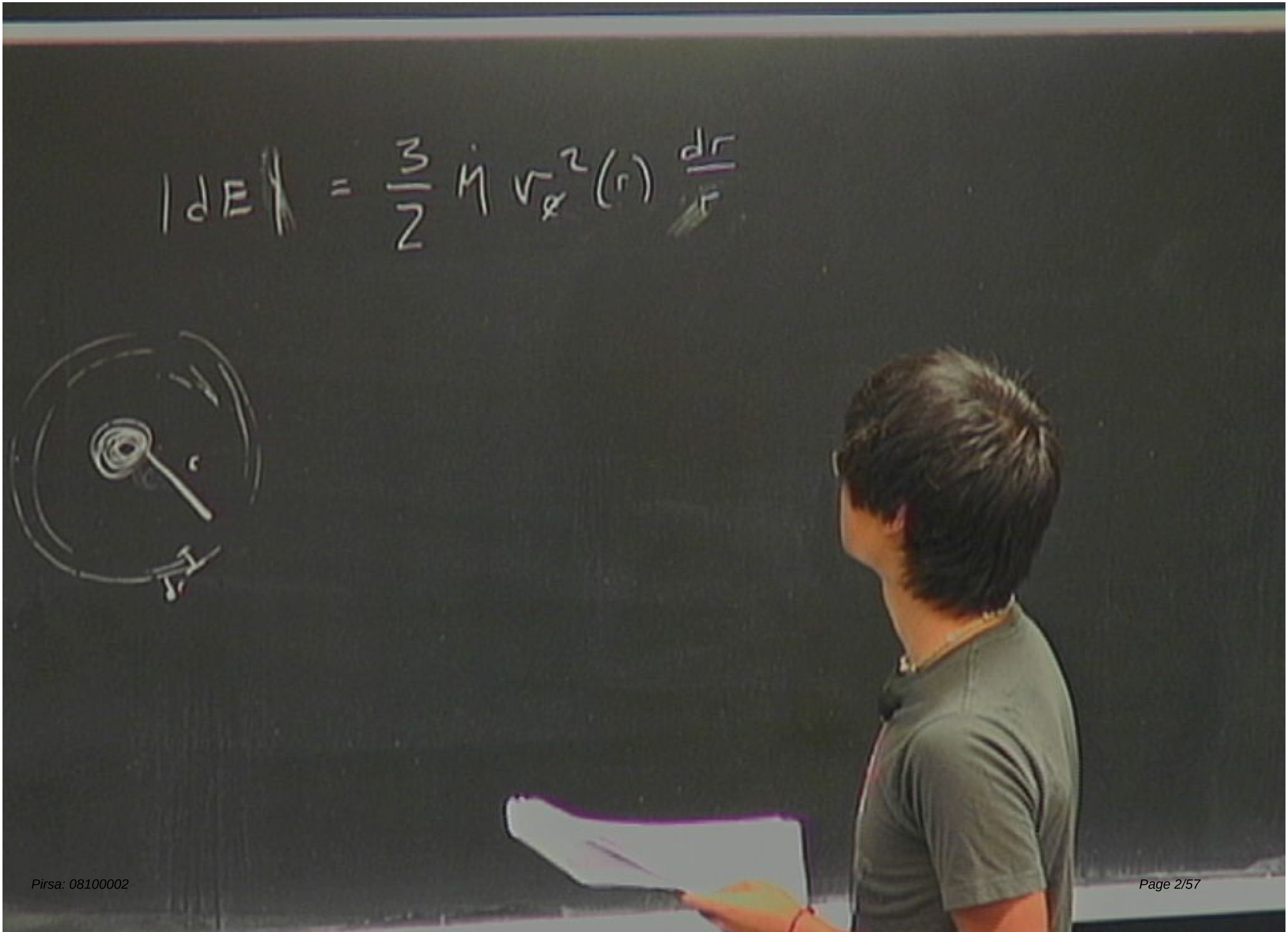
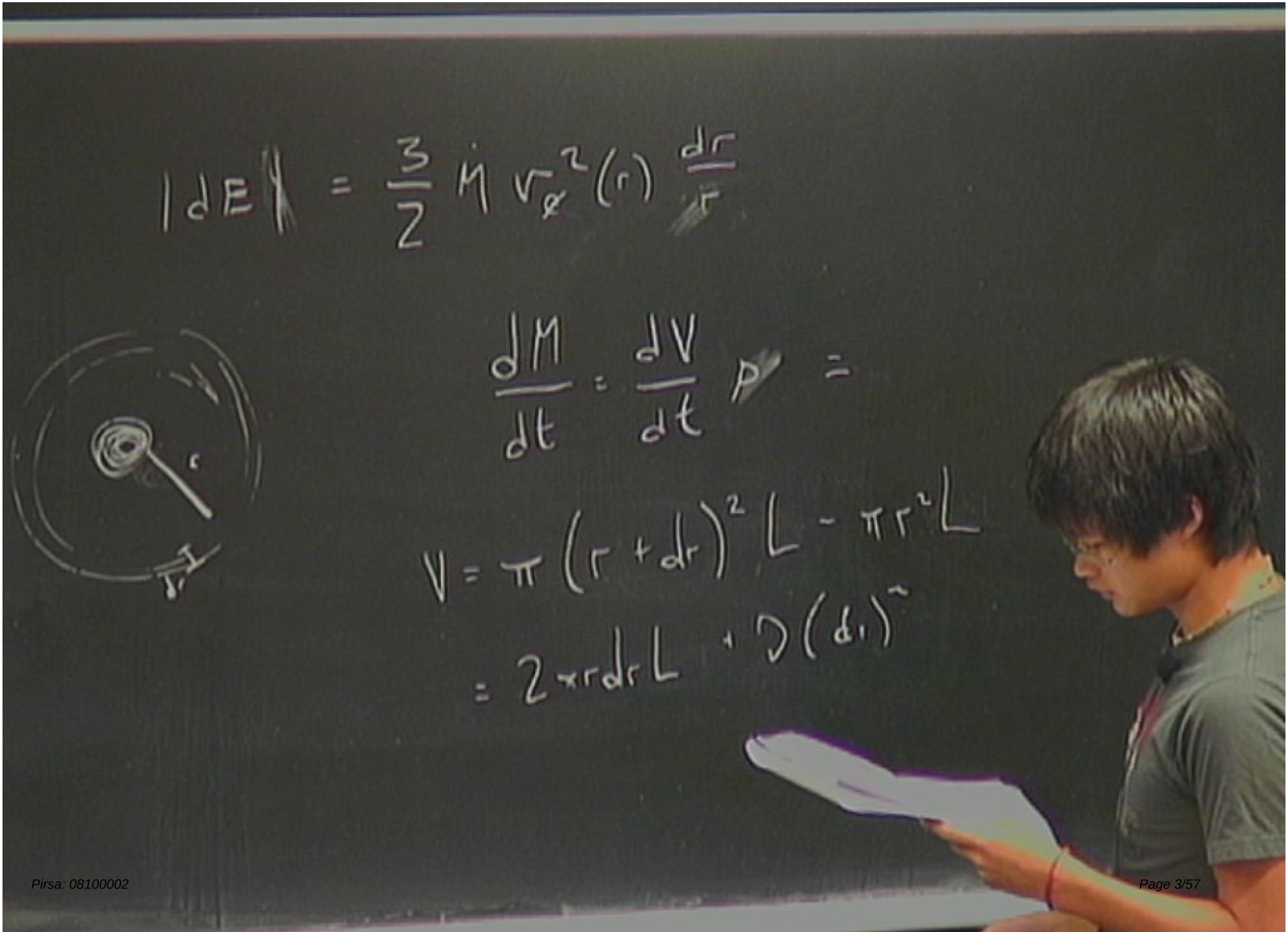
Title: Astrophysics and Cosmology through Problems - 6A

Date: Oct 09, 2008 10:00 AM

URL: http://pirsa.org/08100002

Abstract: This course is aimed at advanced undergraduate and beginning graduate students, and is inspired by a book by the same title, written by Padmanabhan. Each session consists of solving one or two pre-determined problems, which is done by a randomly picked student. While the problems introduce various subjects in Astrophysics and Cosmology, they do not serve as replacement for standard courses in these subjects, and are rather aimed at educating students with hands-on analytic/numerical skills to attack new problems.


$|dE| = \frac{3}{2} \dot{M} v_\phi^2(r) \frac{dr}{r}$


$|dE| = \frac{3}{2}\dot{M}\, v_\phi^2(r)\, \frac{dr}{r}$
$\frac{dM}{dt} = \frac{dV}{dt}\rho =$
$V = \pi(r+dr)^2 L - \pi r^2 L$
$= 2\pi r\, dr\, L + O(dr)^2$

$$|dE| = \frac{3}{2}\dot{M}\, v_\phi^2(r)\,\frac{dr}{r}$$

$$\frac{dM}{dt} = \frac{dV}{dt}\rho = -2\pi r v_r L \rho$$

$$V = \pi (r+dr)^2 L - \pi r^2 L$$

$$= 2\pi r\, dr\, L + O(dr)^2$$

ϕ - comp:

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r}\right) = \eta\left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{\partial^2 v_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2}\right)$$

r - comp:

$$v_r\frac{\partial v_r}{\partial r} - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{GM}{r^2}$$

ϕ - comp:

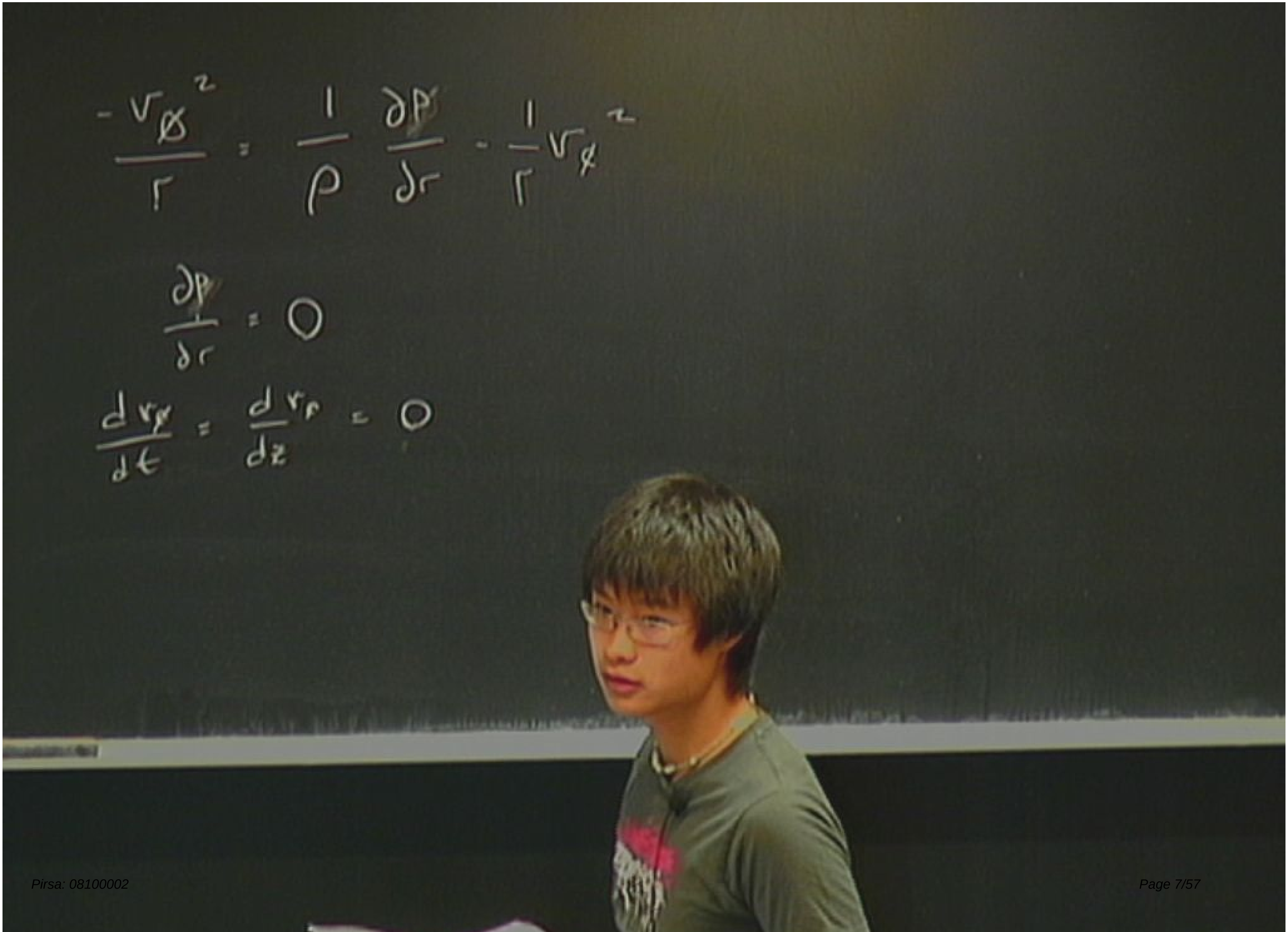
$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r}\right) = \eta\left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{\partial^2 v_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2}\right)$$

r - comp:

$$v_r\frac{\partial v_r}{\partial r} - \frac{v_\phi^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{GM}{r^2}$$

$$v_\phi^2 = \frac{GM}{r}$$

$$\frac{\partial v_r}{\partial r} = 0$$


$-\frac{v_\phi^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{1}{r}v_\phi^2$
$\frac{\partial p}{\partial r} = 0$
$\frac{d v_\phi}{dt} = \frac{d v_r}{dz} = 0$

$$\frac{\partial p}{\partial r} = 0$$

$$\frac{dv_\phi}{dt} = \frac{dv_r}{dz} = 0$$

$$\frac{d}{dr} v_\phi = -\frac{1}{2}\sqrt{\frac{GM}{r^3}}$$

ρ

$$\rho\left(\frac{-v_r}{2}\sqrt{\frac{GM}{r^3}} + v_r\sqrt{\frac{GM}{r^3}}\right) = \eta\left(\frac{3}{4}\sqrt{\frac{GM}{r^5}} - \frac{1}{2}\sqrt{\frac{GM}{r^5}} - \sqrt{\frac{GM}{r^5}}\right)$$

$$\frac{v_r}{2}\rho = \frac{\eta}{r}\left(-\frac{3}{4}\right)$$

$$v_r = -\frac{3}{2}\frac{\eta}{\rho r}$$

$$|dE| = \frac{3}{2}\dot{M}\, v_\phi^2(r)\,\frac{dr}{r}$$

$$\frac{dM}{dt} = \frac{dV}{dt}\rho = -2\pi r v_r L \rho = 3\pi L \eta$$

$$V = \pi (r+dr)^2 L - \pi r^2 L$$

$$= 2\pi r\, dr\, L + O(dr)^2$$

$$S(r) \simeq 2\pi r L \left[\rho v_r \left(\frac{1}{2} v^2 + \psi \right) - \sigma_{ri} v^i \right] \quad , \quad \psi = \frac{-GM}{r}$$

$$S(r) \simeq 2\pi r L \left[\rho v_r \left(\frac{1}{2} v^2 + \psi \right) - \sigma_{ri} v^i \right] , \quad \psi = \frac{-GM}{r}$$

$$dE = \left[S(r+dr) - S(r) \right] = \frac{\partial S}{\partial r} dr$$

$$\frac{dE}{dr} = \frac{\partial}{\partial r} \left[\dot{M} \left(\frac{1}{2} v^2 + \psi \right) + 2\pi r L v^i \sigma_{ri} \right]$$

$$S(r) \simeq 2\pi r L \left[\rho v_r \left(\frac{1}{2} v^2 + \psi \right) - \sigma_{ri} v^i \right] \quad , \quad \psi = \frac{-GM}{r}$$

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$$v_\phi \gg v_r , v_z$$

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$$dE = \left[S(r+dr)-S(r)\right] = \frac{\partial S}{\partial r}dr$$

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$$\frac{dE}{dr} = \frac{\partial}{\partial r}\left[\dot{M}\left(\frac{1}{2}v^2+\psi\right)+2\pi r L v^i\sigma_{ri}\right] \quad , \quad v^2 \simeq v_\phi^2$$

$$\frac{1}{2}v^2+\psi = \frac{1}{2}\frac{GM}{r}-\frac{GM}{r}$$

$$= -\frac{GM}{2r}$$

$$v^i\sigma_{ri} = v^r\sigma_{rr} + v^\phi\sigma_{r\phi}$$

$$= 2\eta v_r\frac{\partial v_r}{\partial r} + \eta v_\phi\left(\frac{\partial v_\phi}{r}-\frac{v_\phi}{r}\right)$$

$$S(r) \simeq 2\pi r L \left[\rho v_r \left(\tfrac{1}{2} v^2 + \psi \right) - \sigma_{ri} v^i \right] \quad , \quad \psi = \frac{-GM}{r}$$

$$dE = \left[S(r+dr) - S(r) \right] = \frac{\partial S}{\partial r} dr$$

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$$\frac{dE}{dr} = \frac{\partial}{\partial r} \left[\dot{M} \left(\tfrac{1}{2} v^2 + \psi \right) + 2\pi r L v^i \sigma_{ri} \right] \quad , \quad v^2 \simeq v_\phi^2$$

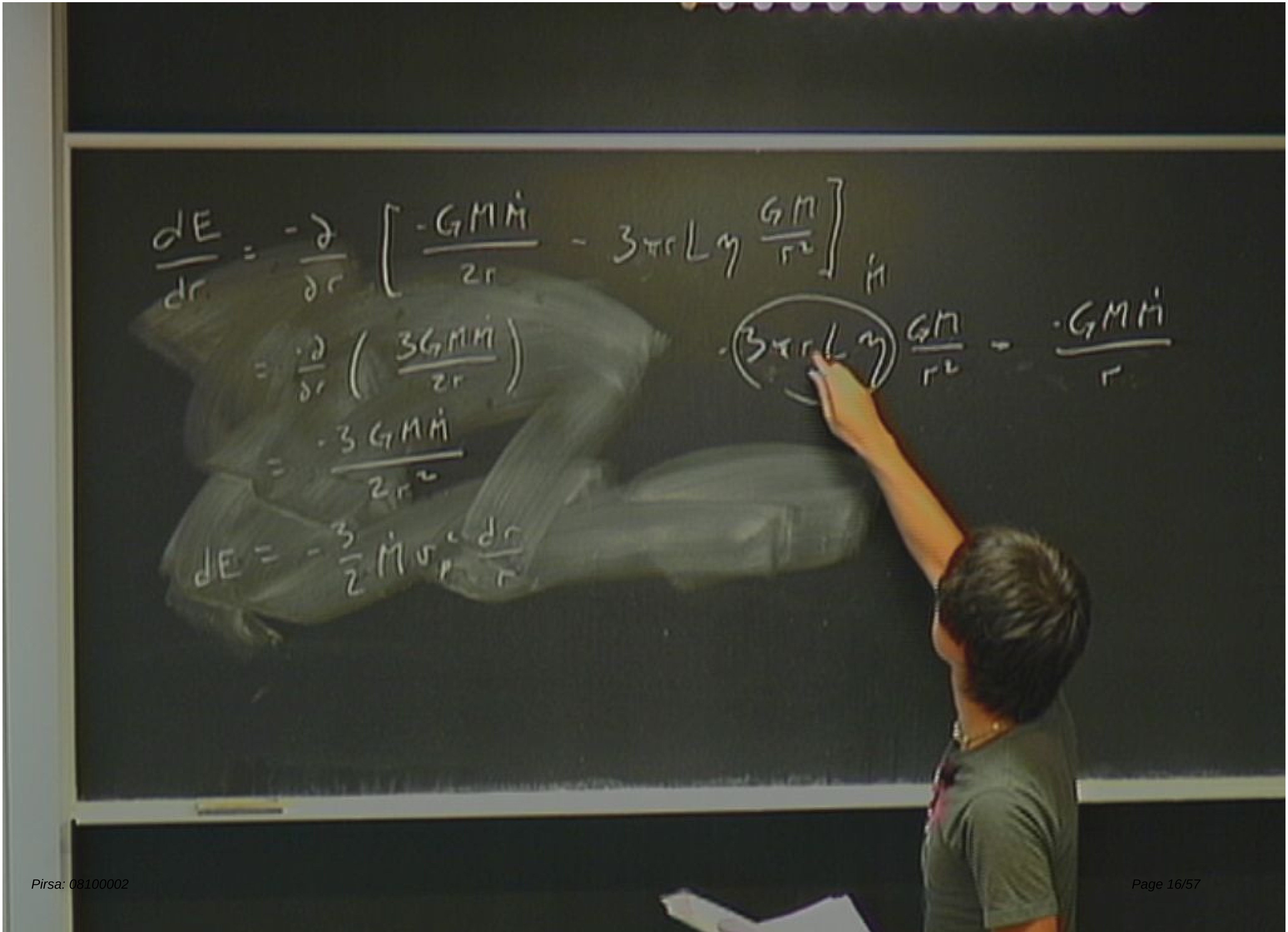
$$\tfrac{1}{2} v^2 + \psi = \frac{1}{2} \frac{GM}{r} - \frac{GM}{r} = -\frac{GM}{2r}$$

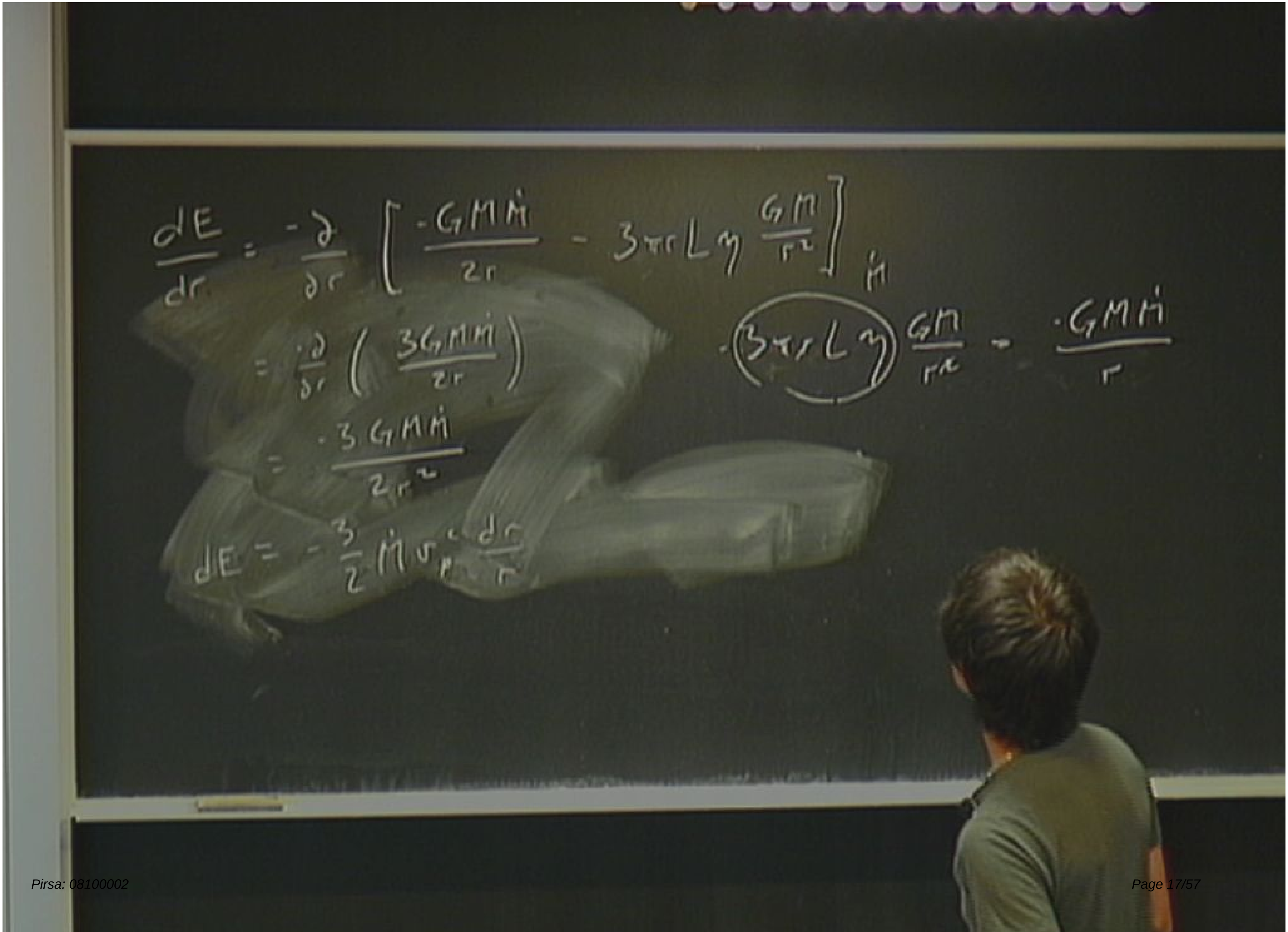
$$v^i \sigma_{ri} = v^r \sigma_{rr} + v^\phi \sigma_{r\phi}$$

$$= 2\eta v^r \frac{\partial v_r}{\partial r} + \eta v^\phi \left(\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} \right)$$

$$= \sqrt{\frac{GM}{r}} \eta \left(-\frac{1}{2} \sqrt{\frac{GM}{r^3}} - \sqrt{\frac{GM}{r^3}} \right)$$

$$= -\frac{3}{2} \eta \frac{GM}{r^2}$$


$\frac{dE}{dr} = -\frac{\partial}{\partial r}\left[\frac{-GM\dot{M}}{2r} - 3\pi r L \eta \frac{GM}{r^2}\right]$ $\dot{M}$
$-3\pi r L \eta \frac{GM}{r^2} = \frac{-GM\dot{M}}{r}$
$= -\frac{\partial}{\partial r}\left(\frac{3GM\dot{M}}{2r}\right)$
$= \frac{-3GM\dot{M}}{2r^2}$
$dE = -\frac{3}{2}\dot{M} v_p^2 \frac{dr}{r}$


$\frac{dE}{dr} = -\frac{\partial}{\partial r}\left[\frac{-GM\dot{M}}{2r} - 3\pi r L \eta \frac{GM}{r^2}\right]$
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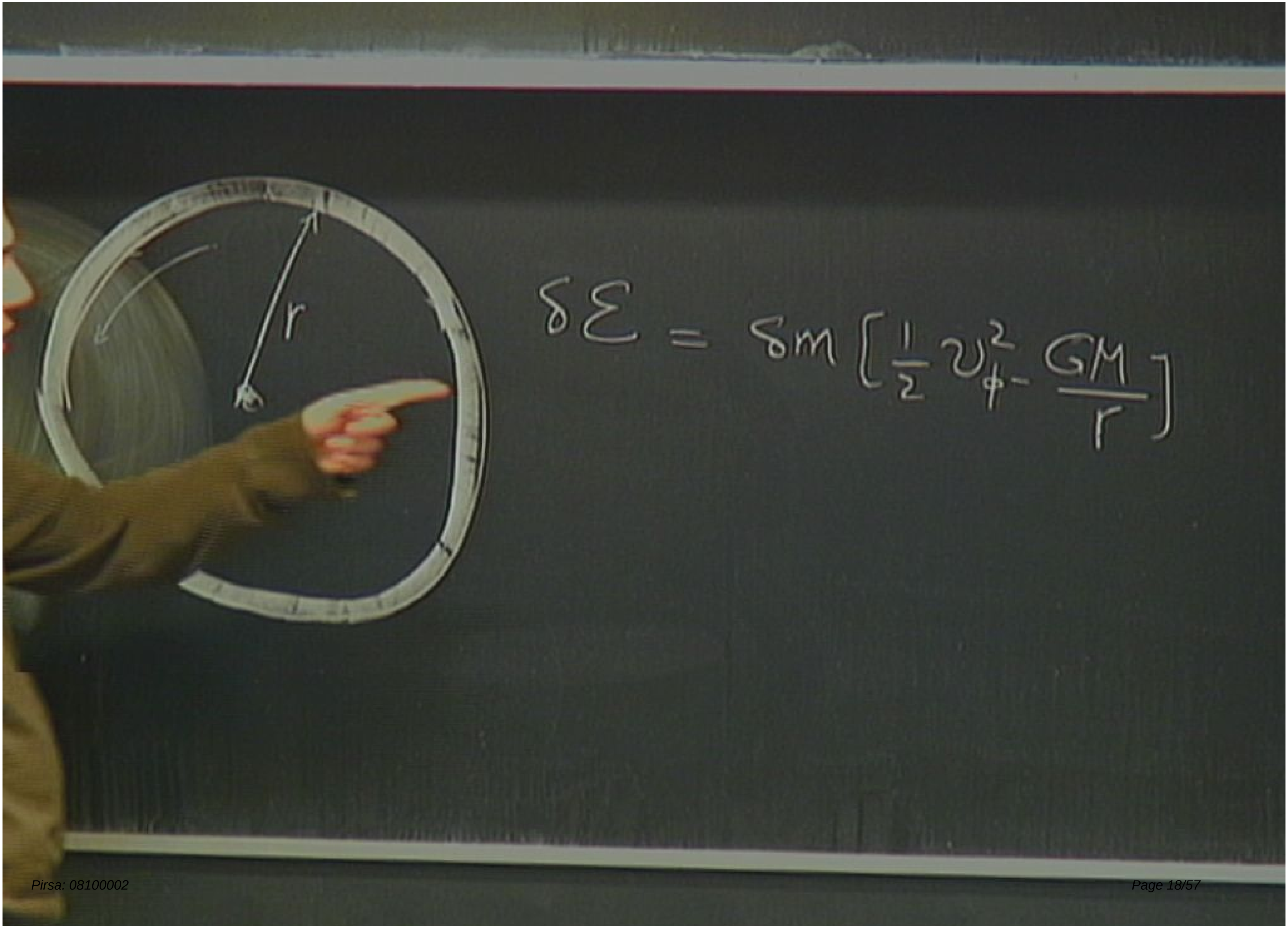
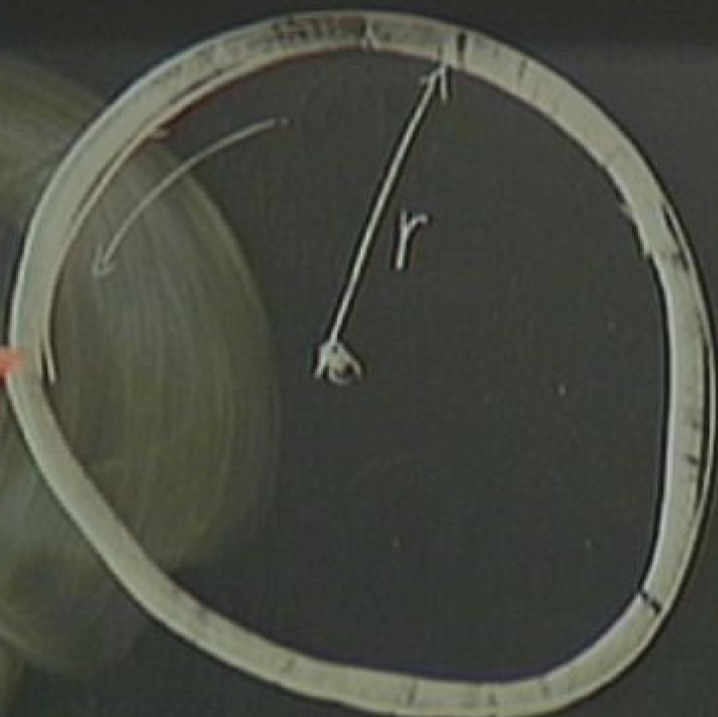
$= -\frac{\partial}{\partial r}\left(\frac{3GM\dot{M}}{2r}\right)$
$= \frac{-3GM\dot{M}}{2r^2}$
$dE = -\frac{3}{2}\dot{M} v_p^2 \frac{dr}{r}$
$-(3\pi r L \eta)\frac{GM}{r^2} = \frac{-GM\dot{M}}{r}$

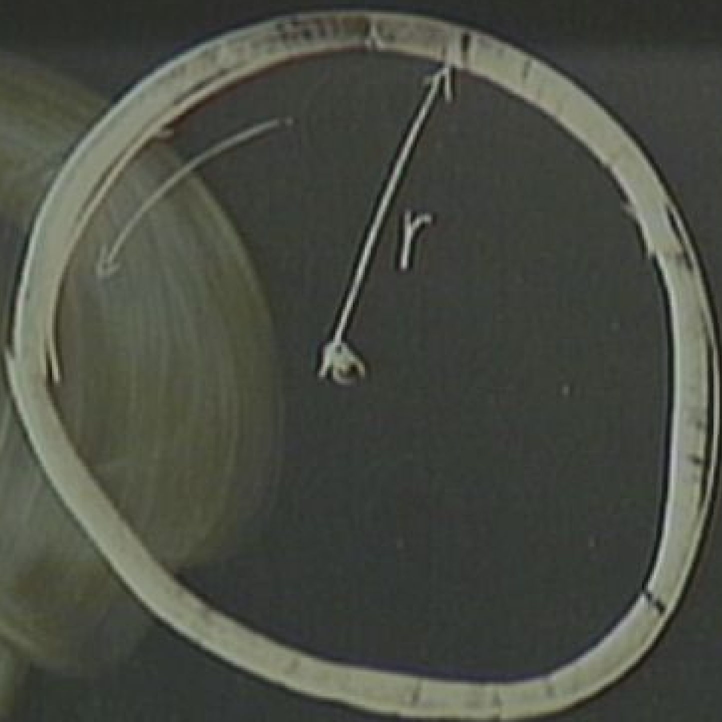

r
$\delta \mathcal{E} = \delta m \left[\frac{1}{2} v_\phi^2 - \frac{GM}{r} \right]$



$$v_\phi^2 = \frac{GM}{r}$$

$$\delta \mathcal{E} = \delta m \left[\frac{1}{2} v_\phi^2 - \frac{GM}{r} \right]$$

$$= -\frac{1}{2}\, \delta m \times \frac{GM}{r}$$



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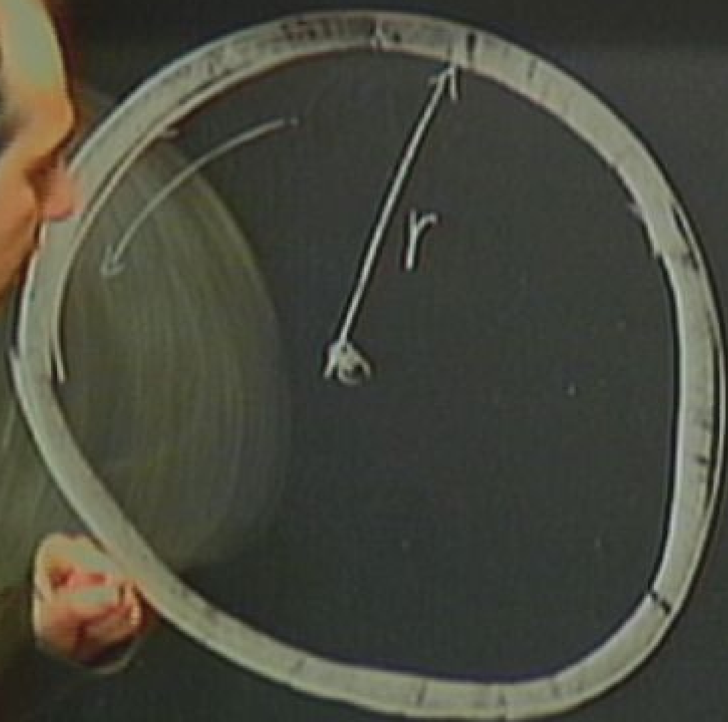

$$v_\phi^2 = \frac{GM}{r}$$

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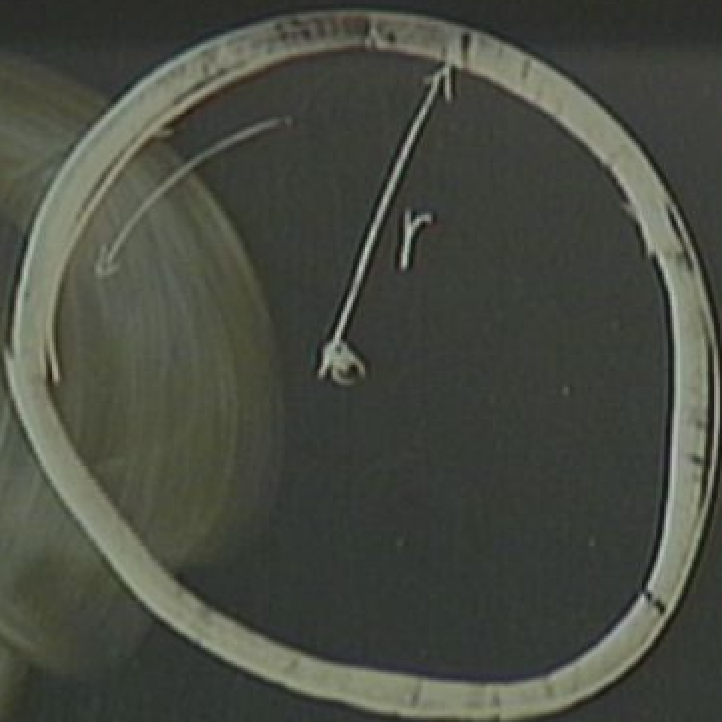
$$= -\frac{1}{2}\,\delta m \times \frac{GM}{r}$$

$$\delta\dot{\mathcal{E}} = \frac{1}{2}\,\frac{GM\,\delta m}{r^2} \times \dot{r} = \frac{GM\,\delta m\,v_r}{2r^2}$$



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$$= -\frac{1}{2}\,\delta m \times \frac{GM}{r}$$

$$\delta\dot{\mathcal{E}} = \frac{1}{2}\,\frac{GM\,\delta m}{r^2} \times \dot{r} = \frac{GM\,\delta m\,V_r}{2r^2}$$

$$\begin{cases} \dot{M} = -2\pi r L \rho v_r \\ \delta m = 2\pi r\, dr \cdot L \cdot \rho \end{cases}$$

$$v_\phi^2 = \frac{GM}{r}$$

$$\delta \mathcal{E} = \delta m \left[\frac{1}{2} v_\phi^2 - \frac{GM}{r} \right]$$

$$= -\frac{1}{2}\, \delta m \cdot \frac{GM}{r}$$

$$\boxed{\delta \dot{\mathcal{E}} = \frac{1}{2} \frac{GM\,\delta m}{r^2} \cdot \dot{r} = \frac{GM\,\delta m\, v_r}{2r^2}}$$

$$\begin{cases} \dot{M} = -2\pi r L \rho v_r \\ \delta m = 2\pi r\, \delta r \cdot L \cdot \rho \end{cases}$$

$$\delta\dot{\mathcal{E}} = -\frac{GM}{2r^2} \cdot \frac{\dot{M}}{\cancel{2\pi r L \rho}} \cdot \cancel{2\pi r L \rho}\, \delta r$$

$$= -\frac{GM\dot{M}}{2r^2}\delta r$$

$$v_\phi^2 = \frac{GM}{r}$$

$$\delta\mathcal{E} = \delta m \left[\frac{1}{2} v_\phi^2 - \frac{GM}{r}\right]$$

$$= -\frac{1}{2}\,\delta m \cdot \frac{GM}{r}$$

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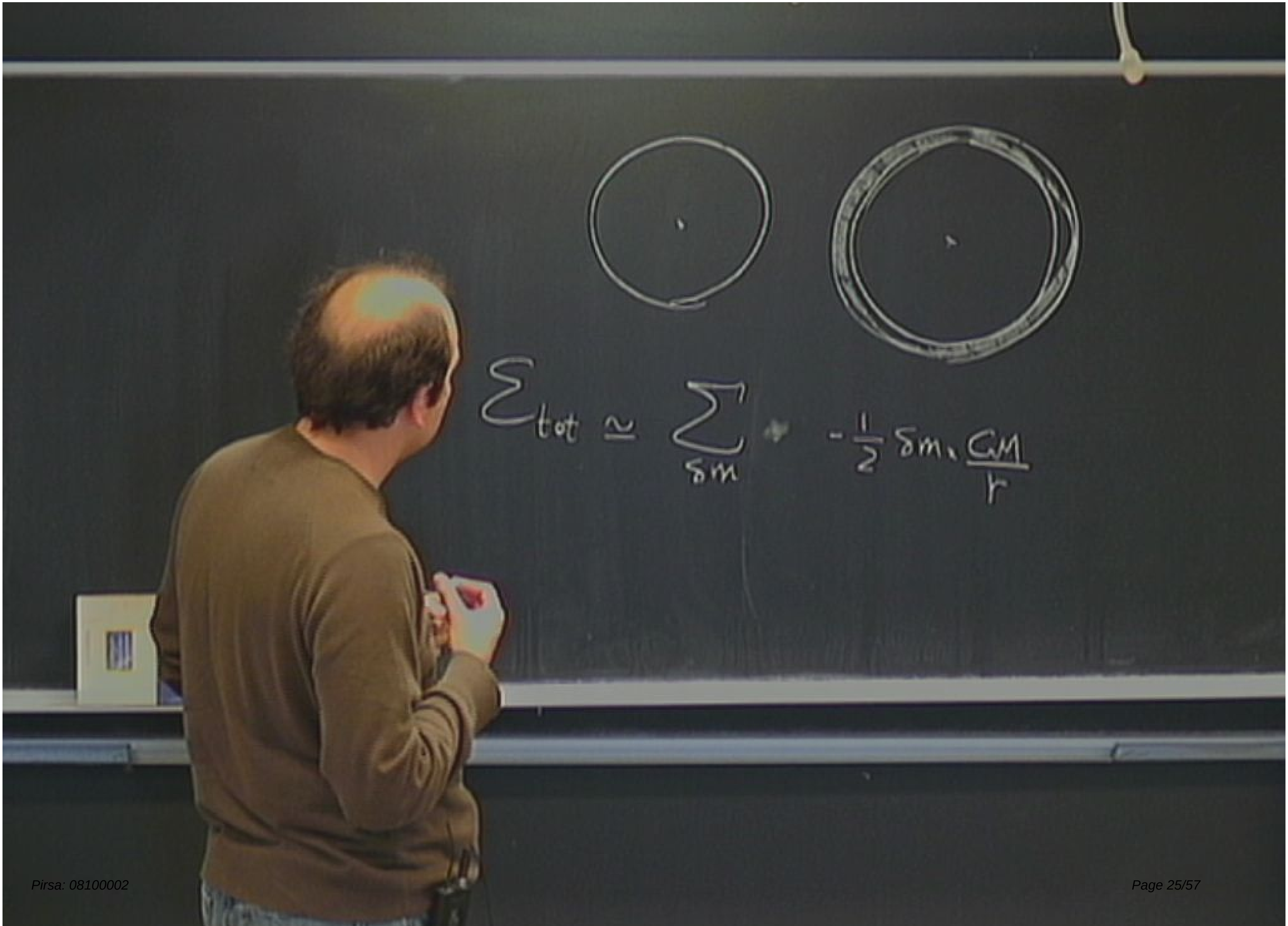
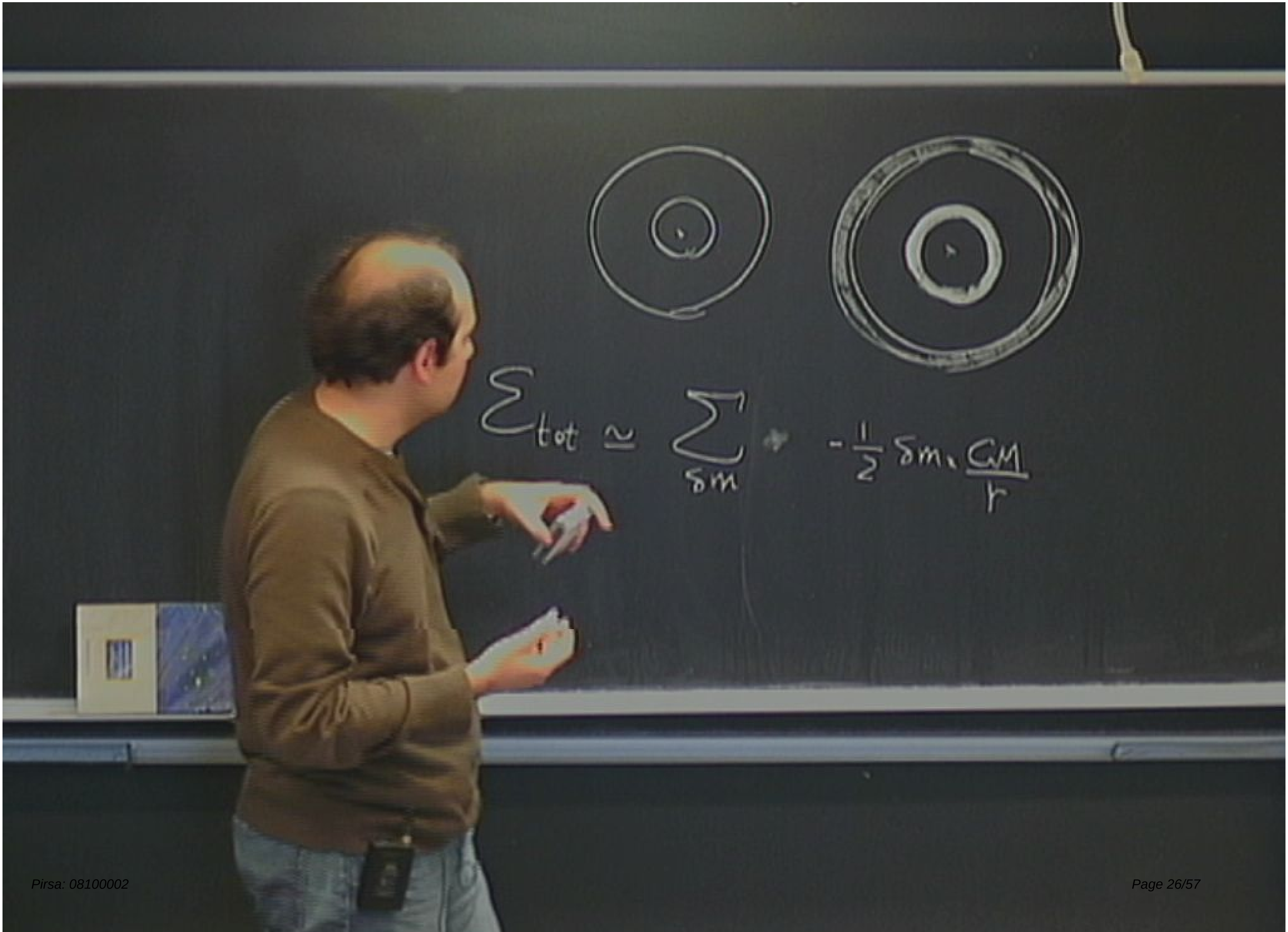

$\mathcal{E}_{tot} \simeq \sum_{\delta m} -\frac{1}{2}\delta m \cdot \frac{GM}{r}$


$E_{tot} \simeq \sum_{\delta m} -\frac{1}{2}\delta m \cdot \frac{GM}{r}$

$$\mathcal{E}_{tot} \simeq \sum_{\delta m} -\frac{1}{2}\,\delta m \cdot \frac{GM}{r}$$

$$\Delta \mathcal{E}_{tot} = -\frac{1}{2}\,\delta m\,\frac{GM}{r_{in}} + \frac{1}{2}\,\delta m\,\frac{GM}{r_{out}}$$

$$= \int \frac{GM}{r}\,\eta \left(-\frac{1}{2}\int \frac{GM}{r^2} - \int \frac{GM}{r^2} \right)$$

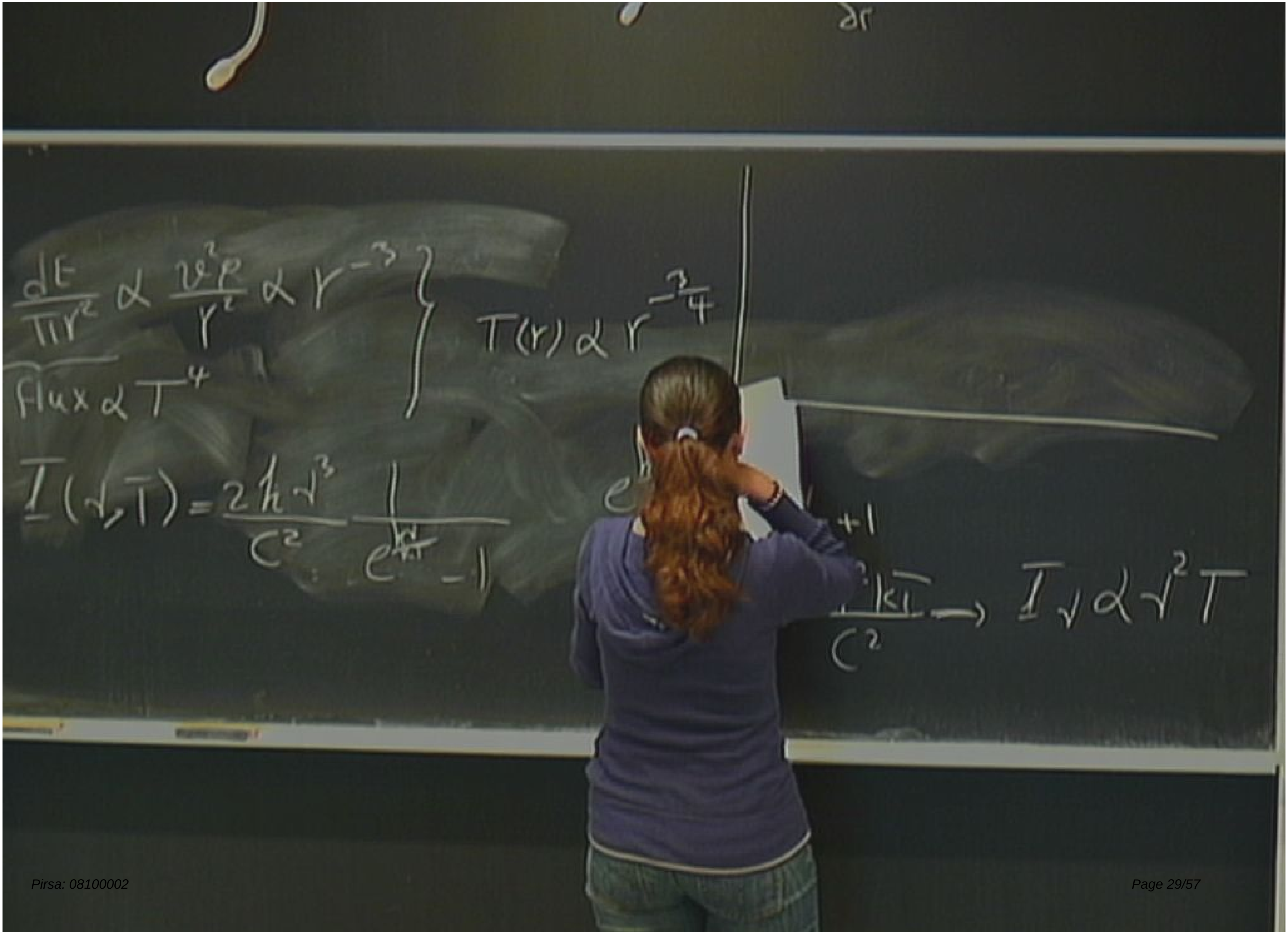
$$= \frac{3}{2}\,\eta\,\frac{GM}{r^2}$$

$$\mathcal{E}_{tot} \simeq \sum_{\delta m} -\frac{1}{2}\,\delta m \times \frac{GM}{r}$$

$$\Delta \mathcal{E}_{tot} = -\frac{1}{2}\,\delta m\,\frac{GM}{r_{in}} + \frac{1}{2}\,\delta m\,\frac{GM}{r_{out}} = \frac{1}{2}\,\delta m \int \frac{GM}{r^2}\,dr$$

$$= \int \frac{GM}{r}\,\eta \left(-\frac{1}{2}\int \frac{GM}{r^3} - \int \frac{GM}{r^3} \right)$$

$$= \frac{3}{2}\,\eta\,\frac{GM}{r^2}$$



$\frac{dE}{\pi r^2} \propto \frac{v^3 \rho}{r^2} \propto r^{-3}$
flux $\propto T^4$
$T(r) \propto r^{-\frac{3}{4}}$
$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$
$I_\nu \propto \nu^2 T$

∂r

$$\frac{dE}{\pi r^2} \propto \frac{v^2 \rho}{r^2} \propto r^{-3}$$

$$\text{flux} \propto T^4$$

$$T(r) \propto r^{-\frac{3}{4}}$$

$$r_{max} \propto \nu^{-\frac{4}{3}}$$

$$I_\nu \propto \int^{r_{max}}$$

$$I(\nu,T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$e^{\frac{h\nu}{kT}} \sim \frac{h\nu}{kT} + 1$$

$$I(\nu,T) = \frac{2h\nu^2 kT}{c^2} \longrightarrow \quad \propto \nu^2 T$$

$$\frac{dE}{\pi r^2} \propto \frac{v^2 \rho}{r^2} \propto r^{-3}$$

$$\text{flux} \propto T^4$$

$$\left.\right\} T(r) \propto r^{-\frac{3}{4}}$$

$$r_{max} \propto \nu^{-\frac{4}{3}}$$

$$I_\nu \propto \int^{r_{max}}$$

$$I(\nu,T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$$

$$e^{\frac{h\nu}{kT}} \sim \frac{h\nu}{kT}+1$$

$$I(\nu,T) = \frac{2h\nu^2 kT}{c^2} \longrightarrow I_\nu \propto \nu^2 T$$

$h\nu$

$$\frac{dE}{\pi r^2} \propto \frac{v^2 \rho}{r^2} \propto r^{-3}$$

$$\text{flux} \propto T^4$$

$$T(r) \propto r^{-\frac{3}{4}}$$

$$r_{max} \propto \nu^{-\frac{4}{3}}$$

$$I_\nu \propto \int^{r_{max}} \nu^2\, T(r)\, r\, dr \propto \int^{r_{max}} \nu^2 r^{\frac{1}{4}}\, dr \propto \nu^2 r_{max}^{\frac{5}{4}}$$

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$e^{\frac{h\nu}{kT}} \sim \frac{h\nu}{kT} + 1$$

$$h\nu \sim kT \Rightarrow h\nu \sim k r^{-\frac{3}{4}}$$

$$I(\nu, T) = \frac{2h\nu^2 kT}{c^2} \longrightarrow I_\nu \propto \nu^2 T$$

∂r

$$\frac{dE}{\pi r^2} \propto \frac{v^3 \rho}{r^2} \propto r^{-3}$$

$$\text{flux} \propto T^4$$

$$\Bigg\} \quad T(r) \propto r^{-\frac{3}{4}}$$

$$r_{max} \propto \nu^{-\frac{4}{3}}$$

$$I_\nu \propto \int^{r_{max}} \nu^2 \, T(r) \, r \, dr \propto \int^{r_{max}} \nu^2 r^{\frac{1}{4}} dr \propto \nu^2 r_{max}^{\frac{5}{4}} \propto \nu^{\frac{1}{3}}$$

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$e^{\frac{h\nu}{kT}} \sim \frac{h\nu}{kT} + 1$$

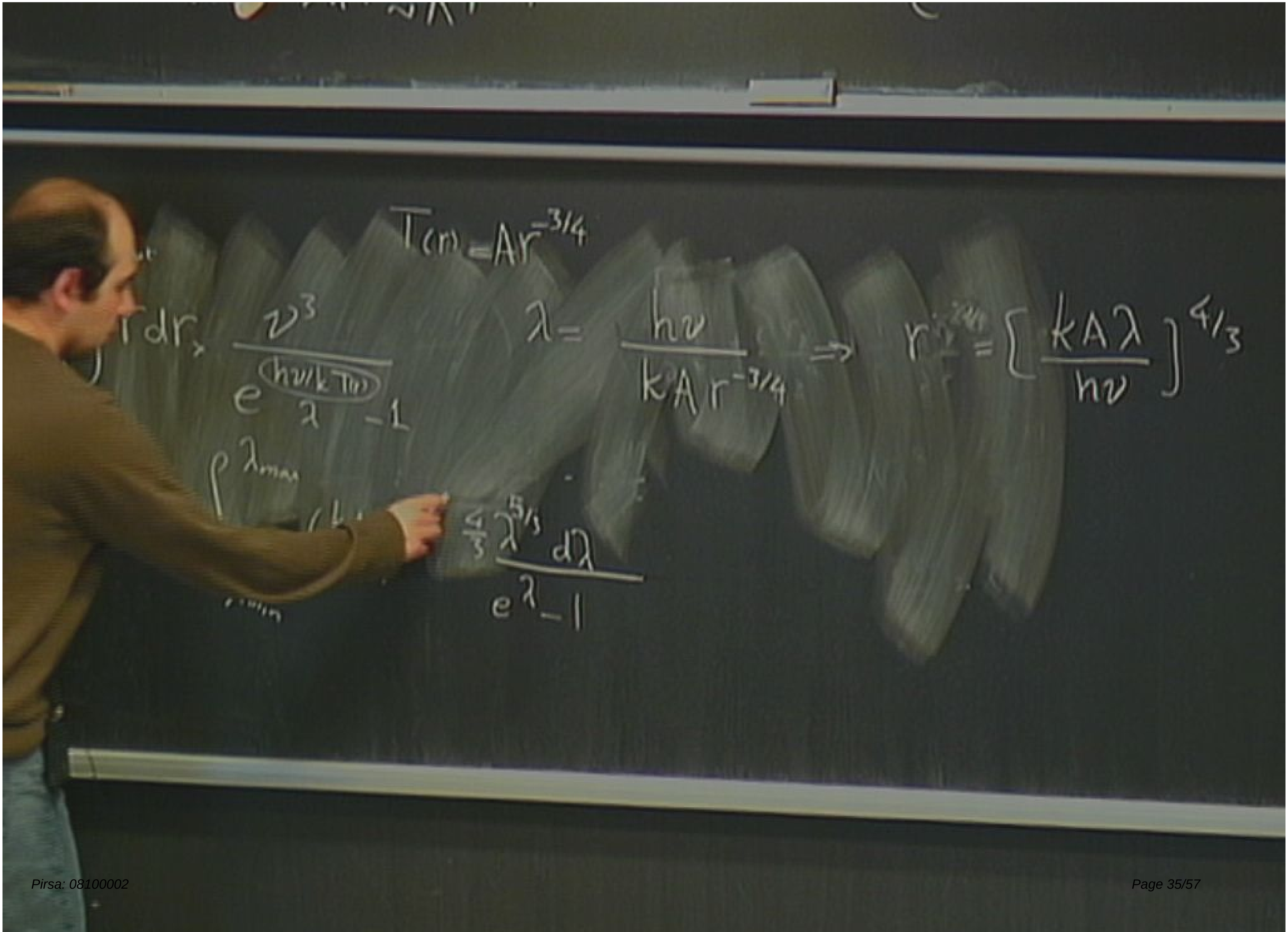
$$I(\nu, T) = \frac{2h\nu^2 kT}{c^2} \longrightarrow I_\nu \propto \nu^2 T$$

$$h\nu \sim kT \Rightarrow h\nu \sim k r^{-\frac{3}{4}}$$

$$T(r) = A r^{-3/4}$$

$$\int_{r_{in}}^{r_{out}} r\,dr \times \frac{\nu^3}{e^{h\nu/kT(r)} - 1}$$

$$= \nu^3$$


$T(r) = Ar^{-3/4}$
$\lambda = \frac{h\nu}{kAr^{-3/4}} \Rightarrow$
$= \left[\frac{kA\lambda}{h\nu} \right]^{4/3}$
$\frac{\nu^3}{e^{h\nu/kT(r)} - 1}$
λ_{max}
$\frac{4}{3} \lambda^{5/3} \frac{d\lambda}{e^{\lambda} - 1}$

$$T(r) = A r^{-3/4}$$

$$\int_{r_{in}}^{r_{out}} r\,dr \times \frac{\nu^3}{e^{h\nu/kT(r)} - 1} \qquad \lambda = \frac{h\nu}{kAr^{-3/4}} \Rightarrow r = \left[\frac{kA\lambda}{h\nu}\right]^{4/3}$$

$$= \nu^3 \int_{\lambda_{min}}^{\lambda_{max}} \left(\frac{kA}{h\nu}\right)^{8/3} \left(\frac{4}{3}\right) \frac{\lambda^{5/3}\,d\lambda}{e^{\lambda} - 1}$$

$$T(r) = A r^{-3/4}$$

$$\int_{r_{in}}^{r_{out}} r\,dr \times \frac{\nu^3}{e^{\overset{\lambda}{h\nu/kT(r)}} - 1}$$

$$\lambda = \frac{h\nu}{kAr^{-3/4}} \Rightarrow r = \left[\frac{kA\lambda}{h\nu}\right]^{4/3}$$

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$$T(r) = A r^{-3/4}$$

$$\int_{r_{in}}^{r_{out}} r\,dr \times \frac{\nu^3}{e^{h\nu/kT(r)} - 1} \qquad \lambda = \frac{h\nu}{kAr^{-3/4}} \Rightarrow r = \left[\frac{kA\lambda}{h\nu}\right]^{4/3}$$

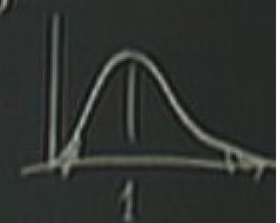
$$= \nu^3 \int_{\lambda_{min}}^{\lambda_{max}} \left(\frac{kA}{h\nu}\right)^{8/3} \left(\frac{4}{3}\right) \frac{\lambda^{5/3}\,d\lambda}{e^\lambda - 1} = \nu^{1/3} \int_{k r_{out}^{3/4} \times \nu}^{k r_{in}^{3/4} \times \nu} \frac{\lambda^{5/3}\,d\lambda}{e^\lambda - 1}$$

$$T(r) = A r^{-3/4}$$

$$\int_{r_{in}}^{r_{out}} r\,dr \times \frac{\nu^3}{e^{h\nu/kT(r)} - 1}$$

λ

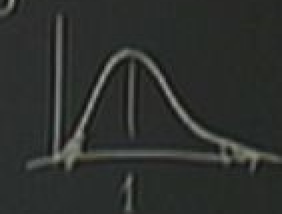
$$\lambda = \frac{h\nu}{kAr^{-3/4}} \Rightarrow r = \left[\frac{kA\lambda}{h\nu}\right]^{4/3}$$



$$= \nu^3 \int_{\lambda_{min}}^{\lambda_{max}} \left(\frac{kA}{h\nu}\right)^{8/3} \left(\frac{4}{3}\right) \frac{\lambda^{5/3}\, d\lambda}{e^{\lambda} - 1} = \nu^{1/3} \int_{kr_{in}^{3/4}\nu}^{kr_{out}^{3/4}\nu} \frac{\lambda^{5/3}\, d\lambda}{e^{\lambda} - 1}$$

$$h\nu \sim kV \Rightarrow h\nu \sim k r^{-\frac{3}{4}} \qquad C^2$$

$$\int_{r_{in}} r\,dr\, \frac{\nu^3}{e^{\frac{h\nu/kT(r)}{\lambda}} - 1} \qquad \lambda = \frac{h\nu}{kAr^{-3/4}} \Rightarrow r = \left[\frac{kA\lambda}{h\nu}\right]^{4/3}$$



$$= \nu^{?} \int_{\lambda_{min}}^{\lambda_{max}} \left(\frac{kA}{h\nu}\right)^{8/3} \left(\frac{4}{3}\right) \frac{\lambda^{5/3}\, d\lambda}{e^{\lambda} - 1} = \nu^{1/3} \int_{k r_{in}^{3/4} \ll \nu}^{k r_{max}^{3/4} \gg \nu} \frac{\lambda^{5/3}\, d\lambda}{e^{\lambda} - 1} \quad \begin{cases} r_{max}^{3/4}\,\nu \gg k^{-1} \\ r_{min}^{3/4}\,\nu \ll k^{-1} \end{cases}$$

$$L - L$$
$$kT - ML^2T^{-2}$$
$$P - T$$
$$M - M$$

$$\frac{kT\,P^2}{ML^2}$$

$$L \propto \sqrt{\frac{kT P^2}{M_*}}$$

$$\sqrt{\frac{kT\,r^3}{M_* GM}}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$P \simeq \frac{r^{3/2}}{\sqrt{GM}}$$

$$\frac{L}{r} = \sqrt{\frac{kTr}{GM^2}}$$

r_{out}

$L - L$

$KT - ML^2T^{-2}$

$P - T$

$M - M$

$$\frac{KT\,P^2}{ML^2}$$

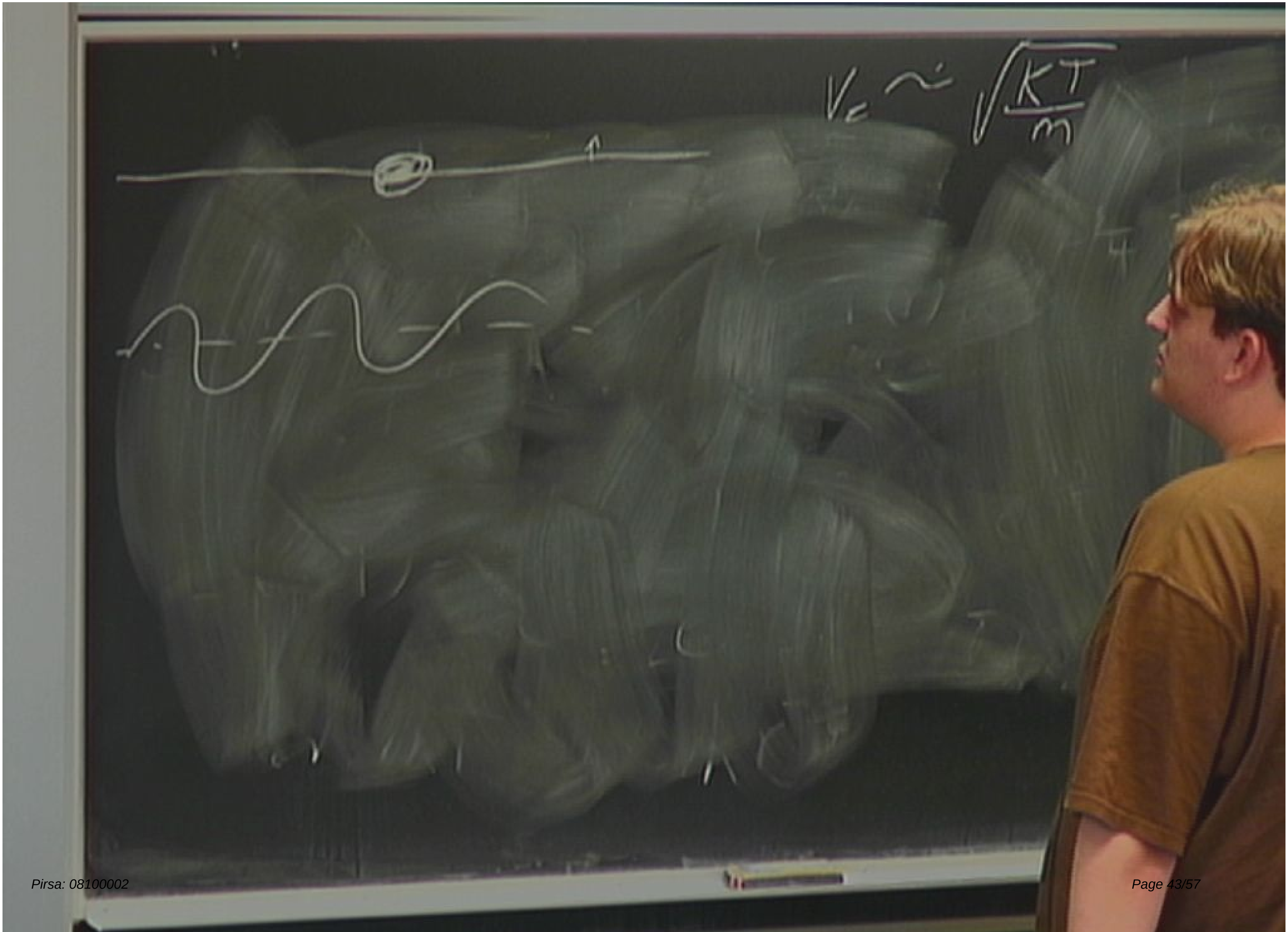
$$L \propto \sqrt{\frac{KTP^2}{M_*}}$$

$$\sqrt{\frac{KT\,r^3}{M_*\,GM}}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$P \hat{=} \frac{r^{3/2}}{\sqrt{GM}}$$

$$\frac{L}{r} = \sqrt{\frac{KT\,r}{GM M_*}}$$


$V_c \sim \sqrt{\frac{KT}{m}}$

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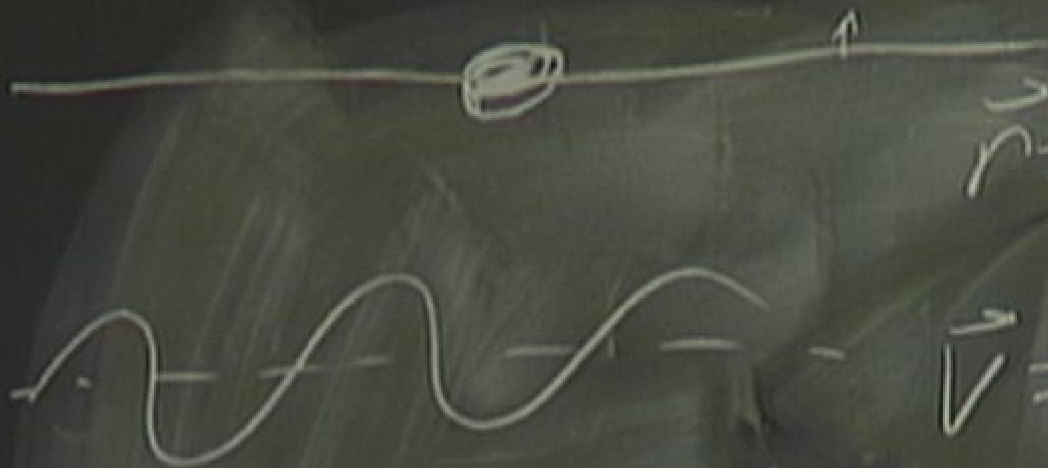
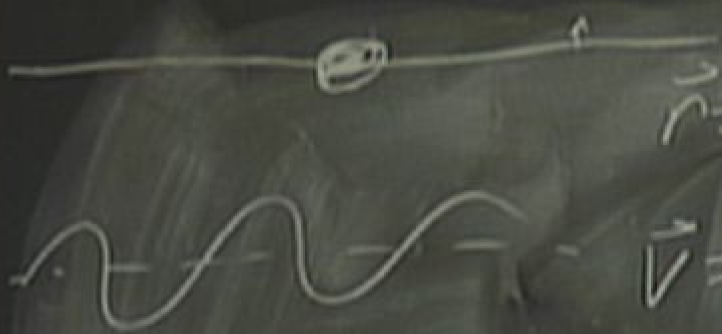


$$\vec{r} = (r, \phi, 0)$$

$$\vec{v} = \left(0, \sqrt{\frac{GM}{r}}, \sqrt{\frac{KT}{m}}\right)$$



$$v_z \sim \sqrt{\frac{kT}{m}}$$

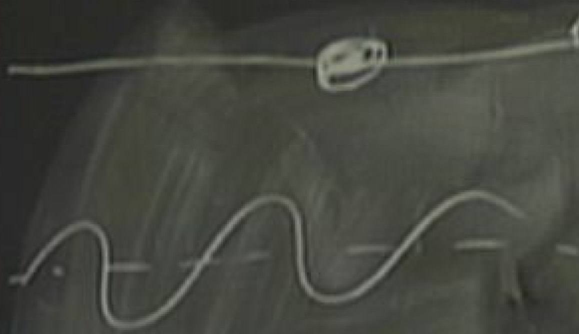
$$\vec{r} = (r, \phi, 0)$$

$$\vec{v} = \left(0, \sqrt{\frac{GM}{r}}, \sqrt{\frac{kT}{m}}\right)$$

$$\psi = \frac{GM}{\sqrt{r^2+z^2}}$$

$$= \frac{GM}{r}\left(1+\frac{z^2}{r^2}\right)^{-1/2}$$

$$\sim \frac{GM}{r}\left(1-\frac{1}{2}\frac{z^2}{r^2}+O(z^4)\right)$$



$$v_c \sim \sqrt{\frac{KT}{m}}$$

$$\vec{r} = (r, \phi, 0)$$

$$\vec{v} = \left(0, \sqrt{\frac{GM}{r}}, \sqrt{\frac{KT}{m}}\right)$$

$$\psi = -\frac{GM}{\sqrt{r^2+z^2}}$$

$$= -\frac{GM}{r}\left(1+\frac{z^2}{r^2}\right)^{-1/2}$$

$$\sim -\frac{GM}{r}\left(1-\frac{1}{2}\frac{z^2}{r^2}+O(z^4)\right)$$

$$\psi \sim + \frac{GMz^2}{r^3}$$

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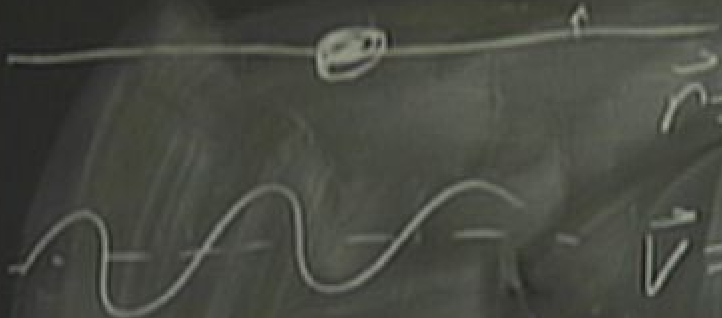
$$z = \sqrt{\frac{KTr^3}{GMm}}$$



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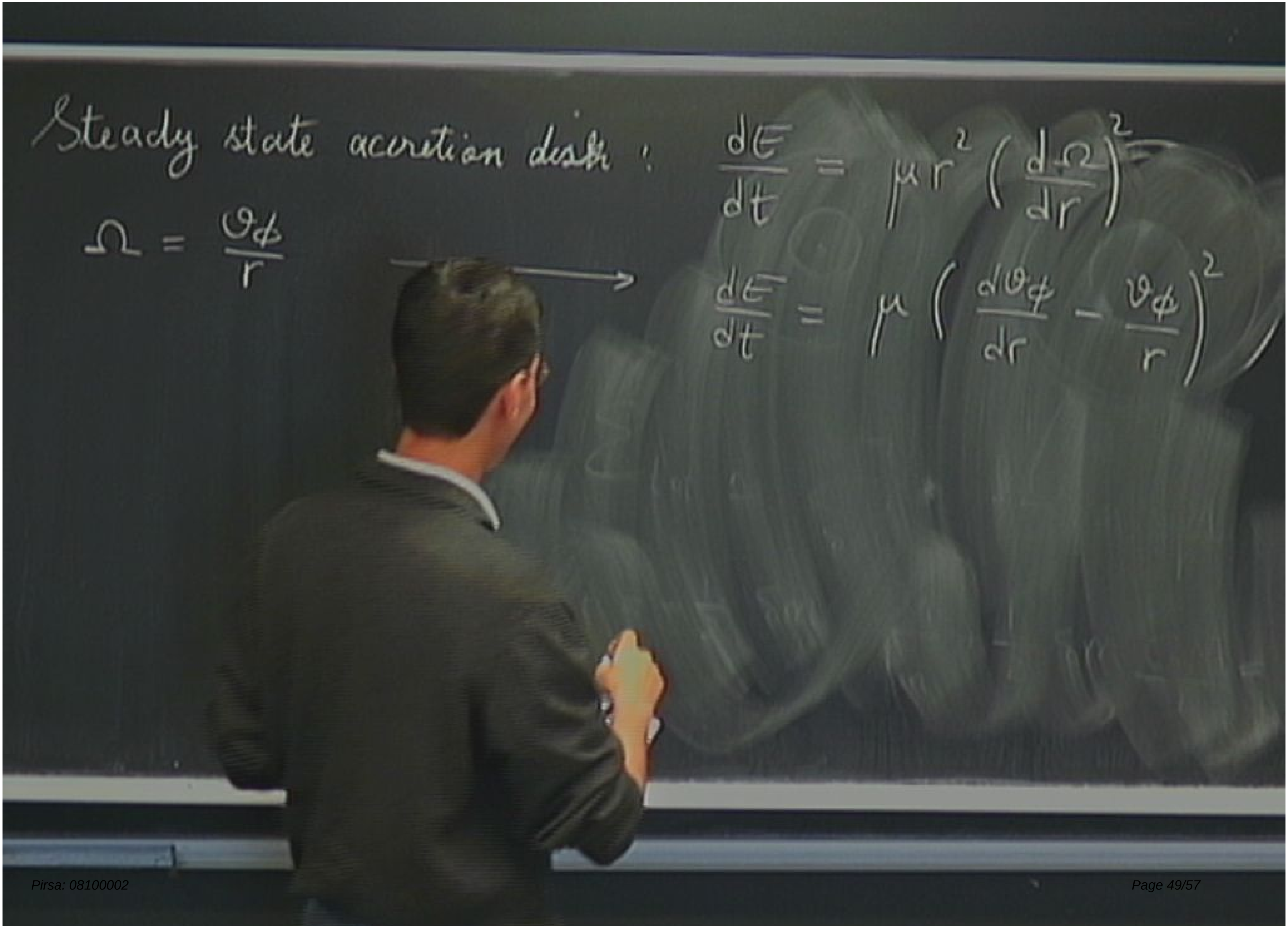
Steady state accretion disk:
$\Omega = \frac{v_\phi}{r}$
$\frac{dE}{dt} = \mu r^2 \left(\frac{d\Omega}{dr}\right)^2$
$\frac{dE}{dt} = \mu \left(\frac{dv_\phi}{dr} - \frac{v_\phi}{r}\right)^2$

Steady state accretion disk : $\frac{dE}{dt} = \mu r^2 \left(\frac{d\Omega}{dr} \right)^2$

$\Omega = \frac{v_\phi}{r} \longrightarrow \frac{dE}{dt} = \underbrace{\mu \left(\frac{d v_\phi}{dr} - \frac{v_\phi}{r} \right)^2}_{\mu \left(\frac{d v_\lambda}{d\lambda} - \frac{v_\lambda}{\lambda} \right)^2}$

Turbulence : $\frac{dE}{dt} \simeq \frac{v_\lambda^3}{\lambda}$

$v_\lambda \cong U \left(\frac{\lambda}{L} \right)^{1/3}$

Steady state accretion disk : $\frac{1}{\rho}\frac{dE}{dt} = \frac{1}{\rho}\mu r^2 \left(\frac{d\Omega}{dr}\right)^2$

$$\Omega = \frac{v_\phi}{r} \longrightarrow \frac{1}{\rho}\frac{dE}{dt} = \underbrace{\frac{1}{\rho}\mu\left(\frac{dv_\phi}{dr} - \frac{v_\phi}{r}\right)^2}_{\frac{1}{\rho}\mu\left(\frac{dv_\lambda}{d\lambda} - \frac{v_\lambda}{\lambda}\right)^2}$$

Turbulence :
$$\frac{dE}{dt} = \frac{v_\lambda^3}{\lambda}$$

$$v_\lambda \simeq U\left(\frac{\lambda}{L}\right)^{1/3} = \underbrace{UL^{-1/3}}_{\alpha}\,\lambda^{1/3}$$

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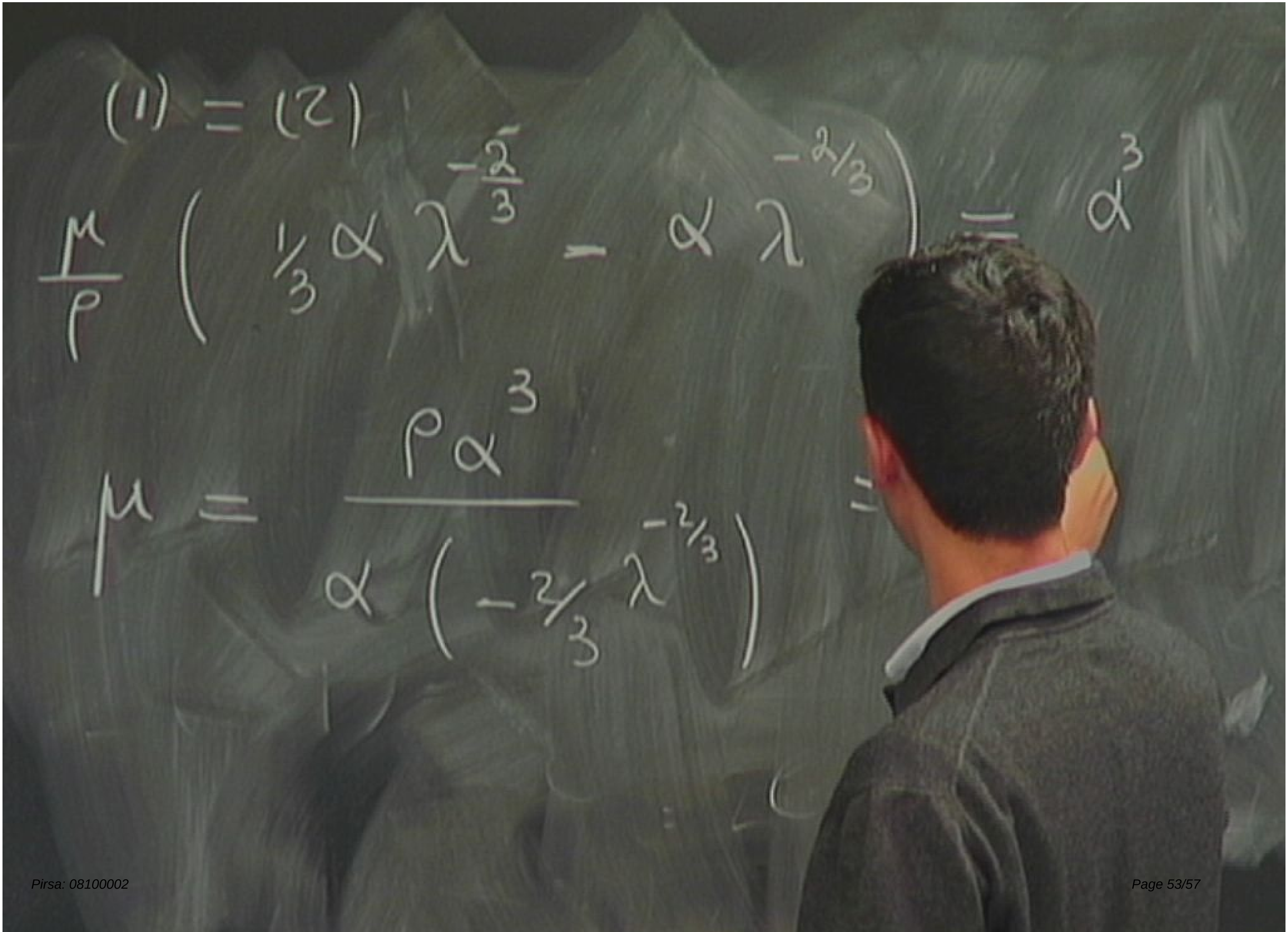
$\Omega = \frac{v_\phi}{r}$ $\longrightarrow$ $\frac{1}{\rho}\frac{dE}{dt} = \frac{1}{\rho}\mu \underbrace{\left(\frac{dv_\phi}{dr} - \frac{v_\phi}{r}\right)^2}$

$\frac{1}{\rho}\mu\left(\frac{dv_\lambda}{d\lambda} - \frac{v_\lambda}{\lambda}\right)^2$ (1)

Turbulence : $\frac{dE}{dt} = \frac{v_\lambda^3}{\lambda}$ (2)

$v_\lambda \cong U\left(\frac{\lambda}{L}\right)^{1/3} = \underbrace{UL^{-1/3}}_{\alpha}\lambda^{1/3}$


(1) = (2)
$\frac{\mu}{\rho}\left(\frac{1}{3}\alpha\lambda^{-\frac{2}{3}} - \alpha\lambda^{-2/3}\right) = \alpha^3$
$\mu = \frac{\rho\alpha^3}{\alpha\left(-\frac{2}{3}\lambda^{-2/3}\right)} =$

$$(1) = (2)$$

$$\frac{\mu}{\rho}\left(\tfrac{1}{3}\alpha\lambda^{-\frac{2}{3}} - \alpha\lambda^{-2/3}\right) = \alpha^3$$

$$\mu = \frac{\rho\alpha^3}{\alpha\left(-\tfrac{2}{3}\lambda^{-2/3}\right)} = -\frac{3\rho U^2 L^{-2/3}}{2}$$

$(1) = (2)$

$$\frac{\mu}{\rho}\left(\tfrac{1}{3}\alpha\lambda^{-\frac{2}{3}} - \alpha\lambda^{-2/3}\right)^2 = \alpha^3$$

$$\mu = \frac{\rho\alpha^3}{\alpha^2\left(-\tfrac{2}{3}\lambda^{-2/3}\right)^2} = -\frac{3\rho u^2 L}{2}\lambda^{2/3}$$

4/3

$$(1) = (2)$$

$$\frac{\mu}{\rho}\left(\tfrac{1}{3}\alpha\lambda^{-\frac{2}{3}} - \alpha\lambda^{-2/3}\right)^2 = \alpha^3$$

$$\mu = \frac{\rho\alpha^3}{\alpha^2\left(-\tfrac{2}{3}\lambda^{-2/3}\right)^2} = \frac{\rho\, U L^{-1/3}}{4/9}\lambda^{4/3}$$

$$(1) = (2)$$

$$\frac{\mu}{\rho}\left(\frac{1}{3}\alpha\lambda^{-\frac{2}{3}} - \alpha\lambda^{-2/3}\right)^2 = \alpha^3$$

$$\mu = \frac{\rho\alpha^3}{\alpha^2\left(-\frac{2}{3}\lambda^{-2/3}\right)^2} = \frac{\rho U L^{-1/3}}{4/9}\lambda^{4/3}$$

$$\boxed{\mu = \alpha_{ss} \times \rho \times c_s \times L}$$