Title: On the time-energy uncertainty relation

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Abstract: In contrast to Heisenberg\'s position-momentum uncertainty relation, the status of the time-energy uncertainty relation has always remained dubious, For example, it is often said that \'time\' in quantum theory is not an observable and not represented by a self-adjoint operator. I will review the background of the problem and propose a view on the uncertainty relations in which the cases of position-momentum and time-energy can be treated in the same way.

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Heisenberg's 1927 paper

Problems for the time-energy commutation rule

Questions

Dynamical variables versus space-time coordinates in classical mechanics

Three options

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Heisenberg's 1927 paper

- Heisenberg introduced his famous uncertainty relation for position and momentum in the form

$$q_1 p_1 \sim h$$

Where q_1, p_1 are (unspecified) measures of uncertainty and argued this was intimately related to the commutation rule

$$pq - qp = -i\hbar$$

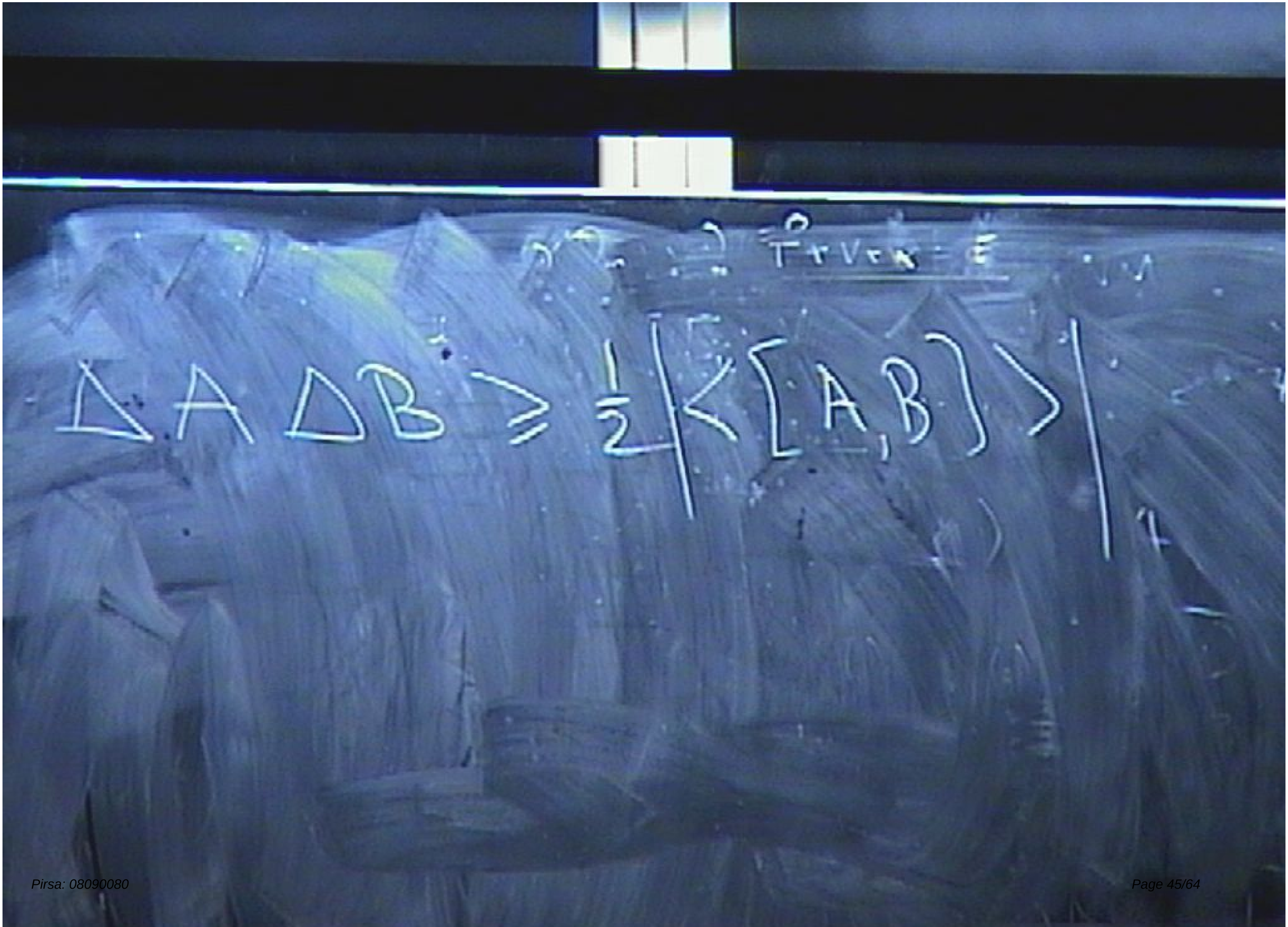
- In the same paper, he introduced a similar relation for time and energy,

$$t_1 E_1 \sim h$$

as a direct consequence of the familiar commutation rule

$$Et - tE = -i\hbar \quad \text{or} \quad Jw - wJ = -i\hbar$$

- However, this "familiar" rule soon turned out to be problematic.


$\Delta A \Delta B \geq \frac{1}{2} |\langle [A,B] \rangle|$

The problem with $[H, T] = -i\hbar$

Theorem (Pauli (1933)-Allcock (1969))

Let $\hat{H}$ be bounded from below. Let $\{|\tau\rangle, \tau \in \mathbb{R}\}$ be a family of (improper) states such that $\forall |\Psi\rangle$:

$$|\langle \tau | e^{it\hat{H}} | \Psi \rangle|^2 = |\langle \tau + t | \Psi \rangle|^2$$

then

$$\langle \tau + t | \tau \rangle \neq 0 \quad \forall \tau, t \in \mathbb{R}$$

A forteriori, the states $|\tau\rangle$ cannot be (improper) eigenstates of a self-adjoint operator T.

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Von Neumann:

> *"... an essential weakness, which is in fact the chief weakness of quantum mechanics: its non-relativistic character, which distinguishes time t from the three space coordinates x, y, z, ... In fact, while all other quantities (especially those x, y, z closely connected to t by the Lorentz transformation) are represented by operators, there corresponds to the time an ordinary number-parameter t just as in classical mechanics."*

Other formulations:

- Energy cannot be precisely measured in a very short time (Landau & Peierls)
- Energy conservation can be violated, if only for a very short time. (Landau & Lifshitz)

These are both artifacts of first-order perturbation theory.

Mandelstam & Tamm

$$\Delta H \tau_A \geq \frac{1}{2}\hbar$$

where

$$\tau_A = \frac{\left|\frac{d < A >}{dt}\right|}{\Delta A}$$

Is often seen as unsatisfactory because τ_A is not a quantum "uncertainty".

Questions

- does relativity require that time is represented as an observable?
- Is it really impossible to find an observable for time?
- Do we need commutation relations in order to have an uncertainty relation?

Let's first go back to classical mechanics. . .

Dynamical variables versus space-time coordinates

Classical Hamiltonian particle mechanics:
Canonical positions and momenta $(\mathbf{q}_1, \ldots, \mathbf{q}_n; \mathbf{p}_1, \ldots, \mathbf{p}_n) \in \Gamma$,
Hamiltonian: $H(\mathbf{q}_1, \ldots \mathbf{q}_n; \mathbf{p}_1, \ldots, \mathbf{p}_n, t)$
Equations of motion:

$$\dot{f}(\mathbf{p}, \mathbf{q}) = \{f, H\} := \sum_i \frac{\partial f}{\partial \mathbf{q}_i}\frac{\partial H}{\partial \mathbf{p}_i} - \frac{\partial f}{\partial \mathbf{p}_i}\frac{\partial H}{\partial \mathbf{q}_i}$$

The system is embedded in space-time: $\mathbb{E}^3 \times \mathbb{T}$ with coordinates $(\mathbf{x}, t)$.

$$\mathbf{q}_i : \Gamma \longrightarrow \mathbb{E}^3, \quad \mathbf{q}_i(\gamma) = \mathbf{x} \in \mathbb{E}^3$$

Note: $\mathbf{q}_i$ are dynamical variables (depend on the state of the system), while $\mathbf{x}$ and t are just a label of a point in space.

It is wrong to say that in QM spatial coordinates are quantized (get represented by observables) in quantum theory; this happens only to dynamical variables.

Three options

1. Treat t as an additional dynamical variable. New phase space $(\mathbf{q}_1, \ldots \mathbf{q}_n, t; \mathbf{p}_1, \ldots, \mathbf{p}_n, p_t) \in \Gamma'$,
   New Hamiltonian
   $$H'(\mathbf{q}_1, \ldots, \mathbf{q}_n, t; \mathbf{p}_1, \ldots \mathbf{p}_n, p_t)$$
   with
   $$p_t = -H(\mathbf{q}_1, \ldots \mathbf{q}_n; \mathbf{p}_1, \ldots, \mathbf{p}_n, t)$$
   Consequence: the new set of dynamical variables are not independent.
   this leads to a constrained Hamiltonian system, and quantization is not obvious. (And does not necessarily bring us closer to relativity)

2. 'Clock variables'. Consider a multiply periodic system (e.g. a classical model of an atom)
   Then there is a canonical transformation from (**p**,**q**) to action/angle variables: $(w_1, \ldots w_n; J_1, \ldots, J_n)$ such that

$$\{w_i, w_k\} = \{J_i, J_k\} = 0, \quad \{w_i, J_k\} = \delta_{ik}$$

   with the solutions:

$$\begin{aligned} J_i(t) &= \text{const} \\ w_i(t) &= \nu_i(J_1, \ldots J_n)t + \phi_i \end{aligned}$$

   Hence w_i is a dynamical variable that keeps in step with t and can be used as clock variable (or 'internal time').
   - But note there may be many of them!

   Quantizing such a system leads to no problems:

$$[\hat{w}_i, \hat{w}_k] = [\hat{J}_i, \hat{J}_k] = 0, \quad [\hat{w}_i, \hat{J}_k] = i\hbar\delta_{ik}\mathbb{I} \tag{1}$$

Examples:

- the rigid rotator:

$$[\hat{w}, \hat{J}] = i\hbar\mathbb{I}, \quad \text{with} \quad \text{Spec } \hat{J} = \mathbb{Z}, \quad \text{Spec } w = [0, 2\pi]$$

  However,

$$\Delta\hat{w}\Delta\hat{J} \geq \frac{\hbar}{2}$$

  is false.
  Nevertheless, it is true in this case that whenever $\Delta J \longrightarrow 0$, $|\langle w|\Psi\rangle|^2 \longrightarrow 1/(2\pi)$, and that when $\Delta w \longrightarrow 0$, then $|\langle j|\Psi||\rangle^2$ becomes 'uniform over $\mathbb{Z}$.
- the digital clock: Salecker-Wigner (1958)
- 'time of arrival' Allcock 1969, Muga&Leavens (2000)

3. do we need to represent time as an observable before we can formulate an uncertainty relation? Why not work with $(\mathbf{x}, t)$ as a space-time coordinates?

Assume space-time is homogeneous and system is isolated. Then space-time translations form a symmetry group.
Temporal translations are generated by the (time-independent) Hamiltonian $\hat{H}$:

$$\hat{U}(\tau) = e^{i\tau\hat{H}}$$

And (Mandelstam & Tamm, Fleming, Bhattacharrya, Vaidman)

$$\tau\Delta\hat{H} \geq \arccos|\langle\Psi|\hat{U}(\tau)|\Psi\rangle| \quad (2)$$

this means: the time τ by which we have to translate a state, so that its overlap with the original state becomes appreciably less then 1, is inversely proportional to its uncertainty in energy.

- This relation gains considerable strength when we realize that

$$d(\Psi,\Phi) := \arccos |\langle \Psi|\Phi\rangle|$$

is the "statistical distance" between two states that follows from the Fisher-Rao information metric.

- What is more, by purely statistical reasoning, this metric leads to the well-known Cramér-Rao inequality. In our case, (integrating this inequality) this yields,

Theorem

Let T be any observable (self-adjoint operator or POVM) such that

$$\langle T\rangle_{\Psi_\tau} = \langle T\rangle_\Psi + \tau$$

(T is an unbiassed estimator for the parameter τ), then:

$$|\langle \Psi|\Psi_\tau\rangle| \leq \left(1 + \left(\frac{\tau}{\Delta \hat{T}}\right)^2\right)^{-1}$$

combining the two:

$$\tau \Delta \hat{H} \geq \arccos |\langle\Psi|\hat{U}(\tau)|\Psi\rangle| \geq \arccos \left(1 + (\frac{\tau}{\Delta \hat{T}})^2\right)^{-1}$$

By Taylor expansion in τ, this implies

$$\Delta T \Delta H \geq \frac{1}{2}$$

for all such observables T.

The same thing can be done for spatial translations and (total) momentum; or for rotations and (total) angular momentum), etc.

Conclusions

- the confusion about the status of time in QM is (at least in part) due to a failure to distinguish between dynamical variables such as positions and spatial coordinates.
- There is no cogent reason from special relativity to replace time by an operator.
- a satisfactory version of the time/energy or position/momentum uncertainty relations can be presented without relying on a canonical commutation relation, in terms of an uncertainty in energy (or momentum) and parameters of the spatiotemporal translation group. This result also holds for relativistic quantum theory and quantum field theory.
- This uncertainty relation is actually stronger than the usual Heisenberg uncertainty relation; i.e. it implies that relation whenever there is an appropriate time (or position) observable.

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