Title: Wiggling Hilbert Space

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Abstract: After using the complex Hilbert space formalism for quantum theory for so long, it is very easy to begin to take for granted features like projection operators and the projection postulate, the algebra of observables, symmetric transition probabilities, linear evolution, etc.... Over the past 50 years there have been many attempts to gain a better understanding of this formalism by reconstructing it from different kinds of (sometimes) physically motivated assumptions. By looking at how the above features are motivated and used in different reconstructions, it becomes clear just how special and restrictive many of them are. The question is then what a theory which does not have some of these features looks like. Another interesting question is whether there are any reasons to be suspicious of postulating them in reconstructions or when trying to generalize or apply the quantum formalism to untested situations.

“Wiggling” Hilbert Space

Cozmin Ududec

Sept. 23, 2008

Staring at Hilbert space

- Try to understand:
  - Its mathematical structure
  - Physical meaning
  - Origin

- Understand it by deriving it...
  - Exercise tells you about the structure of theory
  - Insight into going *beyond* it?
  - Sleep well at night!

Overview

- Projection operators and state update rule
- Algebra of observables
- Unitary evolution
- Symmetric transition probabilities
- Tensor product
- ....

?

1) What axioms are sufficient for these structures?
2) Do these features/objects have any operational meaning?
3) What are theories which do not have these features like?
4) Generalizations and modifications?

$$|\langle x | y \rangle|^2 = |\langle y | x \rangle|^2$$

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Reconstructions

- Axiomatic reconstruction/derivation:
  1) a set of (physical) principles/postulates
  2) their mathematical representation
  3) derivation of the formalism of the theory

- Careful with axioms!!
  - Not necessarily ultimate truths about logic/probability/nature
  - Axioms can be deceptive even if they look obvious
  - Don't sleep TOO well

Old and New Approaches

- ~3 old approaches/styles/starting points:
  - Logic of experimental propositions –> quantum logic
  - algebras of observables –> C*-algebras
  - convexity – states and measurement outcomes

- New approaches:
  - Information-theoretic tasks and conditions, optimal reasoning and continuity, composite/sub-system conditions, etc...

- 'generalized probability' theories:
  - Many seemingly quantum features are generic!!

Physics as…

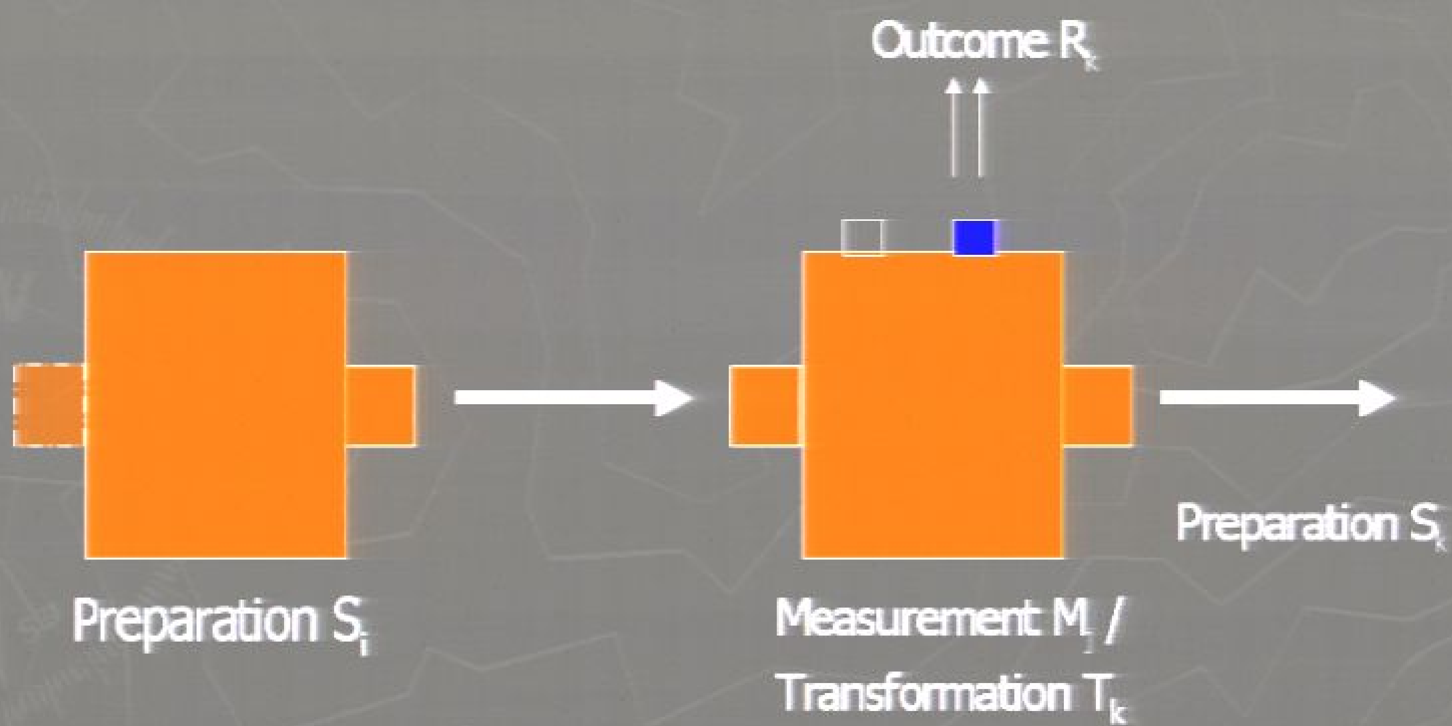
- First goal of any physical theory:
  - Predict measurement outcomes, and correlate experiences
  - Simplify/compress these correlations

- What are the essential features of any physical theory?
  - Look at it from a distance – no Hamiltonians, fields, etc…

Whats left?!

...Little Boxes, and...

- Very common and idealized formalization of what goes on in a lab:
  - S_i is a list of instructions for what to...



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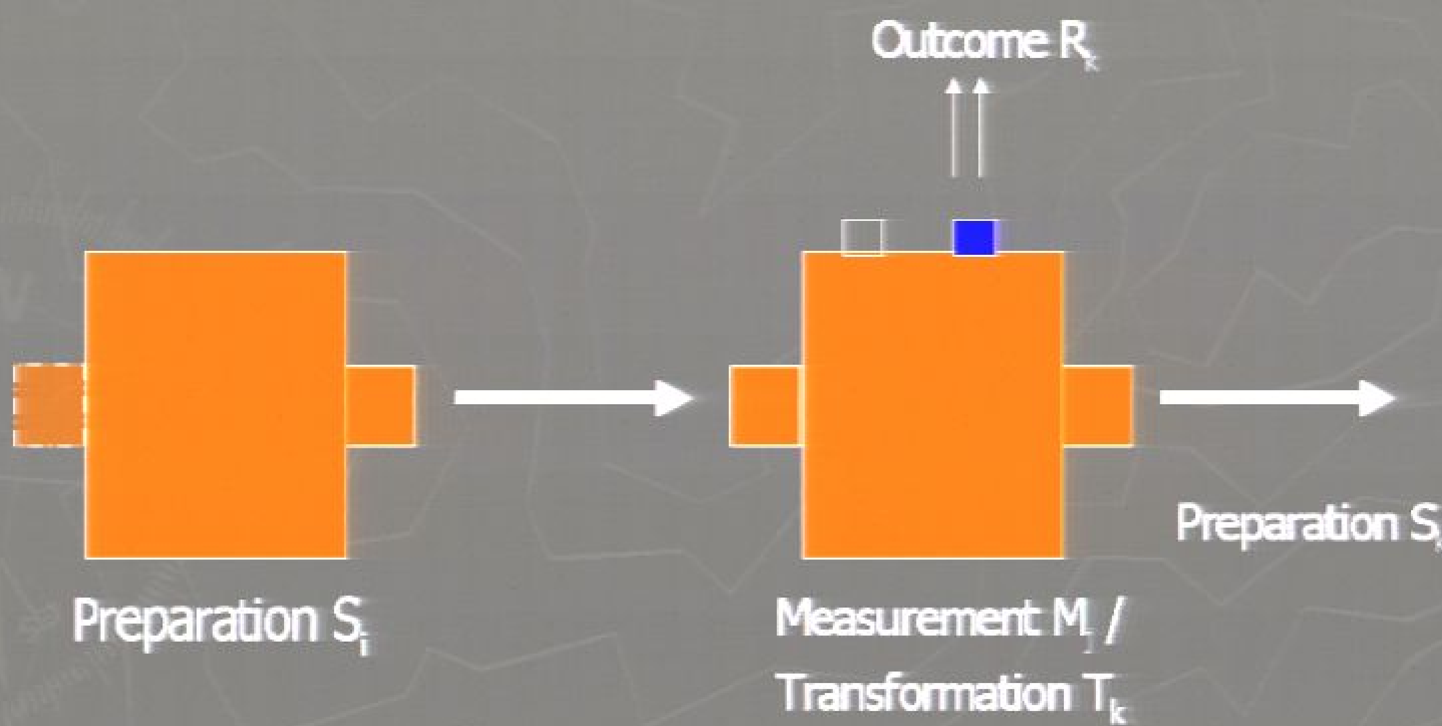


...Probability Data Tables

- S_i=proposition describing some preparation procedure
- R_i=proposition describing 'yes' outcome of measurement M_k
- p_{ij}=probability of outcome

		S_1	S_2	S_3	...
M_1	R_1	p_{11}	p_{12}	p_{13}	...
	R_2	p_{21}	p_{22}	p_{23}	...
	R_3	p_{31}	p_{22}	p_{23}	...
M_2	R_4	p_{41}	p_{22}	p_{23}	...
	...	...	...	...	...

See: Luca Mana – 0403084, 0305117

Compression

- Linear algebra tricks => compression of table into:
  - Given some coordinate (i,j) in the table, we can find p_{ij} as

$$p_{ij} = r_i \, . \, s_j$$

- r_i and s_j are vectors in R^K
  - K is the rank of the matrix
- s_j are elements of *state* space **S**
- r_i are elements of *effect* space **R**
- r_j is a "coordinate" for the (equivalence class of) proposition R_j
- System is "in state s_j" = system prepared by apparatus labelled by vector s_j

Geometry!

- Sets of S vectors and R vectors (as well as transformations) encode the content of the table – the physics!
- Statistics of **anything** leads these kind of structures.
- Qualitative and quantitative features of a system are determined by the geometry of these spaces!

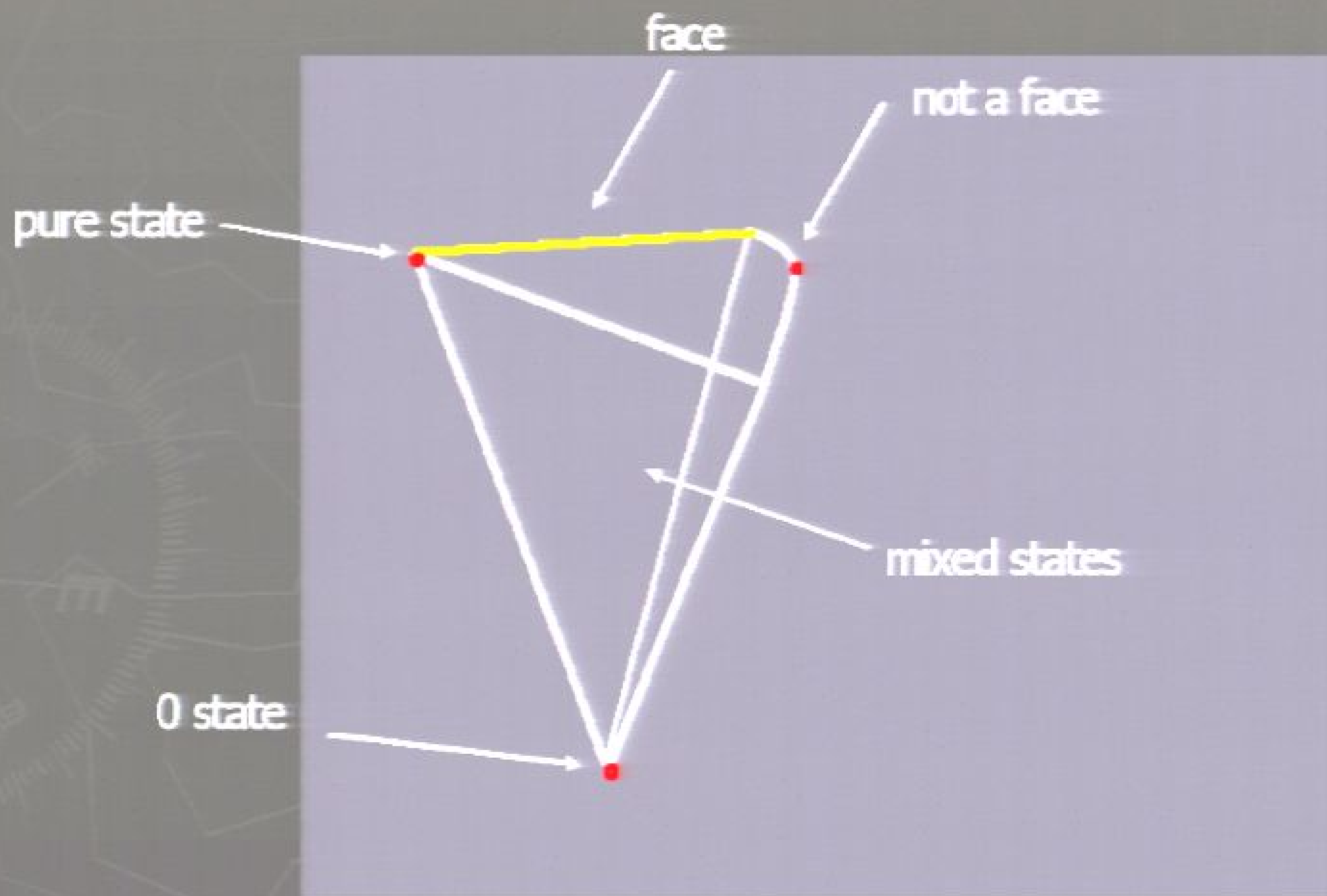
A Bit More Generally

- Physical system modeled in a vector space $\mathbf{R}^k$
  - Un-normalized state space: cone S in $\mathbf{R}^k$
  - Normalized states: S_n in a plane in S
- Outcomes/effects as positive linear functionals on states:
  - Think of $p_{ij} = r_i \,.\, s_j$ as element of R acting on element of S:

    $R=S^*=\{ r \text{ in } \mathbf{R}^k \mid 0<=r(s)<=1 \text{ for all } s \text{ in } S_n \}$
- Order unit: special functional **u** which is 1 on all of S_n, representing "is there a system?"
- Process/transformation/dynamics: positive linear map O: S->S
  - u(Os) : probability that process O occurs

Cones

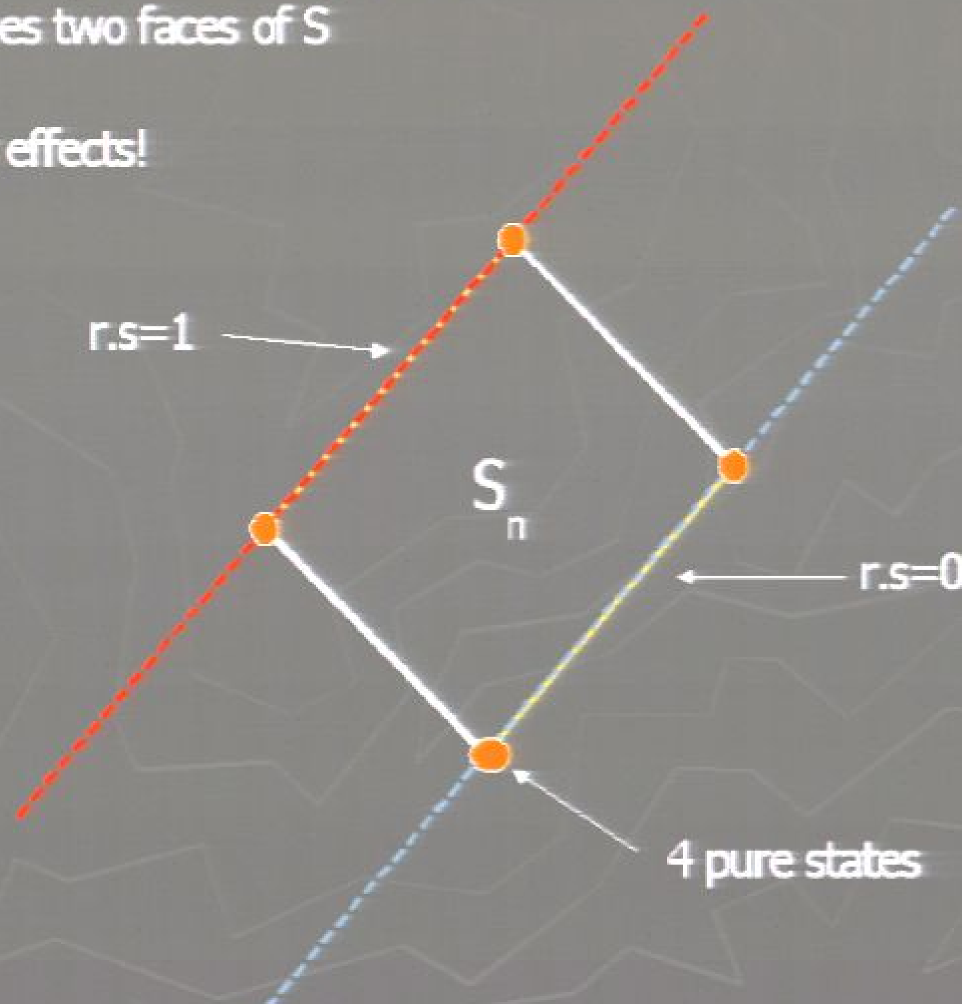
- Cone of states:



Measurement outcomes as planes

Each effect can be thought of as two planes
in S space: determines two faces of S

Note: no algebra for effects!



Filters

- effects <=> operations <=> preparations

- Filter: Operation O_i conditional on some measurement outcome r_i:
  - 1) linear absorbent:
    - $O(s + s') = O(s) + O(s')$ and $u(Os) \le u(s)$
  - 2) neutral to every system to which it is transparent:
    - $u(Os) = u(s) \Rightarrow Os = s$
  - 3) idempotent: $O^2 = O$

- $r_i(s) = u(O_i s)$ – many O_i can give same r_i

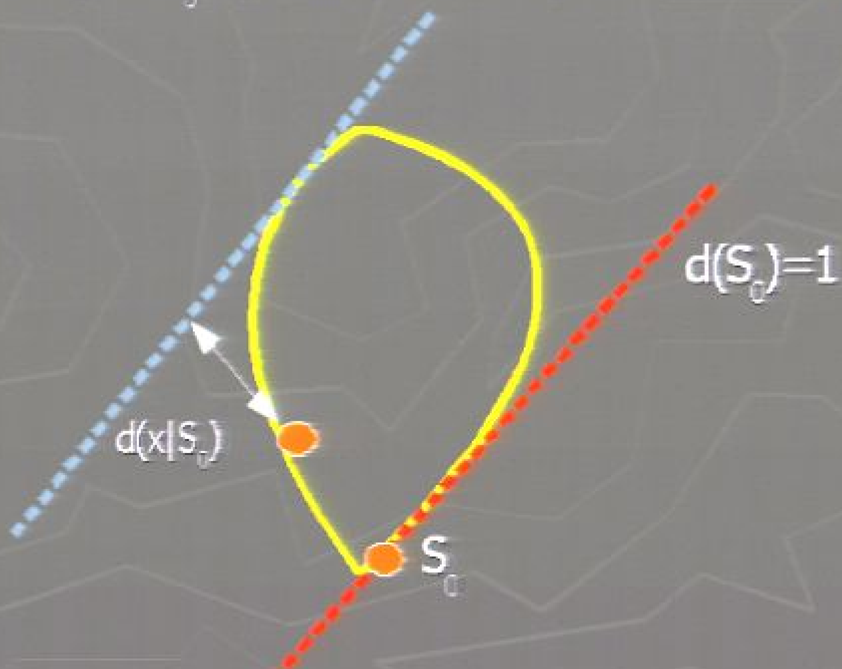
- Filters determine the faces of S: $B_O = \{s \mid Os = s\}$

Detection ratios and transition probabilities

- Take a face S_0 and take $D(S_0)= \{$all $d \mid d(S_0)=1 \}$

- Define detection ratio: $d(x|S_0) = \inf \{ d(x) \mid d$ is in $S_0 \}$

- Any detector registering all S_0 systems, must register <u>at least</u> $d(x|S_0)$ of x systems



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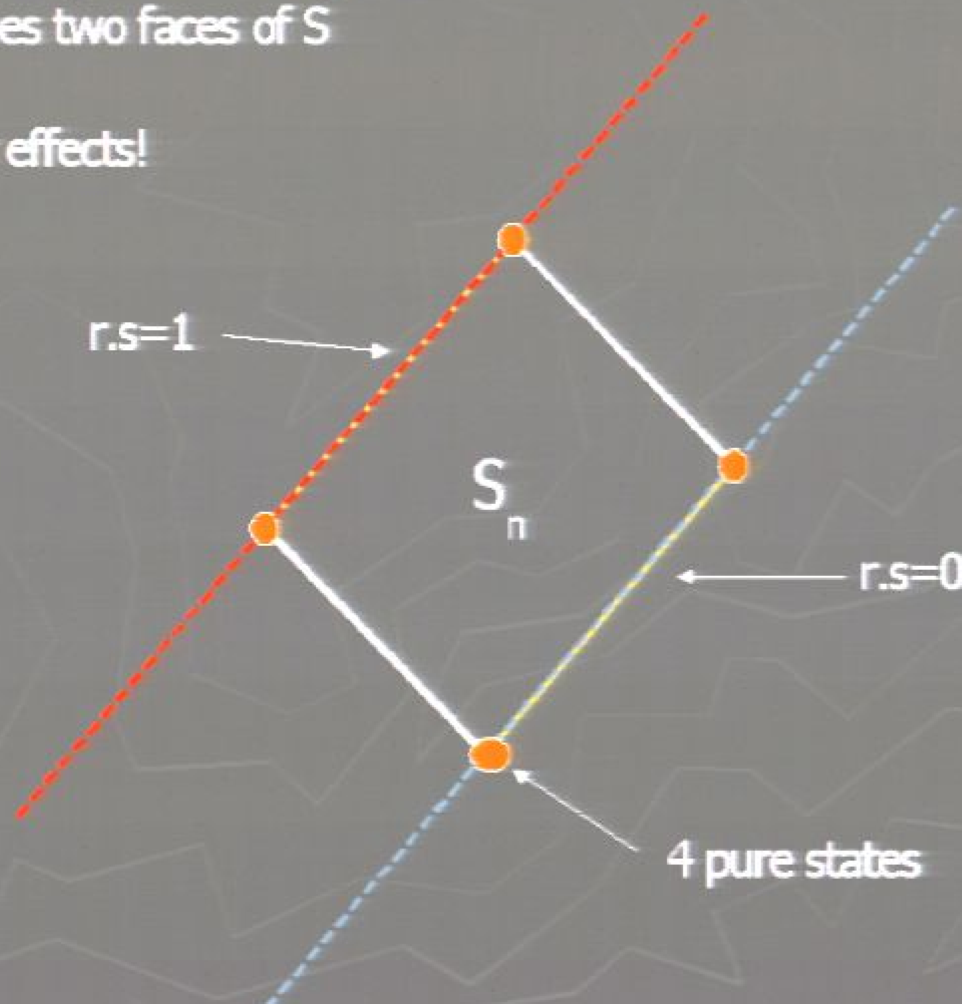

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Impossibility Principle

- Quantum case: Pure state $|a\rangle$, and effect E such that:
  - $Tr[E |a\rangle\langle a|] = 1$
  - For all other states $|b\rangle$, $Tr[E |b\rangle\langle b|] >= | \langle a|b\rangle|^2$
  - Minimal probability of confusing two states
- Hilbert space angle is coded in operational geometry!
- ex: linear and circular polarization of light
- Arent transition probabilities usually associated with dynamics??

Transition Probabilities?

- Take two pure states x, y, and the filters $\{a_x\}$ and $\{a_y\}$
  - a_x, a_y are neutral to x and y, ie: they "verify" that the particles are in the respective states
- Let state x pass through filter a_y
  - $u(a_y x)$?=? transition probability $Pr(y|x)$
- Quantum case: 1-1 relationship between pure states x, and filters a_x identifying them!
- What if there are many a_x ?
- What if there is no unique minimum?
- Is $Pr(y|x)$ symmetric?

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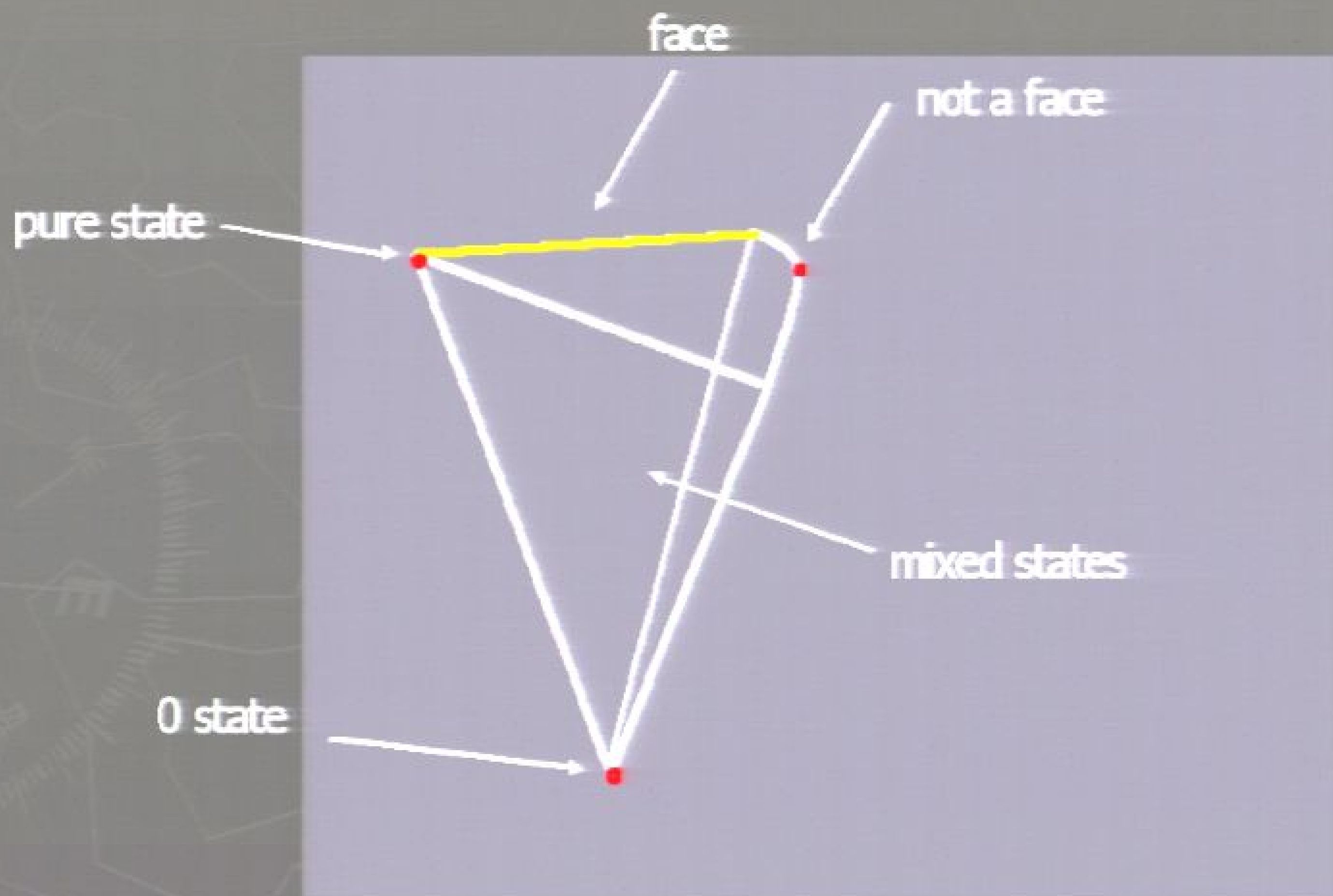

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Haag axioms

- H1) There exits a 1-1 correspondence between pure sources/preparations and pure detectors/effects such that:
  - a) for every pure state s, there exists a pure effect r_s such that:

    $r_s(s)=1$, and $r_s(s')<1$ for all s′ different from s
  - b) $r_s(s')= r_{s'}(s)$

- H2) S is homogeneous, ie: for all s, s′ there is an O with Os=s′

What does this mean?

- Pure states can be prepared and identified!
- probability of some dynamical process?
  - symmetry of corresponding evolution process
- conditional probability?
  - given some information, what is the probability of verifying some other information?
- What if we drop some axioms??

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Example 1

- Box theory:
  - Not strictly convex, only 4 pure states
- Given pure state x, the effect identifying it is not pure!
  - TWO pure functionals for each pure state (almost identifying)
- no unique transition probability between 2 pure states!
  - Detection ratio between any two pure states is 0

Example 2

- Non-elliptic theory:
  - strictly convex, with smooth boundary
- H1a) satisfied
- Asymmetric transition probabilities!
  - Not describable in C*-algebra framework

Consequences

- Theorem: An irreducible space of normalized states S_n satisfies the above axioms if and only if it is a state space of a finite dimensional simple, real Jordan-Banach algebra.

- Jordan algebra: Algebra over a real vector space, such that:
  - 1) xy = yx - commutativity
  - 2) (xy)(xx)=x(y(xx)) – special associativity

1) Self adjoint NxN real matrices
2) Self adjoint NxN complex matrices
3) Self adjoint NxN quaternionic matrices
4) Self adjoint 3x3 octonionic matrices
5) S^n - spin factors

- See Alfsen and Shultz

Geometry again

- Complex projective space:
  - Special symplectic manifold with Riemannian metric
  - Constant (holomorphic sectional) curvature
- Schroedinger evolution = restricted Hamiltonian evolution
- Unitary maps = Symplectic and orthogonal
- Suggests natural modification: non-constant curvature!

- See Ashtekar 9706069

Comments and Questions

- Status of axioms?
  - CBH and information theoretic axioms...
- Why is QT so rigid and symmetric?
- Is there a theory with a strictly convex, smooth boundary, which satisfies $K=N^d$?
- Is it possible to modify these structures while not going back to a classical space?
  - Asymetric transition probabilities?
  - Curvature?

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