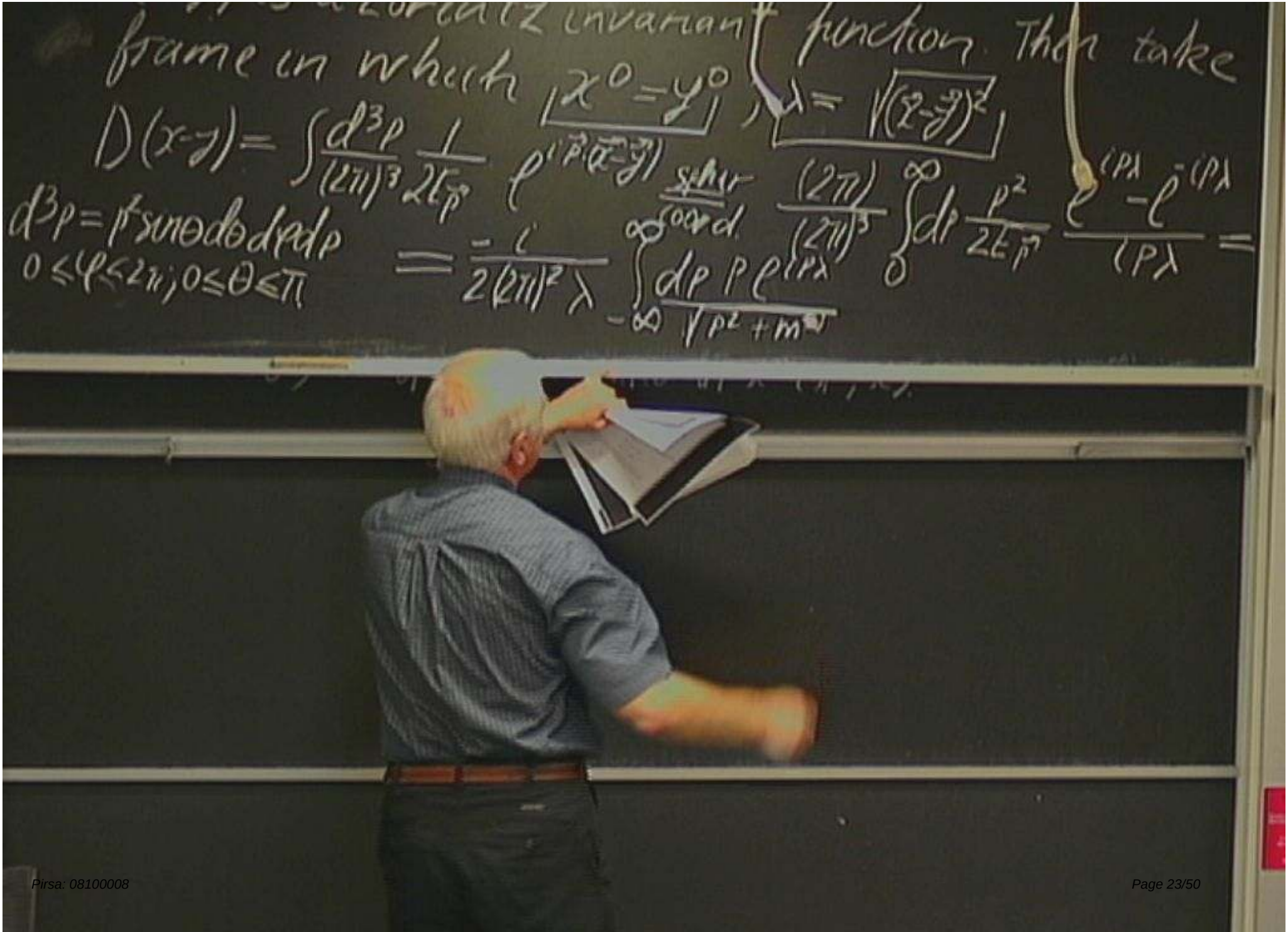
invariant function. Then take
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$0 \le \varphi \le 2\pi; 0 \le \theta \le \pi$
$= \frac{-i}{2(2\pi)^2 \lambda} \int_{-\infty}^{\infty} \frac{dp \, p \, e^{ip\lambda}}{\sqrt{p^2+m^2}}$
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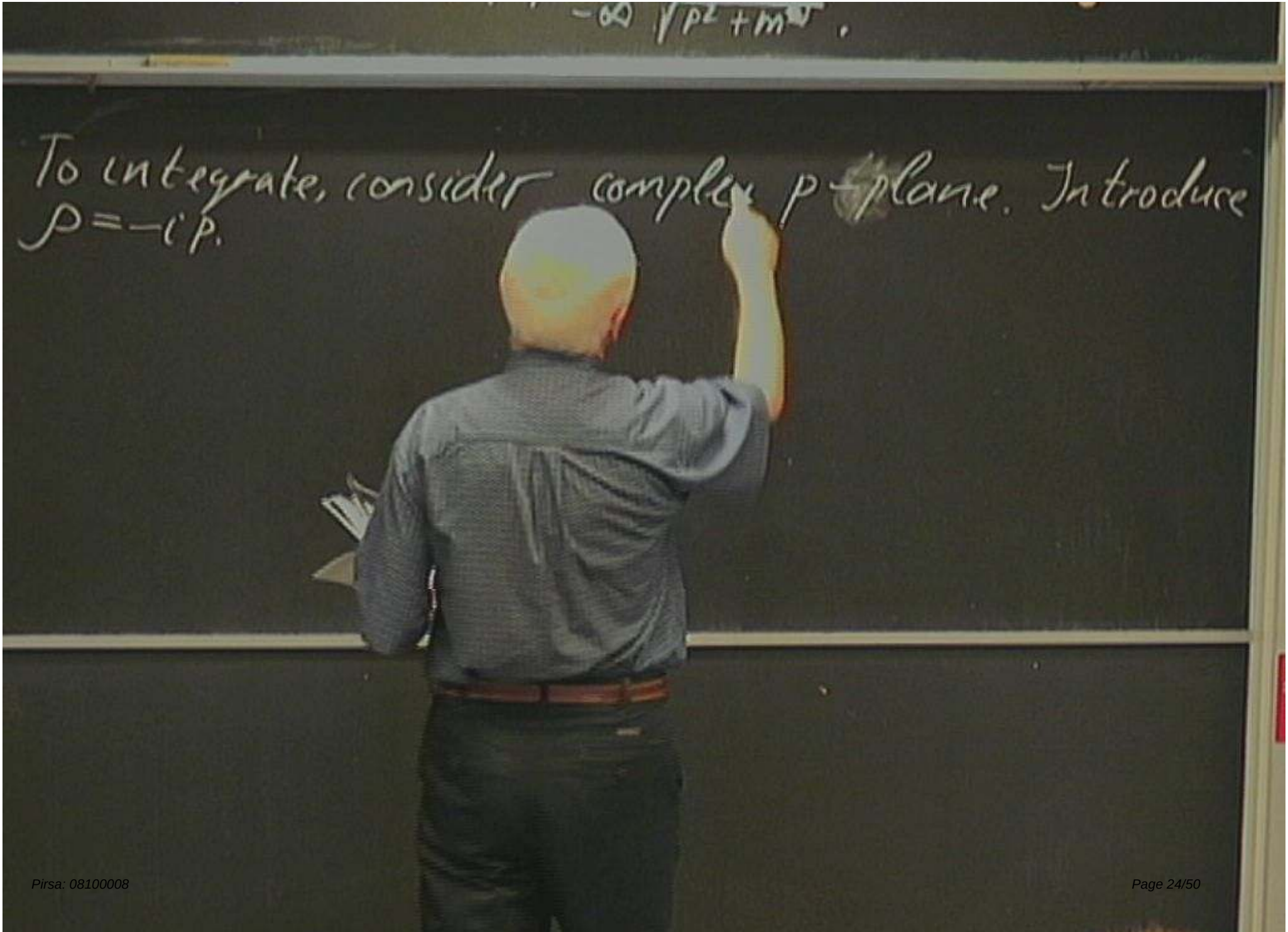
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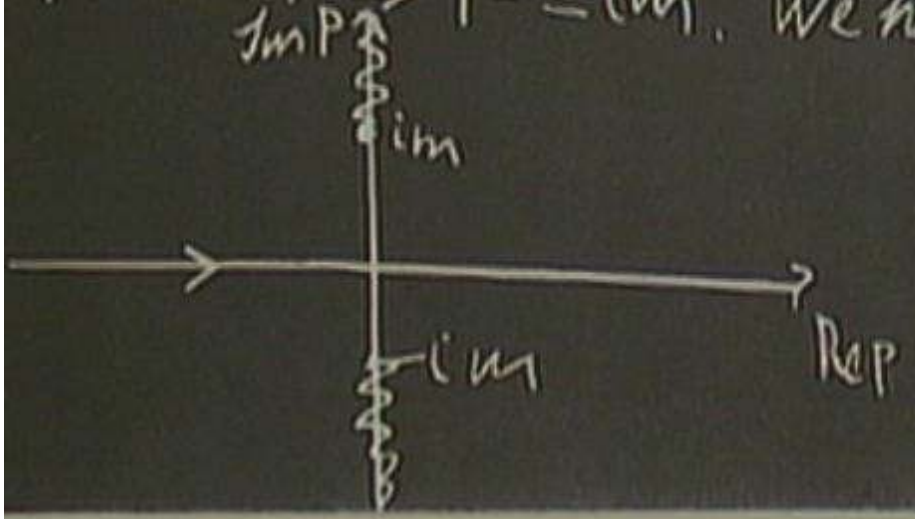

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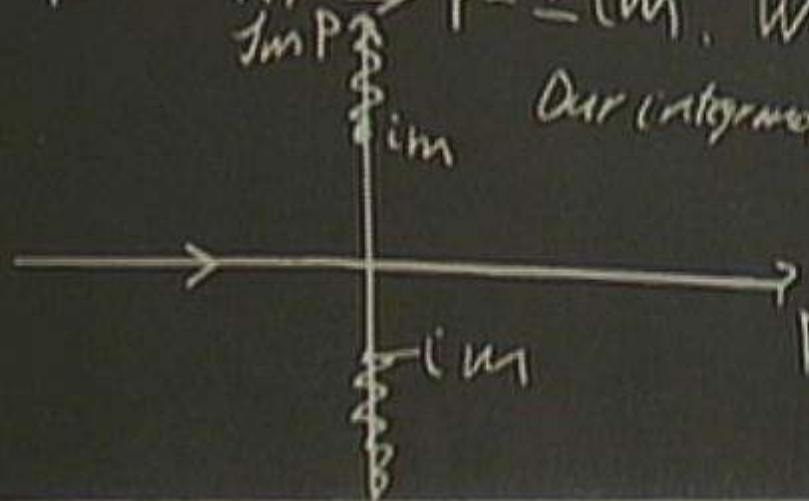

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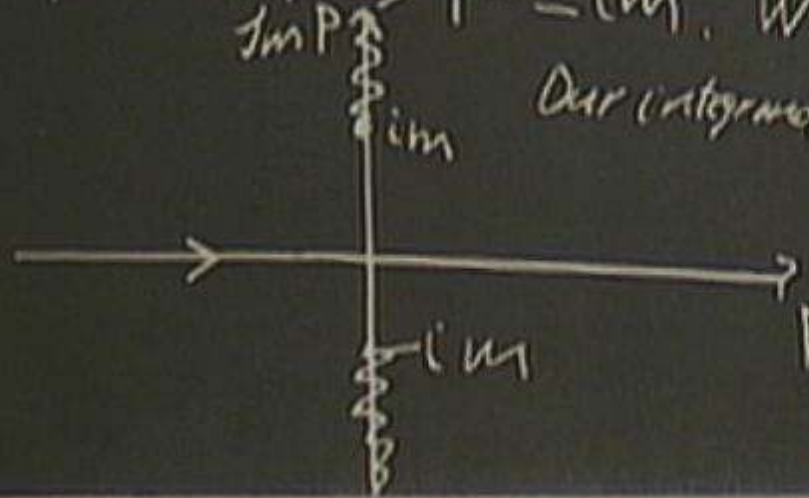

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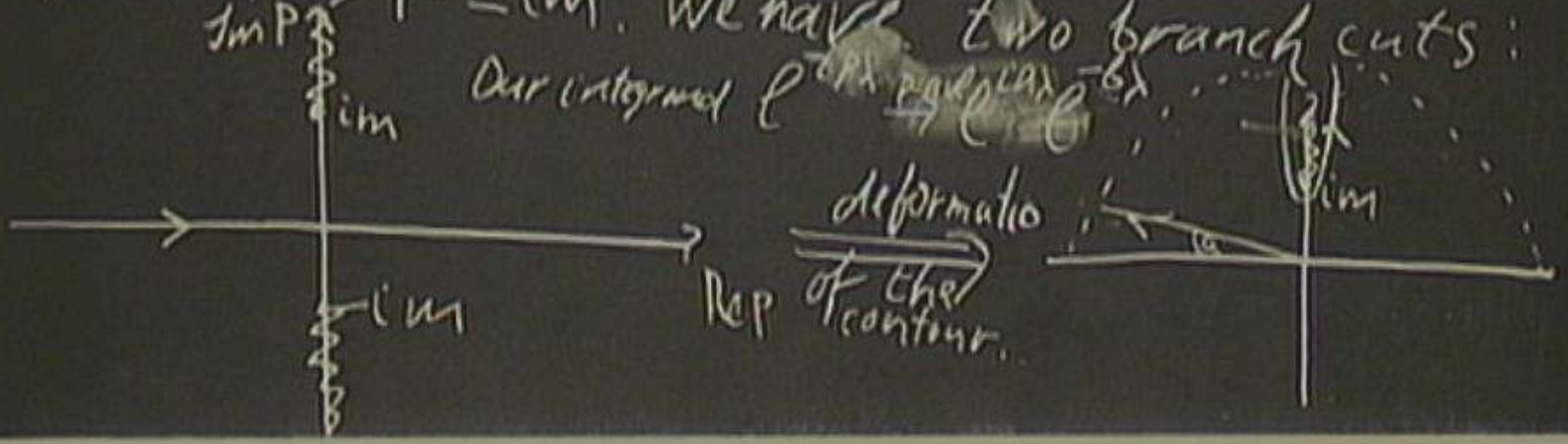
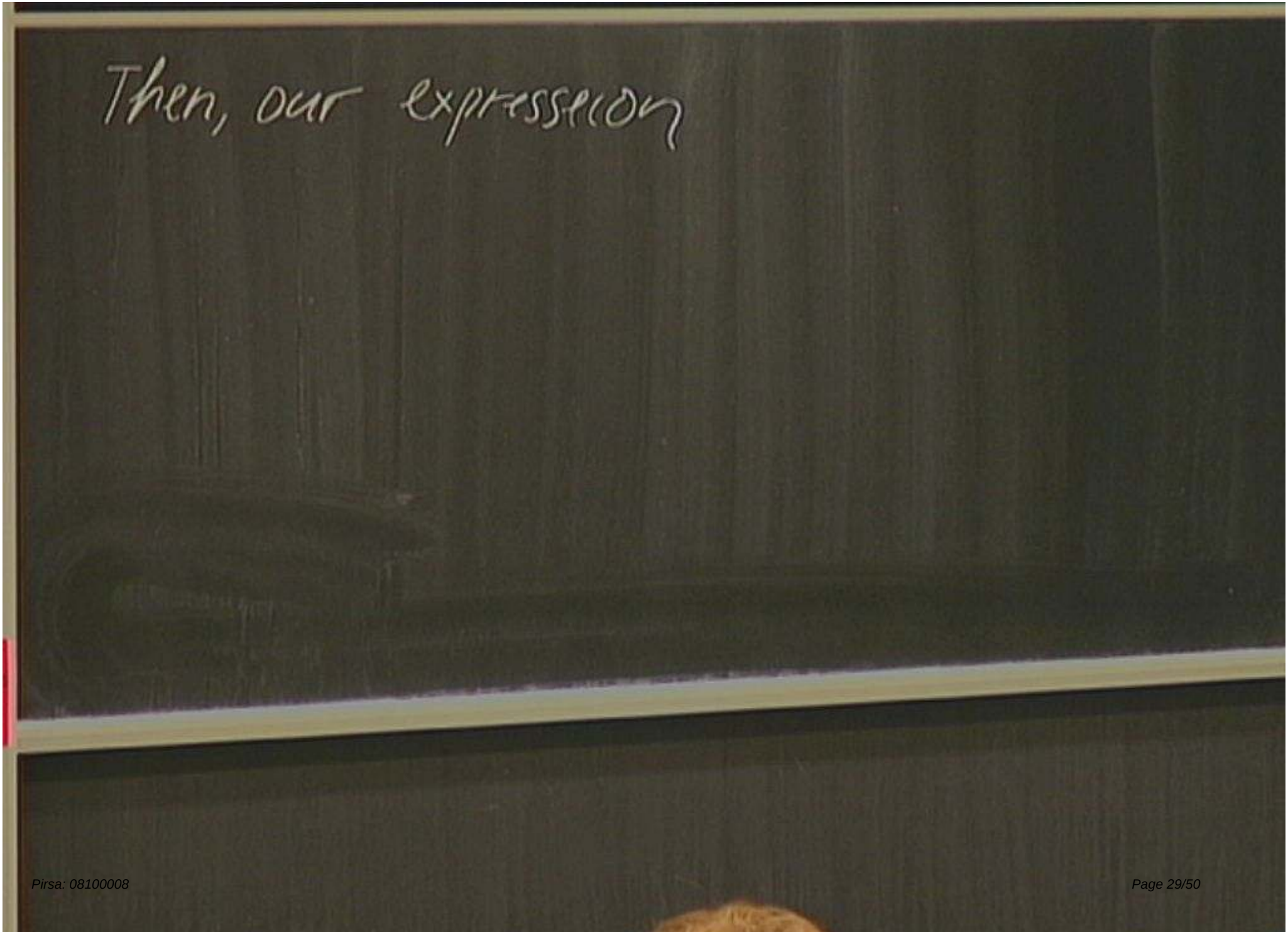


Our integrand e^{-ipx} equals $e^{-\rho x}$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Then, our expression (1) becomes $\frac{1}{4\pi^2 x} \int_m^\infty d\rho \frac{\rho e^{-\rho x}}{\sqrt{\rho^2 - m^2}} \Rightarrow$

$\xrightarrow{x \to \infty}$

Then, our expression (1) becomes $\frac{1}{4\pi^2\lambda}\int_m^\infty d\rho \frac{\rho e^{-\rho\lambda}}{\sqrt{\rho^2-m^2}} \Rightarrow$

$\lambda \to \infty \Rightarrow \sim e^{-m\lambda}$

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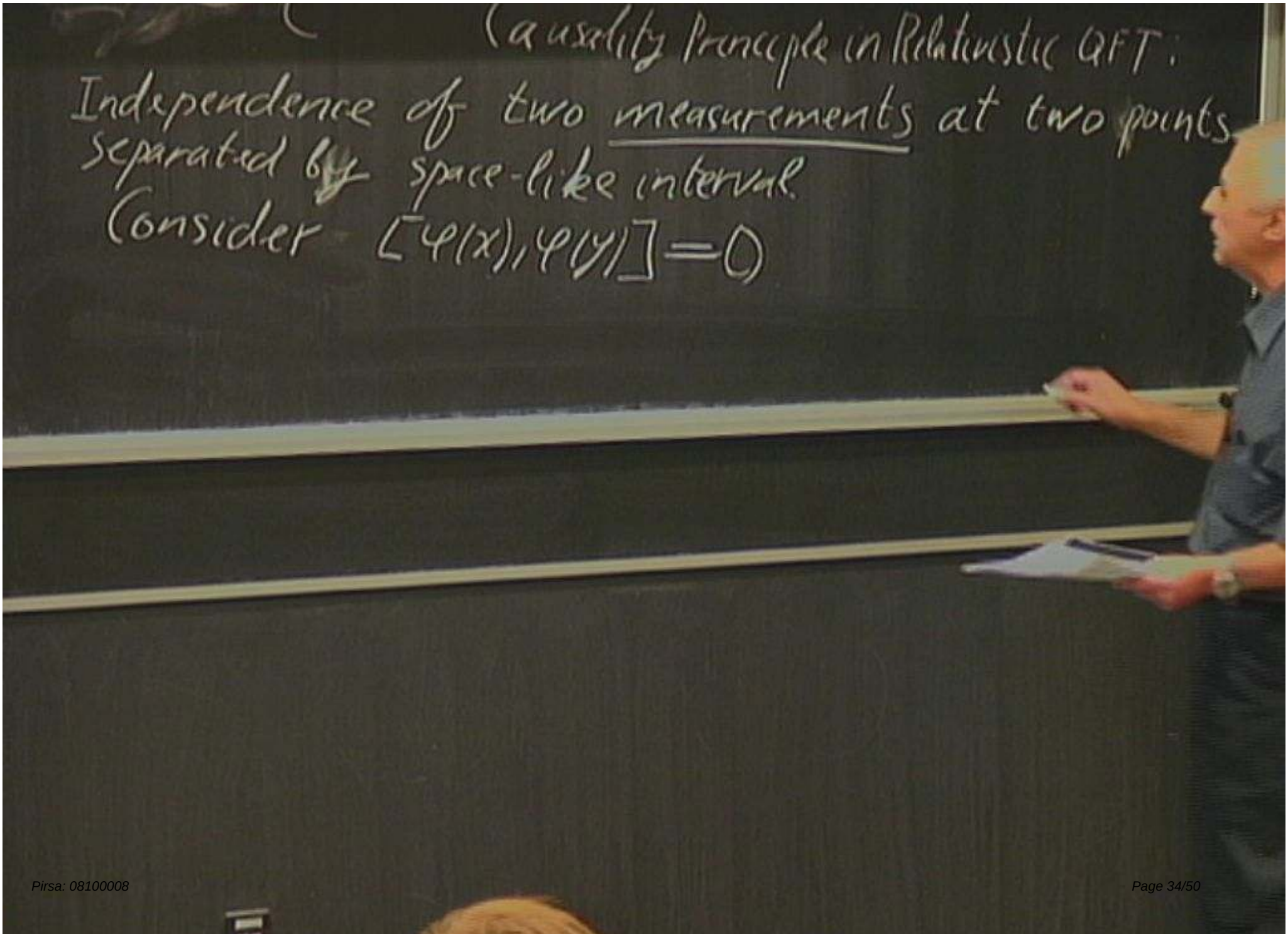
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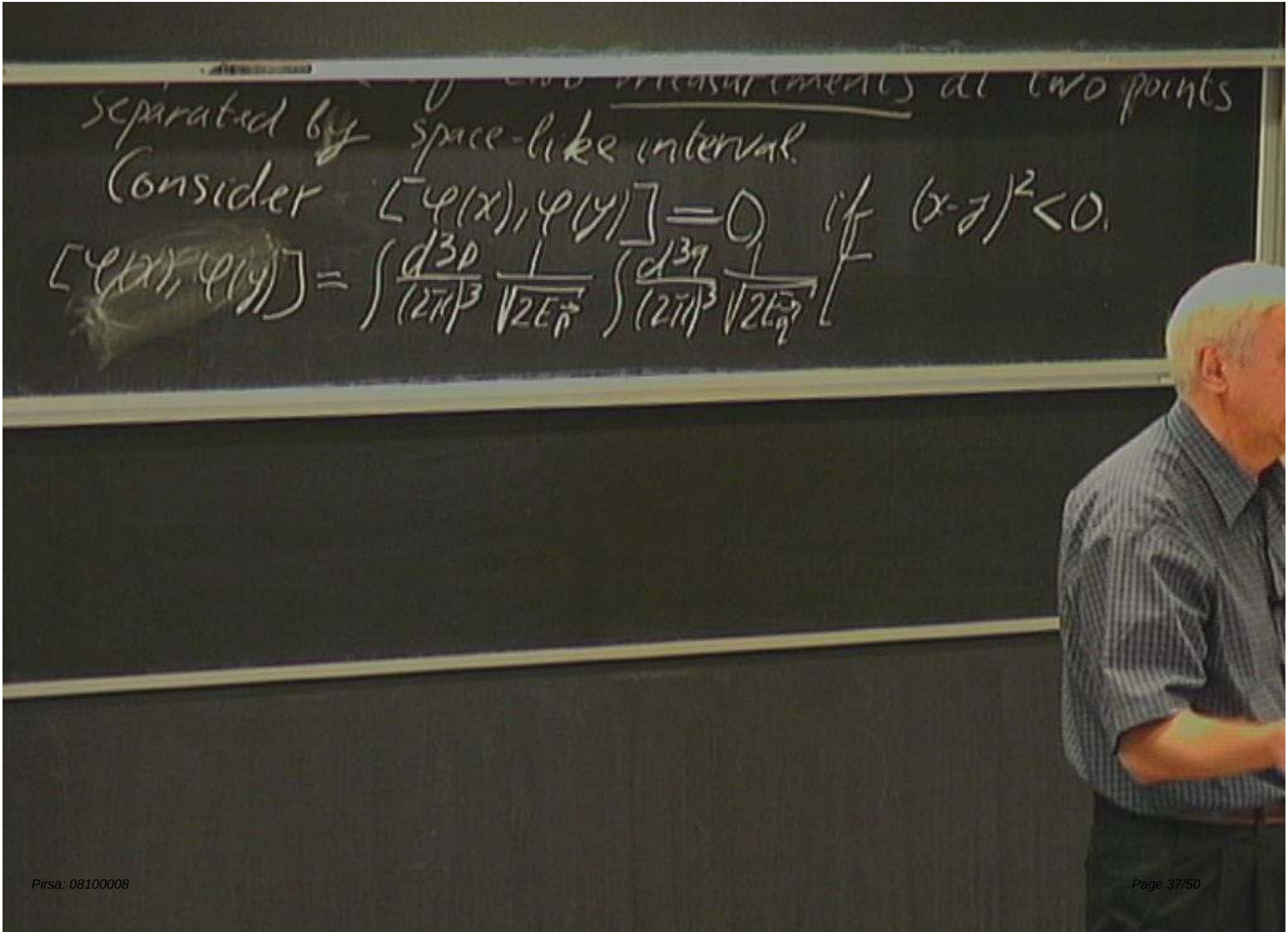
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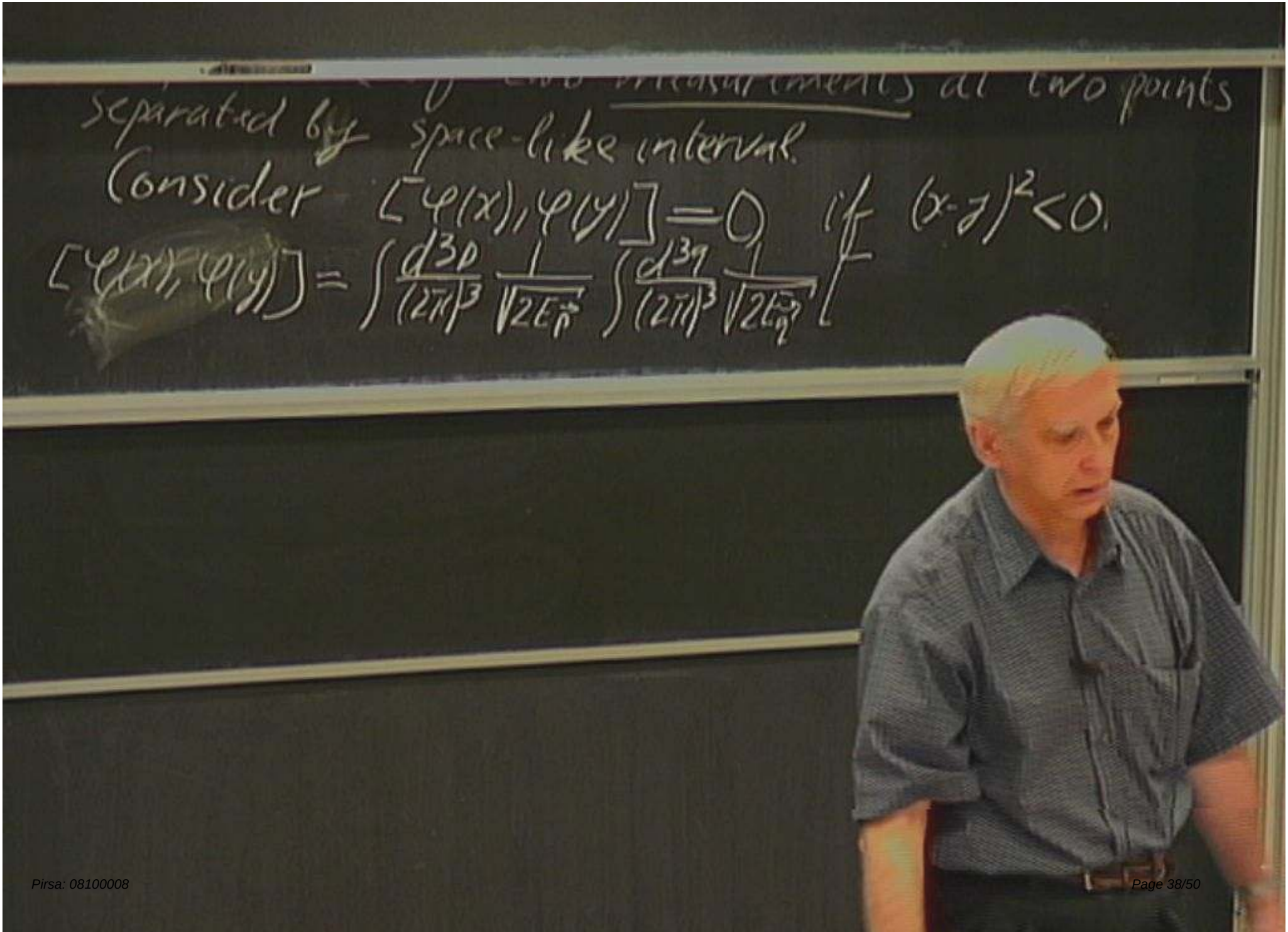
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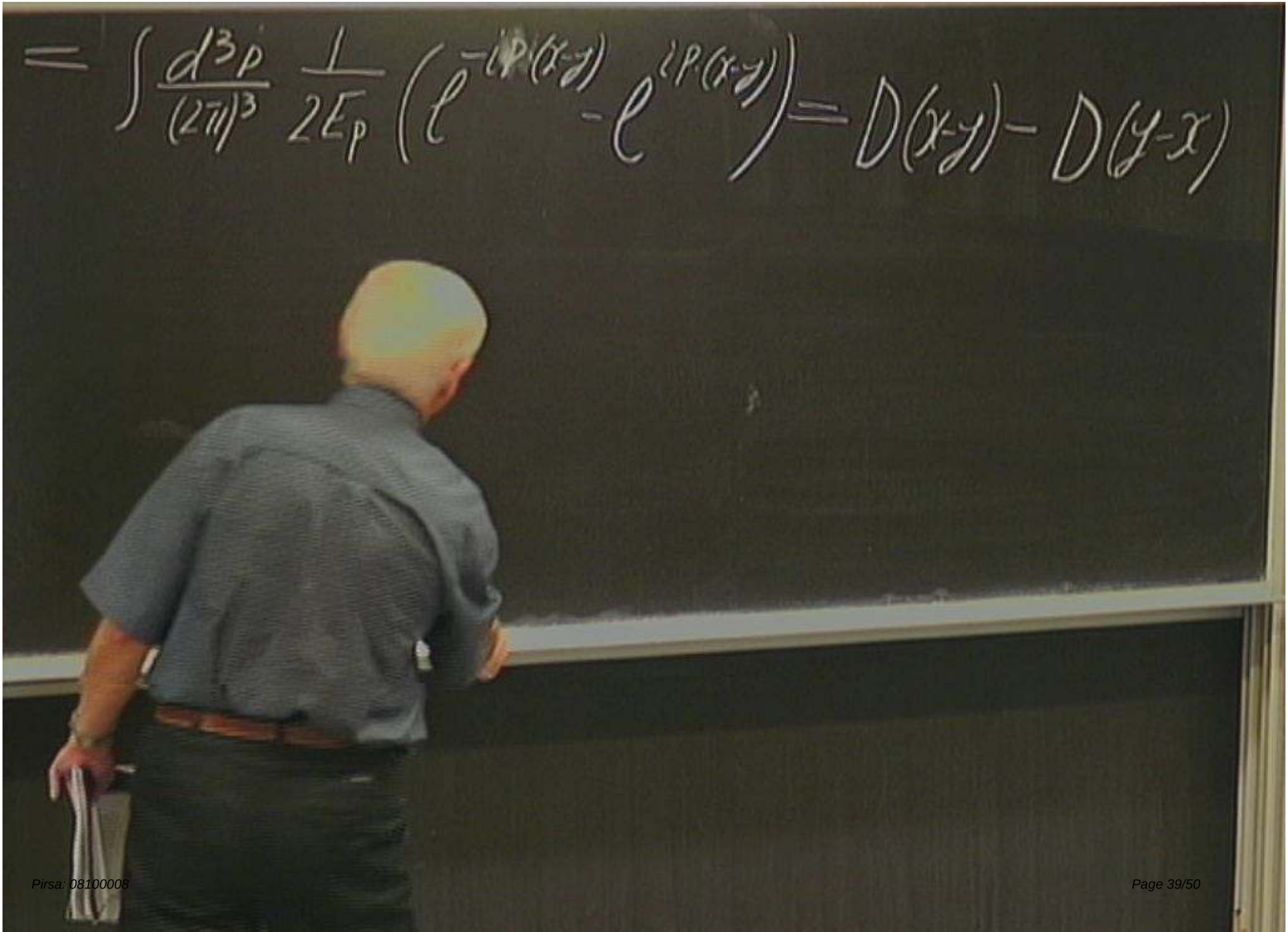
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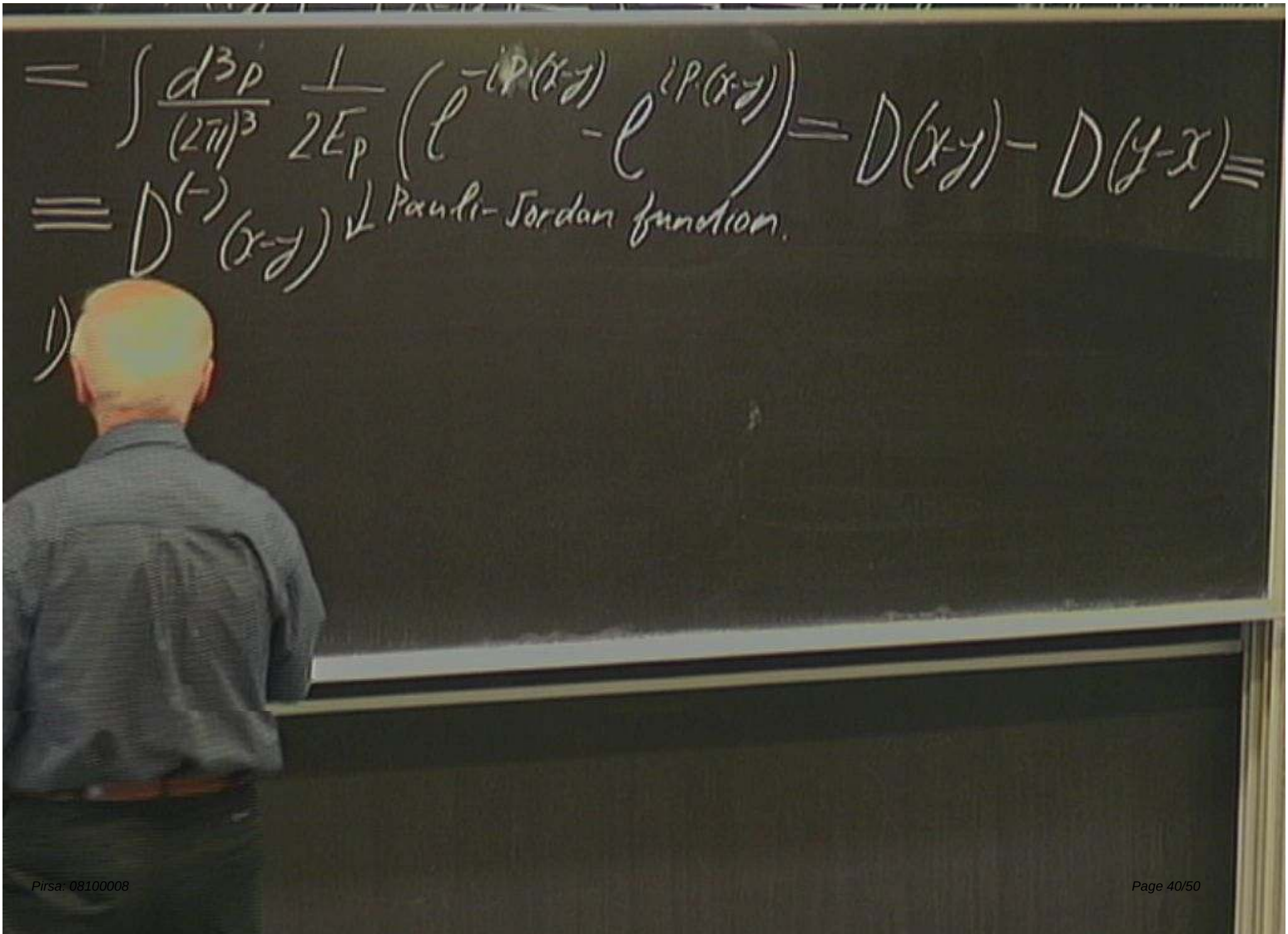
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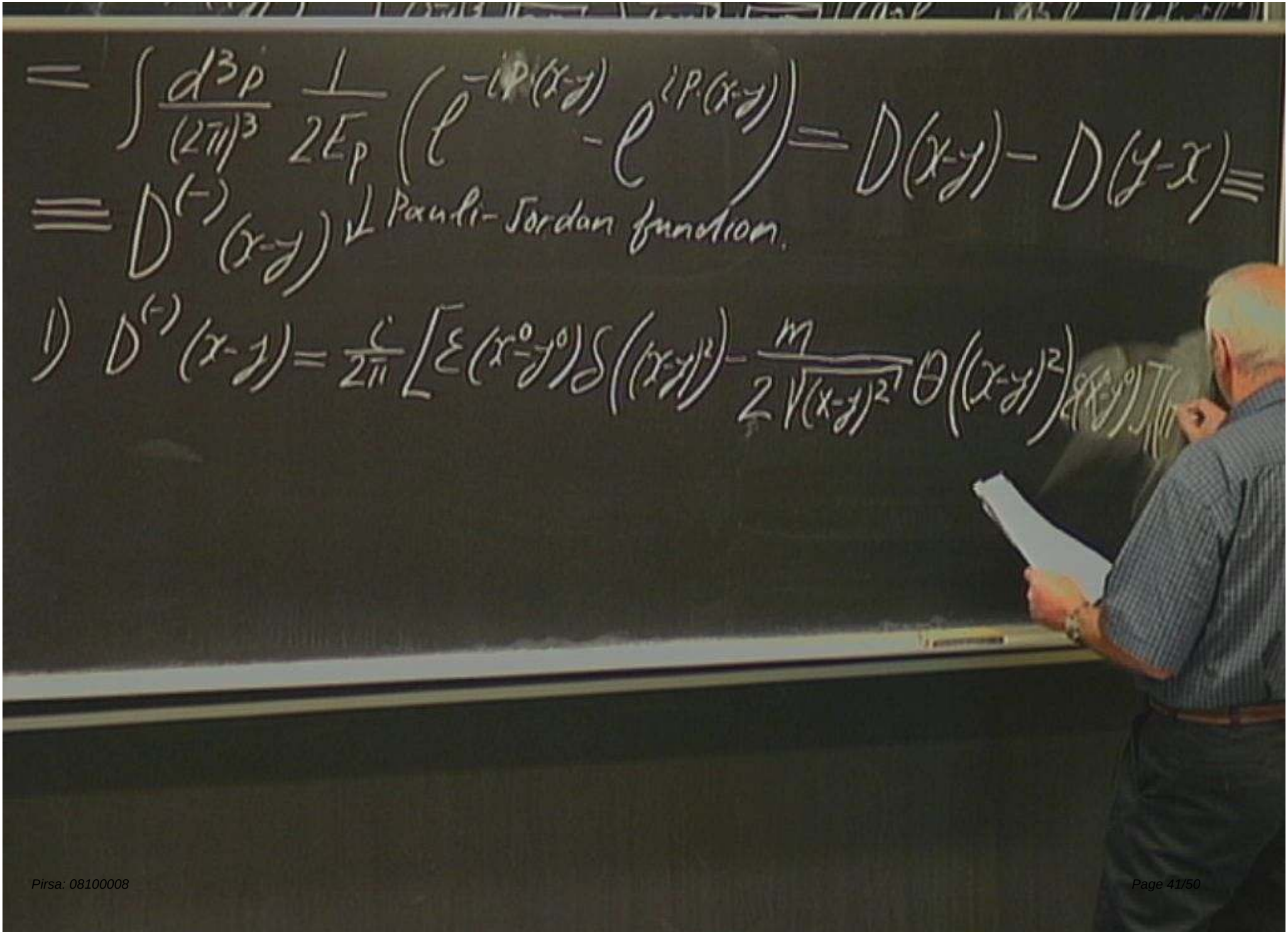
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1) $D^{(-)}(x-y) = \frac{c}{2\pi} \left[\varepsilon(x^0-y^0) \delta((x-y)^2) - \frac{m}{2\sqrt{(x-y)^2}} \theta((x-y)^2) \right]$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) = D(x-y) - D(y-x) \equiv$$

$$\equiv D^{(-)}(x-y)$$ ← Pauli-Jordan function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \varepsilon(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

1) $$D^{(-)}(x-y) = \frac{i}{2\pi} \left[\varepsilon(x^0 - y^0)\, \delta\left((x-y)^2\right) - \frac{m}{2\sqrt{(x-y)^2}}\, \theta\left((x-y)^2\right) J_1\left(m\sqrt{(x-y)^2}\right) \right]$$

Take $(x-y)^2 < 0$ (spatial like interval); Then $D^{(-)}(x-y) = 0$.

Remarks and corrections:

1. ~~$F^{\mu\nu} = x^\mu \int d^3x\, T^{0\nu} - x^\nu \int d^3x\, T^{0\mu}$~~ ⟹ correct

$$J^{\mu\nu} = \int d^3x \left[x^\mu T^{0\nu} - x^\nu T^{0\mu} \right].$$

Remarks and corrections:

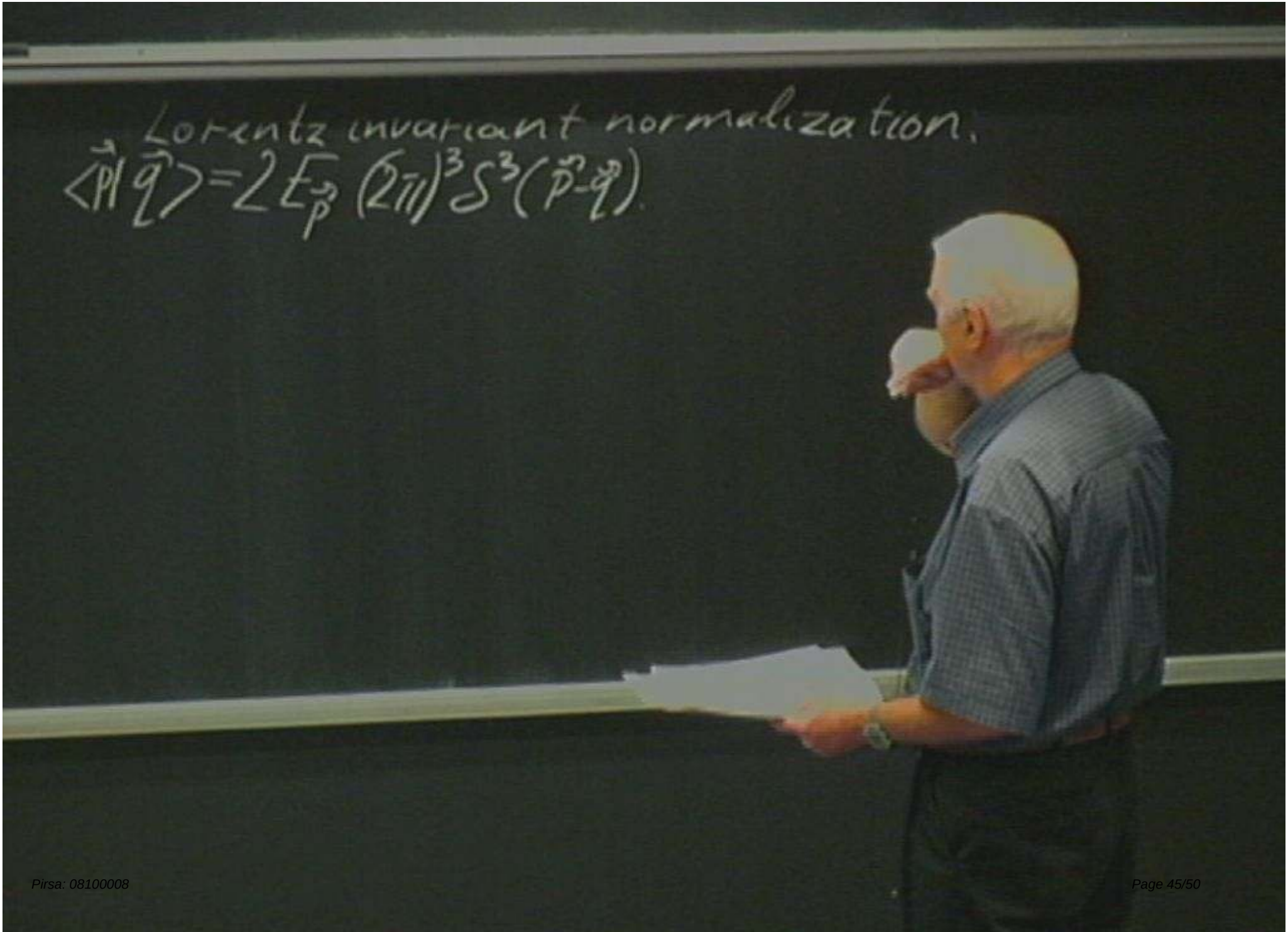
1. ~~$J^{\mu\nu} = x^\mu \int d^3x\, T^{0\nu} - x^\nu \int d^3x\, T^{0\mu}$~~ correct

$$J^{\mu\nu} = \int d^3x \left[x^\mu T^{0\nu} - x^\nu T^{0\mu} \right].$$

2. $U(\Lambda)|0\rangle = |0\rangle$, $\underline{U(a)|0\rangle = |0\rangle}$ $\quad U(a) = e^{i P^\mu a_\mu}$

$P^0|0\rangle \equiv H|0\rangle = 0$, $\vec{P}|0\rangle = 0$

$$P^0 = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}}\, a^+_{\vec{p}} a_{\vec{p}} \qquad \vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^+_{\vec{p}} a_{\vec{p}}$$


Lorentz invariant normalization.
$\langle \vec{p}' | \vec{q} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^3(\vec{p}' - \vec{q})$

Lorentz transf $\langle \vec{p}' | \vec{q}' \rangle = \sqrt{2E_{\vec{p}'}}\sqrt{2E_{\vec{q}'}}\langle 0 | a_{\vec{p}'} a^{\dagger}_{\vec{q}'} | 0 \rangle \Longrightarrow$

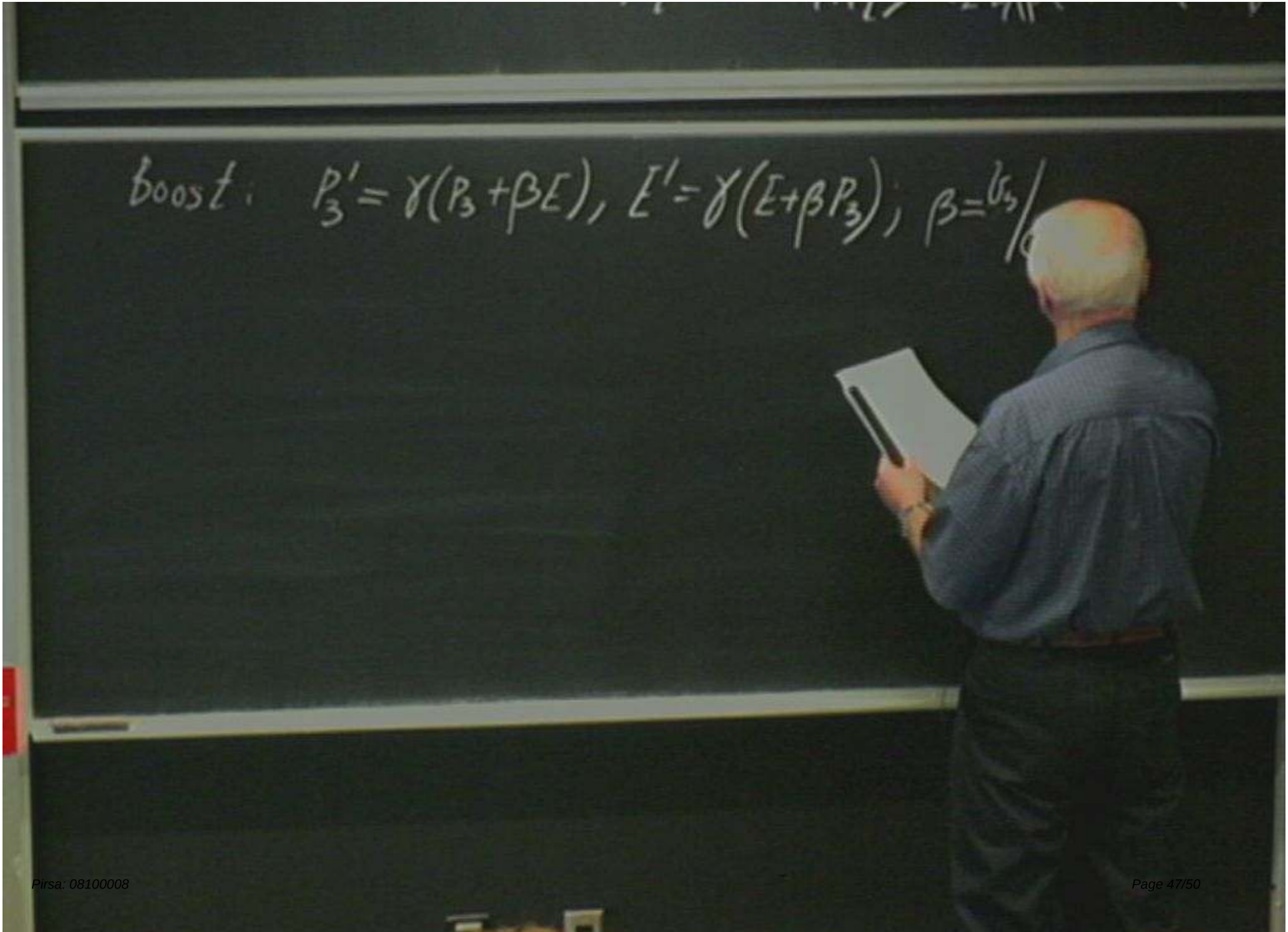
$\Longrightarrow \sqrt{2E_{\vec{p}'}}\sqrt{2E_{\vec{q}'}}\langle \vec{p} | U^{\dagger}(\Lambda) U(\Lambda) | \vec{q} \rangle = \sqrt{2E_{\vec{p}'}}\sqrt{2E_{\vec{q}'}}\langle 0 | a_{\vec{p}} U(\Lambda) U(\Lambda) a^{\dagger}_{\vec{q}} | 0 \rangle =$

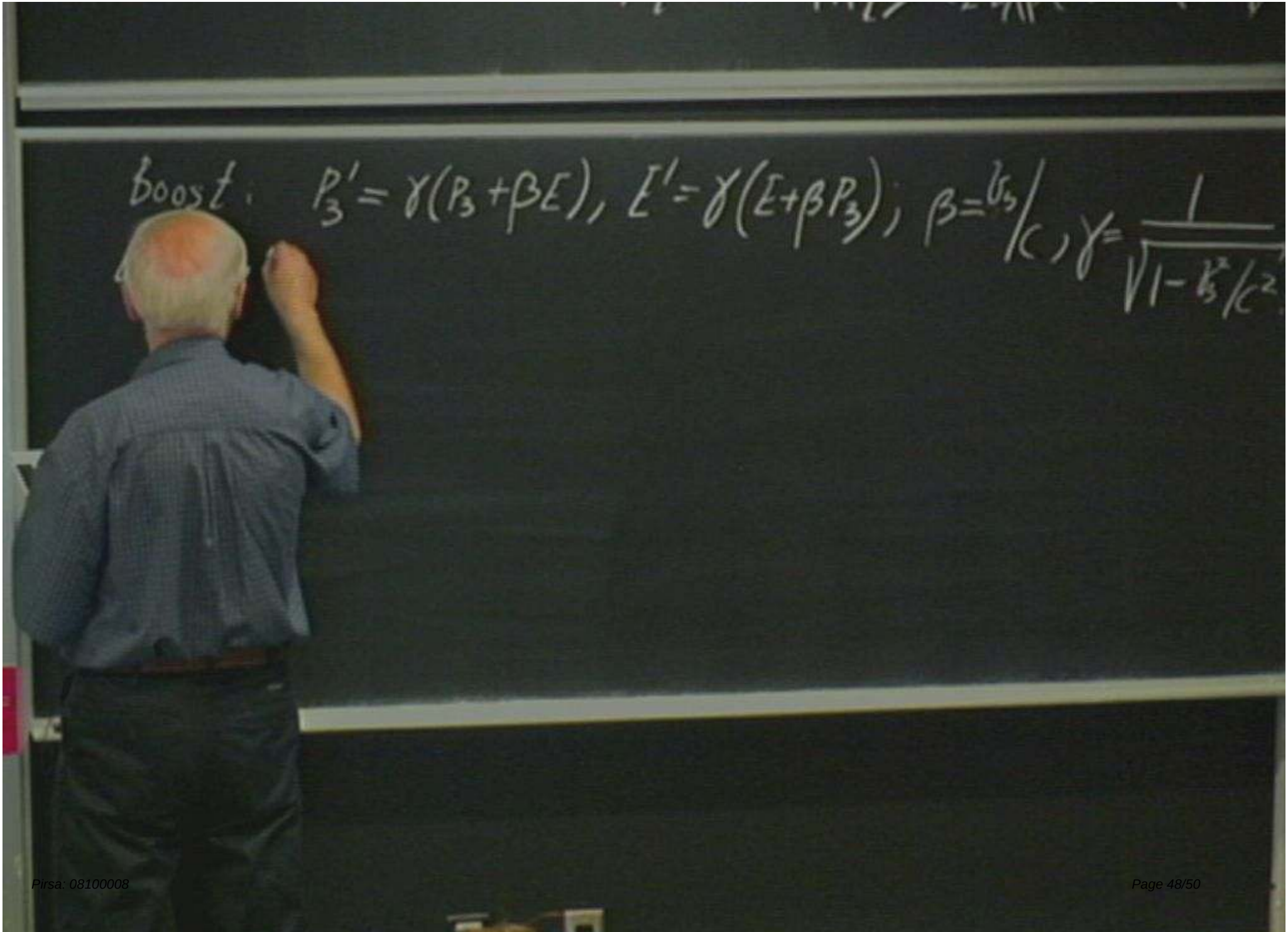
$U^{\dagger}(\Lambda) = U^{-1}(\Lambda)$

$= 2E_p (2\pi)^3 \delta^3(\vec{p} - \vec{q})$

On the other hand, $\langle \vec{p}' | U^{\dagger}(\Lambda) U(\Lambda) | \vec{q}' \rangle = \langle \Lambda \vec{p} | \Lambda \vec{q} \rangle = 2E_{\Lambda p} (2\pi)^3 \delta^3(\vec{\Lambda p} - \vec{\Lambda q})$

$(\sim) (2\pi)^3 \delta(p - q)$


boost: $P_3' = \gamma(P_3 + \beta E)$, $E' = \gamma(E + \beta P_3)$; $\beta = v_3$


boost: $P_3' = \gamma(P_3 + \beta E)$, $E' = \gamma(E + \beta P_3)$; $\beta = v_3/c$, $\gamma = \frac{1}{\sqrt{1 - v_3^2/c^2}}$

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Evaluate $\delta^{(3)}(\vec{p} - \vec{q}) \overset{\text{use}}{=\!=\!=} \delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$

$= \delta^{(3)}(\vec{p}' - \vec{q}') \frac{dp_3'}{dp_3} =$

$= \delta^{(3)}(\vec{p}' - \vec{q}') \gamma\left(1 + \beta \frac{dE}{dp_3}\right) = \delta^3(\vec{p}' - \vec{q}') \frac{\gamma}{E}(E + \beta p_3) = \delta^3(\vec{p}' - \vec{q}') \frac{E'}{E} \Rightarrow$

$\Rightarrow E\delta^{(4)}(\vec{p} - \vec{q}) = E' \delta^{(3)}(\vec{p}' - \vec{q}')$.

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