Title: Quantum reference frames and relationalism in quantum theory

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Abstract: A reference frame can be treated as a physical quantum object internal to the theory. Quantum reference frames whose size, and therefore accuracy, are bounded in some way necessarily limit one\'s ability to prepare states and to perform quantum operations and measurements on a system. The nature of these limitations is similar in many ways to that of decoherence. We investigate how a quantum reference frame of bounded size can be \'dequantized\', i.e., treated as external to the quantum formalism, in such a way as to induce an effective decoherence on any system described relative to it. In particular, we show that this decoherence has an interpretation as a lack of classical information about an ideal (infinite size) reference frame.

Reference frames and relationalism in quantum theory

Stephen Bartlett

The University of Sydney

with

Terry Rudolph (Imperial College London)

Robert Spekkens (Cambridge → Perimeter Institute)

An observation

A common view:
quantum states only contain information about
the intrinsic properties of the system

We submit:
quantum states also contain information about
the extrinsic properties of the system
Specifically: the relation to other systems external to it.

Stephen D Bartlett, Terry Rudolph, and Robert W. Spekkens
"Reference Frames, Superselection Rules, and Quantum Information"
Reviews of Modern Physics, **79,** 555 (2007)

What does it mean to say that the spin is up along the z axis?

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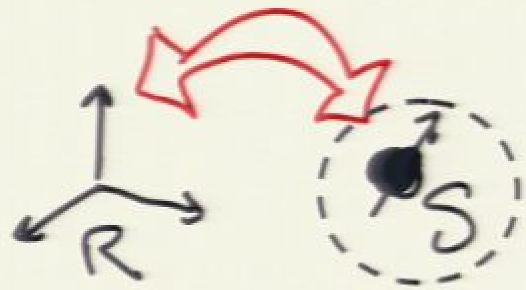
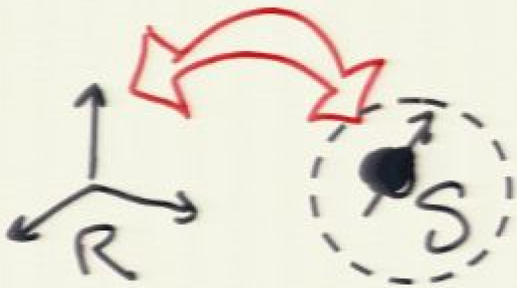


It means spin up relative to another physical system, such as gyroscopes in the lab, that define the z axis (i.e. act as a Cartesian reference frame)

External RF paradigm

$$\rho_S \in \mathcal{L}(\mathcal{H}_S)$$



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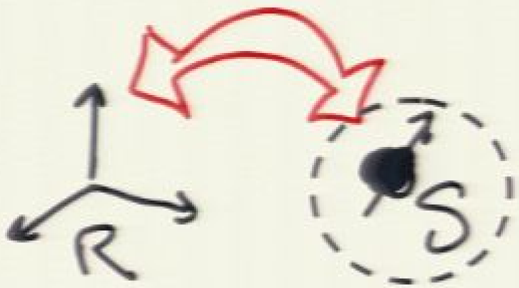
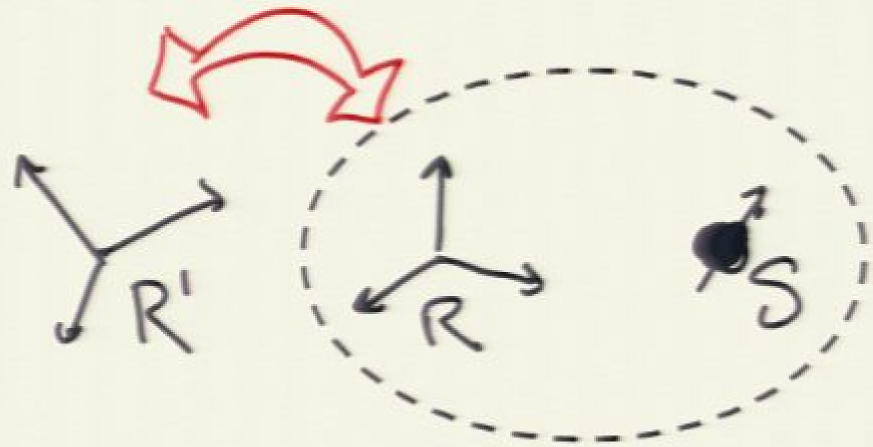


Internal RF paradigm

$$\sigma_{RS} \in \mathcal{L}(\mathcal{H}_R) \otimes \mathcal{L}(\mathcal{H}_S)$$



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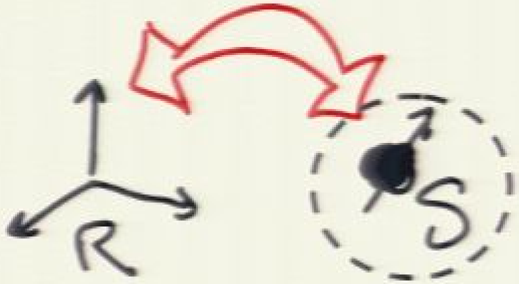

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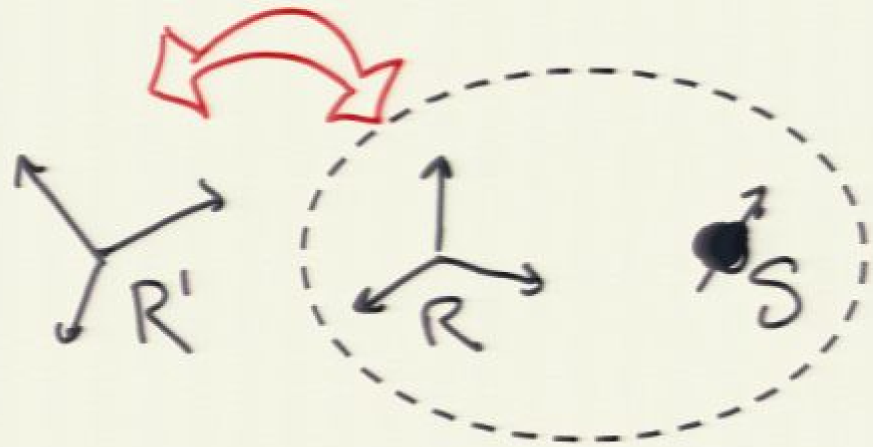

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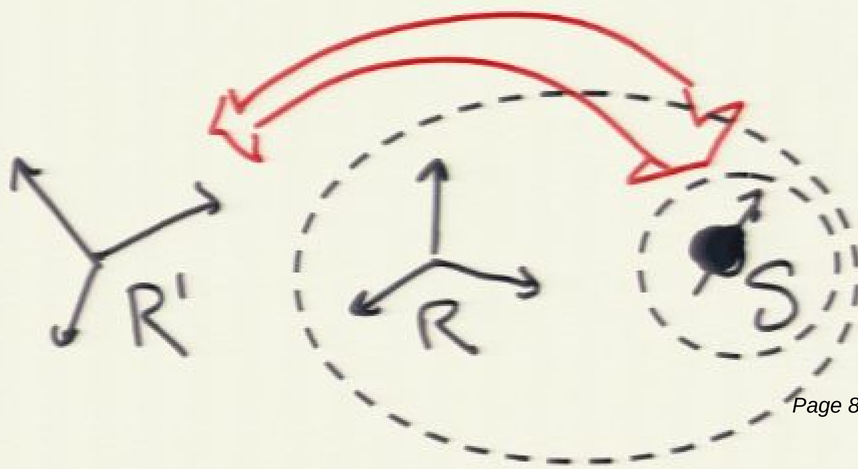

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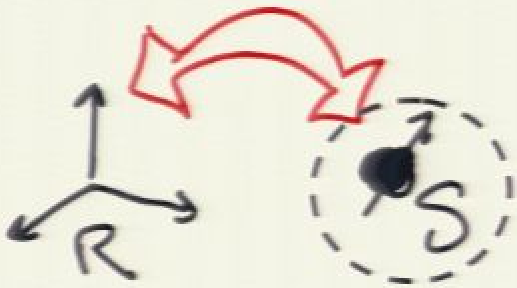

So, the two states
need not be the same

Internal RF paradigm

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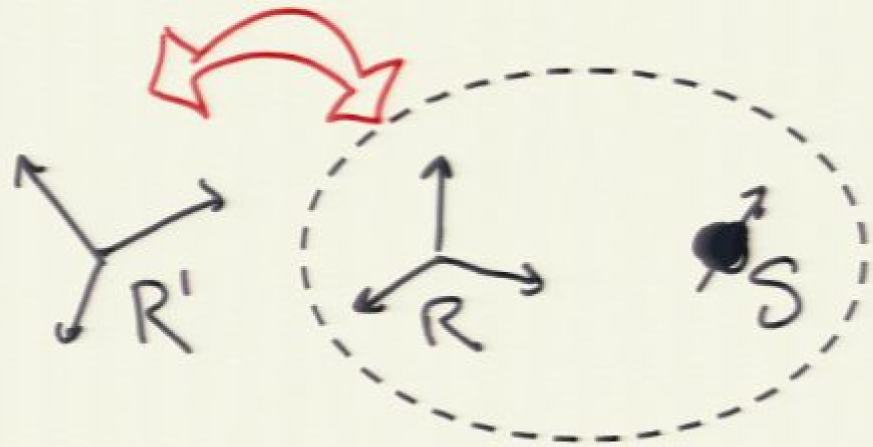

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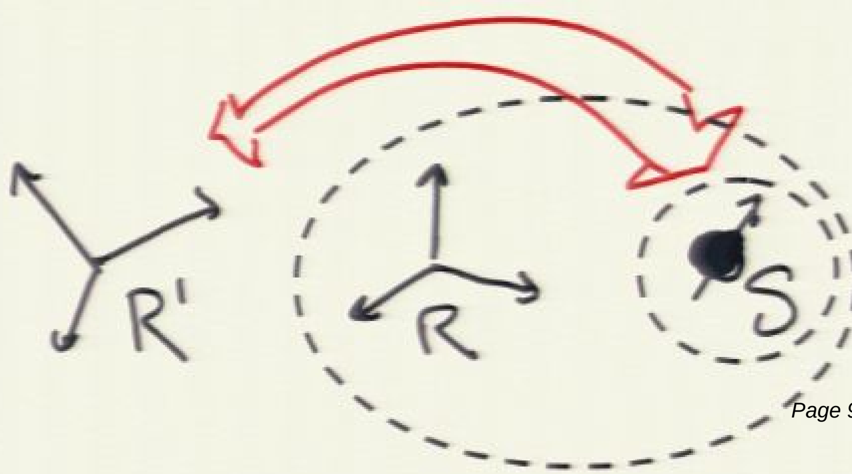

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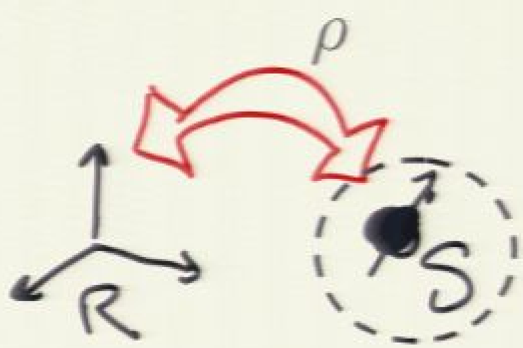

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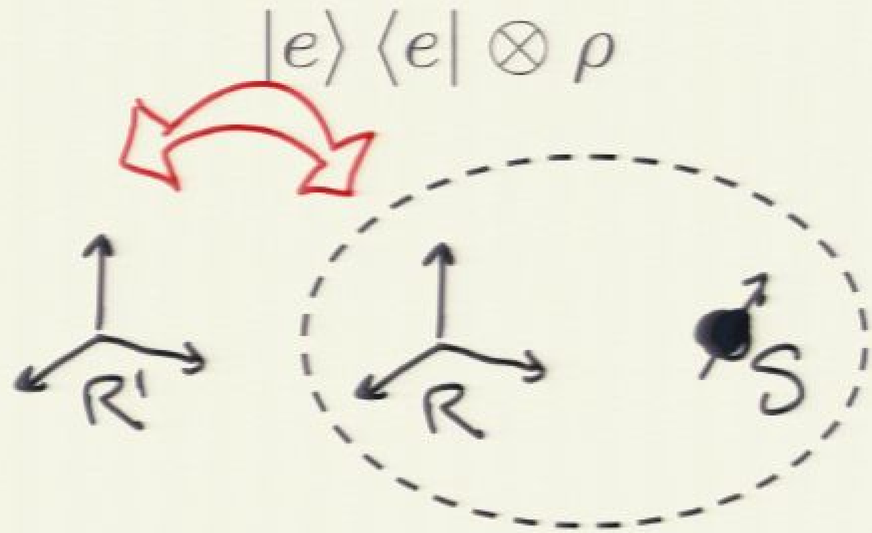
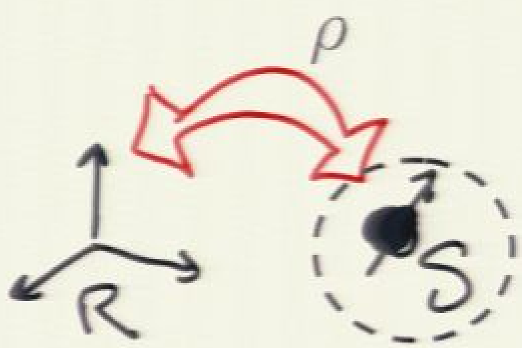
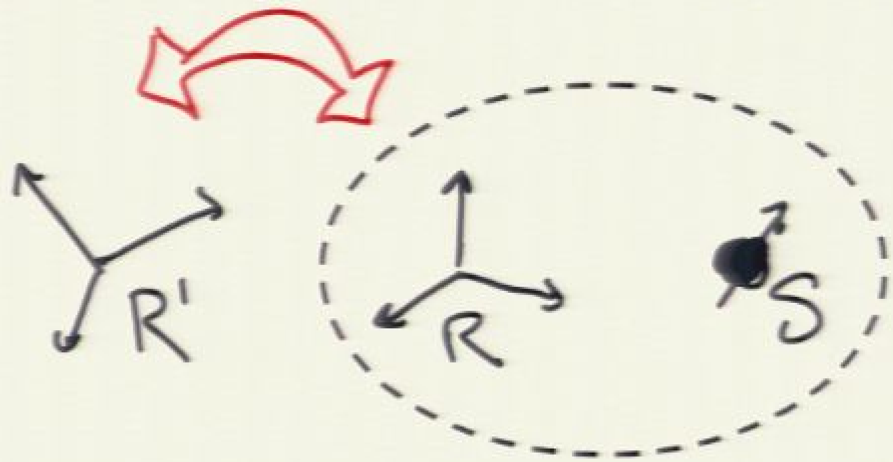


External RF paradigm



Internal RF paradigm

$$U_R(\Omega)\,|e\rangle\,\langle e|\,U_R^{\dagger}(\Omega)\otimes U(\Omega)\rho U^{\dagger}(\Omega)$$



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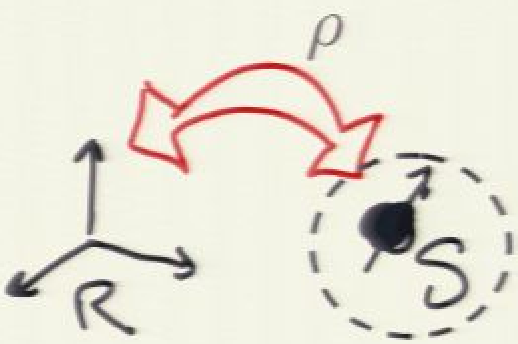

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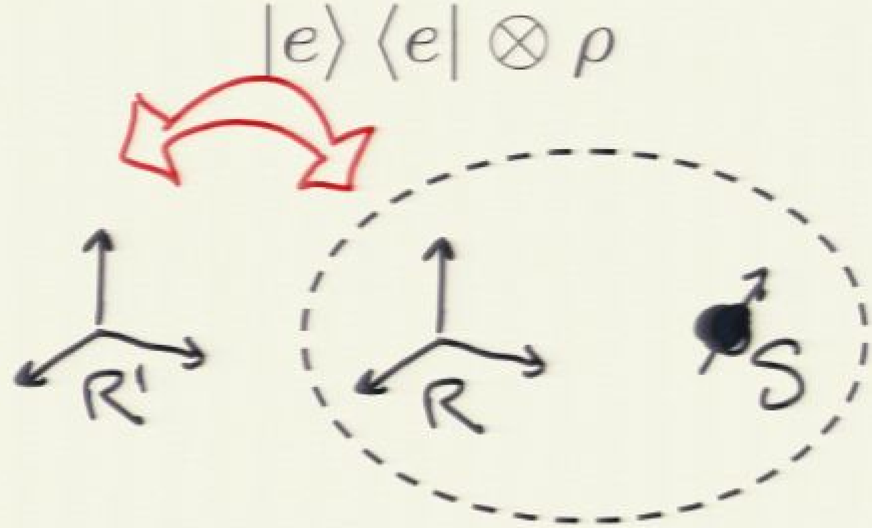

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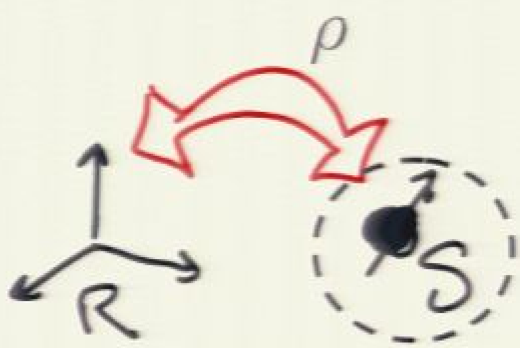

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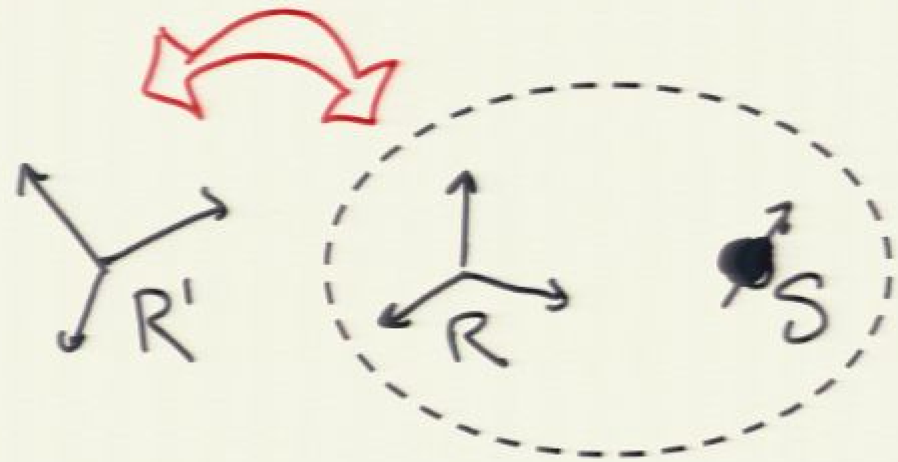
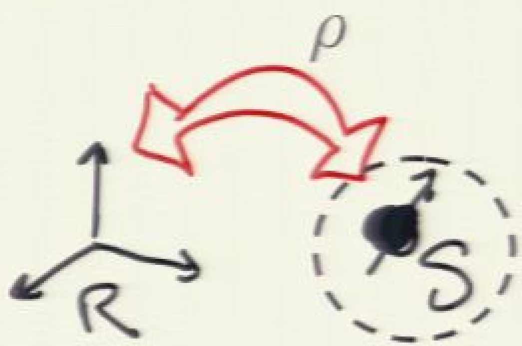
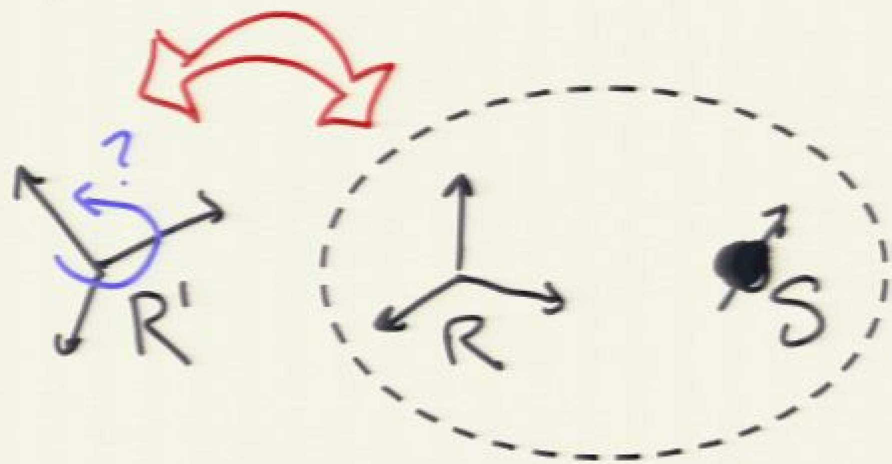


External RF paradigm

Internal RF paradigm

$$\frac{1}{d}\int d\Omega U_R(\Omega)\,|e\rangle\langle e|\,U_R^\dagger(\Omega)\otimes U(\Omega)\rho U^\dagger($$





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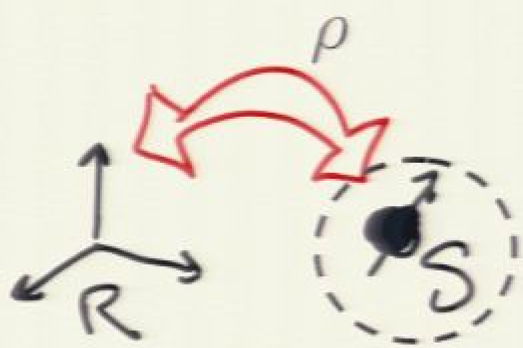
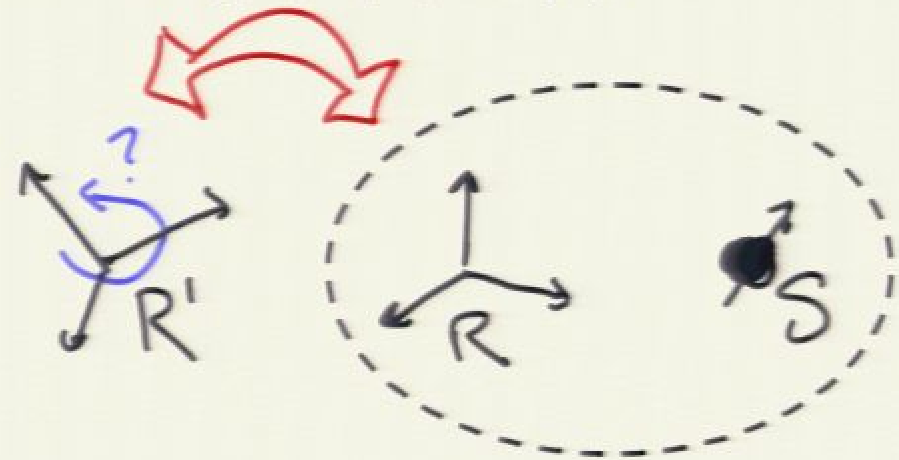


Internal RF paradigm

$$d^{-1}\mathcal{G}(|e\rangle\langle e| \otimes \rho)$$



$$\mathcal{G}(\cdot) \equiv \int d\Omega (U_R(\Omega) \otimes U(\Omega))(\cdot)(U_R^\dagger(\Omega) \otimes U^\dagger($$

State of RS is rotationally-invariant

No coherence between eigenstates of J^2

Question

What properties of the state $|e\rangle$
are required for it to serve as a 'good' RF?

Result:

External RF paradigm is just as good as internal RF paradigm
→ One can even model bounded-size RF effects
These appear as effective decoherence for relational degrees of freedom

Page and Wootters, PRD **27,** 2885 (1983).
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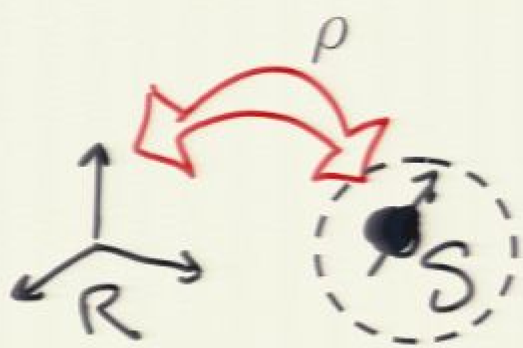
External RF paradigm



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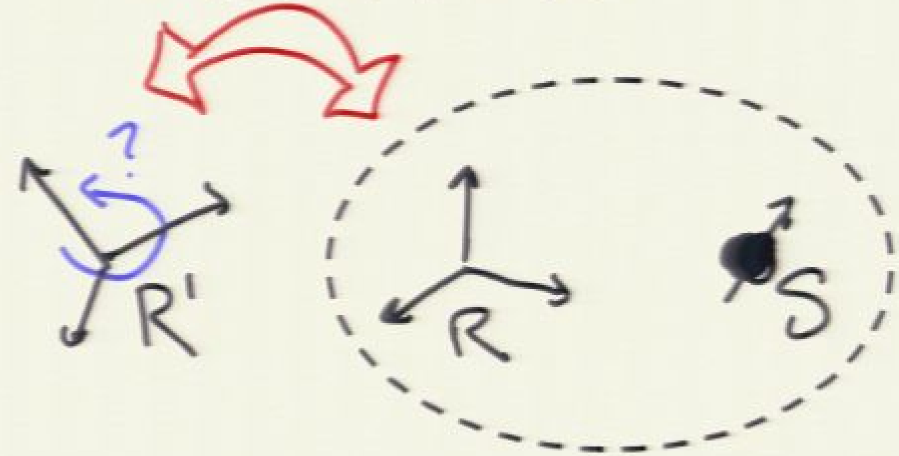

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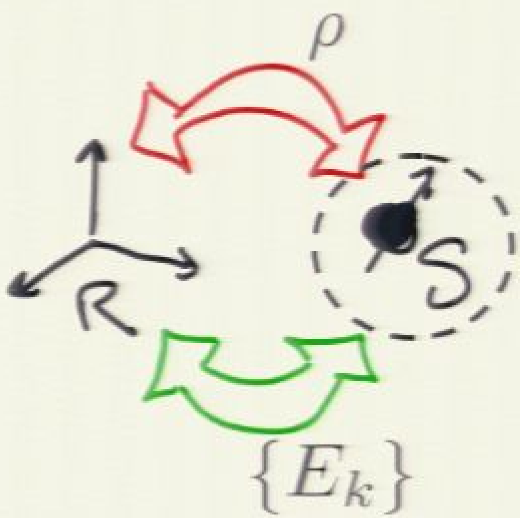
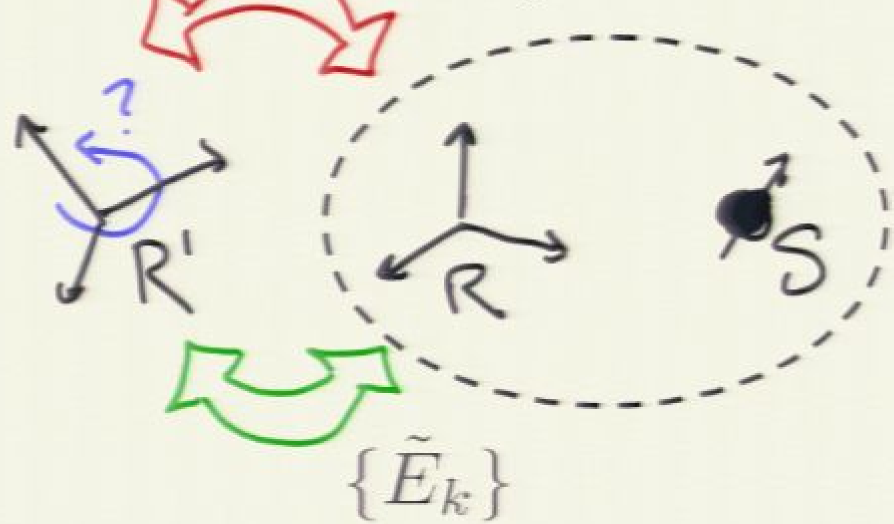
External RF paradigm

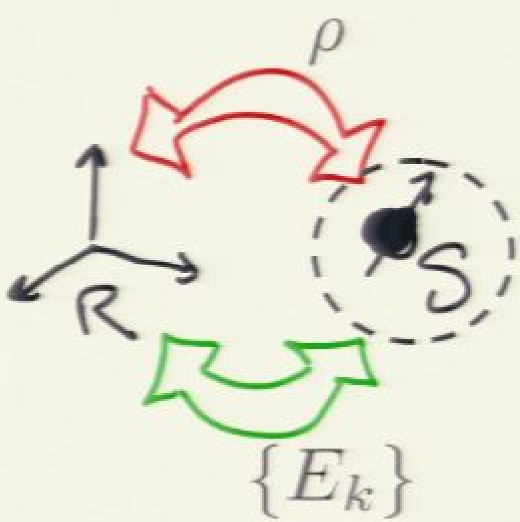


Internal RF paradigm

$$\tilde{\rho} = \frac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho)$$



External RF paradigm



defined on $\mathcal{H}_S$

$\mathrm{Tr}_S[\rho E_k]$

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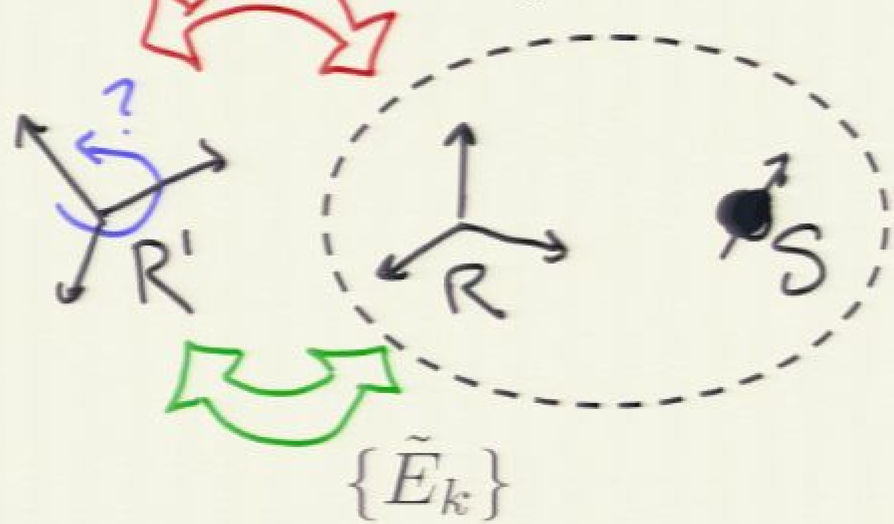

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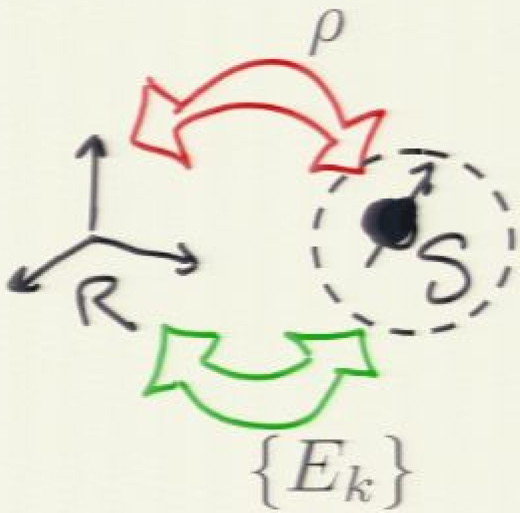

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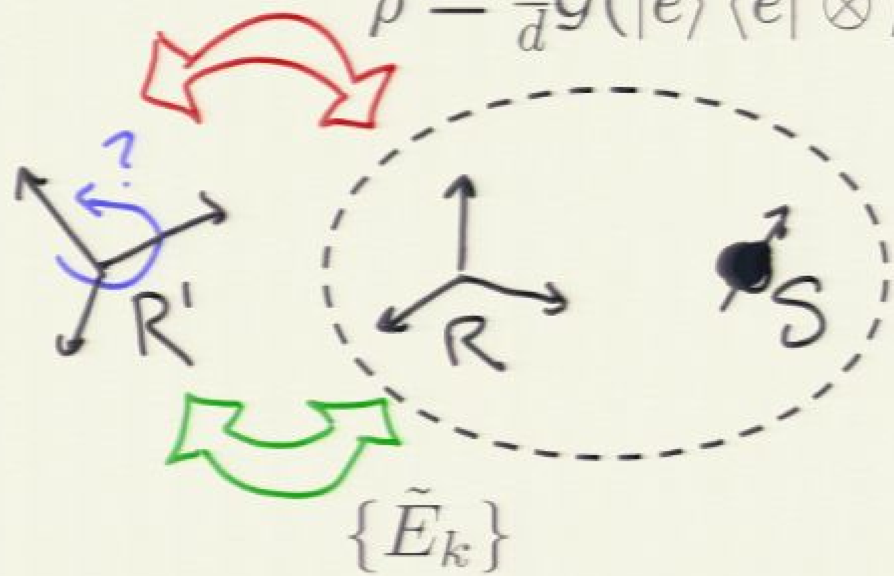

defined on $\mathcal{H}_R \otimes \mathcal{H}_S$

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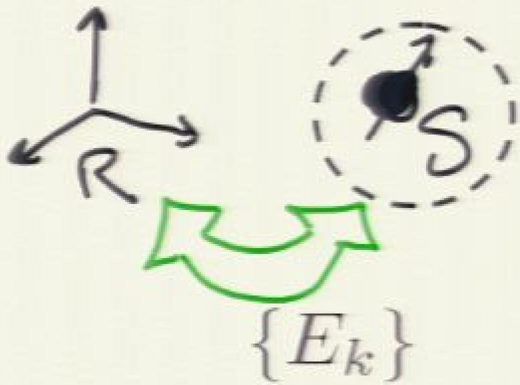
$\mathrm{Tr}_{RS}[\tilde{\rho}\tilde{E}_k]$

Can we find a measurement scheme such that

$$\mathrm{Tr}_{RS}[\tilde{\rho}\tilde{E}_k] = \mathrm{Tr}_S[\rho E_k] \quad ?$$

External RF paradigm



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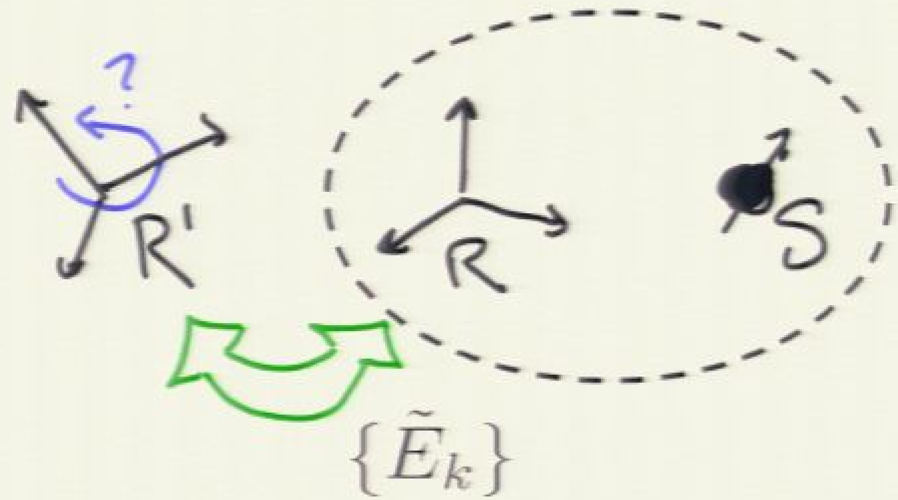

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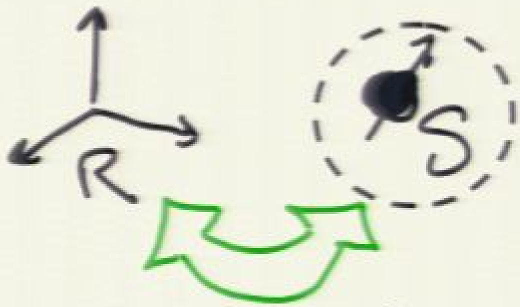
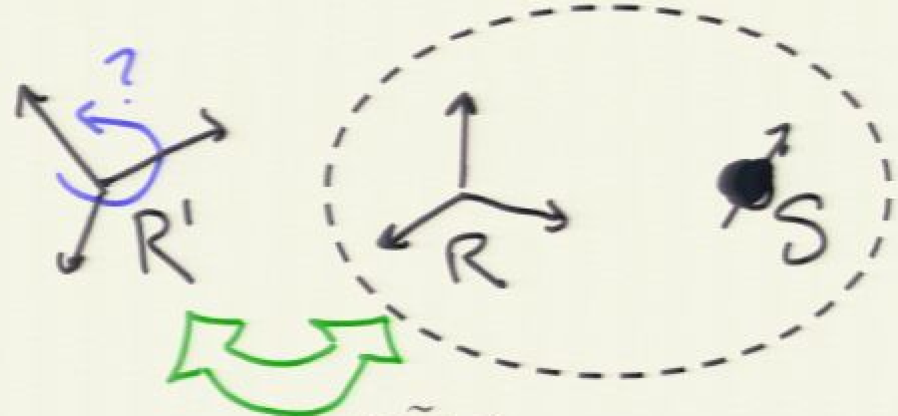


$\{E_k\}$

Internal RF paradigm

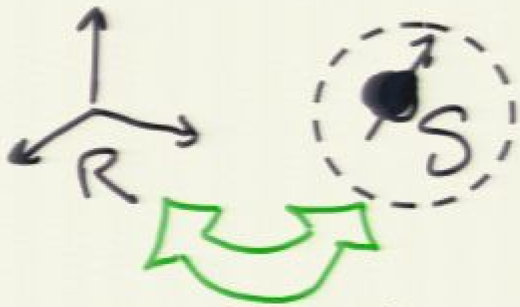


$\{\tilde{E}_k\}$

Measure a covariant POVM on R

$$\{U_R(\Omega)|e\rangle\langle e|U_R^\dagger(\Omega)\}_\Omega$$

External RF paradigm



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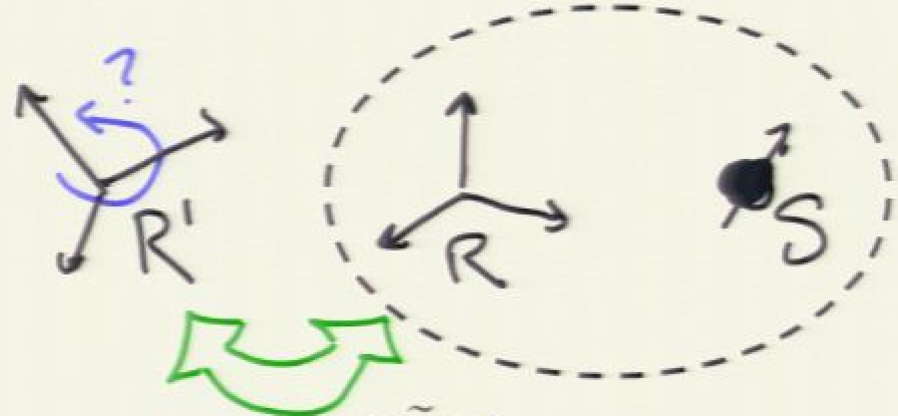

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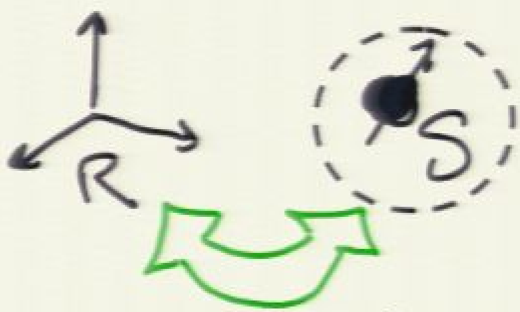
Upon obtaining Ω, measure on S

$$\{U(\Omega)E_k U^\dagger(\Omega)\}_k$$

External RF paradigm



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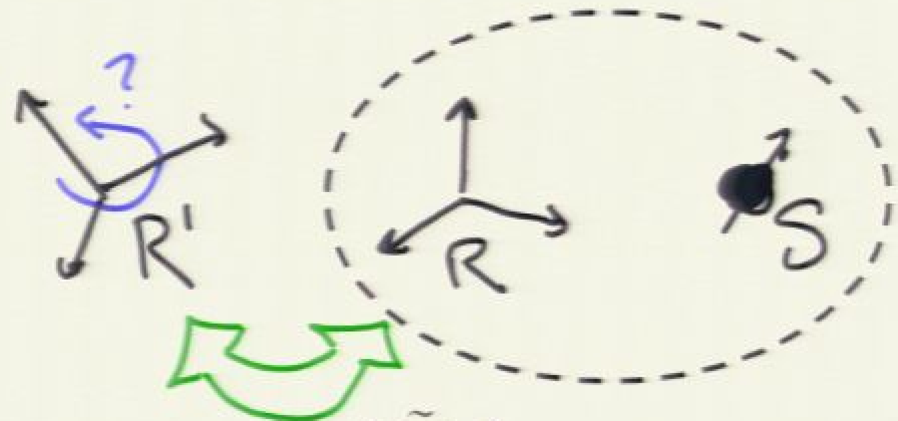
Internal RF paradigm



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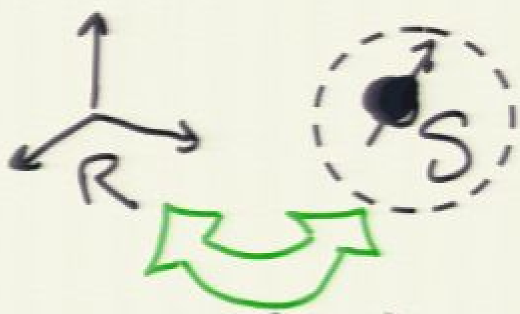
But R' has an unknown orientation

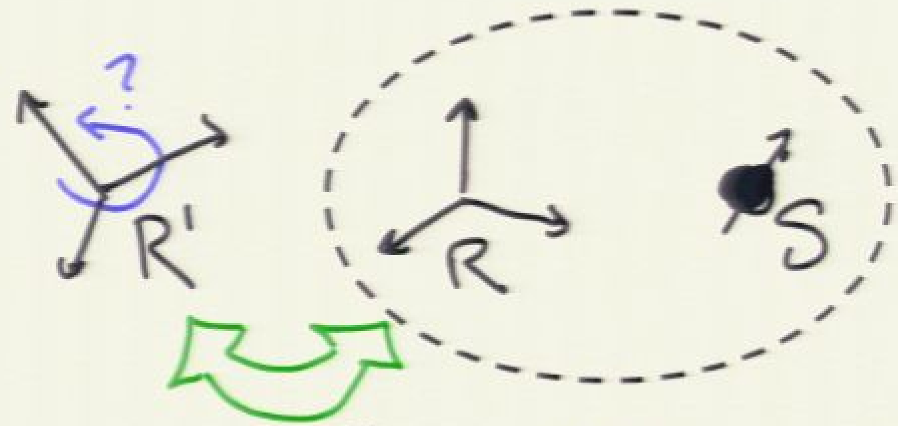
$$\tilde{E}_k = \int d\Omega U_R(\Omega)|e\rangle\langle e|U_R^\dagger(\Omega) \otimes U(\Omega)E_k U^\dagger($$

External-R paradigm



$\{E_k\}$

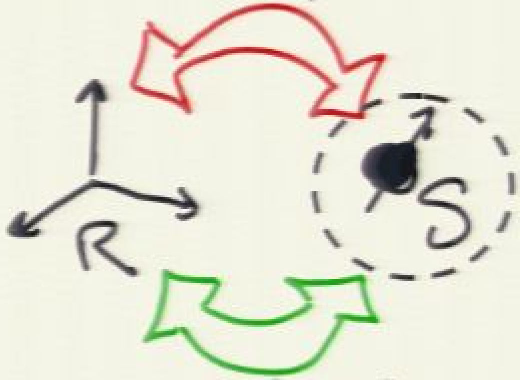
Internal-R paradigm



$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}$

External RF paradigm

ρ

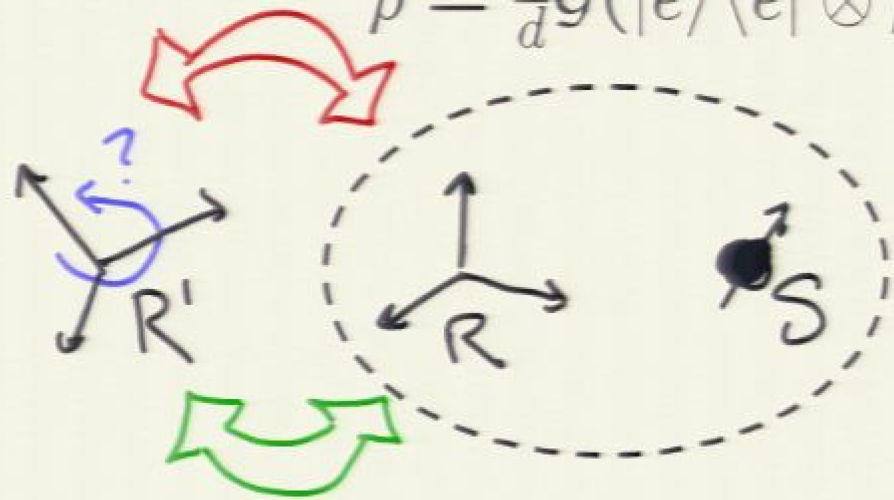


$\{E_k\}$

$\mathrm{Tr}_S[\rho E_k]$

Internal RF paradigm

$\tilde{\rho} = \frac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho)$



$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}$

$\mathrm{Tr}_{RS}[\tilde{\rho}\tilde{E}_k]$

$$\mathrm{Tr}_{RS}[\tilde{\rho}\tilde{E}_k]$$
$$= \mathrm{Tr}_{RS}[\, \tfrac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho)\; \mathcal{G}(|e\rangle\langle e| \otimes E_k)]$$
$$= \tfrac{1}{d}\int d\Omega d\Omega' \; |\langle e|U_R(\Omega\Omega'^{-1})|e\rangle|^2 \; \mathrm{Tr}_S[U(\Omega\Omega'^{-1})\rho U^\dagger(\Omega\Omega'^{-1})E$$
$$= \tfrac{1}{d}\int d\Omega \; |\langle e|U_R(\Omega)|e\rangle|^2 \; \mathrm{Tr}_S[U(\Omega)\rho U^\dagger(\Omega)E_k]$$
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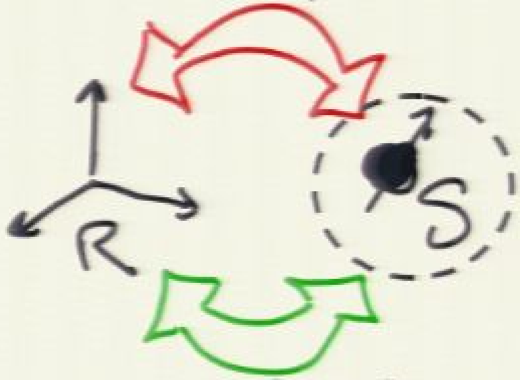
External RF paradigm

ρ



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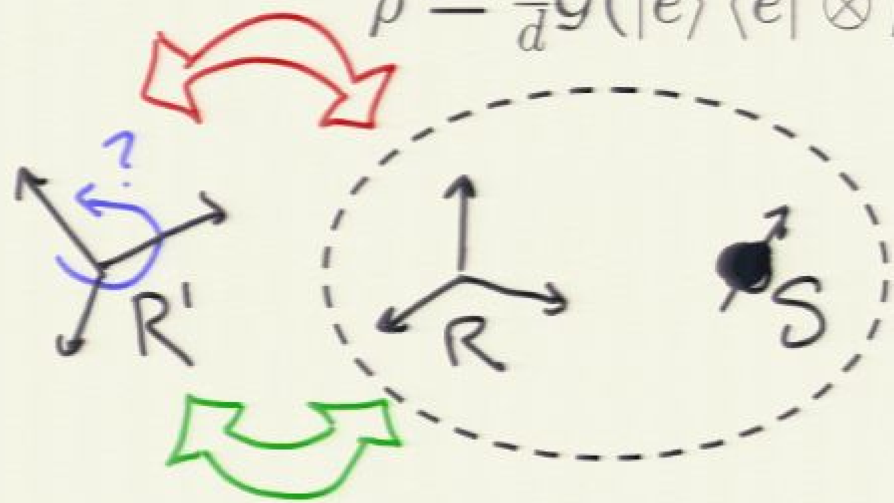

$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}$

$\mathrm{Tr}_{RS}[\tilde{\rho}\tilde{E}_k]$

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$$= \mathrm{Tr}_{RS}[\ \tfrac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho)\ \mathcal{G}(|e\rangle\langle e| \otimes E_k)]$$
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$$= \tfrac{1}{d}\int d\Omega\ |\langle e|U_R(\Omega)|e\rangle|^2\ \mathrm{Tr}_S[U(\Omega)\rho U^\dagger(\Omega)E_k]$$
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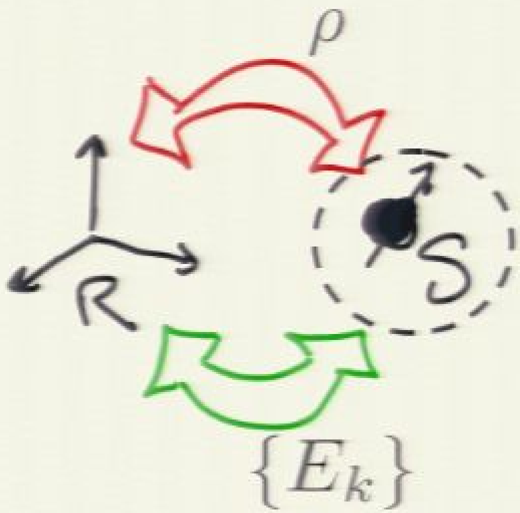
$$= \mathrm{Tr}_S[\mathcal{D}(\rho)\ E_k]$$

External RF paradigm



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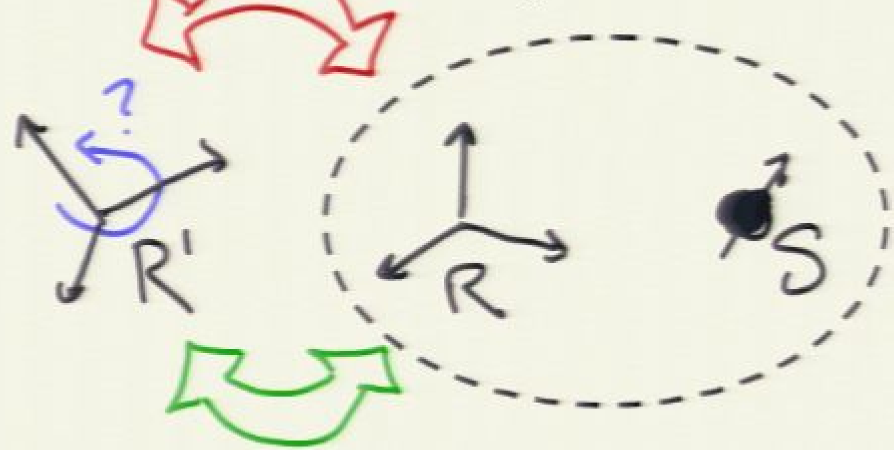


$$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}$$

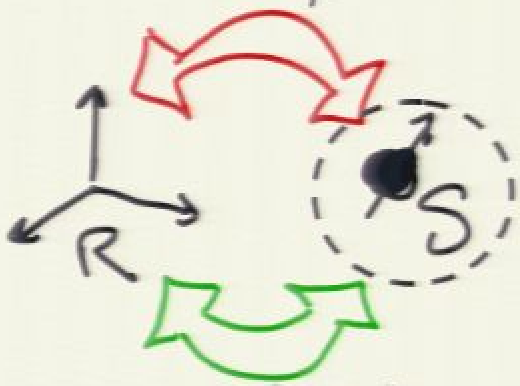
$$\mathrm{Tr}_{RS}[\tilde{\rho}\tilde{E}_k] = \mathrm{Tr}_S[\mathcal{D}(\rho)E_k]$$

where

$$\mathcal{D}(\cdot) = \frac{1}{d}\int d\Omega |\langle e|U_R(\Omega)|e\rangle|^2 U(\Omega)(\cdot)U^\dagger(\Omega$$

External RF paradigm

ρ



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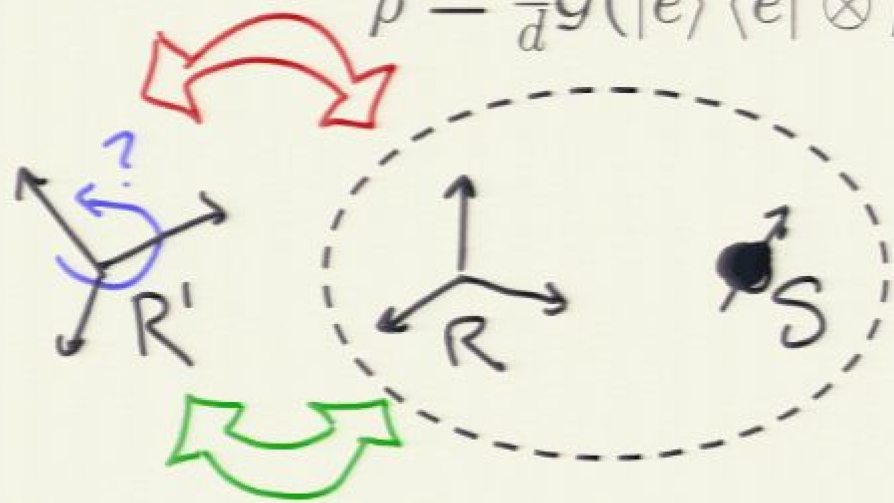

$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}$

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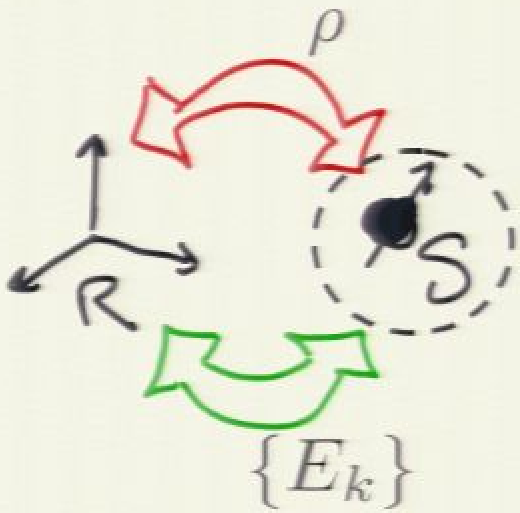
where

$\mathcal{D}(\cdot) = \frac{1}{d}\int d\Omega |\langle e|U_R(\Omega)|e\rangle|^2 U(\Omega)(\cdot)U^\dagger(\Omega)$

RF of unbounded-size: $\mathcal{D} = \mathrm{id}$

RF of bounded-size: $\mathcal{D} \neq \mathrm{id}$

External RF paradigm	Internal RF paradigm



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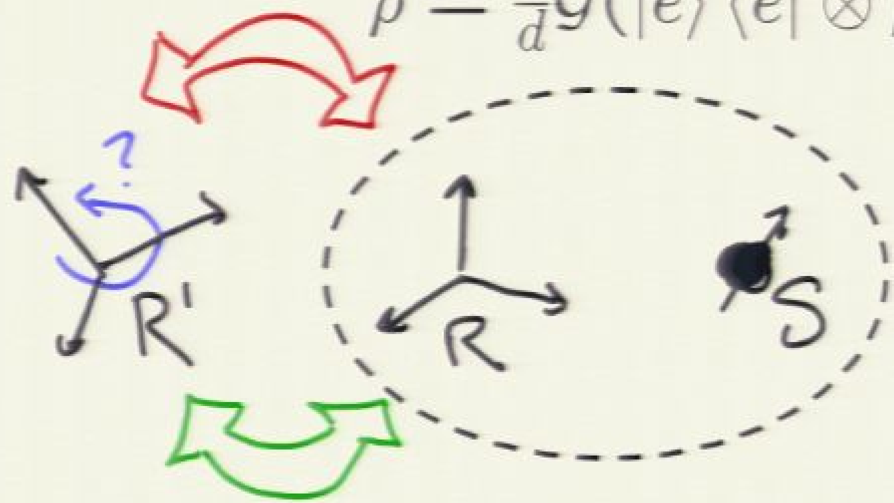

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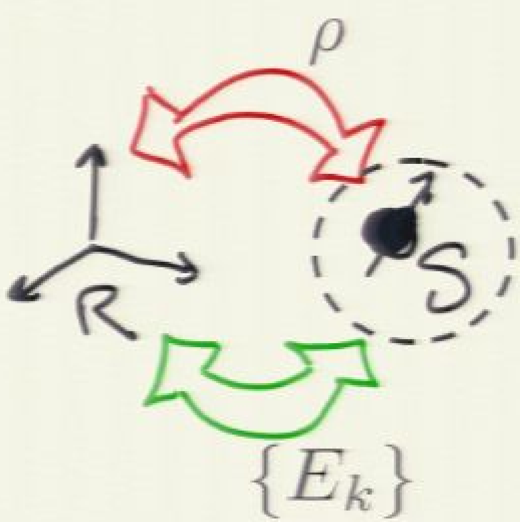

$\mathrm{Tr}_S[\mathcal{D}(\rho) E_k]$

Internal RF paradigm

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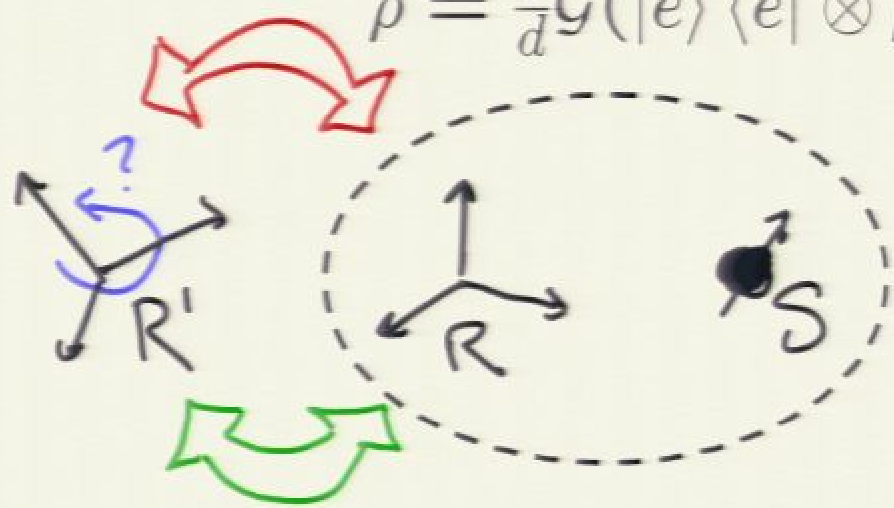

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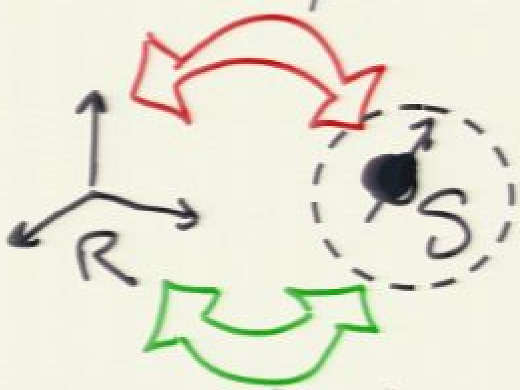
External RF paradigm

ρ



$\{E_k\}$

$$\text{Tr}_S[\mathcal{D}(\rho)E_k]$$

Internal RF paradigm

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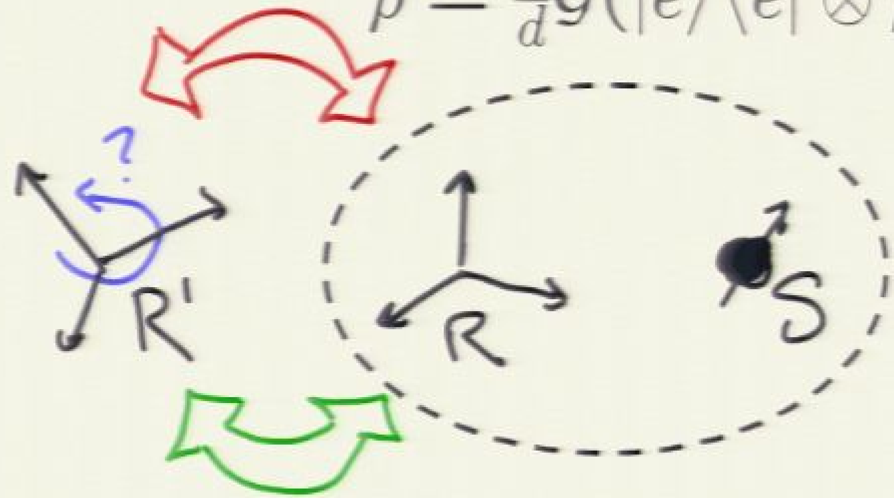

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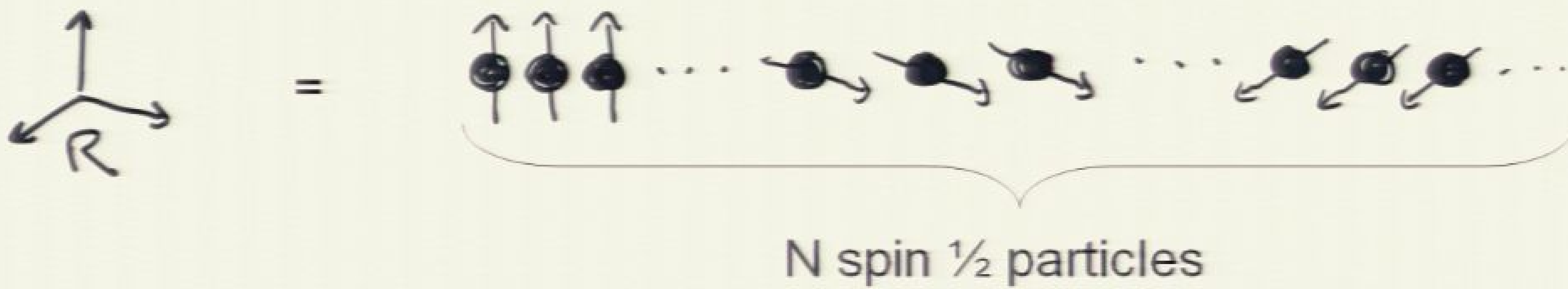
Bounded-size RF → Effective Decoherence

Bounded-size RF as effective decoherence: an example

For a given RF state, we can determine the decoherence map by treating it internally and thereafter treat it externally



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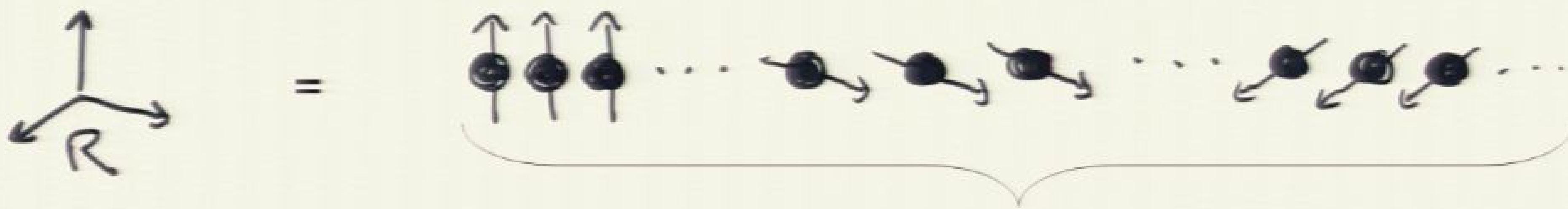

N spin ½ particles

But this is not the optimal state for sending a Cartesian frame

$$\frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1)$$

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$$\begin{aligned}
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= (0 \oplus 1) \otimes (0 \oplus 1) \\
&= 0 \oplus 1 \oplus 1 \oplus (0 \oplus 1 \oplus 2) \\
&= 0^2 \oplus 1^3 \oplus 2
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= (0 \oplus 1) \otimes (0^2 \oplus 1^3 \oplus 2) \\
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$$|j, m, \alpha\rangle \in \quad \mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2$$

Define a new tensor product structure as follows:

$$|j, m\rangle \otimes |j, \alpha\rangle \equiv |j, m, \alpha\rangle$$

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$\text{span}\{|j, m\rangle, m = -j, \ldots, j\}$ $\quad$ $\text{span}\{|j, \alpha\rangle, \alpha = 1, \ldots, d_{\mathcal{N}_j}\}$

“Virtual bits” in classical information theory

Standard factorization into Cartesian product of 2 bits

$$(a, b) \in \{0, 1\} \times \{0, 1\}$$

Novel factorization into Cartesian product of 2 bits

$$(a, c) \in \{0, 1\} \times \{0, 1\}$$

where $c = a \oplus b$ is a **virtual bit**

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"Virtual subsystems" in quantum theory

Standard factorization into tensor product of 2 qubits

$H_A \otimes H_B$

Novel factorization into tensor product of 2 qubits

$H_C \otimes H_D$ where C and D are **virtual subsystems**

$$|\Phi\rangle_C|+\rangle_D = |\Phi,+\rangle \equiv |0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B$$

$$|\Phi\rangle_C|-\rangle_D = |\Phi,-\rangle \equiv |0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B$$

$$|\Psi\rangle_C|+\rangle_D = |\Psi,+\rangle \equiv |0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B$$

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$$\underbrace{\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2}_{N} = \bigoplus_{j=0}^{N/2} \mathcal{M}_j \otimes \mathcal{N}_j$$

span$\{|j,m\rangle, m = -j, \ldots, j\}$

Representation spaces

$$d_{\mathcal{M}_j} = (2j+1)$$

span$\{|j,\alpha\rangle, \alpha = 1, \ldots, d_{\mathcal{N}_j}\}$

Multiplicity spaces

$$d_{\mathcal{N}_j} = \binom{N}{N/2-j} \frac{2j+1}{N/2+j+1}$$

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For *N*=*20*

j	$d_{\mathcal{M}_j}$	$d_{\mathcal{N}_j}$
10	21	1
9	19	19
8	17	170
7	15	950
6	13	3705
5	11	10659
4	9	23256
3	7	38760
2	5	48450
1	3	41990
0	1	16796

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span$\{|j,m\rangle, m=-j,\ldots,j\}$ — **Representation spaces**

span$\{|j,\alpha\rangle, \alpha=1,\ldots,d_{\mathcal{N}_j}\}$ — **Multiplicity spaces**

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For *N=20*

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Near-optimal state for encoding a Cartesian frame

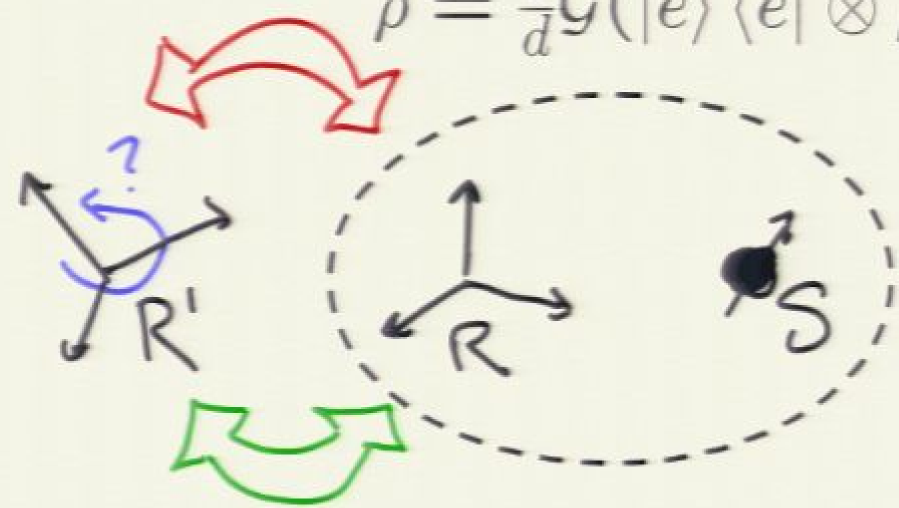
Let j_R be the largest value of j such that $d_{\mathcal{N}_j} \geq d_{\mathcal{M}}$

Define:

$$|e\rangle = \sum_{j=0}^{j_R} \sqrt{\frac{2j+1}{d_R^*}} \sum_{m=-j}^{j} |j,m\rangle \otimes |\phi_{j,m}\rangle$$

Internal RF paradigm

$$\tilde{\rho} = \frac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho)$$



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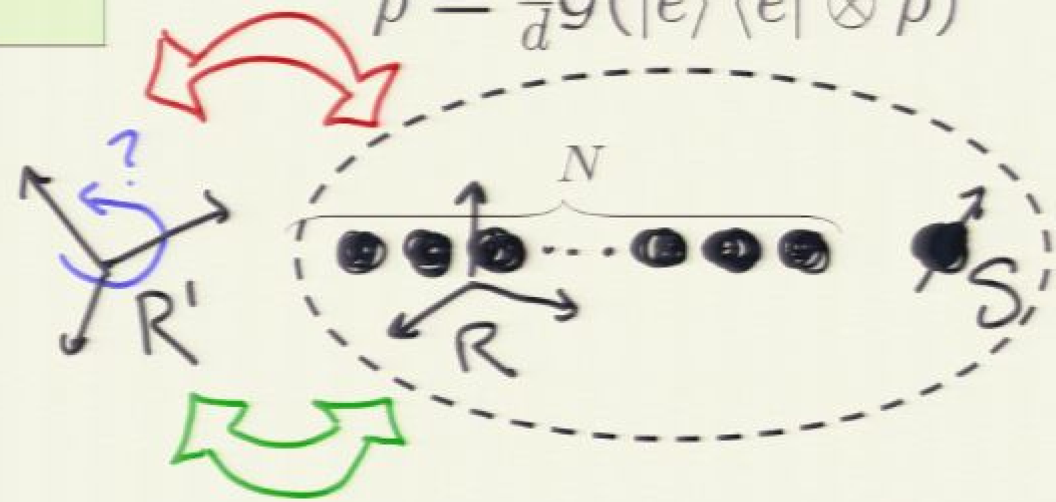

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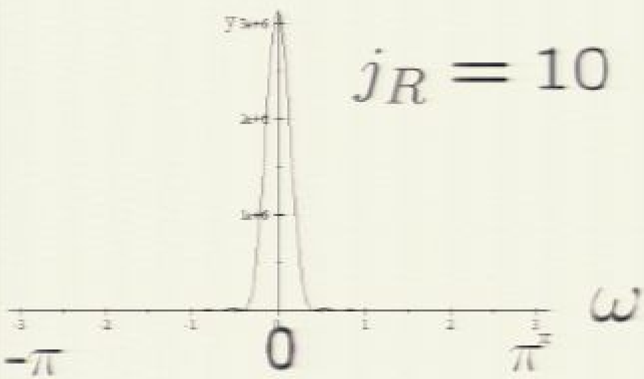
$$|\langle e|U_R(\Omega)|e\rangle|^2$$



$$U_R(\Omega) = U(\Omega)^{\otimes N}$$

$$U(\Omega) = e^{i\omega \mathbf{n}\cdot\mathbf{J}}$$

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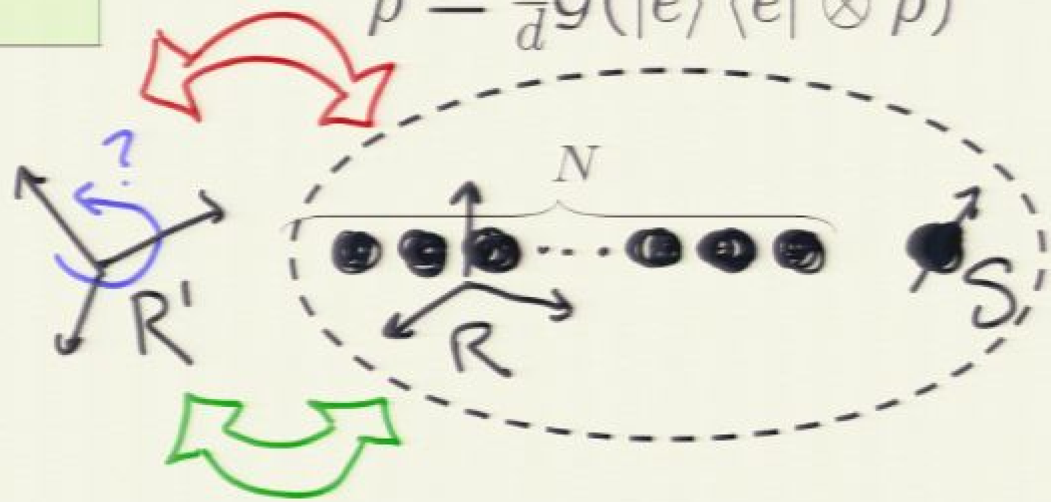

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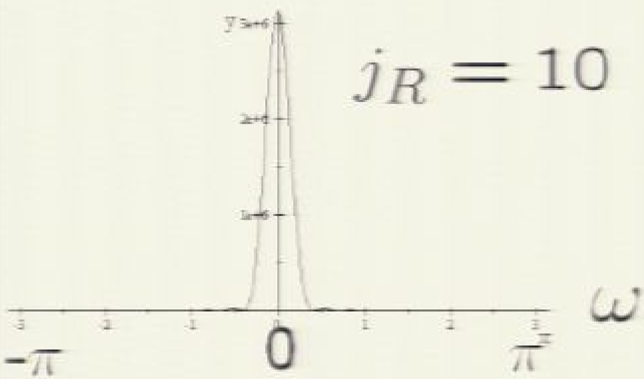

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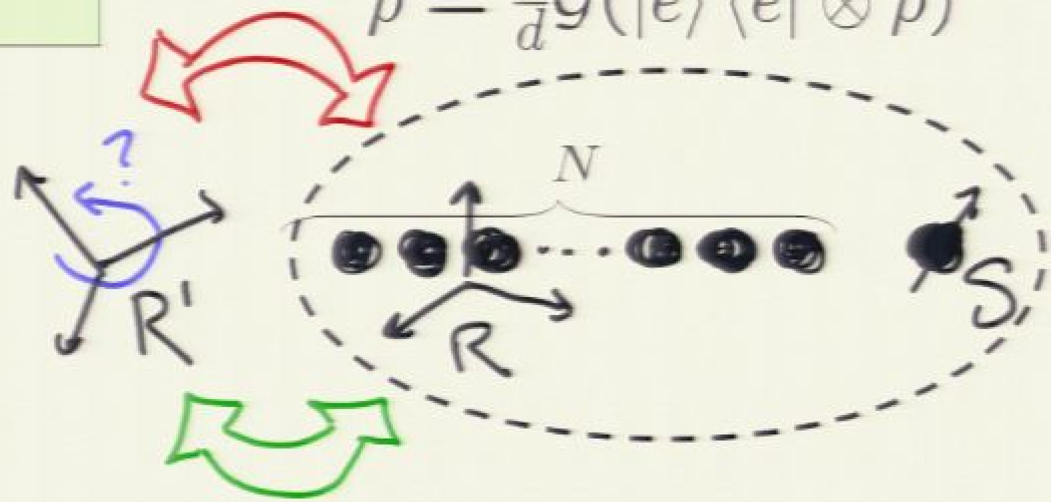

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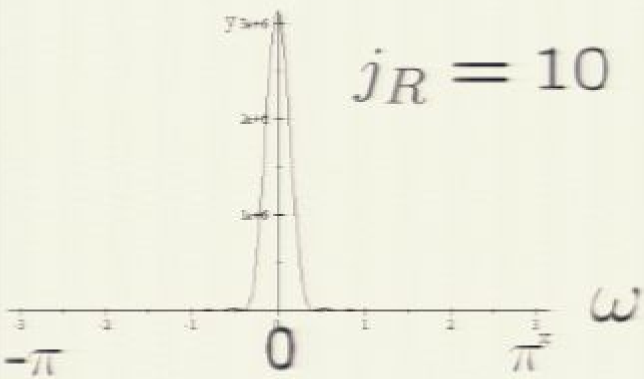
$$\mathcal{D}(\cdot) = \frac{1}{d}\int d\Omega |\langle e|U_R(\Omega)|e\rangle|^2 U(\Omega)(\cdot)U^\dagger(\Omega$$

$$\mathcal{D}[\rho] = \left(\frac{j_R}{j_R+1}\mathrm{id} + \frac{1}{j_R+1}\mathcal{G}\right)[\rho]$$

$$|e\rangle = \sum_{j=0}^{j_R} \sqrt{\frac{2j+1}{d_R^*}} \sum_{m=-j}^{j} |j,m\rangle \otimes |\phi_{j,m}\rangle$$

Internal RF paradigm

$$|\langle e|U_R(\Omega)|e\rangle|^2$$



$$j_R = 10$$

$$U_R(\Omega) = U(\Omega)^{\otimes N}$$

$$U(\Omega) = e^{i\omega \mathbf{n}\cdot\mathbf{J}}$$

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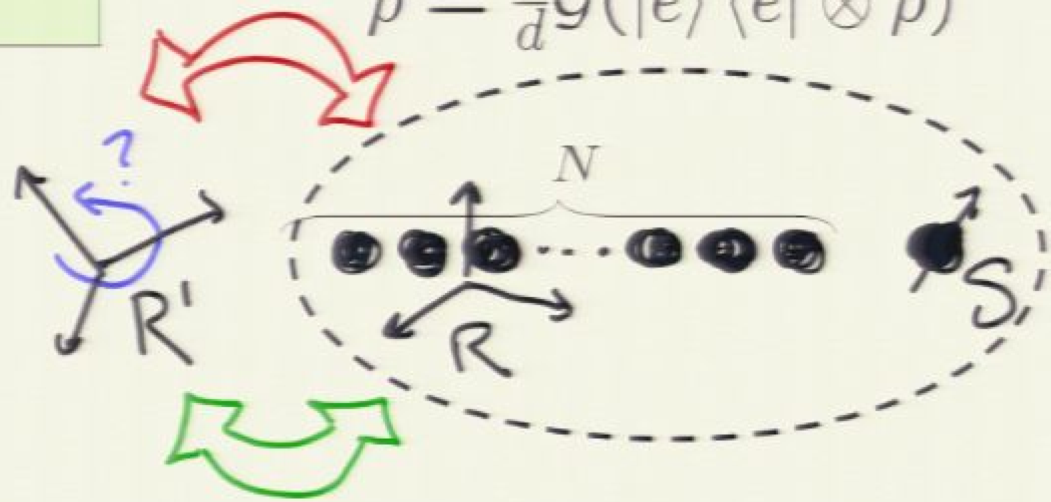

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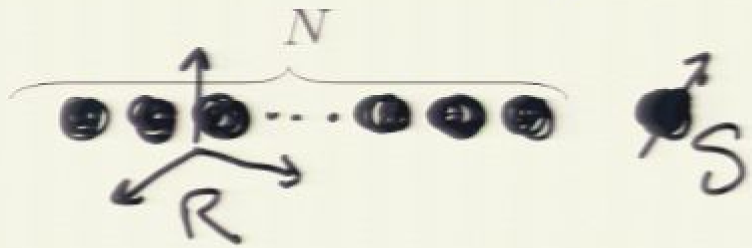
Limit of unbounded RF

$$\mathcal{D} \to \mathrm{id} \ \text{as} \ j_R \to \infty$$

$$\mathcal{D}[\rho] = (\frac{j_R}{j_R+1}\mathrm{id} + \frac{1}{j_R+1}\mathcal{G})[\rho]$$

Where does the relation between R and S live in the Hilbert space?



$$\mathcal{H}_{RS} = \underbrace{\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2}_{N} \otimes \mathcal{C}_2$$

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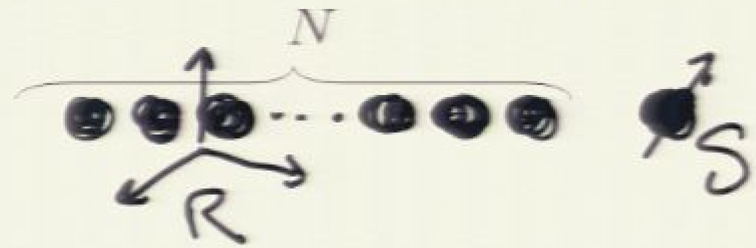
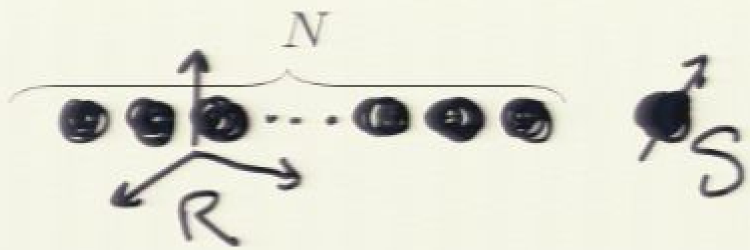


$$\mathcal{H}_{RS} = \underbrace{\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2}_{N} \otimes \mathcal{C}_2$$

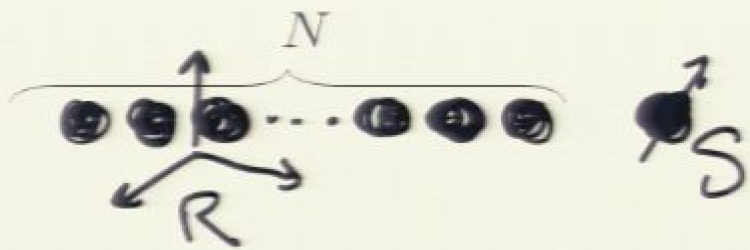
The relational degrees of freedom between R and S are

- Invariant under collective rotations

- Invariant under changes of the internal d.o.f.s of R



$$\underbrace{\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2}_{N} \otimes \mathcal{C}_2 = \bigoplus_{J=1/2}^{j_R+1/2} \mathcal{M}_{RS}^{(J)} \otimes \mathcal{N}_{RS}^{(J)}$$



$$\underbrace{\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2}\ \otimes\ \mathcal{C}_2 \quad = \bigoplus_{J=1/2}^{j_R+1/2} \mathcal{M}_{RS}^{(J)} \otimes \mathcal{N}_{RS}^{(J)}$$

$$\mathcal{H}'_R = \bigoplus_{j=0}^{j_R} \mathcal{M}_R^{(j)} \otimes \mathcal{N}_R'^{(j)}$$

Invariant under collective rotations

Relations among components of R

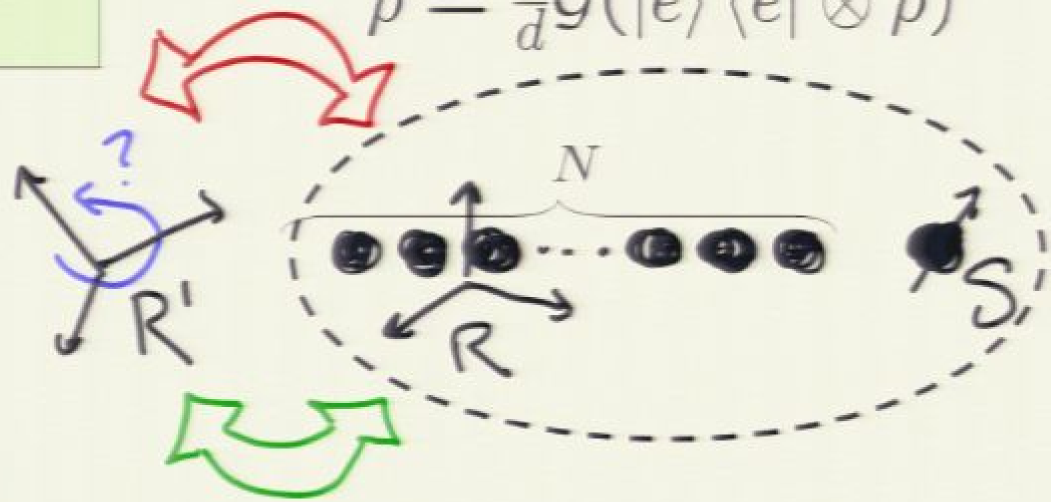
$$\mathcal{N}_{RS}^{(J)} = \begin{cases} \mathcal{N}_R'^{(J+\frac{1}{2})} \oplus \mathcal{N}_R'^{(J-\frac{1}{2})}, & J < j_R + \frac{1}{2} \\ \mathcal{N}_R'^{(j_R)}, & J = j_R + \frac{1}{2}. \end{cases}$$

$$|e\rangle = \sum_{j=0}^{j_R} \sqrt{\frac{2j+1}{d_R^*}} \sum_{m=-j}^{j} |j,m\rangle \otimes |\phi_{j,m}\rangle$$

$$|e\rangle = \sum_{j=0}^{j_R} \sqrt{\frac{2j+1}{d_R^*}} \sum_{m=-j}^{j} |j,m\rangle \otimes |\phi_{j,m}\rangle$$

Internal RF paradigm

$$\tilde{\rho} = \frac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho)$$



$$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}_k$$

Recall:

$$\mathcal{H}_{RS} = \bigoplus_{J=1/2}^{j_R+1/2} \mathcal{M}_{RS}^{(J)} \otimes \mathcal{N}_{RS}^{(J)}$$

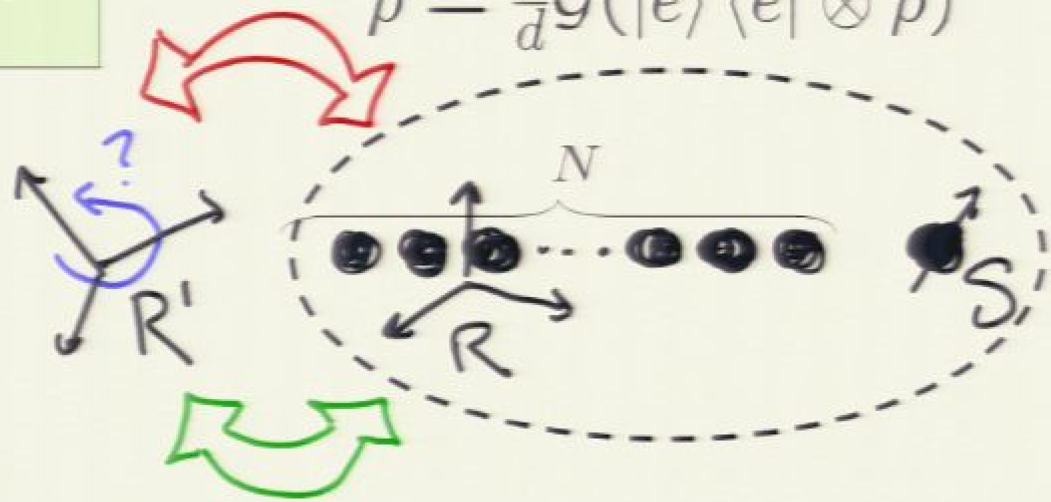
$$\mathcal{N}_{RS}^{(J)} = \begin{cases} \mathcal{L}_{RS}^{(J)} \otimes \mathcal{L}_{RS}^{(\frac{1}{2})}, & J < j_R + \frac{1}{2} \\ \mathcal{N}_R^{(j_R)}, & J = j_R + \frac{1}{2}. \end{cases}$$

There is a perfect copy of ρ and E_k in each $\mathcal{L}_{RS}^{(\frac{1}{2})}$

Internal RF paradigm

$$|e\rangle = \sum_{j=0}^{j_R} \sqrt{\frac{2j+1}{d_R^*}} \sum_{m=-j}^{j} |j,m\rangle \otimes |\phi_{j,m}\rangle$$

$$\tilde{\rho} = \frac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho)$$



$$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}_k$$

Recall:

$$\mathcal{H}_{RS} = \bigoplus_{J=1/2}^{j_R+1/2} \mathcal{M}_{RS}^{(J)} \otimes \mathcal{N}_{RS}^{(J)}$$

$$\mathcal{N}_{RS}^{(J)} = \begin{cases} \mathcal{L}_{RS}^{(J)} \otimes \mathcal{L}_{RS}^{(\frac{1}{2})} & J < j_R + \frac{1}{2} \\ \mathcal{N}_R^{(j_R)}, & J = j_R + \frac{1}{2}. \end{cases}$$

There is a perfect copy of ρ and E_k in each $\mathcal{L}_{RS}^{(\frac{1}{2})}$

Probabilities of different J

$$p_J = \begin{cases} \frac{(2J+1)^2}{d_{R^\star}}, & J < j_R + \frac{1}{2}, \\ \frac{(2j_R+1)(j_R+1)}{d_{R^\star}}, & J = j_R + \frac{1}{2}. \end{cases}$$

Different sorts of RFs ←→ different groups

The results are similar for other groups

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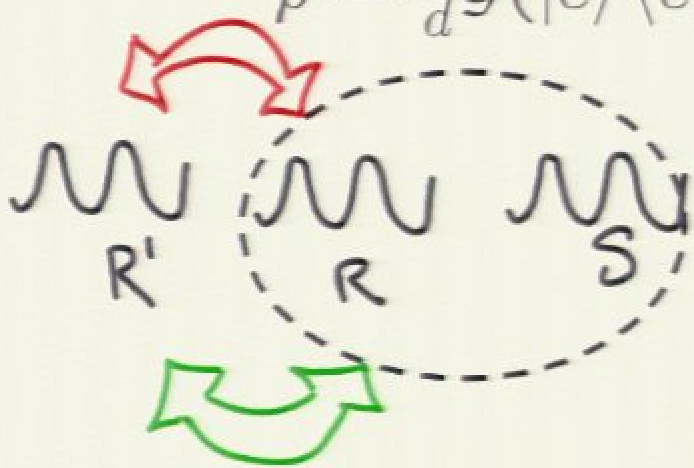

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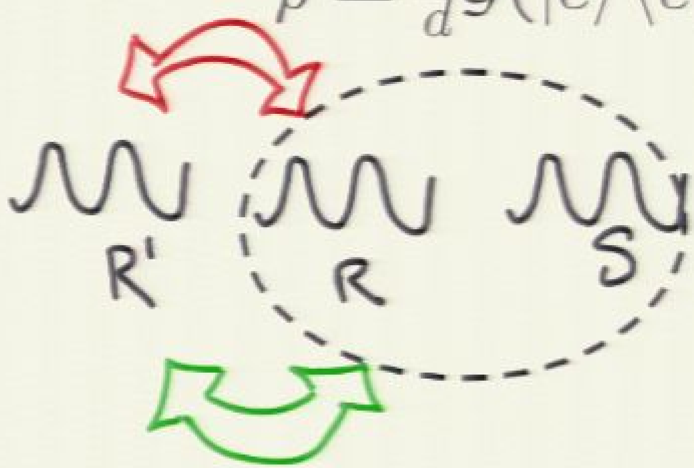

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One finds:

$$\mathcal{D}[\rho] = (\frac{N_R}{N_R+1}\mathrm{id} + \frac{1}{N_R+1}\mathcal{G})[\rho$$

Conclusions

Quantum states describe extrinsic as well as intrinsic properties, equivalently, relational rather than absolute degrees of freedom

External RF paradigm is just as good as internal RF paradigm
→ One can even model bounded-size RF effects
These appear as effective decoherence for relational degrees of freedom