

Title: Quantum reference frames and relationalism in quantum theory

Date: Sep 29, 2008 04:00 PM

URL: <http://pirsa.org/08090075>

Abstract: A reference frame can be treated as a physical quantum object internal to the theory. Quantum reference frames whose size, and therefore accuracy, are bounded in some way necessarily limit one's ability to prepare states and to perform quantum operations and measurements on a system. The nature of these limitations is similar in many ways to that of decoherence. We investigate how a quantum reference frame of bounded size can be 'dequantized', i.e., treated as external to the quantum formalism, in such a way as to induce an effective decoherence on any system described relative to it. In particular, we show that this decoherence has an interpretation as a lack of classical information about an ideal (infinite size) reference frame.

# **Reference frames and relationalism in quantum theory**

**Stephen Bartlett**

**The University of Sydney**

with

Terry Rudolph (Imperial College London)

Robert Spekkens (Cambridge → Perimeter Institute)

## An observation

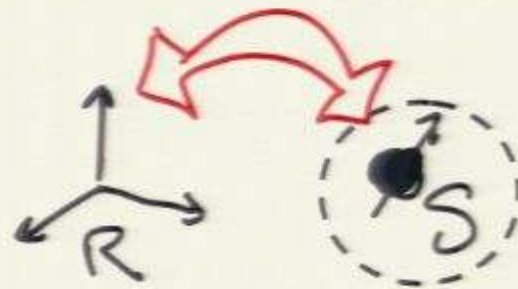
A common view:  
quantum states only contain information about  
the **intrinsic properties** of the system

We submit:  
quantum states also contain information about  
the **extrinsic properties** of the system  
Specifically: the relation to other systems external to it.

Stephen D Bartlett, Terry Rudolph, and Robert W. Spekkens  
“Reference Frames, Superselection Rules, and Quantum Information”  
*Reviews of Modern Physics*, **79**, 555 (2007)

What does it mean to say  
that the spin is up along the  
z axis?

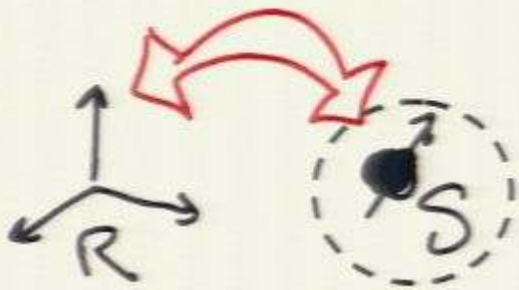
What does it mean to say that the spin is up along the z axis?



It means spin up **relative to another physical system**, such as gyroscopes in the lab, that define the z axis (i.e. act as a Cartesian reference frame)

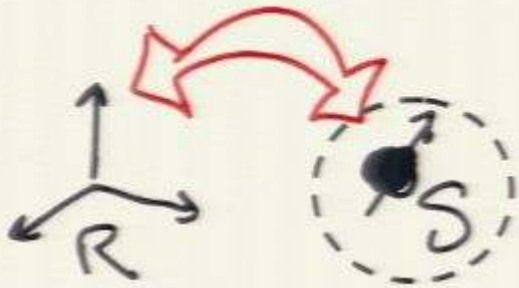
# External RF paradigm

$$\rho_S \in \mathcal{L}(\mathcal{H}_S)$$



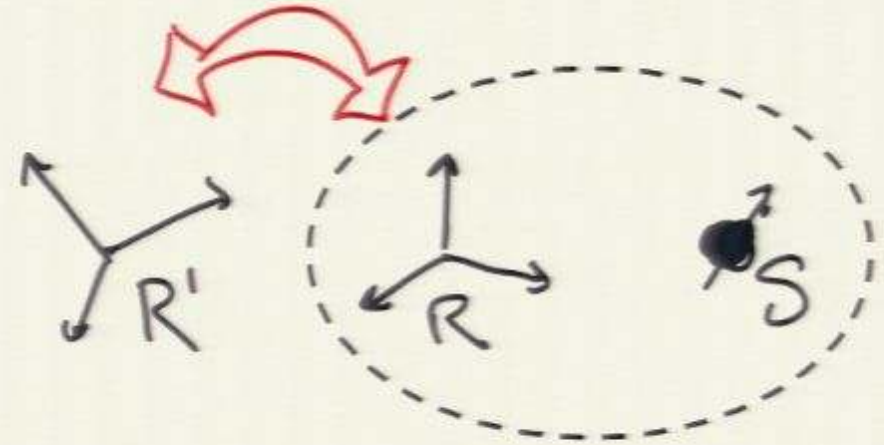
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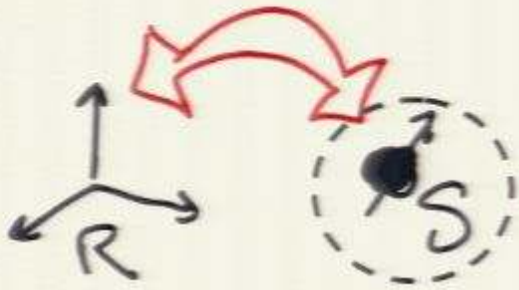
## Internal RF paradigm

$$\sigma_{RS} \in \mathcal{L}(\mathcal{H}_R) \otimes \mathcal{L}(\mathcal{H}_S)$$



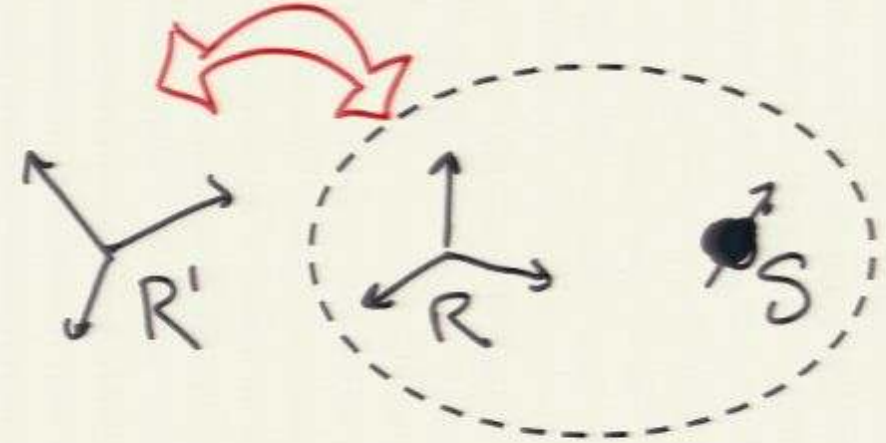
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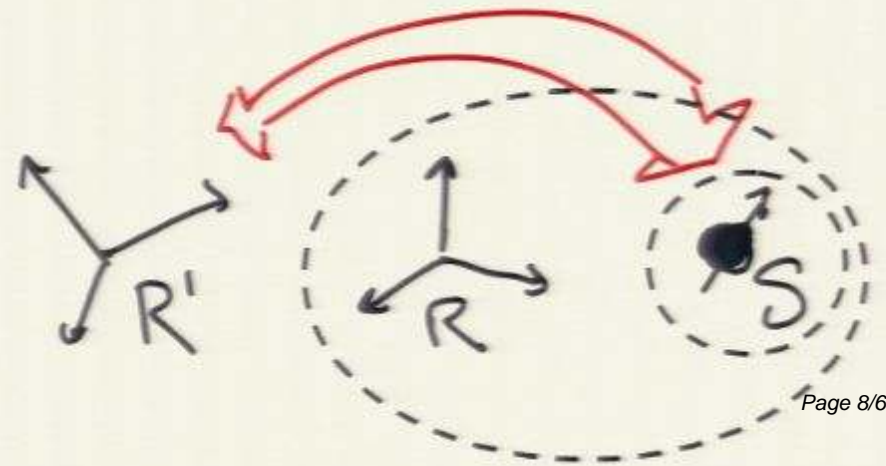


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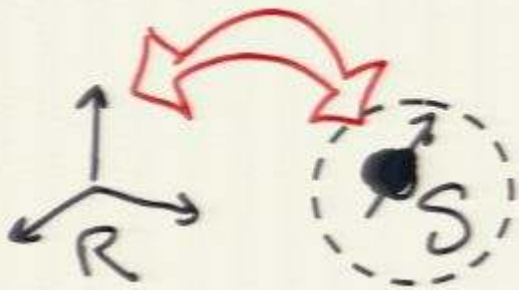


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## External RF paradigm

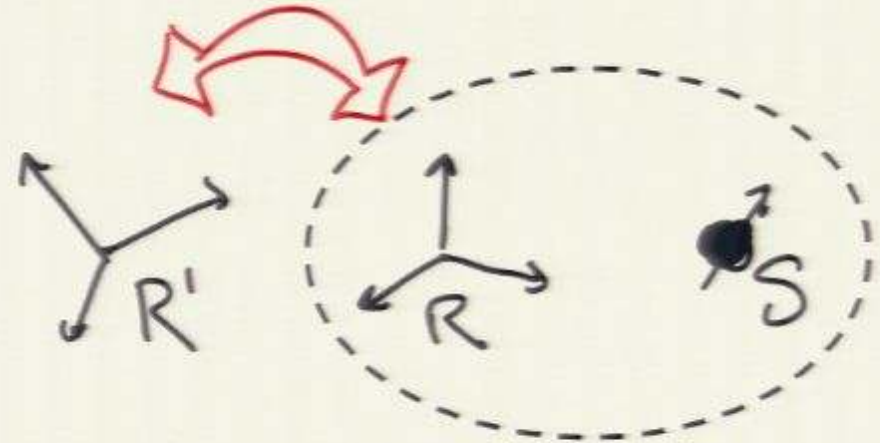
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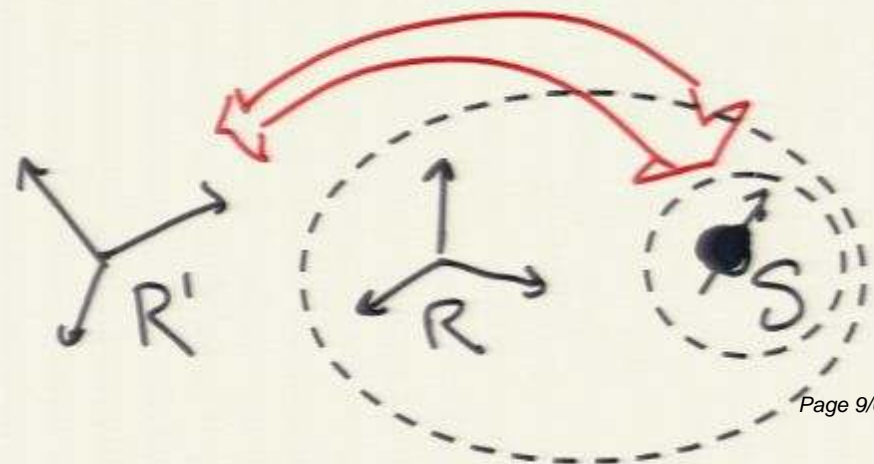
So, the two states  
need not be the same

## Internal RF paradigm

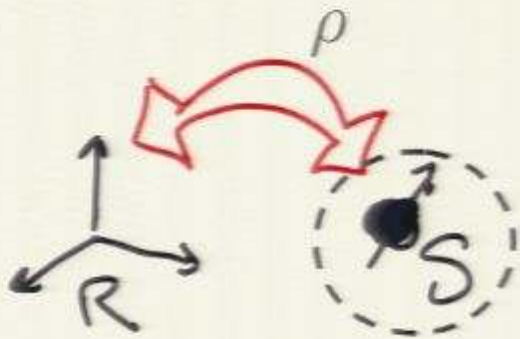
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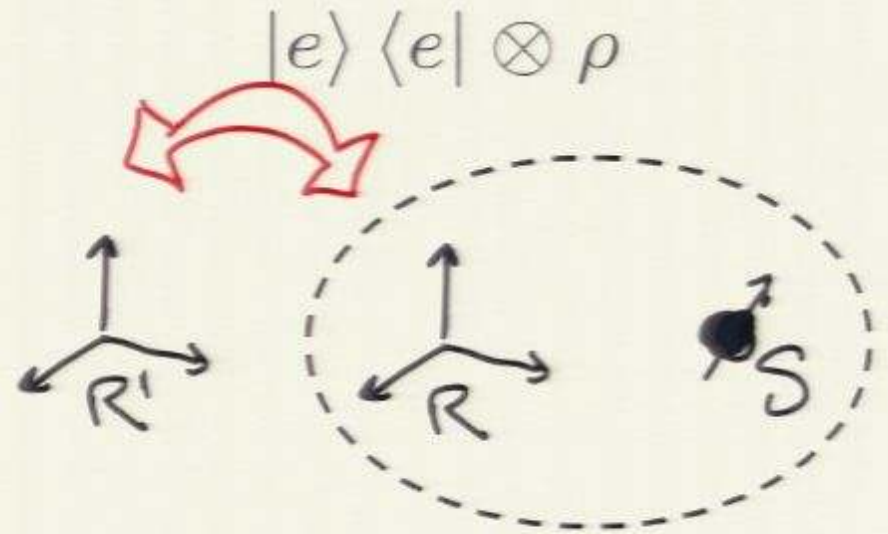
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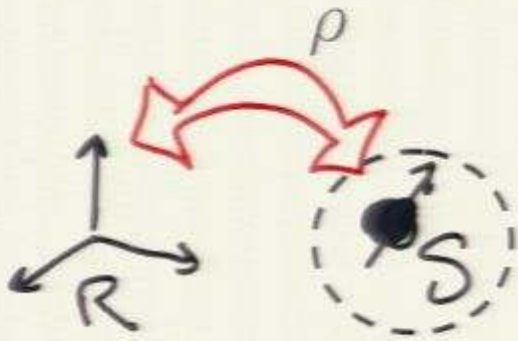
## External RF paradigm



## Internal RF paradigm

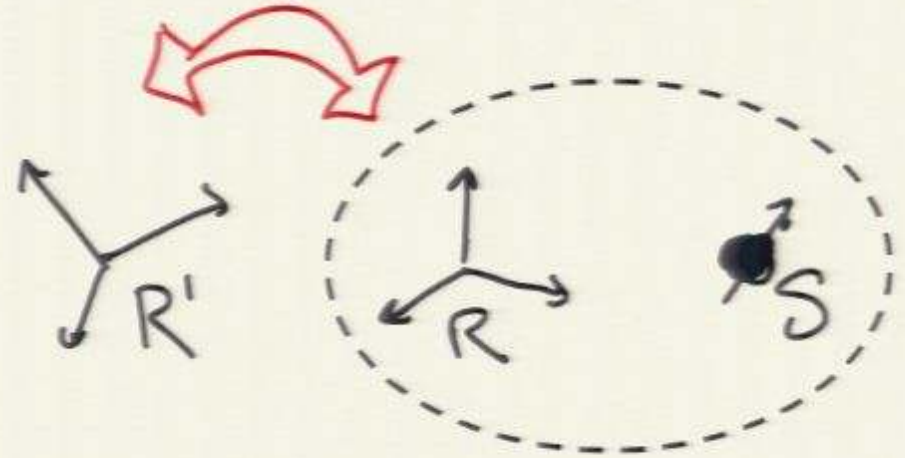


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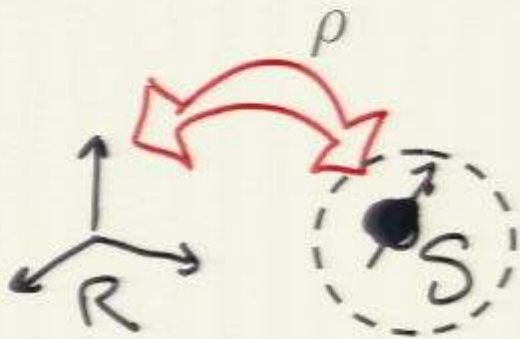


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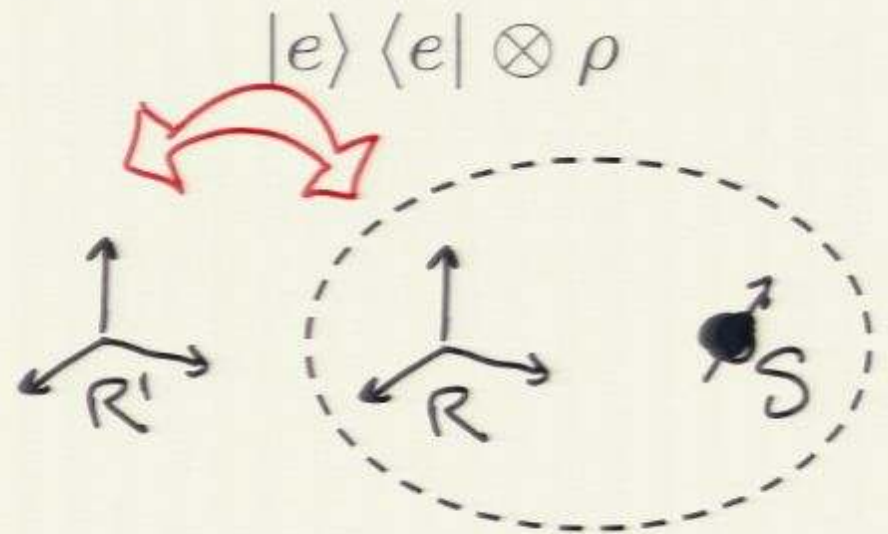
$$U_R(\Omega) |e\rangle \langle e| U_R^\dagger(\Omega) \otimes U(\Omega) \rho U^\dagger(\Omega)$$



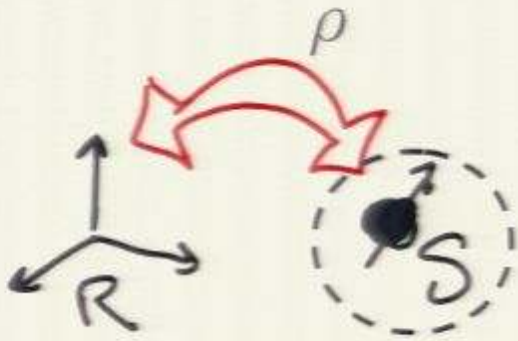
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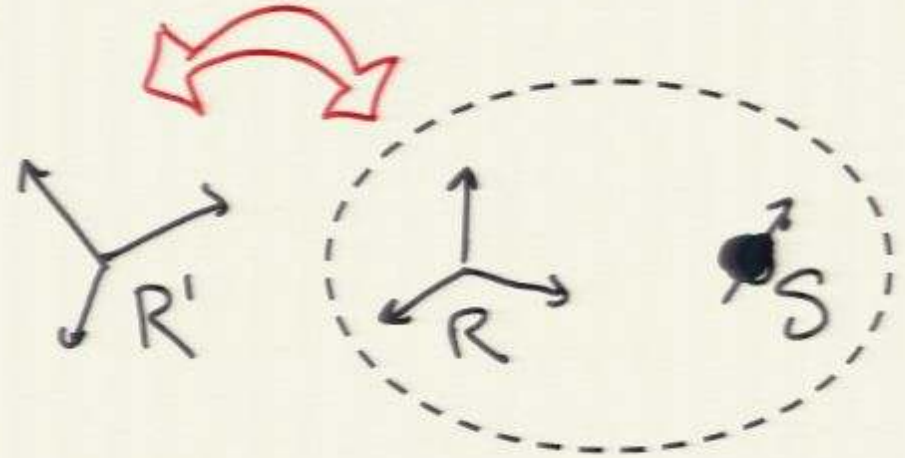


## External RF paradigm



## Internal RF paradigm

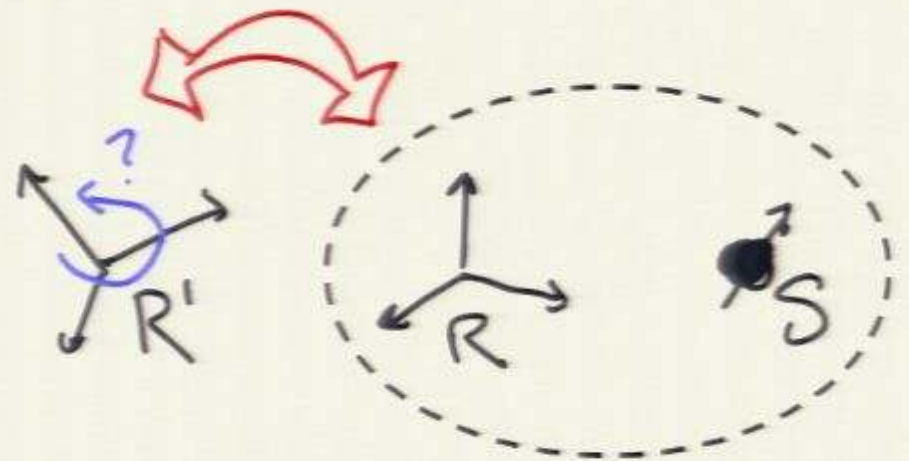
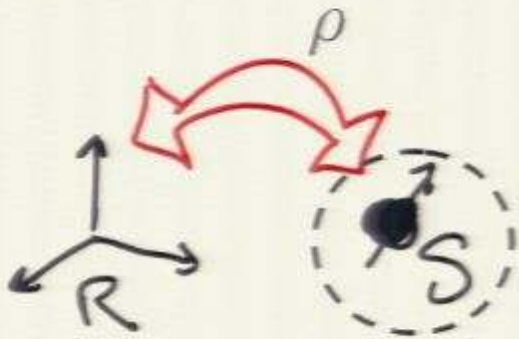
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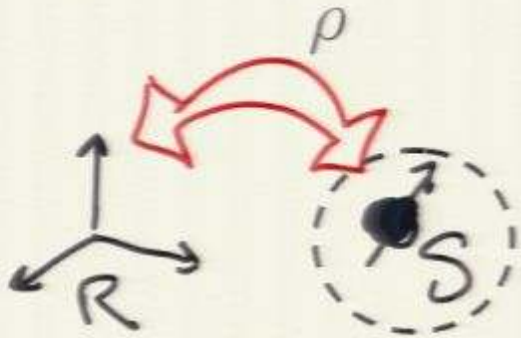
External RF paradigm

Internal RF paradigm

$$\frac{1}{d} \int d\Omega U_R(\Omega) |e\rangle \langle e| U_R^\dagger(\Omega) \otimes U(\Omega) \rho U^\dagger(\Omega)$$

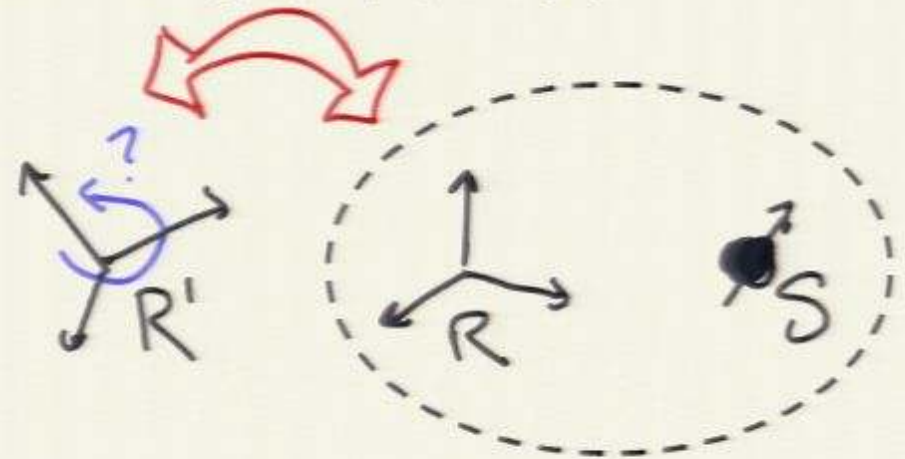


## External RF paradigm



## Internal RF paradigm

$$d^{-1} \mathcal{G}(|e\rangle \langle e| \otimes \rho)$$



$$\mathcal{G}(\cdot) \equiv \int d\Omega (U_R(\Omega) \otimes U(\Omega)) (\cdot) (U_R^\dagger(\Omega) \otimes U^\dagger(\Omega))$$

State of RS is **rotationally-invariant**

No coherence between eigenstates of  $J^2$

## Question

What properties of the state  $|e\rangle$  are required for it to serve as a 'good' RF?

## Result:

External RF paradigm is just as good as internal RF paradigm

→ One can even model **bounded-size** RF effects

These appear as **effective decoherence** for relational degrees of freedom

Page and Wootters, PRD **27**, 2885 (1983).

Milburn, PRA **44**, 5401 (1991).

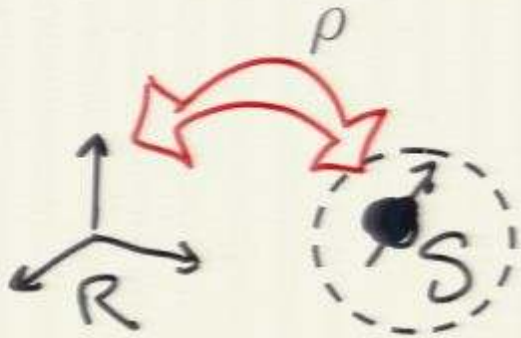
Gambini, Porto, Pullin, PRL **93**, 240401 (2004).

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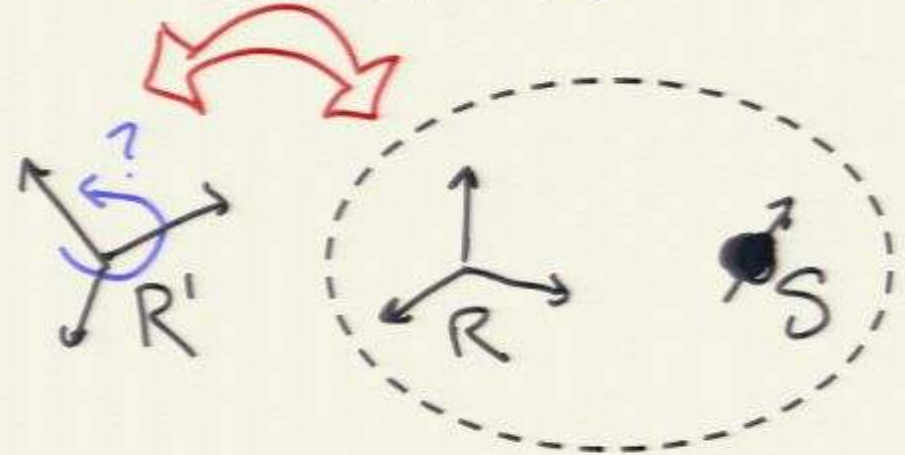
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## External RF paradigm



## Internal RF paradigm

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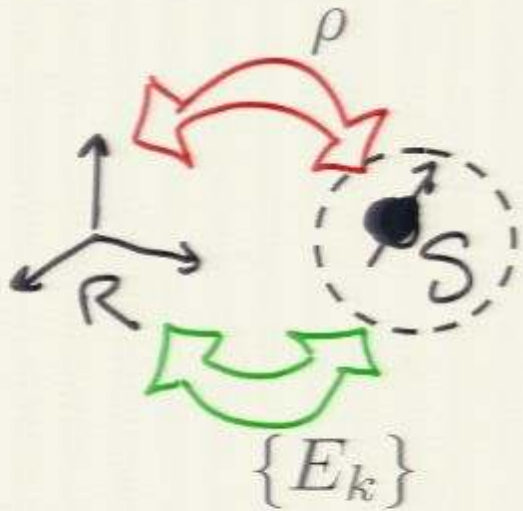
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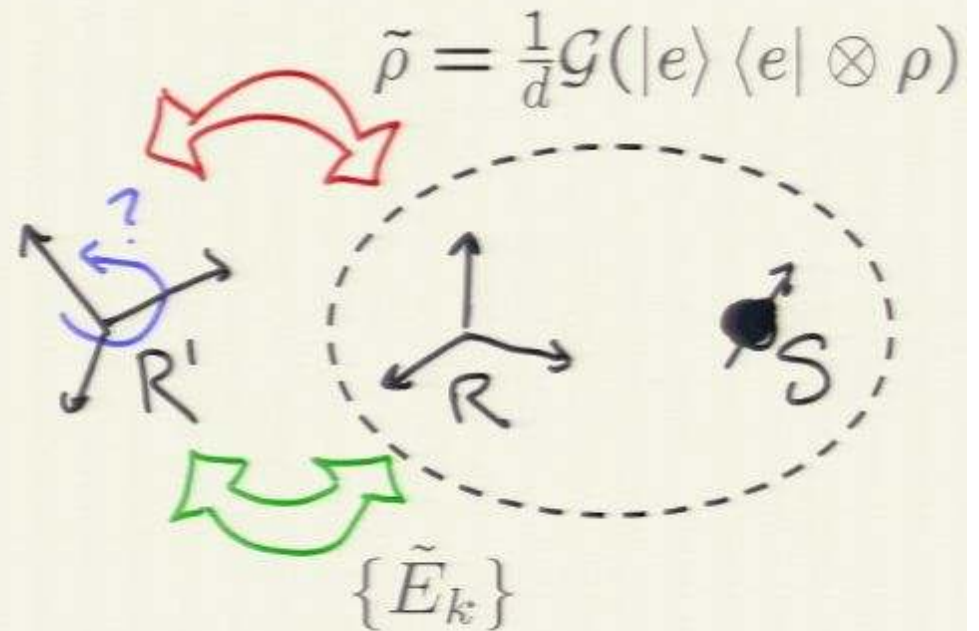
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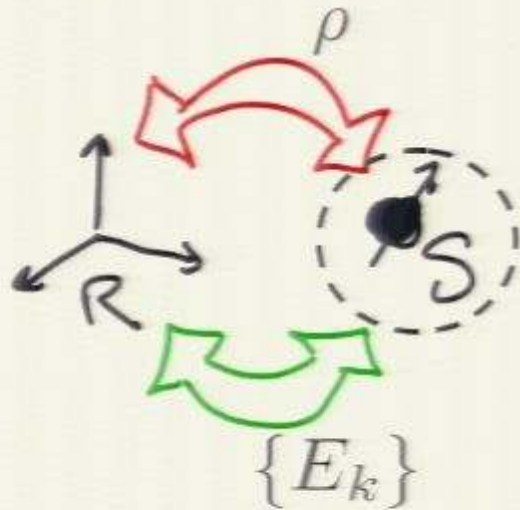
## External RF paradigm



## Internal RF paradigm



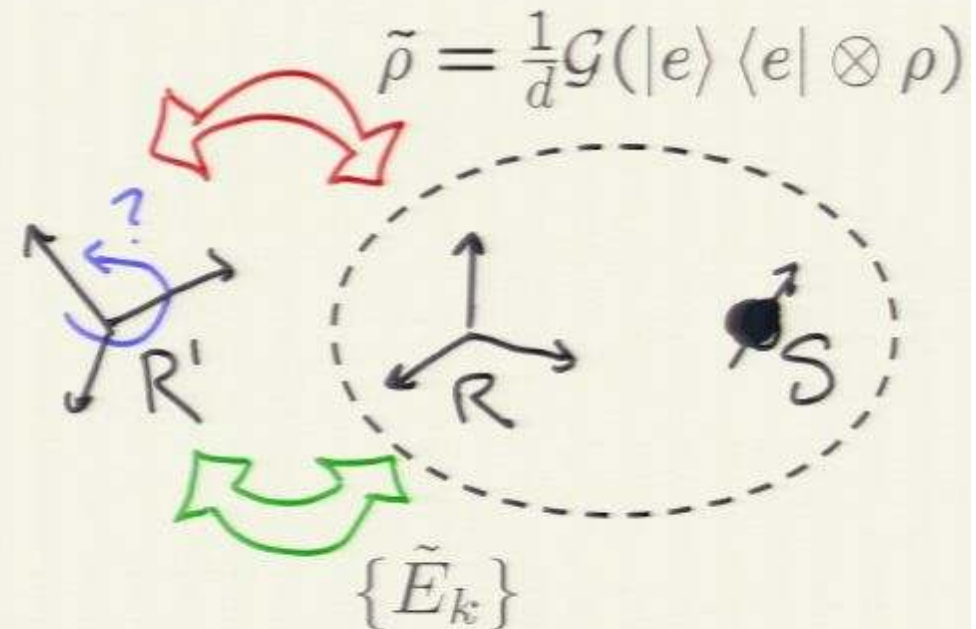
## External RF paradigm



defined on  $\mathcal{H}_S$

$$\text{Tr}_S[\rho E_k]$$

## Internal RF paradigm

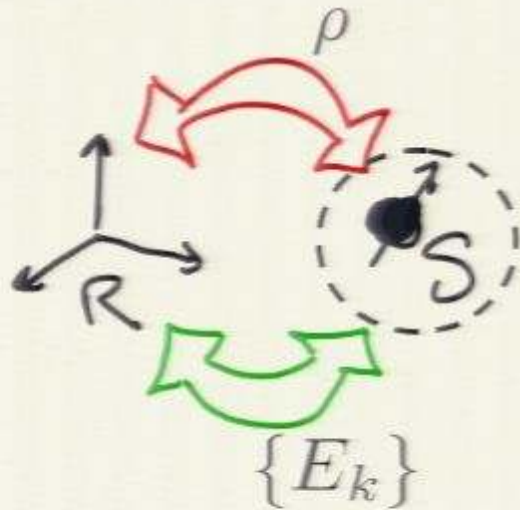


defined on  $\mathcal{H}_R \otimes \mathcal{H}_S$

and rotationally-invariant

$$\text{Tr}_{RS}[\tilde{\rho} \tilde{E}_k]$$

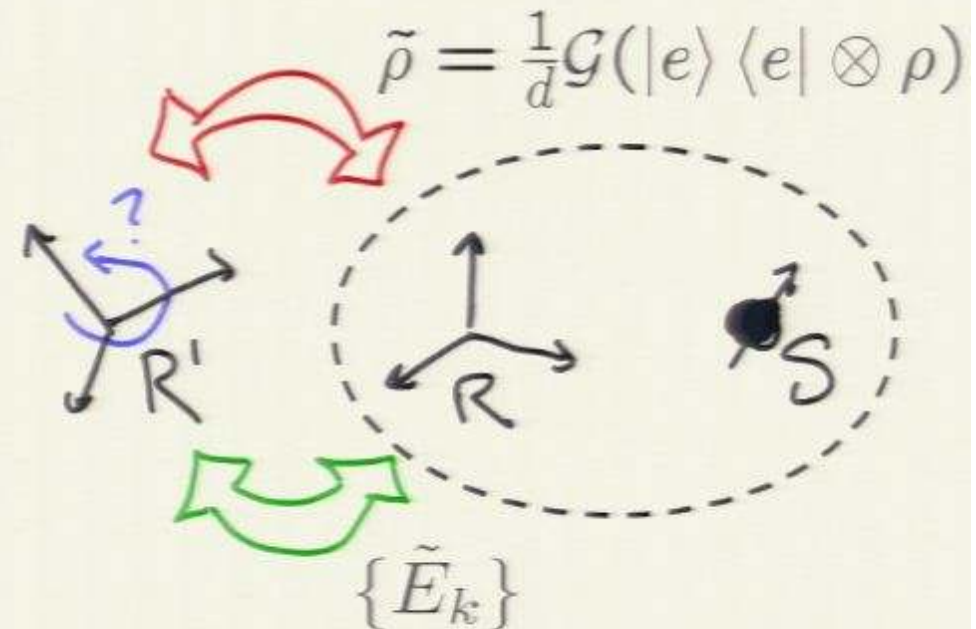
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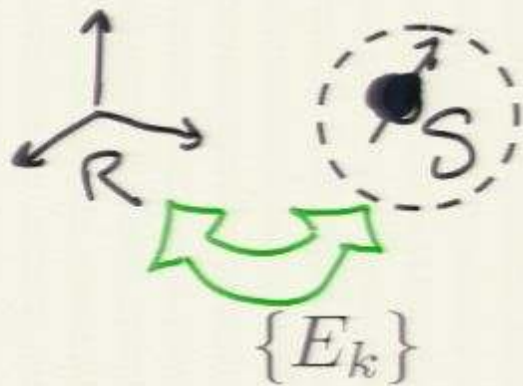
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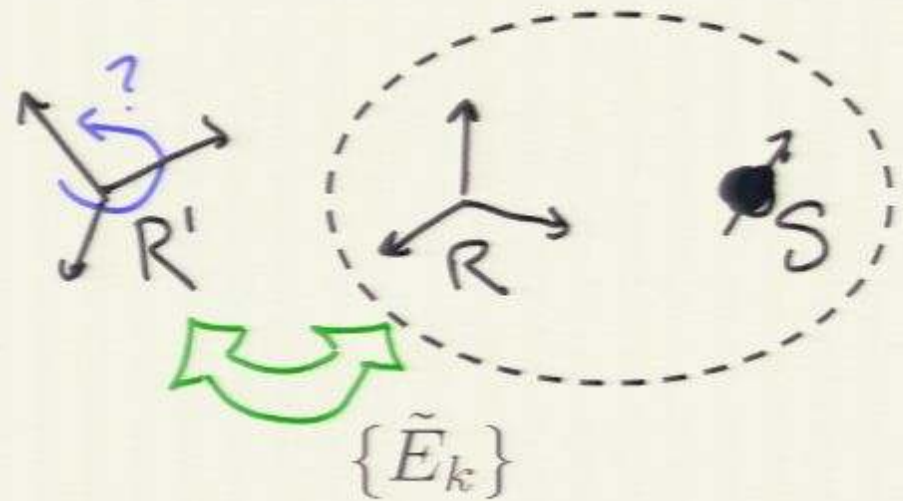
Can we find a measurement scheme such that

$$\text{Tr}_{RS}[\tilde{\rho} \tilde{E}_k] = \text{Tr}_S[\rho E_k] \quad ?$$

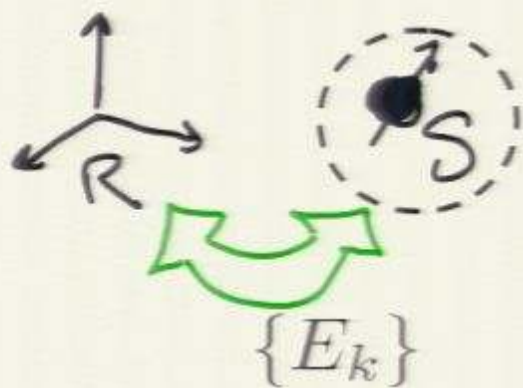
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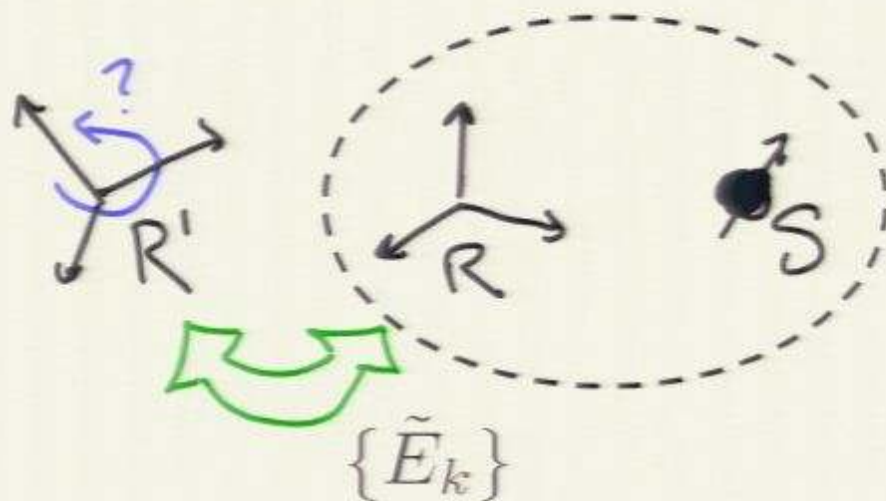
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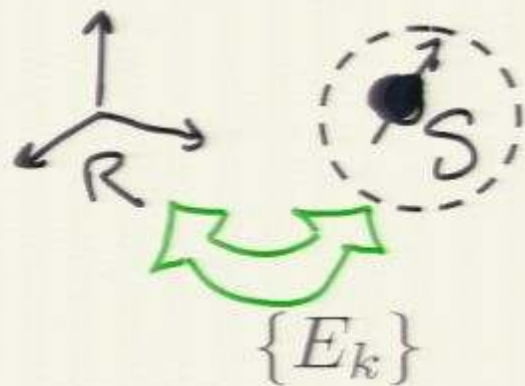
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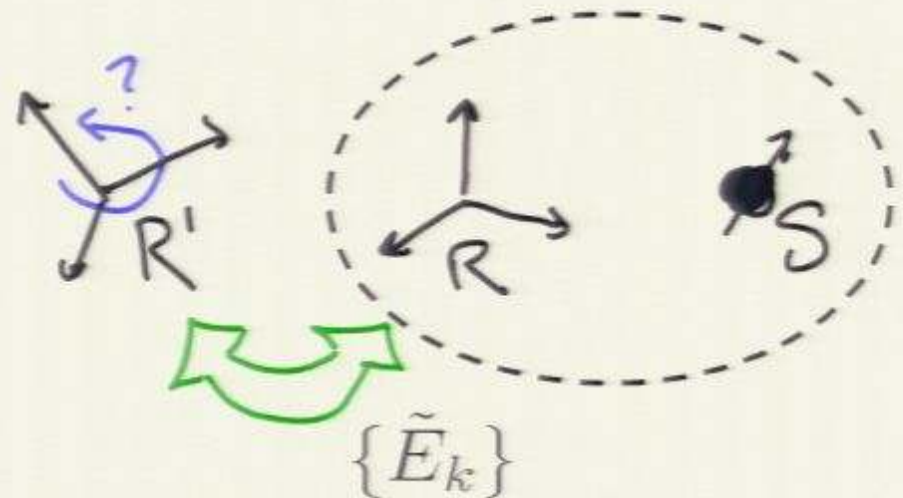
Measure a covariant POVM on R

$$\{U_R(\Omega)|e\rangle\langle e|U_R^\dagger(\Omega)\}_\Omega$$

## External RF paradigm



## Internal RF paradigm



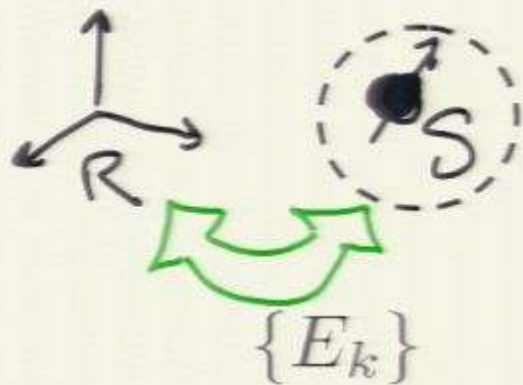
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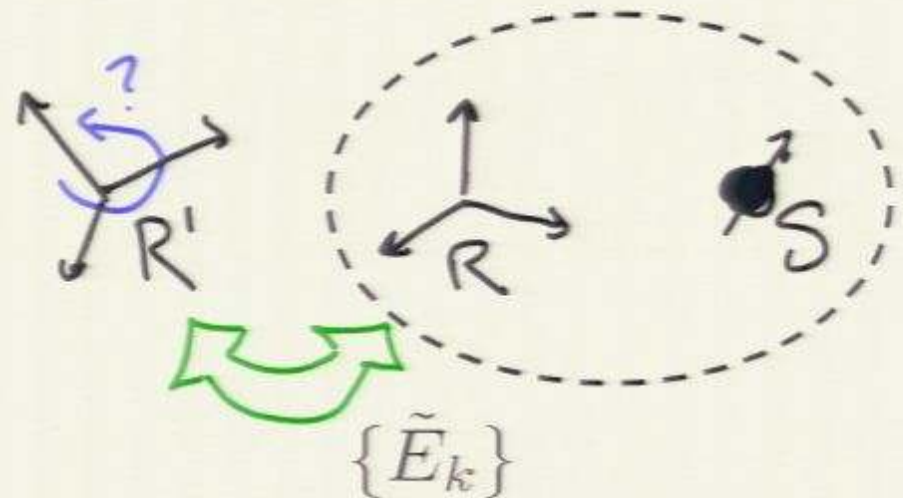
Upon obtaining  $\Omega$ , measure on S

$$\{U(\Omega)E_kU^\dagger(\Omega)\}_k$$

## External RF paradigm



## Internal RF paradigm



Measure a covariant POVM on R

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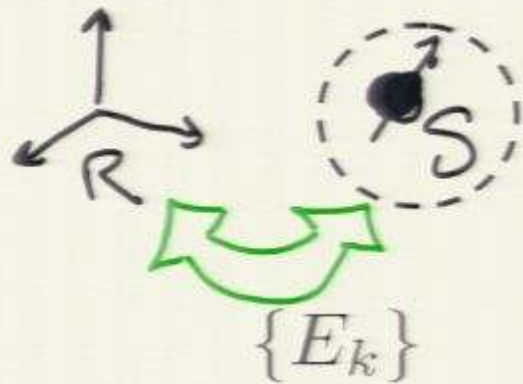
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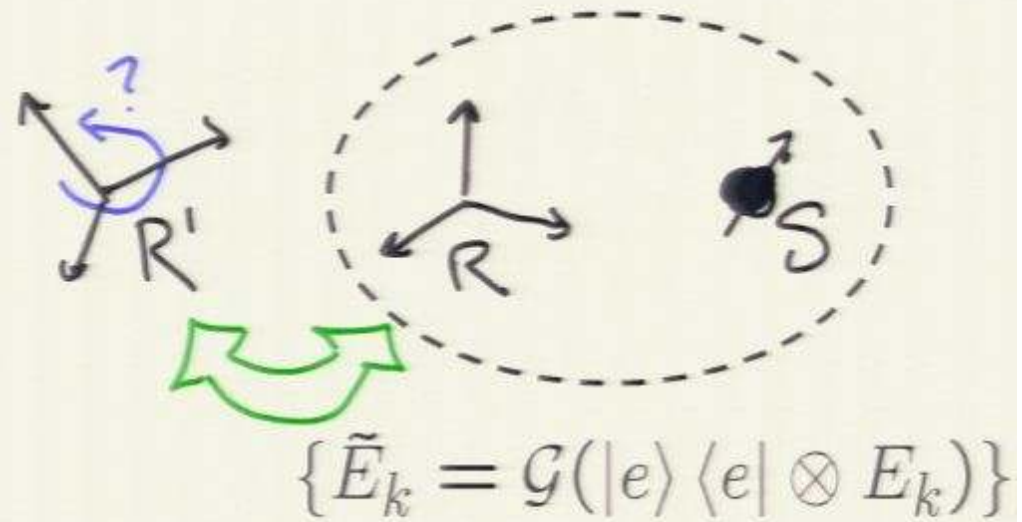
But R' has an unknown orientation

$$\tilde{E}_k = \int d\Omega U_R(\Omega)|e\rangle\langle e|U_R^\dagger(\Omega) \otimes U(\Omega)E_kU^\dagger(\Omega)$$

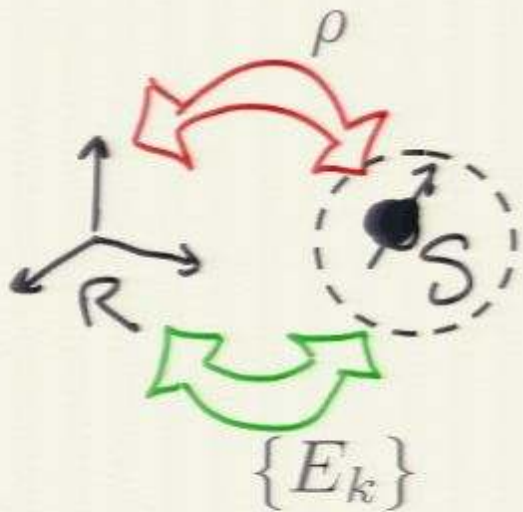
## External-R paradigm



## Internal-R paradigm

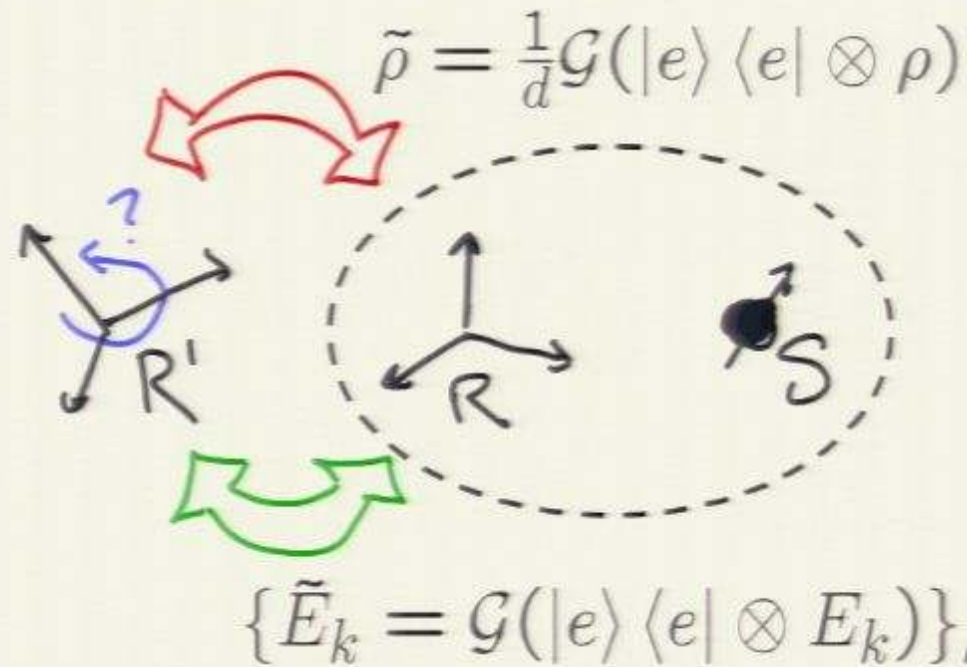


## External RF paradigm



$$\text{Tr}_S[\rho E_k]$$

## Internal RF paradigm

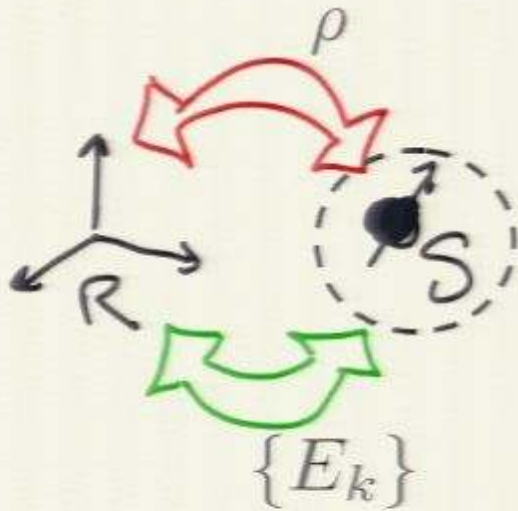


$$\{\tilde{E}_k = \mathcal{G}(|e\rangle \langle e| \otimes E_k)\}$$

$$\text{Tr}_{RS}[\tilde{\rho} \tilde{E}_k]$$

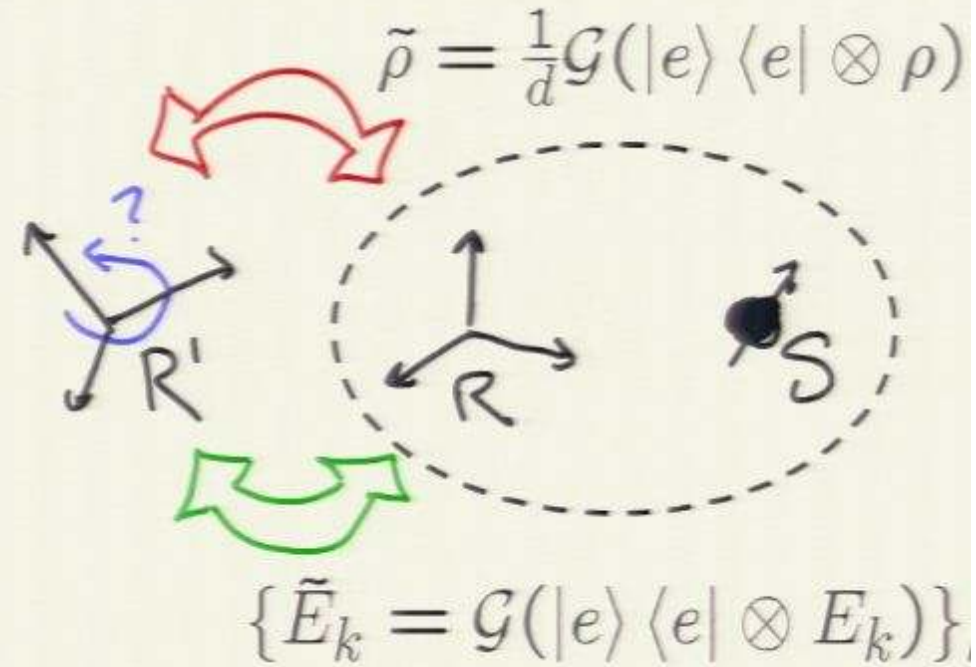
$$\begin{aligned}
& \text{Tr}_{RS}[\tilde{\rho}\tilde{E}_k] \\
&= \text{Tr}_{RS}\left[\frac{1}{d}\mathcal{G}(|e\rangle\langle e| \otimes \rho) \mathcal{G}(|e\rangle\langle e| \otimes E_k)\right] \\
&= \frac{1}{d} \int d\Omega d\Omega' |\langle e|U_R(\Omega\Omega'^{-1})|e\rangle|^2 \text{Tr}_S[U(\Omega\Omega'^{-1})\rho U^\dagger(\Omega\Omega'^{-1})E_k] \\
&= \frac{1}{d} \int d\Omega |\langle e|U_R(\Omega)|e\rangle|^2 \text{Tr}_S[U(\Omega)\rho U^\dagger(\Omega)E_k] \\
&= \frac{1}{d} \text{Tr}_S\left[\int d\Omega |\langle e|U_R(\Omega)|e\rangle|^2 U(\Omega)\rho U^\dagger(\Omega) E_k\right] \\
&= \text{Tr}_S[\mathcal{D}(\rho) E_k]
\end{aligned}$$

## External RF paradigm



$$\text{Tr}_S[\rho E_k]$$

## Internal RF paradigm



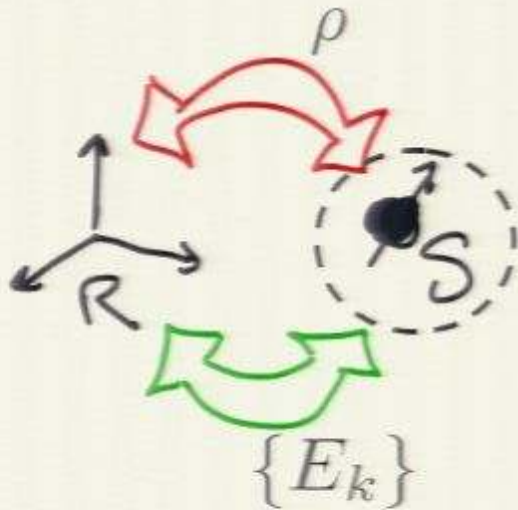
$$\tilde{\rho} = \frac{1}{d} \mathcal{G}(|e\rangle \langle e| \otimes \rho)$$

$$\{\tilde{E}_k = \mathcal{G}(|e\rangle \langle e| \otimes E_k)\}$$

$$\text{Tr}_{RS}[\tilde{\rho} \tilde{E}_k]$$

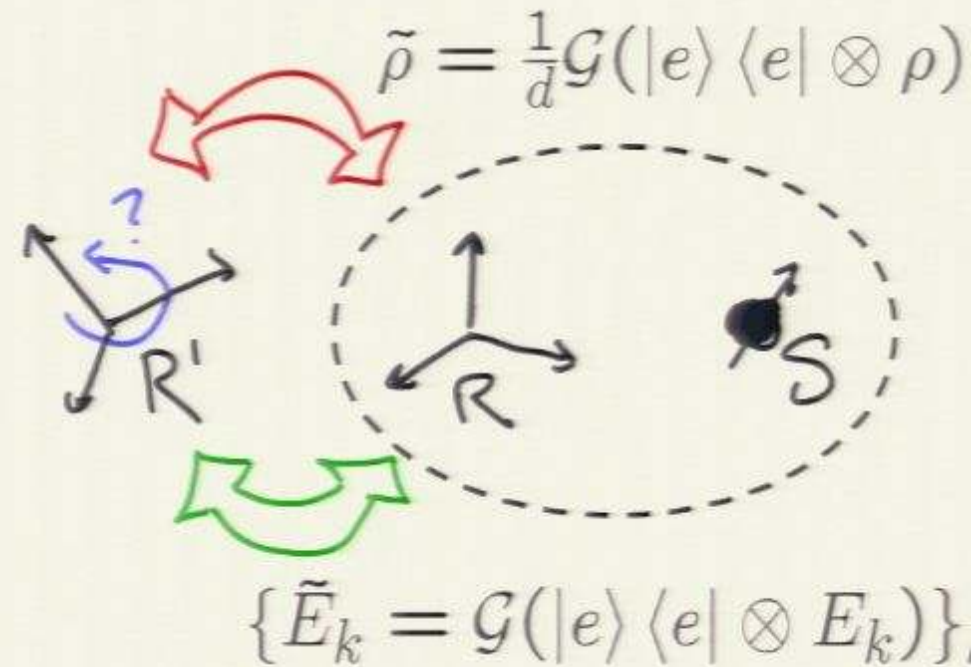
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## External RF paradigm



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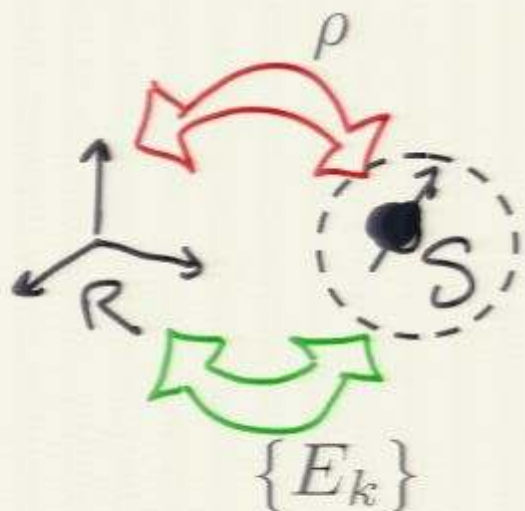
$$\{\tilde{E}_k = \mathcal{G}(|e\rangle\langle e| \otimes E_k)\}$$

$$\text{Tr}_{RS}[\tilde{\rho} \tilde{E}_k] = \text{Tr}_S[\mathcal{D}(\rho) E_k]$$

where

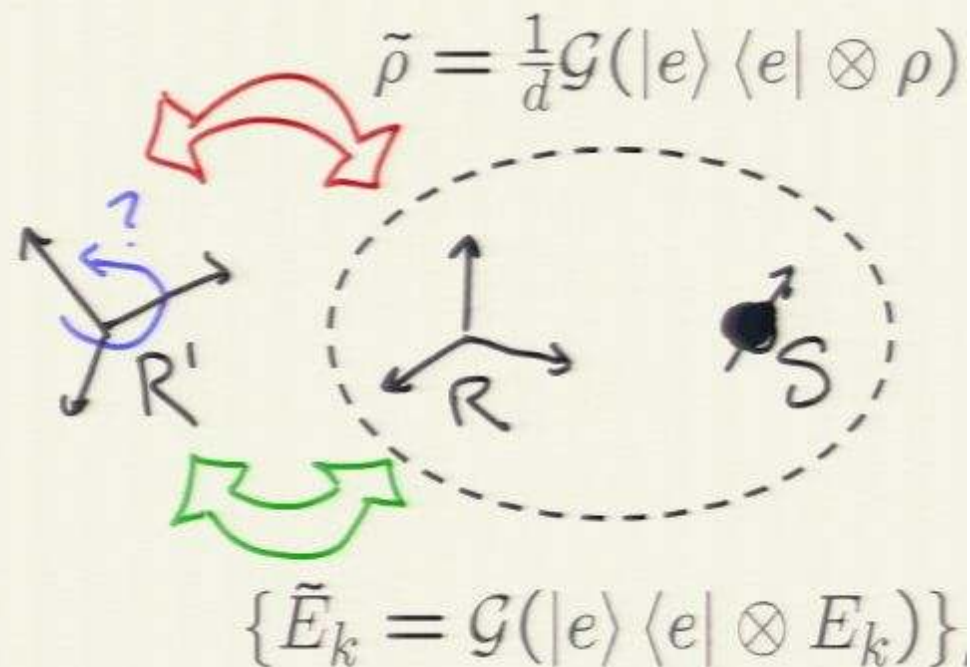
$$\mathcal{D}(\cdot) = \frac{1}{d} \int d\Omega |\langle e|U_R(\Omega)|e\rangle|^2 U(\Omega)(\cdot)U^\dagger(\Omega)$$

## External RF paradigm



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## Internal RF paradigm



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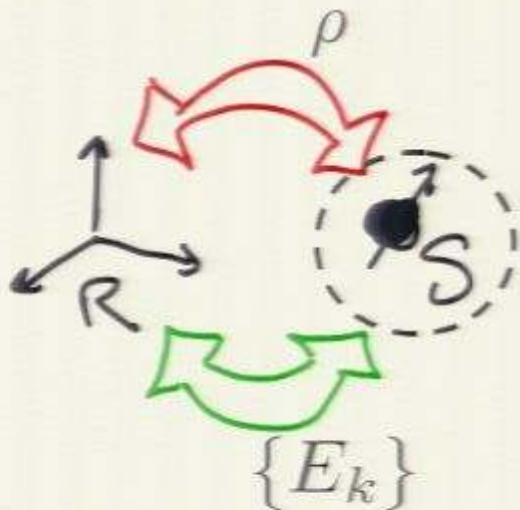
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RF of **unbounded-size**:  $\mathcal{D} = \text{id}$

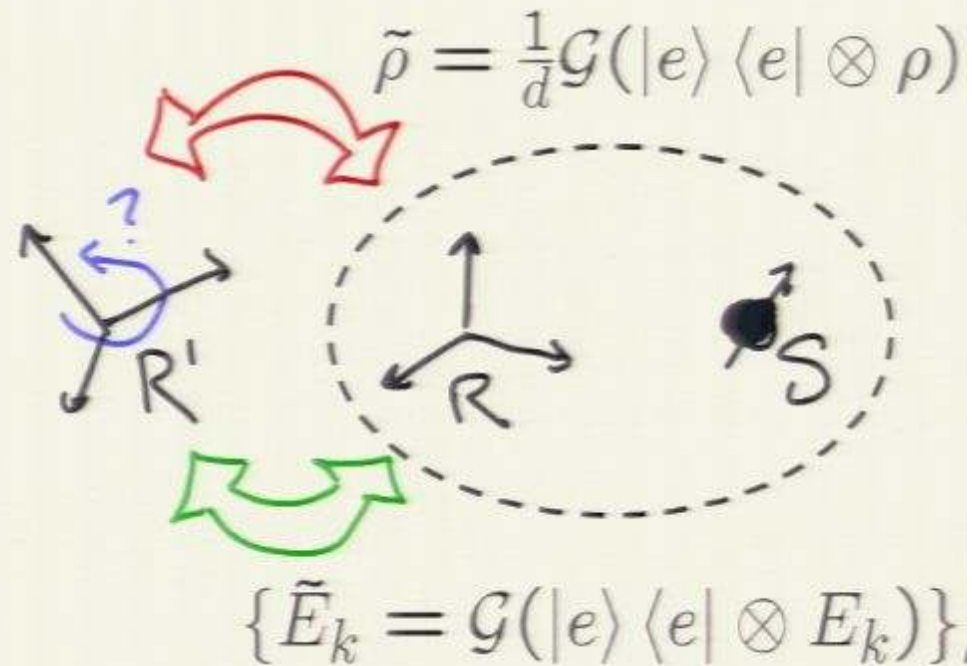
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## External RF paradigm



~~$$\text{Tr}_S[\rho E_k]$$~~

## Internal RF paradigm



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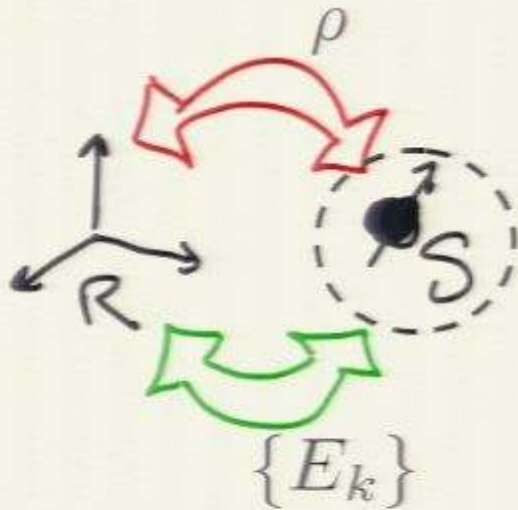
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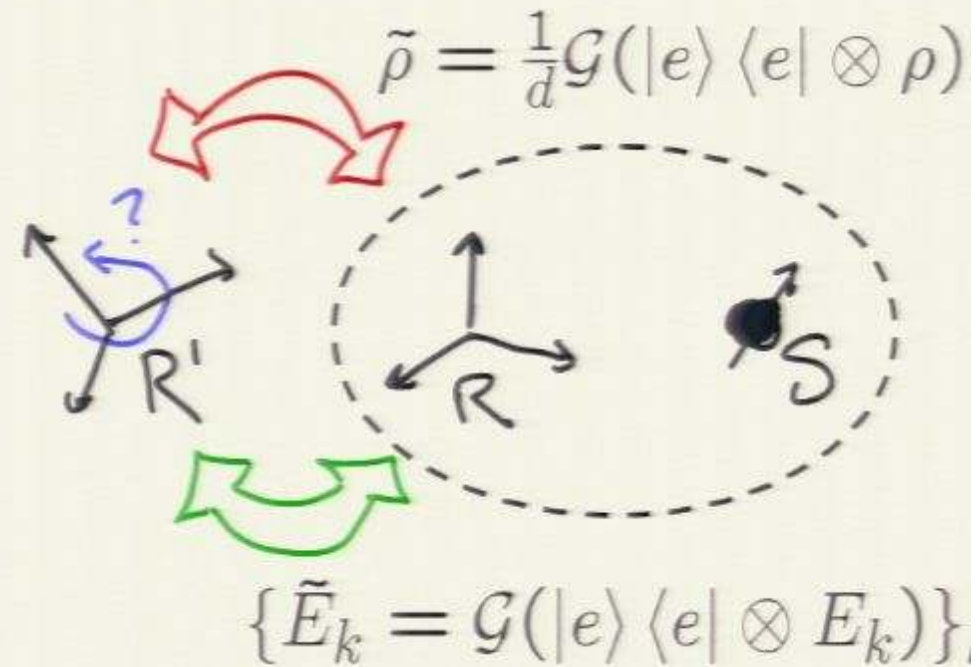
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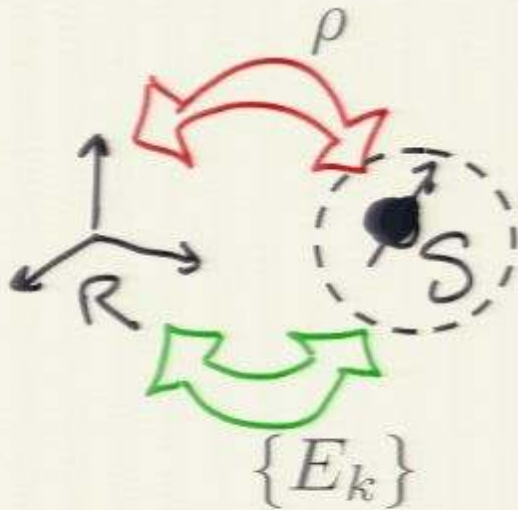
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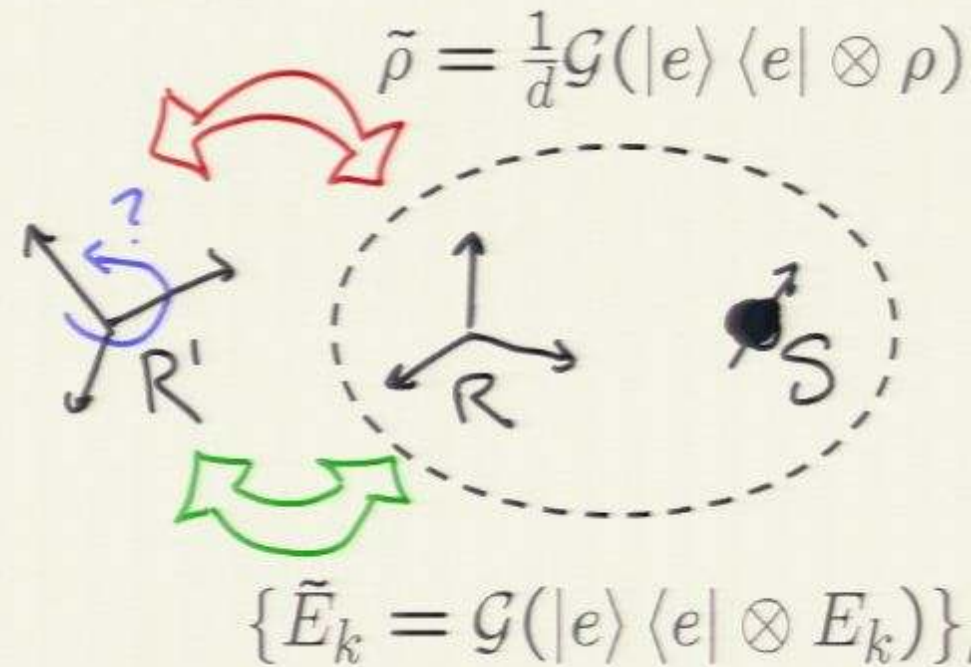
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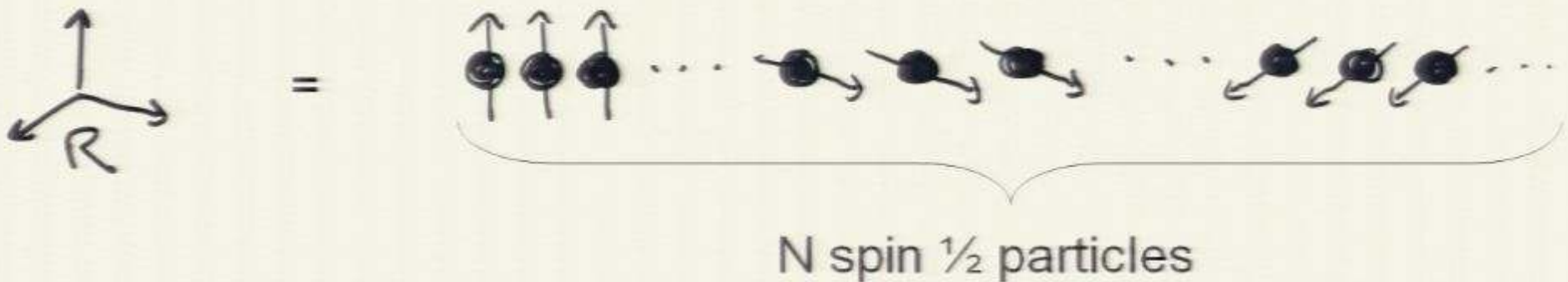
where

$$\mathcal{D}(\cdot) = \frac{1}{d} \int d\Omega |\langle e|U_R(\Omega)|e\rangle|^2 U(\Omega)(\cdot)U^\dagger(\Omega)$$

Bounded-size RF  $\rightarrow$  Effective Decoherence

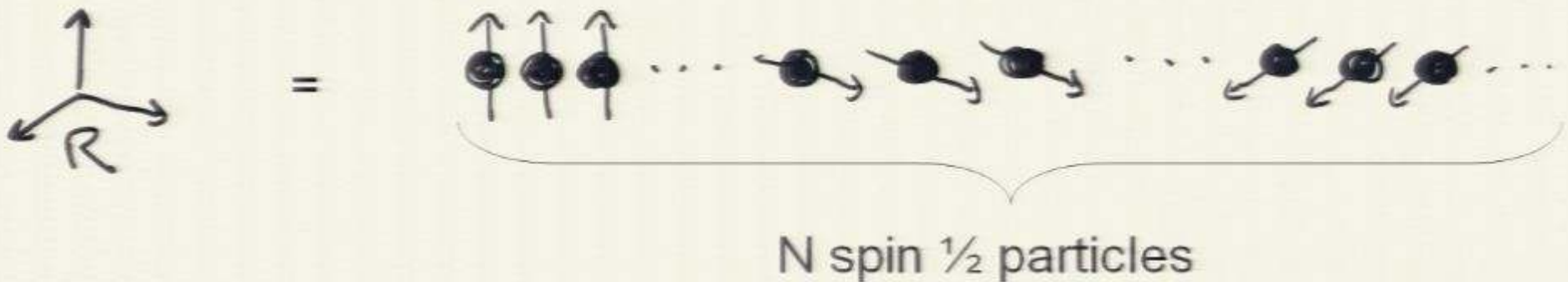
# Bounded-size RF as effective decoherence: an example

For a given RF state, we can determine the decoherence map by treating it internally and thereafter treat it externally



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But this is not the optimal state for  
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Define a new tensor product structure as follows:

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$\text{span}\{|j, m\rangle, m = -j, \dots, j\}$

$\text{span}\{|j, \alpha\rangle, \alpha = 1, \dots, d_{\mathcal{N}_j}\}$

## “Virtual bits” in classical information theory

Standard factorization into Cartesian product of 2 bits

$$(a, b) \in \{0, 1\} \times \{0, 1\}$$

Novel factorization into Cartesian product of 2 bits

$$(a, c) \in \{0, 1\} \times \{0, 1\}$$

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## “Virtual subsystems” in quantum theory

Standard factorization into tensor product of 2 qubits

$$H_A \otimes H_B$$

Novel factorization into tensor product of 2 qubits

$H_C \otimes H_D$  where C and D are **virtual subsystems**

$$|\Phi\rangle_C |+\rangle_D = |\Phi, +\rangle \equiv |0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B$$

$$|\Phi\rangle_C |-\rangle_D = |\Phi, -\rangle \equiv |0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B$$

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$$\underbrace{\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \cdots \otimes \mathcal{C}_2}_N = \bigoplus_{j=0}^{N/2} \mathcal{M}_j \otimes \mathcal{N}_j$$

span $\{|j, m\rangle, m = -j, \dots, j\}$

Representation  
spaces

$$d_{\mathcal{M}_j} = (2j + 1)$$

span $\{|j, \alpha\rangle, \alpha = 1, \dots, d_{\mathcal{N}_j}\}$

Multiplicity  
spaces

$$d_{\mathcal{N}_j} = \binom{N}{N/2 - j} \frac{2j + 1}{N/2 + j + 1}$$

$$\underbrace{C_2 \otimes C_2 \otimes \dots \otimes C_2}_N = \bigoplus_{j=0}^{N/2} \mathcal{M}_j \otimes \mathcal{N}_j$$

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Multiplicity spaces

$$d_{\mathcal{N}_j} = \binom{N}{N/2 - j} \frac{2j + 1}{N/2 + j + 1}$$

For  $N=20$

$j$	$d_{\mathcal{M}_j}$	$d_{\mathcal{N}_j}$
10	21	1
9	19	19
8	17	170
7	15	950
6	13	3705
5	11	10659
4	9	23256
3	7	38760
2	5	48450
1	3	41990
0	1	16796

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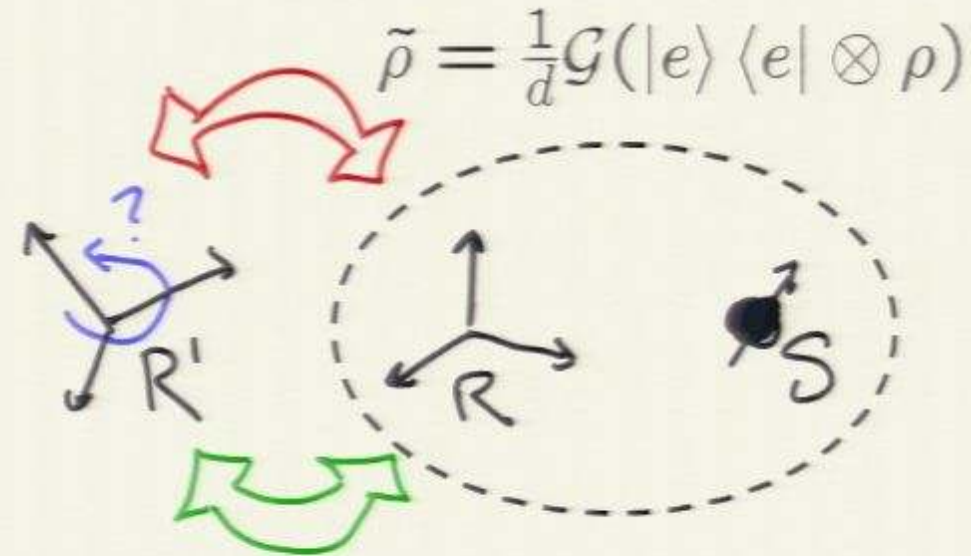
### Near-optimal state for encoding a Cartesian frame

Let  $j_R$  be the largest value of  $j$  such that  $d_{\mathcal{N}_j} \geq d_{\mathcal{M}_j}$

Define:

$$|e\rangle = \sum_{j=0}^{j_R} \sqrt{\frac{2j + 1}{d_R^*}} \sum_{m=-j}^j |j, m\rangle \otimes |\phi_{j,m}\rangle$$

## Internal RF paradigm



$$\{\tilde{E}_k = \mathcal{G}(|e\rangle \langle e| \otimes E_k)\}$$

$$\text{Tr}_{RS}[\tilde{\rho} \tilde{E}_k] = \text{Tr}_S[\mathcal{D}(\rho) E_k]$$

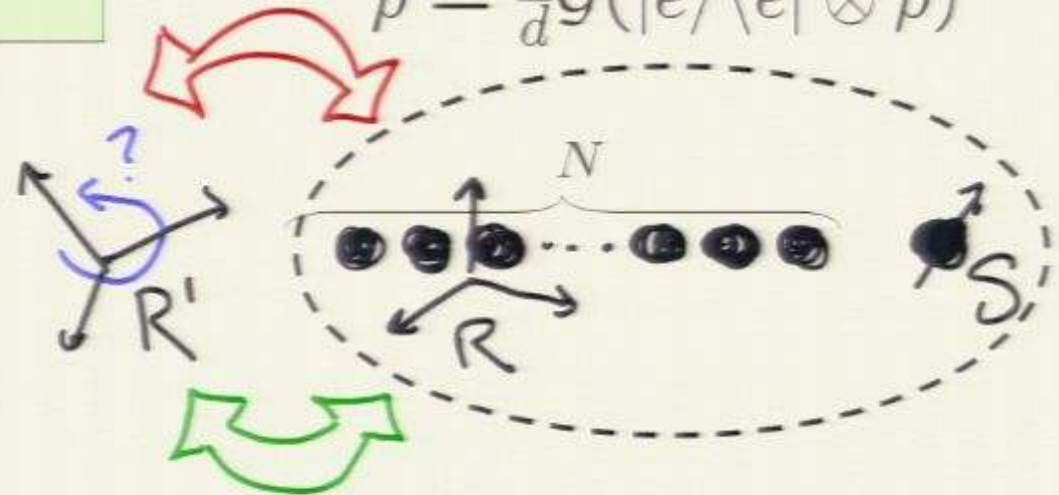
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$$\tilde{\rho} = \frac{1}{d} \mathcal{G}(|e\rangle \langle e| \otimes \rho)$$



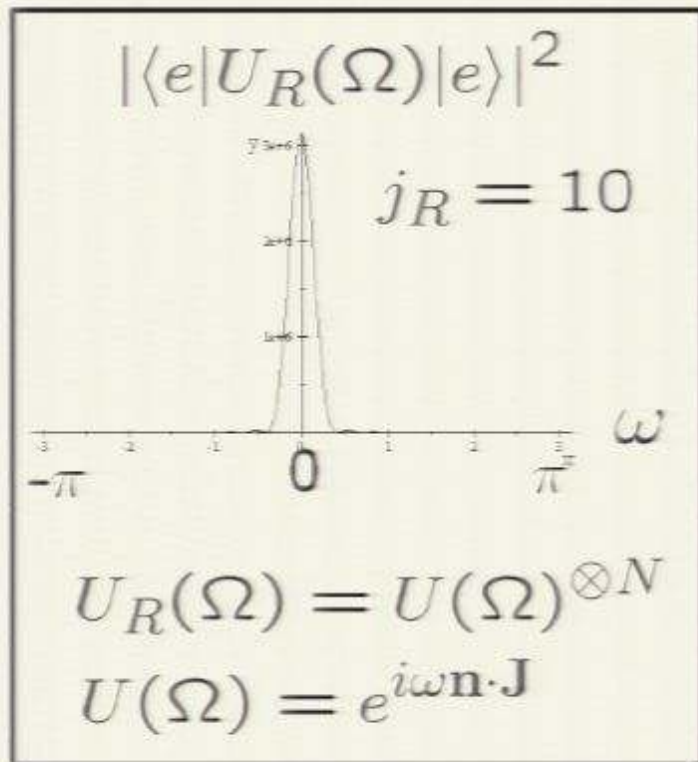
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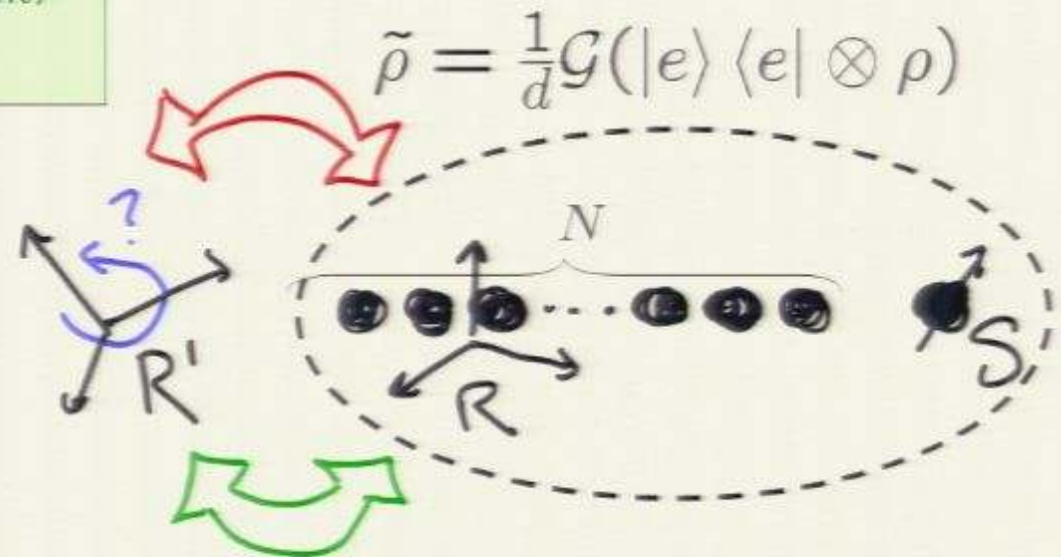
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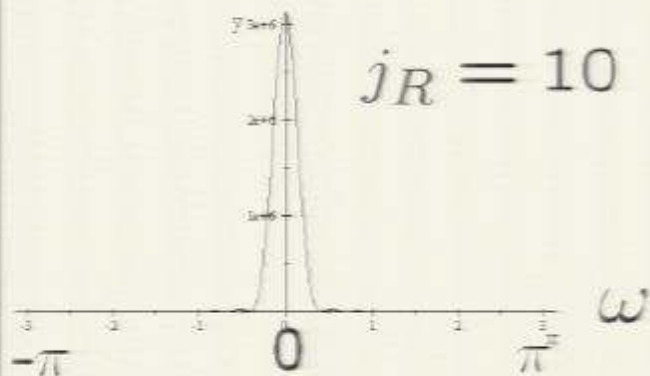
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$$|\langle e|U_R(\Omega)|e\rangle|^2$$

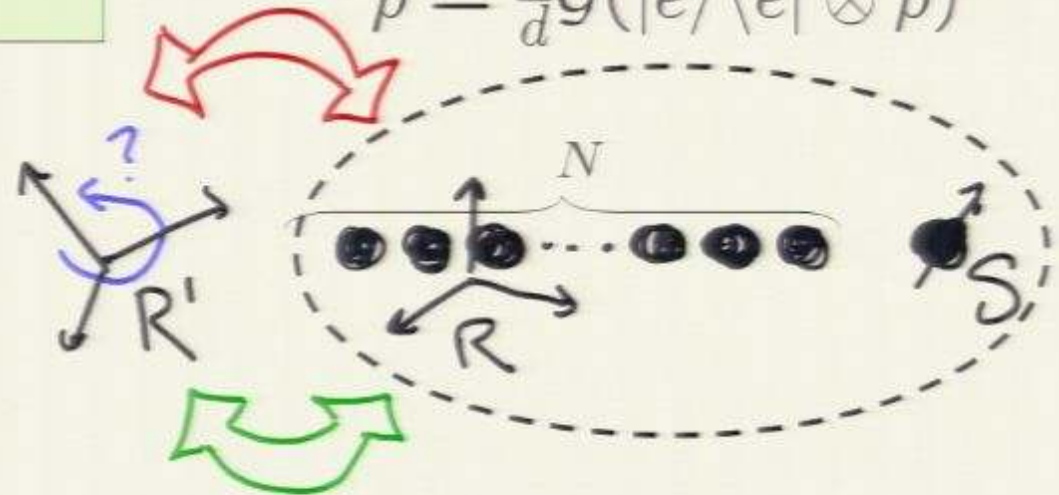


$$U_R(\Omega) = U(\Omega)^{\otimes N}$$

$$U(\Omega) = e^{i\omega \mathbf{n} \cdot \mathbf{J}}$$

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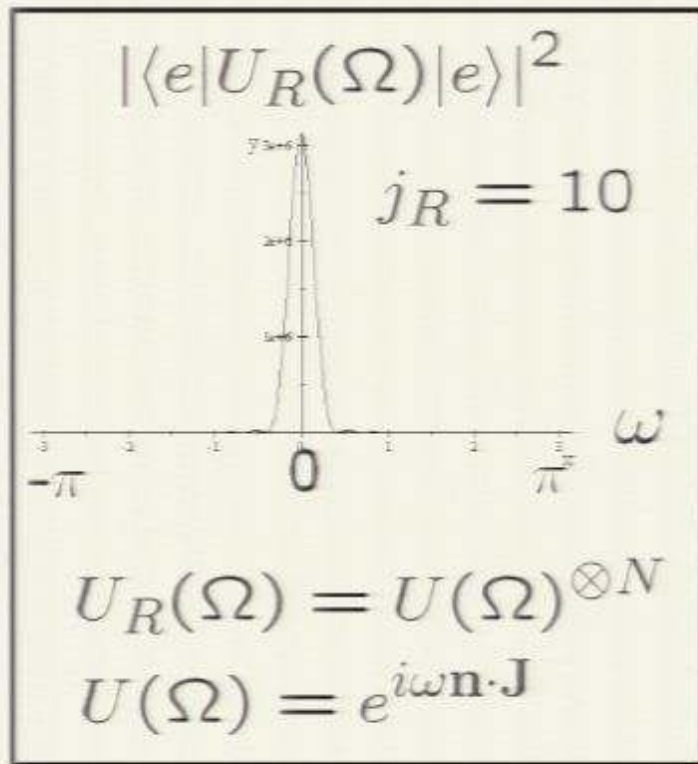
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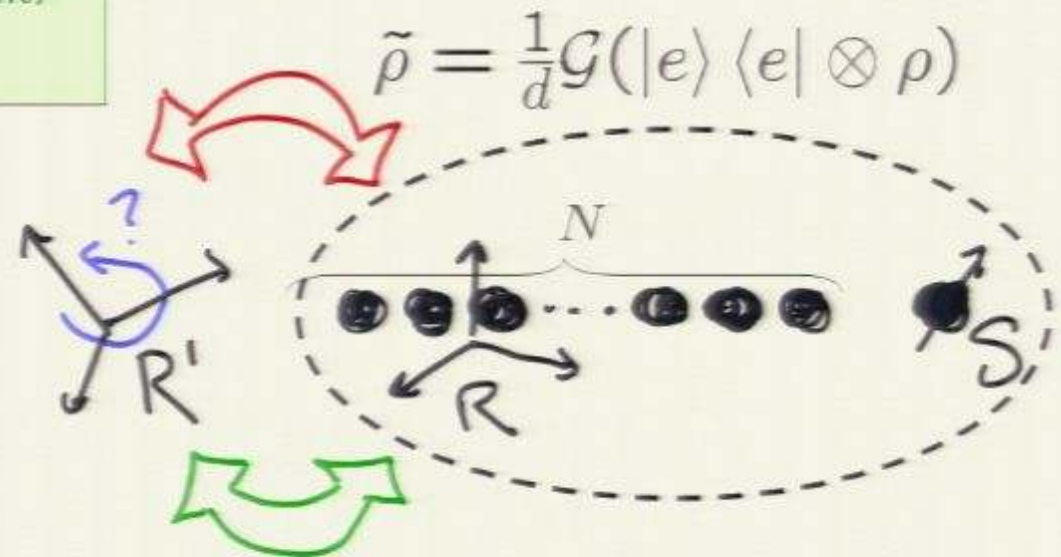
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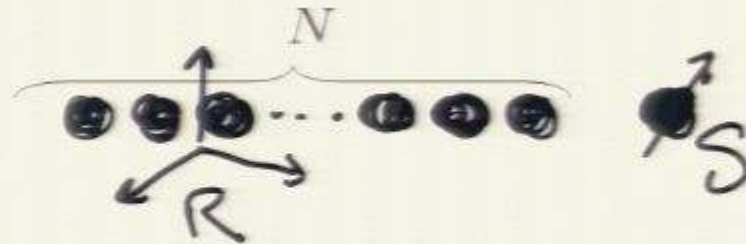
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## Limit of unbounded RF

$$\mathcal{D} \rightarrow \text{id} \text{ as } j_R \rightarrow \infty$$

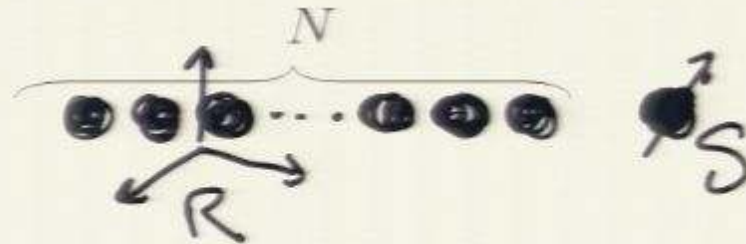
$$\mathcal{D}[\rho] = \left( \frac{j_R}{j_R + 1} \text{id} + \frac{1}{j_R + 1} \mathcal{G} \right) [\rho]$$

Where does the relation between R and S live in the Hilbert space?

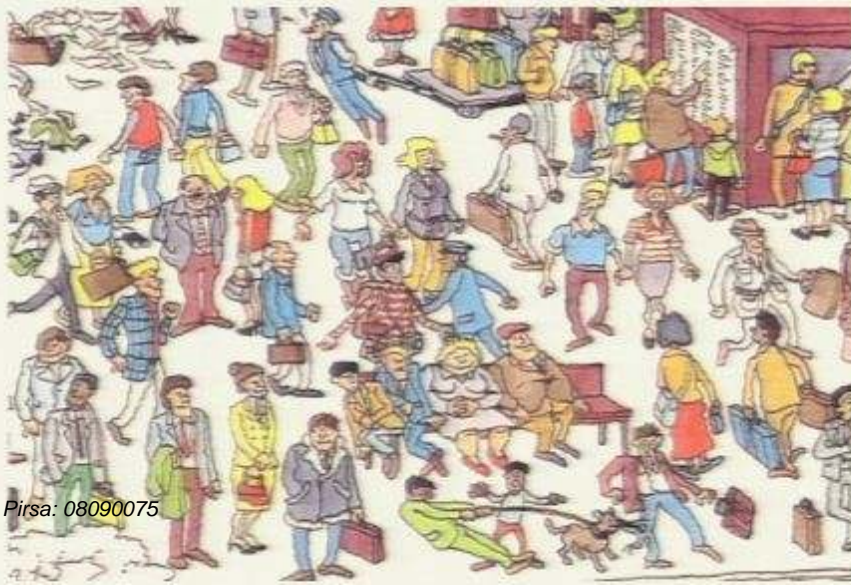


$$\mathcal{H}_{RS} = \underbrace{\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \dots \otimes \mathcal{C}_2}_N \otimes \mathcal{C}_2$$

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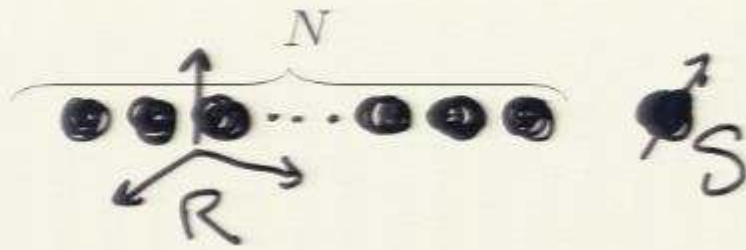
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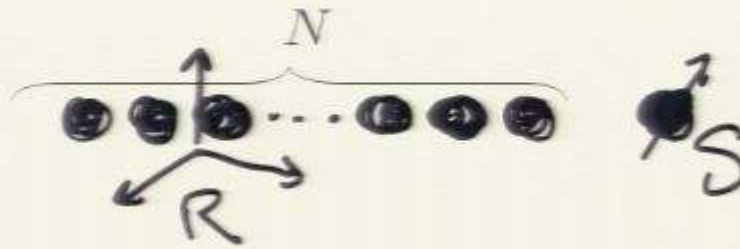
The relational degrees of freedom between R and S are

- Invariant under collective rotations

- Invariant under changes of the internal d.o.f.s of R



$$\underbrace{C_2 \otimes C_2 \otimes \dots \otimes C_2}_{N} \otimes C_2 = \bigoplus_{J=1/2}^{j_R+1/2} \mathcal{M}_{RS}^{(J)} \otimes \mathcal{N}_{RS}^{(J)}$$



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$$\mathcal{H}'_R = \bigoplus_{j=0}^{j_R} \mathcal{M}_R^{(j)} \otimes \mathcal{N}_R^{(j)}$$

Invariant under collective rotations

Relations among components of R

$$\mathcal{N}_{RS}^{(J)} = \begin{cases} \mathcal{N}_R^{(J+\frac{1}{2})} \oplus \mathcal{N}_R^{(J-\frac{1}{2})}, & J < j_R + \frac{1}{2} \\ \mathcal{N}_R^{(j_R)}, & J = j_R + \frac{1}{2}. \end{cases}$$

$$|e\rangle = \sum_{j=0}^{j_R} \sqrt{\frac{2j+1}{d_R^*}} \sum_{m=-j}^j |j, m\rangle \otimes |\phi_{j,m}\rangle$$

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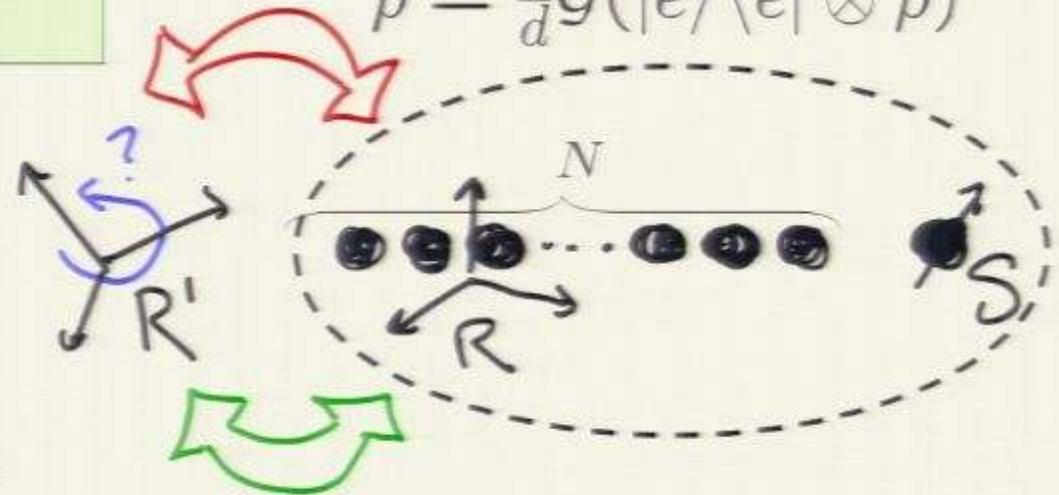
Recall:

$$\mathcal{H}_{RS} = \bigoplus_{J=1/2}^{j_R+1/2} \mathcal{M}_{RS}^{(J)} \otimes \mathcal{N}_{RS}^{(J)}$$

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Internal RF paradigm

$$\tilde{\rho} = \frac{1}{d} \mathcal{G}(|e\rangle \langle e| \otimes \rho)$$



$$\{\tilde{E}_k = \mathcal{G}(|e\rangle \langle e| \otimes E_k)\}_k$$

There is a **perfect copy** of  $\rho$  and  $E_k$  in each  $\mathcal{L}_{RS}^{(\frac{1}{2})}$

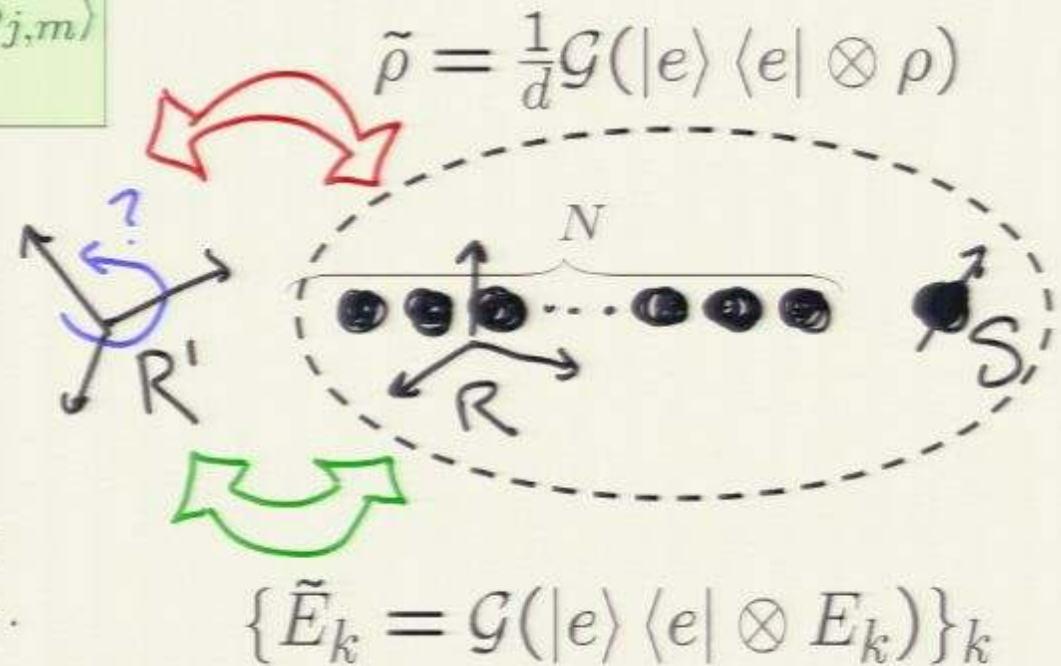
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Internal RF paradigm



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Probabilities of different  $J$

$$p_J = \begin{cases} \frac{(2J+1)^2}{d_{R^*}}, & J < j_R + \frac{1}{2}, \\ \frac{(2j_R+1)(j_R+1)}{d_{R^*}}, & J = j_R + \frac{1}{2}. \end{cases}$$

Different sorts of RFs  $\leftrightarrow$  different groups

The results are similar for other groups

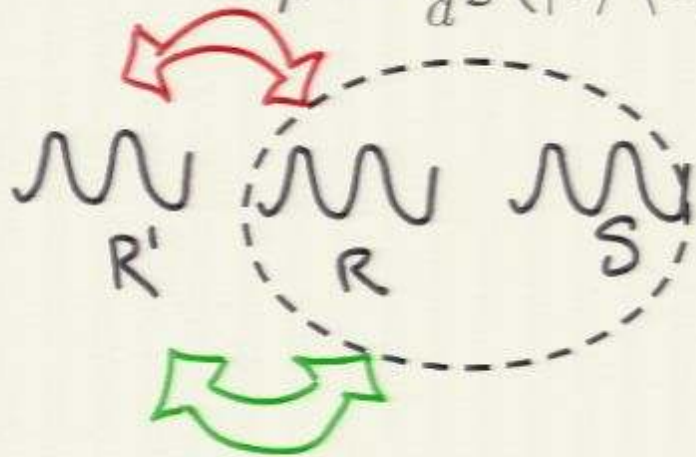
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Ex: bounded-size phase reference

$$\mathcal{G}(\cdot) \equiv \int d\phi U(\phi)(\cdot)U^\dagger(\phi)$$

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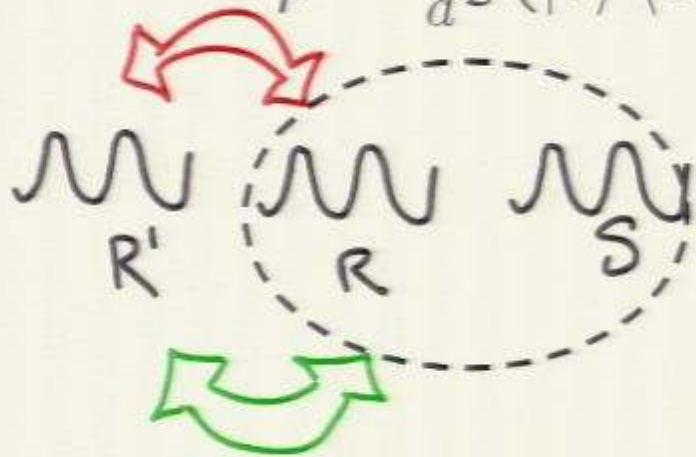
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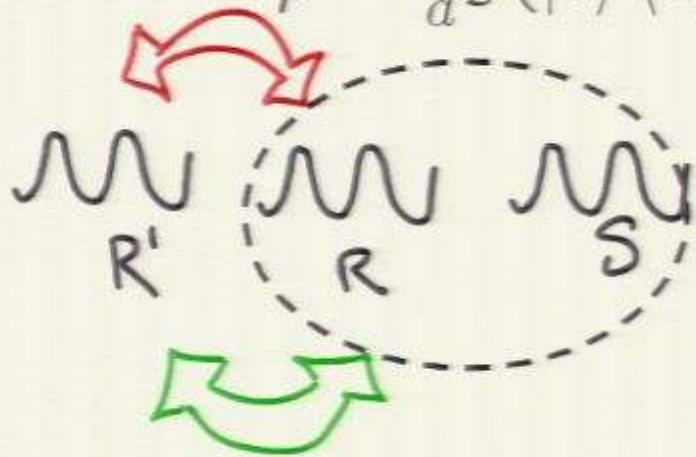
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One finds:

$$\mathcal{D}[\rho] = \left( \frac{N_R}{N_R + 1} \text{id} + \frac{1}{N_R + 1} \mathcal{G} \right) [\rho]$$

# Conclusions

Quantum states describe **extrinsic** as well as intrinsic properties, equivalently, **relational** rather than absolute degrees of freedom

External RF paradigm is just as good as internal RF paradigm

→ One can even model **bounded-size** RF effects

These appear as **effective decoherence** for relational degrees of freedom