Title: Bipartite states of low rank are almost surely entangled

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Abstract: There are many results showing that the probability of entanglement is high for large dimensions. Recently, Arveson showed that the probability of entanglement is zero when the rank of a bipartite state is no larger than half the dimension of the smaller space. We show that that the probability of entanglement is zero when the rank of a bipartite state is no larger than half the maximum of the rank of its two reduced density matrices. Our approach is quite different from that of Arveson and uses a different measure. But both approaches show that the separable states lie in a lower dimensional manifold given a reasonable parameterization of the separable states. This is joint work with Elisabeth Werner, using on characterizations of the extreme points of qubit channels given by Ruskai, Szarek and Werner and the extreme points of entanglement breaking channels given by M. Horodecki, Shor and Ruskai.

Bipartite States of Small Rank are Almost Surely Entangled

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joint with Elsiabeth Werner (CWRU)

Supported by U.S. National Science Foundation

Most results on prob of entang show high if dim suff large

- G. Aubrun and S. J. Szarek

Tensor products of convex sets & volume of separable states on N qudits
Phys. Rev. A. 73, 022109 (2006) arXiv:quant-ph/0503221

- S. J. Szarek, *Phys. Rev.* A 72, 032304 (2005), quant-ph/0310061

The volume of sep (qubit) states is super-doubly-exponentially small

- L. Gurvits, H. Barnum *Phys. Rev.* A quant-ph/0409095

Better bound on exponent of the radius of the multipartite sep. ball

Phys. Rev. A 68, 042312 (2003) quant-ph/0302102

Separable balls around max. mixed multipartite quantum states

Phys. Rev. A 66, 062311 (2002) quant-ph/0204159

Largest sep. balls around the max. mixed bipartite quantum states

- P. Hayden, D.W. Leung, A. Winter, 'Aspects of generic entanglement

Comm. Math. Phys. 265, 95-117 (2006), quant-ph/0407049

Results here $\forall d$ if rank suff small

can assume w.l.o.g. that dim $\mathcal{H}_A \geq$ dim $\mathcal{H}_B$

Arveson: (arXiv:0712.4163) showed $\mathrm{rank}(\gamma_{AB}) \leq \frac{1}{2}\,\mathrm{rank}(\gamma_B)$

$\Rightarrow \gamma_{AB}$ almost surely entangled indep of dim

Claim: • $\mathrm{rank}(\gamma_{AB}) < \mathrm{rank}(\gamma_A) \Rightarrow \gamma_{AB}$ entangled

probably known – easy consequence of well-known results

• $\mathrm{rank}(\gamma_{AB}) = \mathrm{rank}(\gamma_A) \Rightarrow \gamma_{AB}$ almost surely entangled

follows from results in RSW and HSR

Ruskai, Szarek and Werner, A Characterization of CPT Maps on M_2 *Lin. Alg. Appl.* 347, 159187 (2002), quant-ph/0101003

M. Horodecki, Shor and Ruskai, Entanglement breaking channels *Rev. Math. Phys.* 15 ,629641 (2003), quant-ph/030203

First consider $\mathrm{rank}(\rho_{AB}) < d_A$ and $\rho_A = \frac{1}{d_A} I_A$

Reduction criterion for separable $\rho_A \otimes I_B \geq \rho_{AB}$, or

Major. criterion for separable: e-vals of ρ_A majorize e-vals of ρ_{AB}

ρ_{AB} separable $\Rightarrow \|\rho_{AB}\|_\infty \leq \|\rho_A\|_\infty$

But $\mathrm{rank}(\rho_{AB}) < d_A \Rightarrow \|\rho_{AB}\|_\infty > \frac{1}{d_A} = \|\rho_A\|_\infty$

Contradicition – so ρ_{AB} entangled (always, no probability)

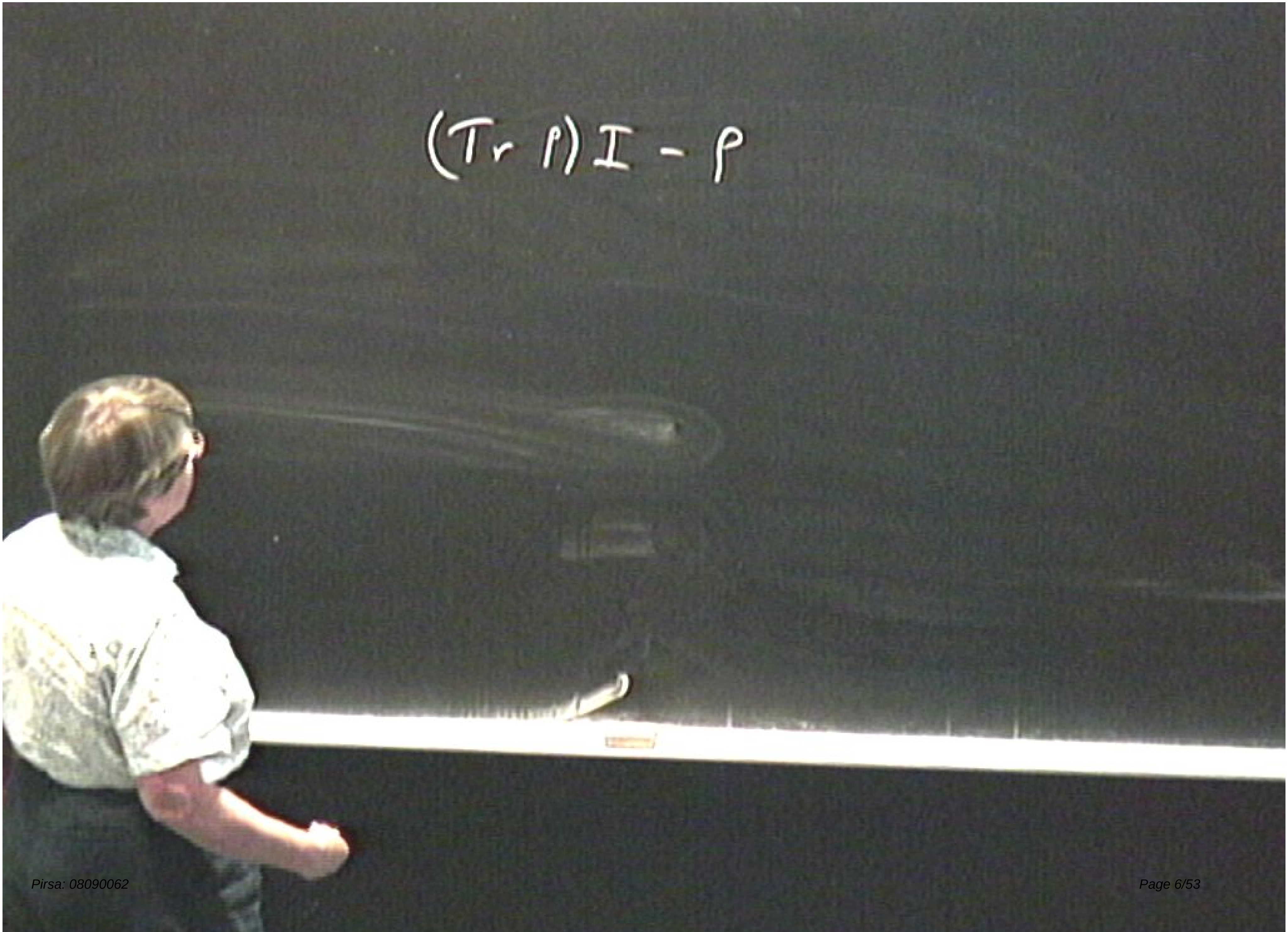
Consider $\mathrm{rank}(\rho_{AB}) < d_A = \mathrm{rank}(\rho_A)$

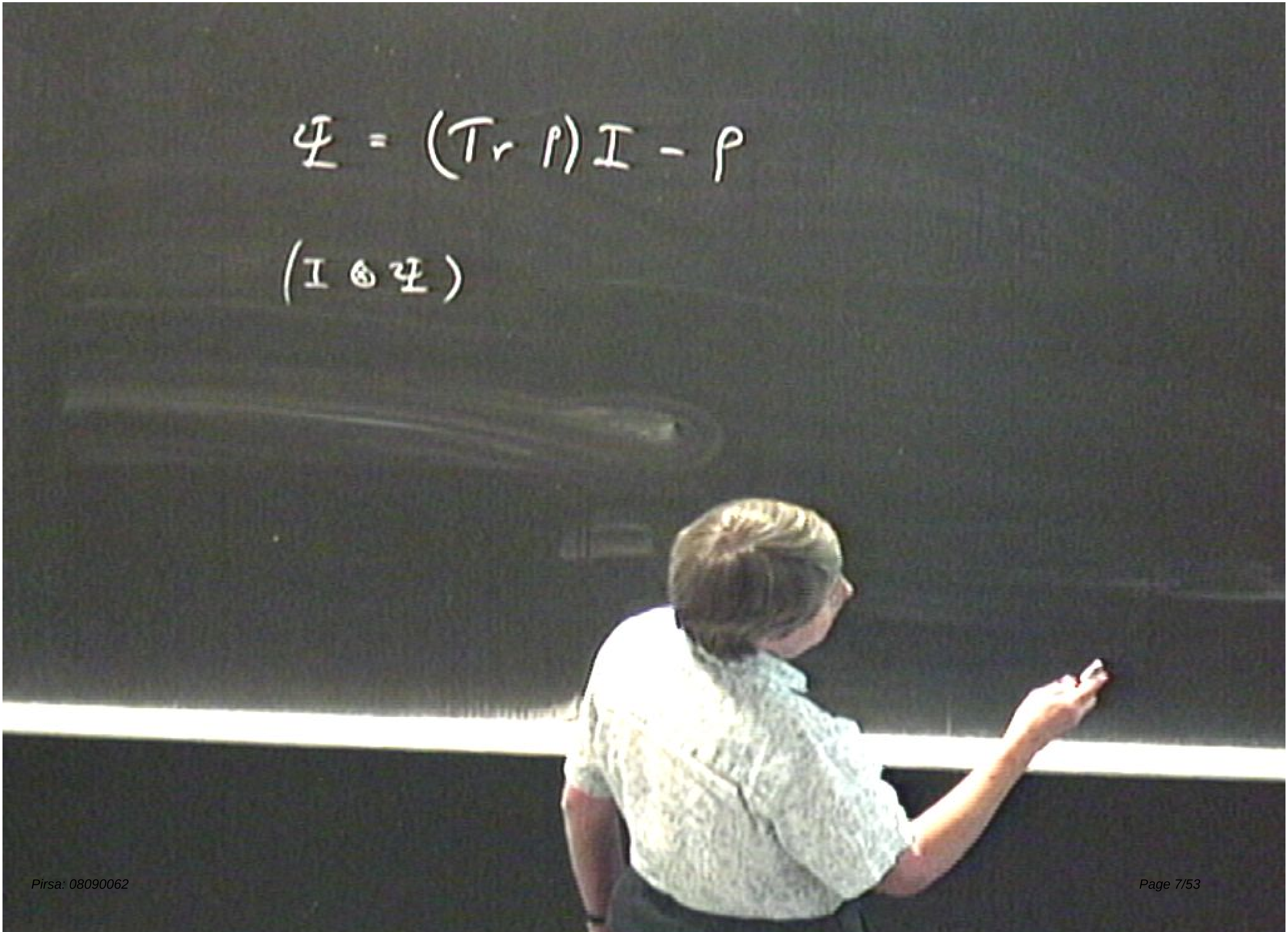
ρ_{AB} sep $\Leftrightarrow \tilde{\rho}_{AB} = \frac{1}{d_A}(\rho_A^{-1/2} \otimes I_B)\rho_{AB}(\rho_A^{-1/2} \otimes I_B)$ sep

But since $\tilde{\rho}_A = \frac{1}{d_A} I_A$ by above, ρ_{AB} entangled

Thm: $\mathrm{rank}(\rho_{AB}) < \mathrm{rank}(\rho_A) \Rightarrow \rho_{AB}$ entangled.

Pf: Consider $\mathcal{H}_A = \mathrm{range}(\rho_A) = [\ker(\rho_A)]^\perp$


(Tr ρ) I − ρ


$\Psi = (\mathrm{Tr}\,\rho) I - \rho$
$(I \otimes \Psi)$

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Bipartite qubit states: recall qubit channels, CPT maps

$$t_{jk} = \text{Tr}\,\sigma_j \Phi(\sigma_k) \qquad T_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_{10} & t_{11} & t_{21} & t_{31} \\ t_{20} & t_{21} & t_{21} & t_{31} \\ t_{30} & t_{e1} & t_{21} & t_{31} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix}$$

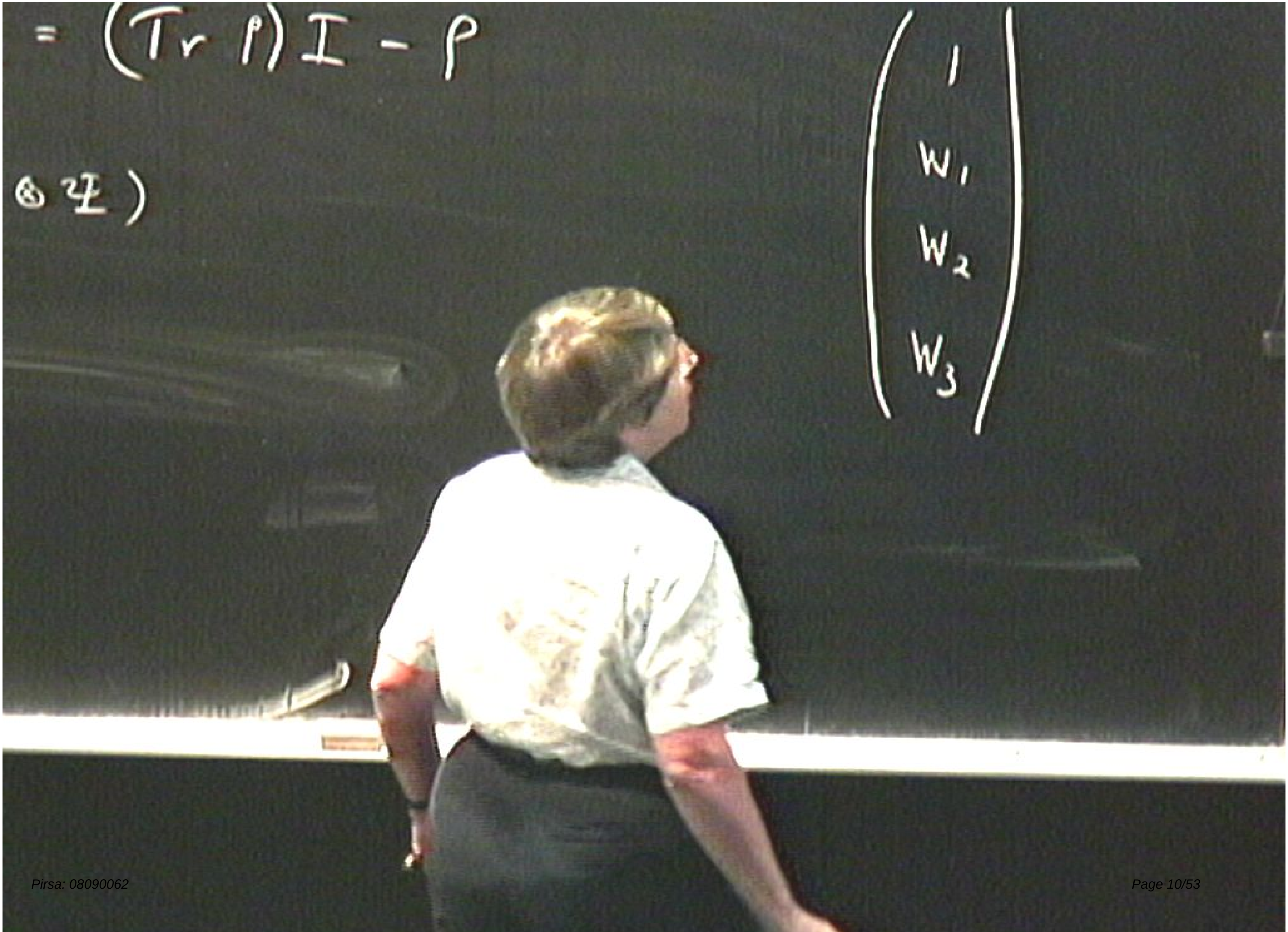
$$\Phi(\rho) = \tfrac{1}{2}\Phi(I + \textstyle\sum_j w_j \sigma_j) = \tfrac{1}{2}\left[I + \textstyle\sum_j (t_{j0} + \textstyle\sum_k t_{jk} w_k)\sigma_j\right]$$

Apply SVD to get $\mathcal{O}_2^\dagger T \mathcal{O}_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\quad \mathcal{O}_m$ real orthog

Equiv to 2×2 U, V , $\det U = \det V = 1$ with $U \simeq \mathcal{O}_1, V = \mathcal{O}_2$

$$\begin{aligned} (\Gamma_V \circ \Phi \circ \Gamma_U)(\rho) &= \tfrac{1}{2} V^\dagger (\Phi[U^\dagger (I + \textstyle\sum_j w_j \sigma_j) U]) V \\ &= \tfrac{1}{2}\left[I + \textstyle\sum_j (t_j + \lambda_j w_j)\sigma_j\right] \end{aligned}$$

$\Gamma_U(\rho) = U^\dagger \rho U$


= (Tr ρ) I − ρ
1
W1
W2
W3

$$\Psi = (\mathrm{Tr}\,\rho)\, I - \rho$$

$$(I \otimes \Psi)$$

$$\begin{matrix} \Phi & & \rho & & \Phi(\rho) \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & & & \\ 1 & & & \end{pmatrix} & & \begin{pmatrix} 1 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix} & = & \begin{pmatrix} 1 \\ \\ \\ \\ \end{pmatrix} \end{matrix}$$

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Aside on SVD:

Any $m \times n$ matrix A. $A^\dagger A$ is self-adjoint

Write $A^\dagger A = V^\dagger D V$ $|A| = V^\dagger \sqrt{D} V$ Then

$A = |A| W = V^\dagger \sqrt{D} U$ with W unitary or part isom. $U = VW$.

Can apply to any linear op on a Hilbert space

- (Schmidt) $\psi(x,y)$ define $A[f(y)] = (Af)(x) = \int \overline{\psi}(x,y) f(y) dy$

  SVD becomes $f(x,y) = \sum_k \mu_k \phi_k(x) \Phi_k(y)$

- $\mathbf{C}_m \otimes \mathbf{C}_n$ iso to $\mathcal{B}(\mathbf{C}_m, \mathbf{C}_n)$ $|f \otimes g\rangle \to |f\rangle\langle g|$

- Can apply to CP $\Phi = |\Phi| \circ \Gamma_U == \Gamma_V \circ \Phi_{\rm diag} \circ \Gamma_U$

Different basis changes on input and output spaces

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Choi matrix or state rep $\sum_{jk} |e_j\rangle\langle e_k| \otimes \Phi(|e_j\rangle\langle e_k|)$ then transforms as $(U \otimes V)^\dagger \Big(\sum_{jk} |e_j\rangle\langle e_k| \otimes \Phi(|e_j\rangle\langle e_k|)\Big)(U \otimes V)$

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$$\text{Choi } \Phi = \frac{1}{4} \begin{pmatrix} 1+t_3+x_3 & t_1-it_2 & 0 & x_1+x_2 \\ t_1-it_2 & 1-t_3-x_3 & x_1-x_2 & 0 \\ 0 & x_1-x_2 & 1+t_3-x_3 & t_1-it_2 \\ x_1+x_2 & 0 & t_1+it_2 & 1-t_3+x_3 \end{pmatrix}$$

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$$\begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix} = \begin{pmatrix} \sqrt{A} & 0 \\ 0 & \sqrt{B} \end{pmatrix} \begin{pmatrix} I & X \\ X^\dagger & I \end{pmatrix} \begin{pmatrix} \sqrt{A} & 0 \\ 0 & \sqrt{B} \end{pmatrix}$$

pos semi-def $\Leftrightarrow X^\dagger X \leq I$, i.e., X a contraction

rank $\leq 2 \Leftrightarrow X$ is unitary $\quad \begin{pmatrix} I & U \\ U^\dagger & I \end{pmatrix} \begin{pmatrix} v \\ -U^\dagger v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

RSW worked out conds for rank 2 – get $t_3 = 0$ but then permute

Note $t_1 = t_2 = 0$ gives

$$\frac{1}{4} \begin{pmatrix} 1+t_3+x_3 & 0 & 0 & x_1+x_2 \\ 0 & 1+t_3-x_3 & x_1-x_2 & 0 \\ 0 & x_1-x_2 & 1-t_3-x_3 & 0 \\ x_1+x_2 & 0 & 0 & 1-t_3+x_3 \end{pmatrix}$$

different block structure , but easy to verify that this case is

pos semi-def $\Leftrightarrow (\lambda_1 \pm \lambda_2)^2 \leq (1 \pm \lambda_3)^2 - t_3^2$

and rank $\leq 2 \Leftrightarrow (\lambda_1 \pm \lambda_2)^2 = (1 \pm \lambda_3)^2 - t_3^2$

Φ with Choi-rank ≤ 2 $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos u & 0 & 0 \\ 0 & 0 & \cos v & 0 \\ \sin u \sin v & 0 & 0 & \cos u \cos v \end{pmatrix}$

Satisfies nasc $(\lambda_1 \pm \lambda_2)^2 \leq (1 \pm \lambda_3)^2 - t_3^2$ with equality

Ext. points have u, v in $[-\pi, \pi]$ but need to avoid double counting

Want to consider $(u, v) \in \overline{\Delta} = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

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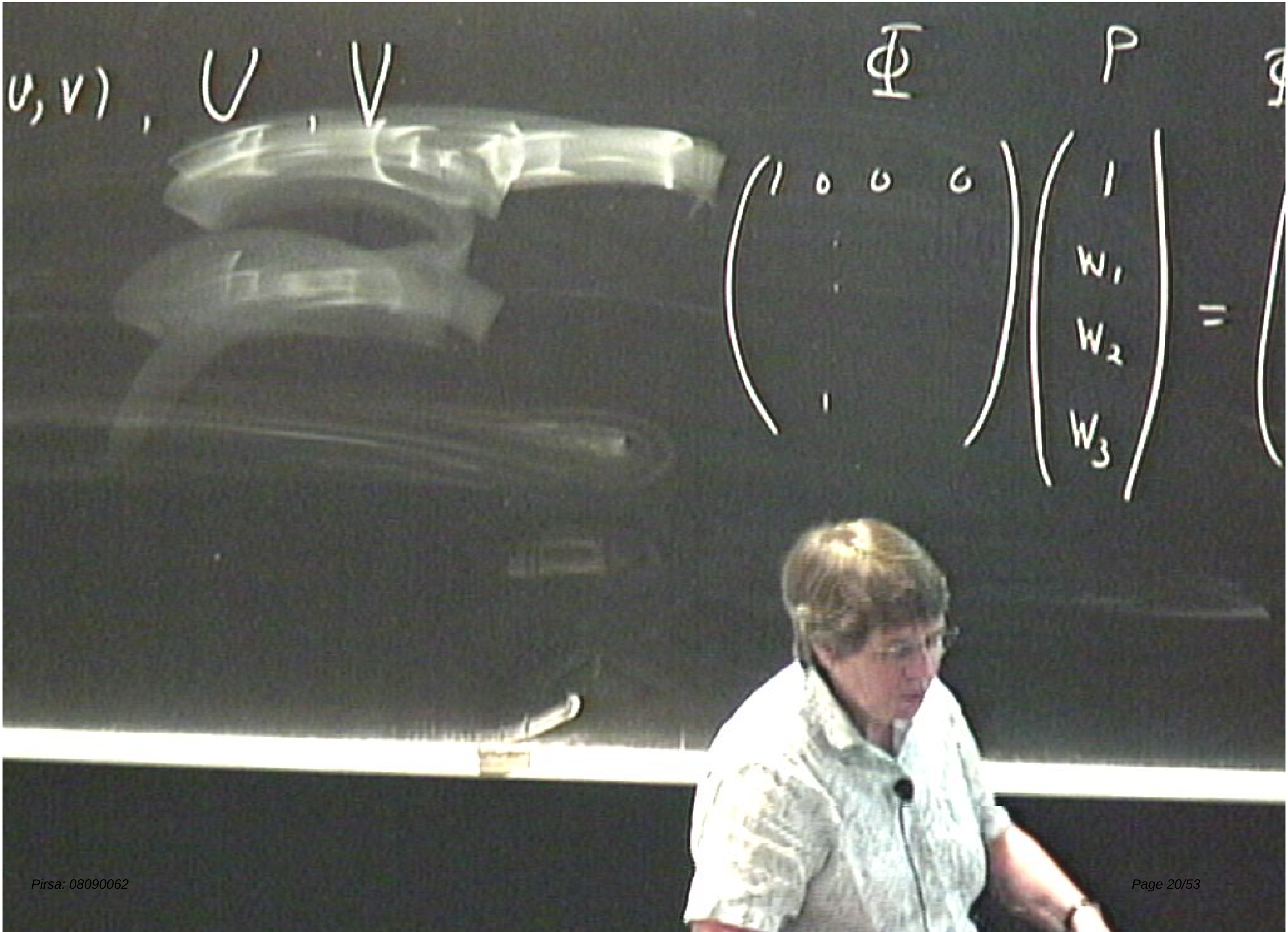
(u, v) , U , V
Φ
P
1 0 0 0
1
W1
W2
W3
=

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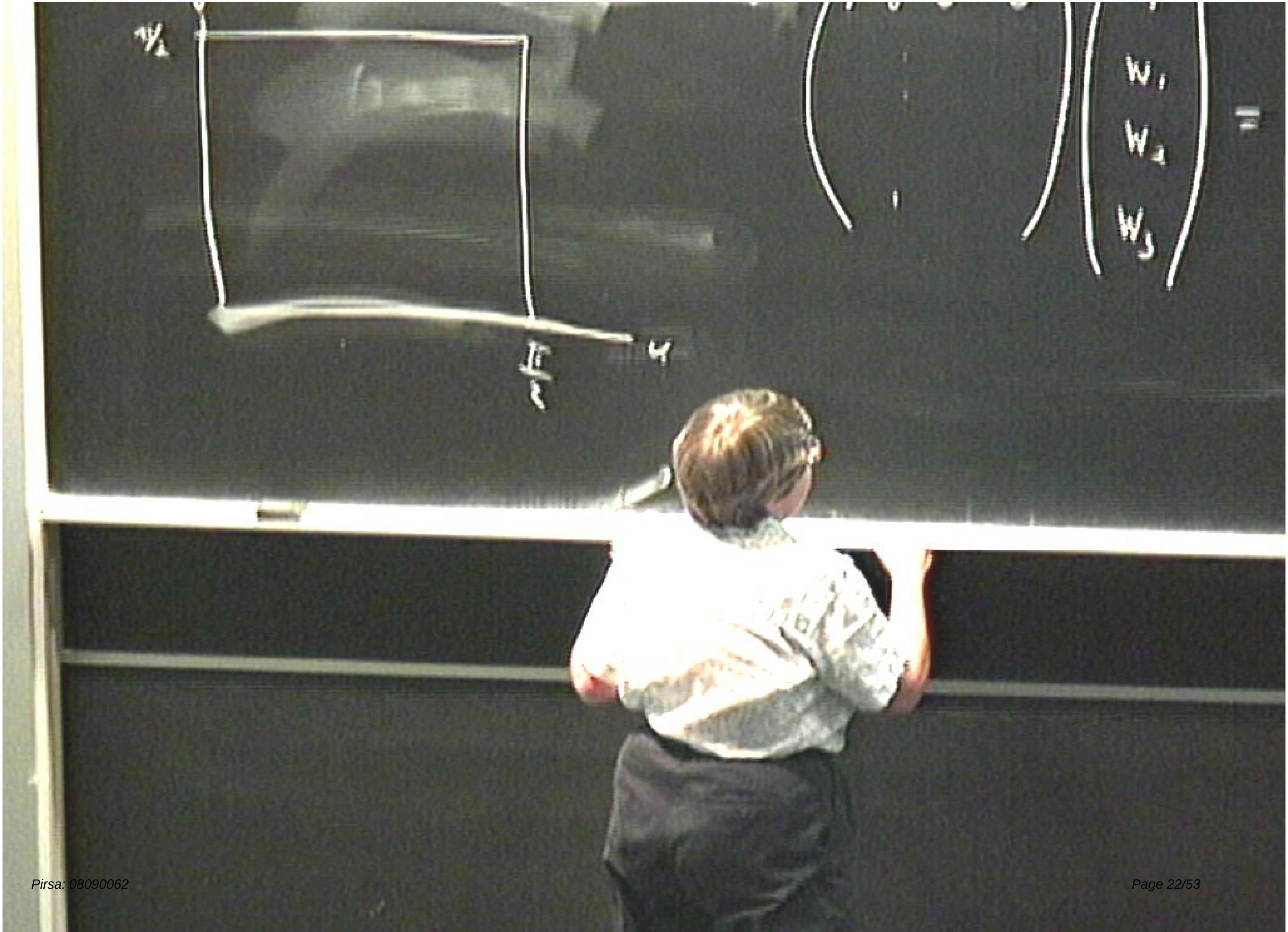
W₁
W₂
W₃

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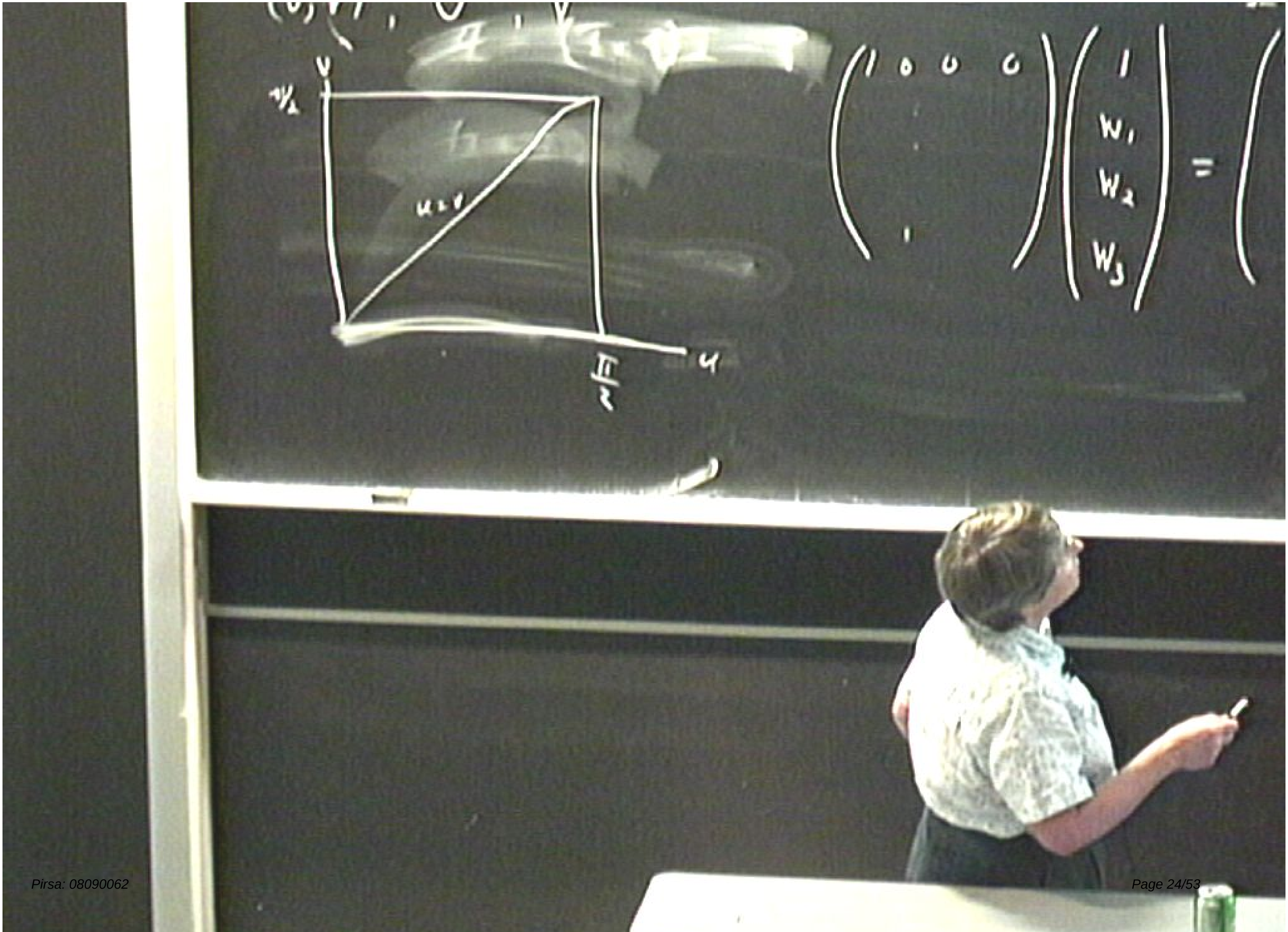

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Want to say: rank 2 CPT isomorphic to $\overline{\Delta} \times SU(2) \times SU(2)$

EB subset iso to $(u = \frac{\pi}{2}, v = \frac{\pi}{2}) \times SU(2) \times SU(2)$

use fact lines have measure 0 wrt rect. $\overline{\Delta}$

to conclude EB channels (or sep states) have measure $0 \cdot 1 \cdot 1 = 0$.

Roughly correct, but technical issues because of invariance of some "lines" (i.e., amp and phase-damping) under certain rotations. Can't really matter because only happens on lines (measure zero)

$SU(2)$ should really be $SU(2)/\mathbf{C}$, i.e., subgp with det $U = +1$.

$\Delta = \{(u, v) : 0 < u \leq \frac{\pi}{2}, 0 < v \leq \frac{\pi}{2}, u \neq v\}$

set of rank 2 CPT qubit Φ iso to

set of rank 2 bipartite qubit states with $\gamma_A = \frac{1}{d_A} I_A$ iso to

$$\begin{aligned} \Delta \times SU(2) \times SU(2) \quad & \bigcup \quad \{(u, u)\}_{u \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2))/\mathcal{R}_z \\ & \bigcup \quad \{(u, 0)\}_{u \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2))/\mathcal{R}_y \\ & \bigcup \quad \{(0, v)\}_{v \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2))/\mathcal{R}_x \\ & \subset \quad \overline{\Delta} \times SU(2) \times SU(2) \end{aligned}$$

Put Lebesque measure m on $\overline{\Delta}$ almost any measure ν on $SU(2)/\mathbf{C}$

use product measure $\mu = m \times \nu \times \nu$ and renorm to get prob meas

Use this to construct prob. measure on rank 2 qubit channels

omit details – somewhat technical

bottom line: EB subset still lines with measure $0 \cdot 1 \cdot 1 = 0$.

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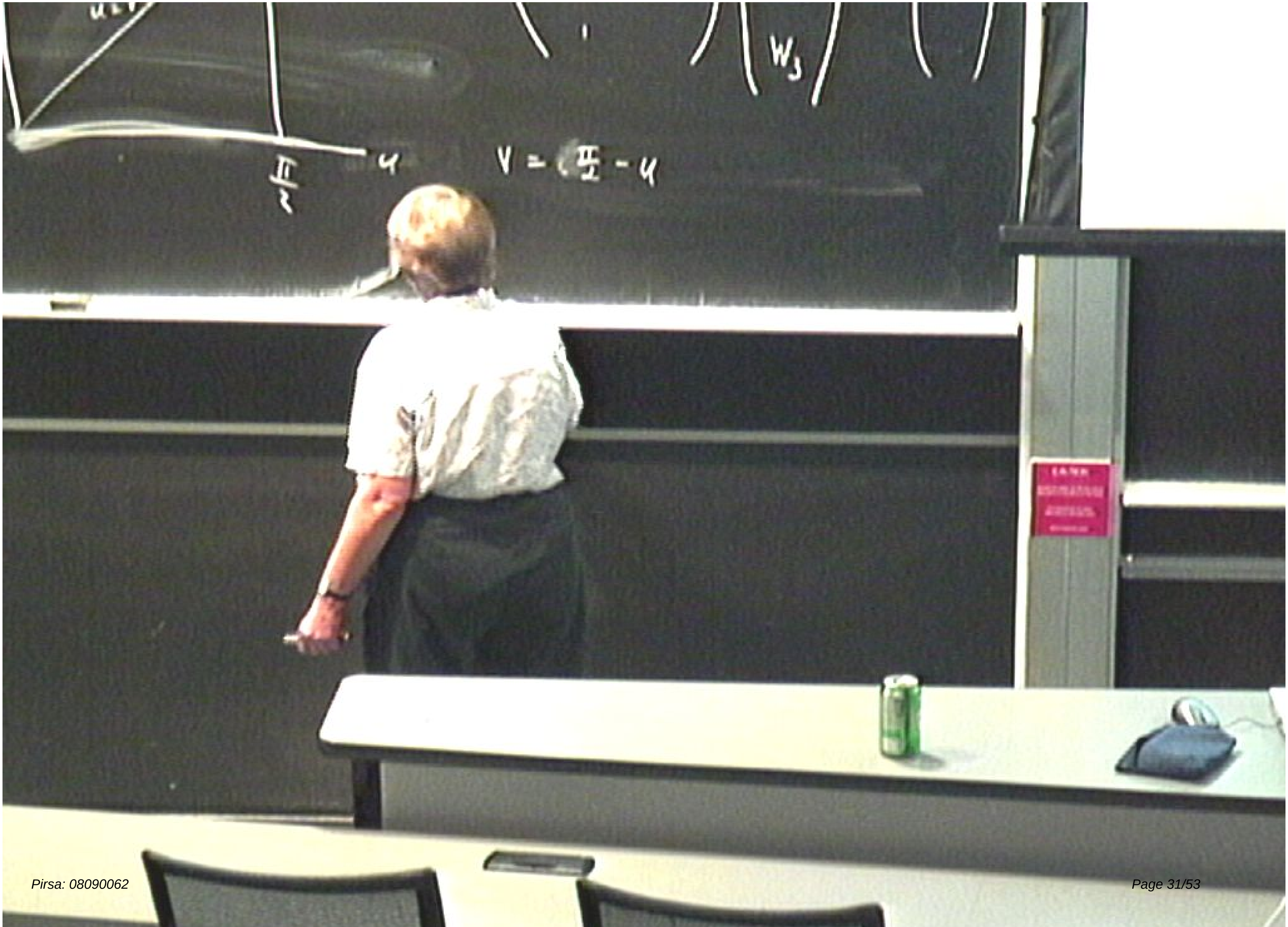
$SU(2)$ should really be $SU(2)/\mathbf{C}$, i.e., subgp with $\det U = +1$.


W3
V = π/2 - u

$$\Delta = \{(u, v) : 0 < u \leq \tfrac{\pi}{2}, 0 < v \leq \tfrac{\pi}{2}, u \neq v\}$$

set of rank 2 CPT qubit Φ iso to

set of rank 2 bipartite qubit states with $\gamma_A = \frac{1}{d_A} I_A$ iso to

$$\begin{aligned} \Delta \times SU(2) \times SU(2) \quad & \bigcup \quad \{(u, u)\}_{u \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2)) / \mathcal{R}_z \\ & \bigcup \quad \{(u, 0)\}_{u \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2)) / \mathcal{R}_y \\ & \bigcup \quad \{(0, v)\}_{v \in (0, \frac{\pi}{2}]} \times (SU(2) \times SU(2)) / \mathcal{R}_x \\ & \subset \quad \overline{\Delta} \times SU(2) \times SU(2) \end{aligned}$$

Put Lebesque measure m on $\overline{\Delta}$ almost any measure ν on $SU(2)/\mathbf{C}$

use product measure $\mu = m \times \nu \times \nu$ and renorm to get prob meas

Use this to construct prob. measure on rank 2 qubit channels

omit details – somewhat technical

bottom line: EB subset still lines with measure $0 \cdot 1 \cdot 1 = 0$.

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Remove restriction $\tilde\rho_A = \frac{1}{d_A} I_A$ as before

$$\tilde\rho_{AB} \to (\rho_A^{-1/2} \otimes I_B)\tilde\rho_{AB}(\rho_A^{-1/2} \otimes I_B)$$

Get equiv. classes $\mathcal{D}_\rho = \{\tilde\rho_{AB} : \tilde\rho_A = \rho\}$

Excluding rank one projections from M_2

$$\{\rho : \mathrm{rank}\ \rho = 2\} \quad \text{iso} \quad \{U^\dagger \begin{pmatrix} x & 0 \\ 0 & 1-x \end{pmatrix} U : x \in (0, \tfrac{1}{2})\} \cup I$$

$$\text{iso} \quad [(0, \tfrac{1}{2}) \times SU(2)] \cup I$$

Extend previous sets $\quad \Delta \times SU(2) \times SU(2) \times (0, \frac{1}{2}) \times SU(2)$

Product measure $\mu = m \times \nu \times \nu \times m_1 \times \nu$

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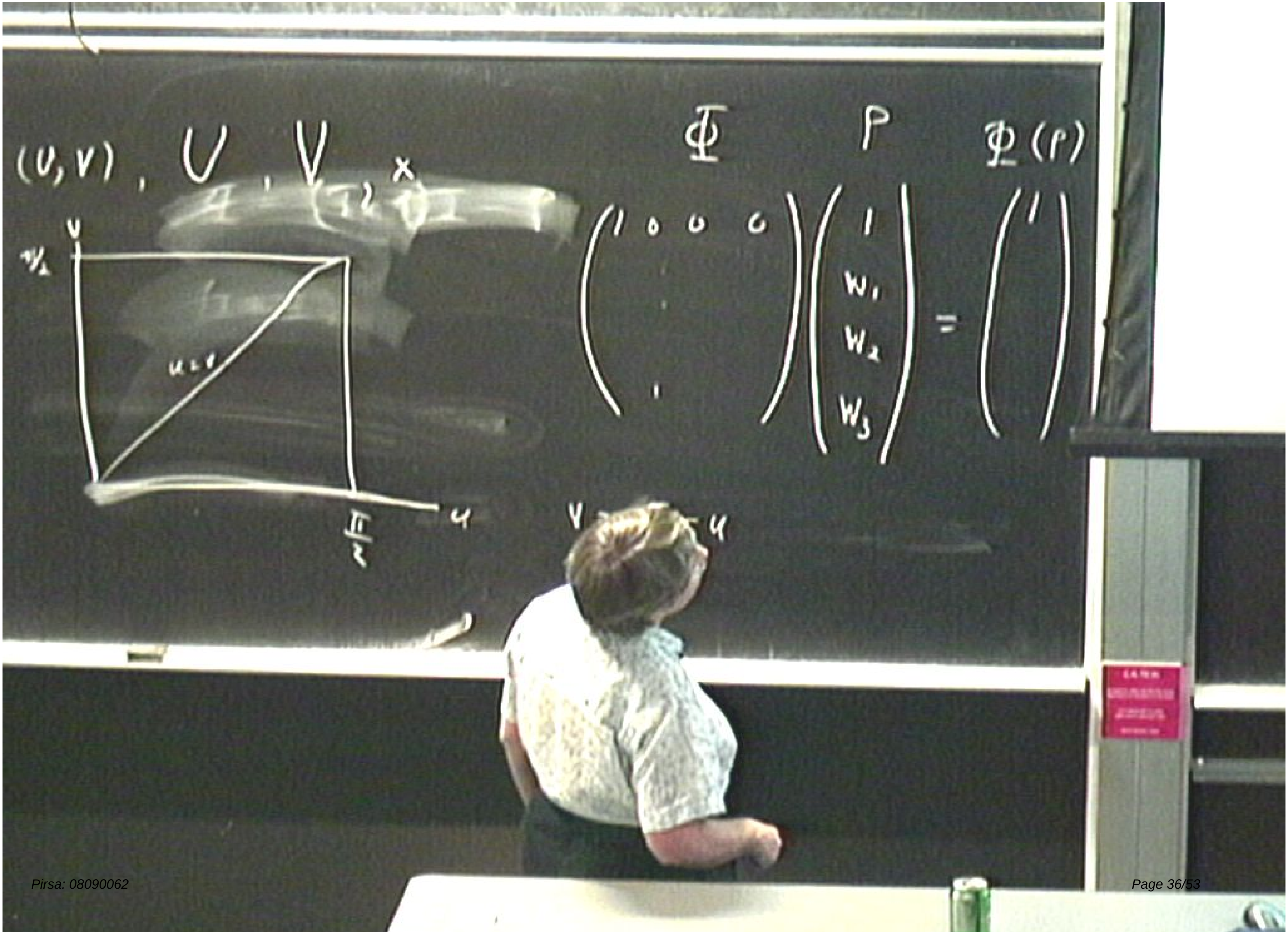
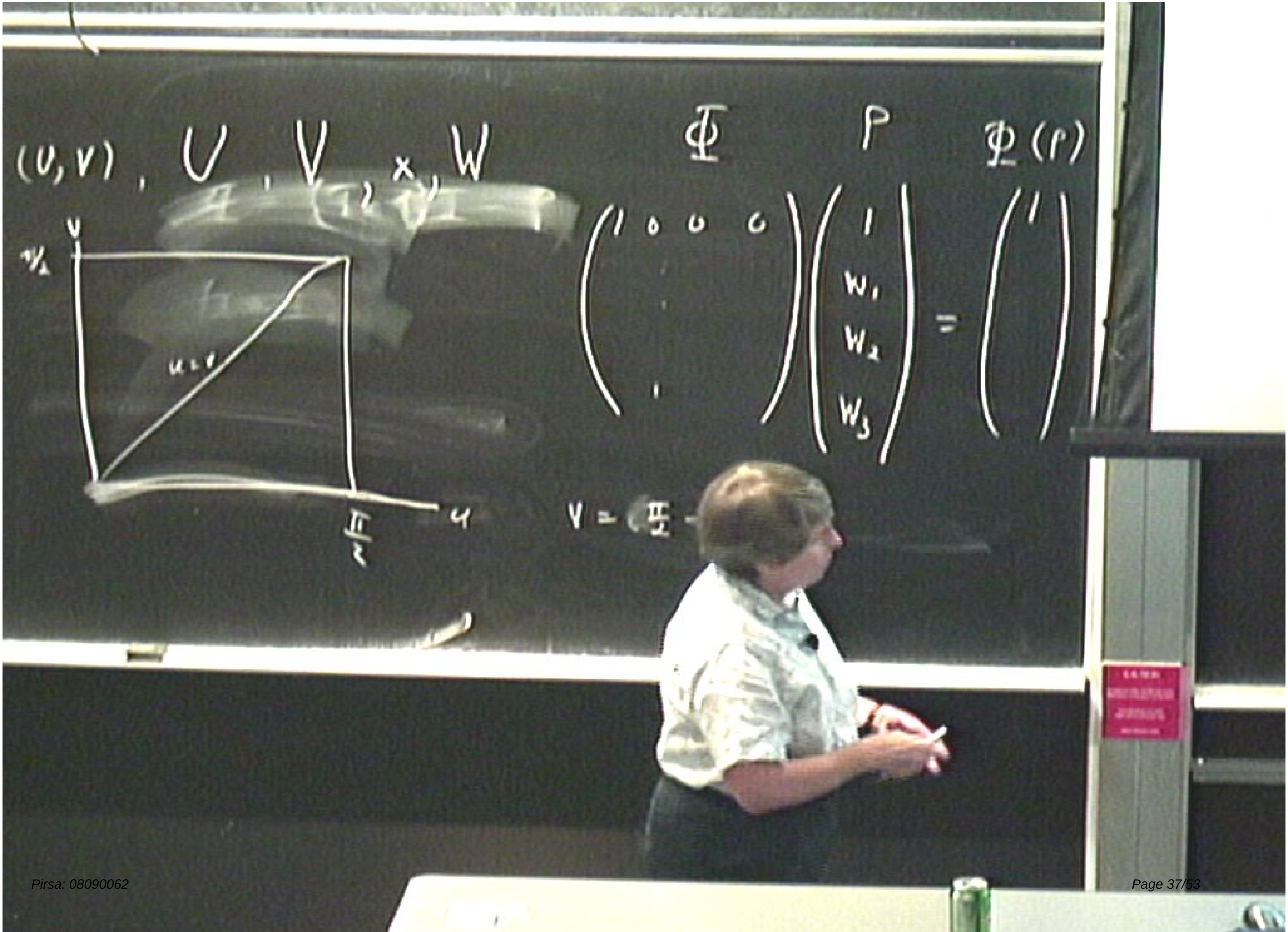

(U,V) , U , V , x
Φ
P
Φ(P)
π/2
u<v
W1
W2
W3
u
π/2


(U, V) , U , V , x , W
Φ
P
Φ(P)
W1
W2
W3
u = v
π/2
v = π/2

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Arveson observed $\mathcal{V}_d$ can be given structure of

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General $\Phi : M_{d_A} \to M_{d_B}$ or bipartite states on $M_{d_A} \otimes M_{d_B}$

Key Lemma: (P. Horodecki, et al) If ρ_{AB} is separable, ρ_{AB} has rank d_A and ρ_A has rank d_A, then ρ_{AB} can be written as a convex comb of product states using at most d_A products.

Cor: If ρ_{AB} is separable and $\rho_A = \frac{1}{d_A} I_A$ then

$$\rho_{AB} = \frac{1}{d_A} \sum_k |g_k\rangle\langle g_k| \otimes |\psi_k\rangle\langle \psi_k|$$

with g_k an O.N. basis for $\mathbf{C}_{d_A}$

Note: ψ_k need not be O.N. or even lin. indep.

ρ_{AB} sep with $\rho_A = \frac{1}{d_A} I_A \Rightarrow \rho_{AB} = \frac{1}{d_A} \sum_k |g_k\rangle\langle g_k| \otimes |\psi_k\rangle\langle\psi_k|$

Let $P_k \equiv |\psi_k\rangle\langle\psi_k|$ and rewrite in block form.

$$\rho_{AB} = \frac{1}{d_A} \begin{pmatrix} P_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & P_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & P_3 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & \dots & P_{d_A} \end{pmatrix}$$

Now consider

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with Q a $2d_B \times 2d_B$ pos semi-def matrix of rank 2 and $\mathrm{Tr}_B\, Q = I_2$.

Then ρ_{AB} is sep $\Leftrightarrow$ Q is sep. Reduced gen $\rho_A = \frac{1}{d_A} I_A$ to $d_A = 2$.

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Param full rank ρ_A by $\{0 < x_1 \leq x_2 \leq \ldots \leq x_{d_A}\}$ and $SU(d_A)$

measure issues with deg e-vals. but again tricky spots have

measure zero wrt ℓ_1 unit sphere in $\mathbf{R}_{d_A}$ or , equiv.,

wrt to (renorm Leb) measure 1 subset of $\mathbf{R}_{d-1}$

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$\Rightarrow$ Can't be true for $\mathrm{rank}(\rho_{AB}) = d_A d_B$.

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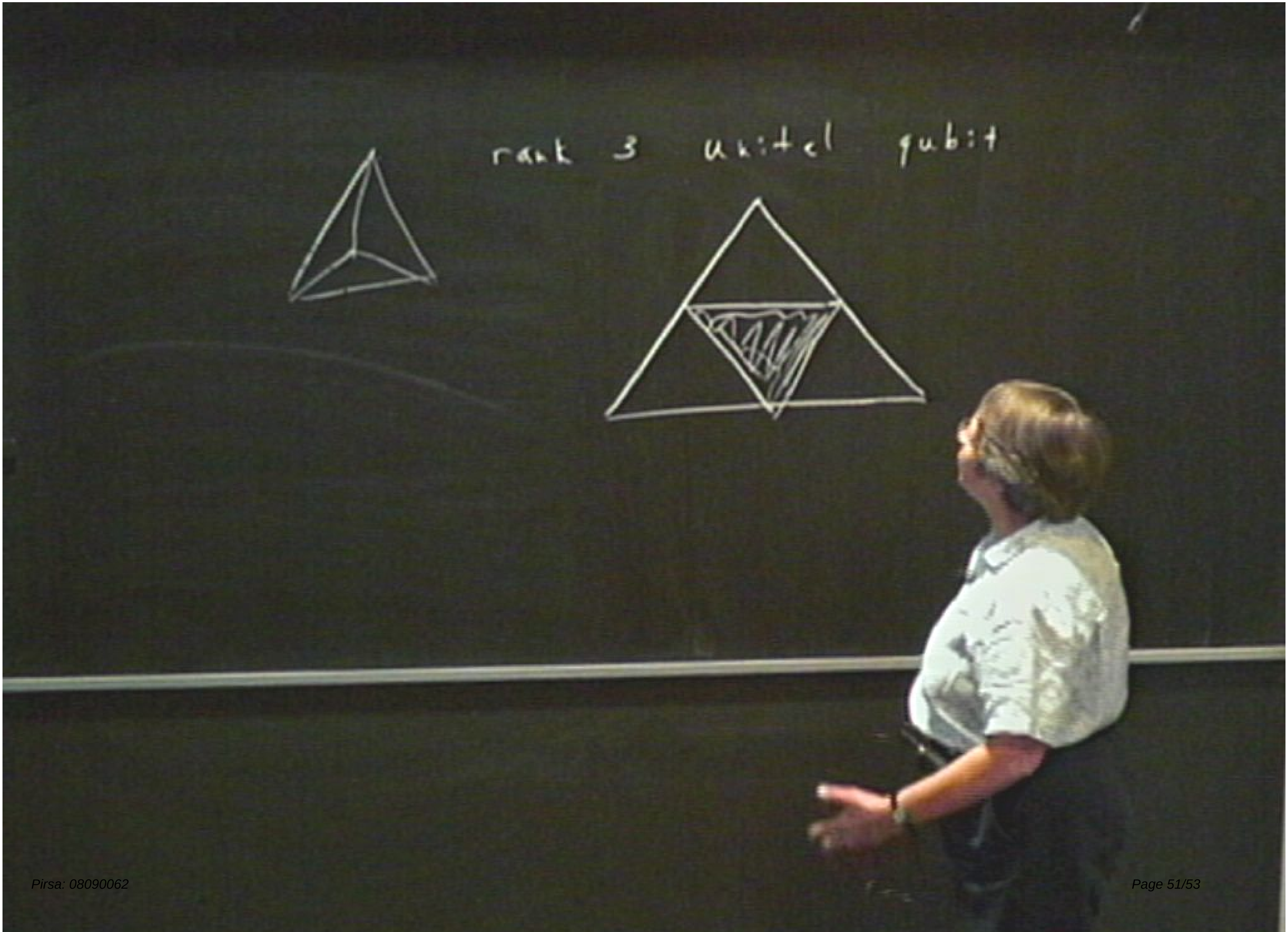
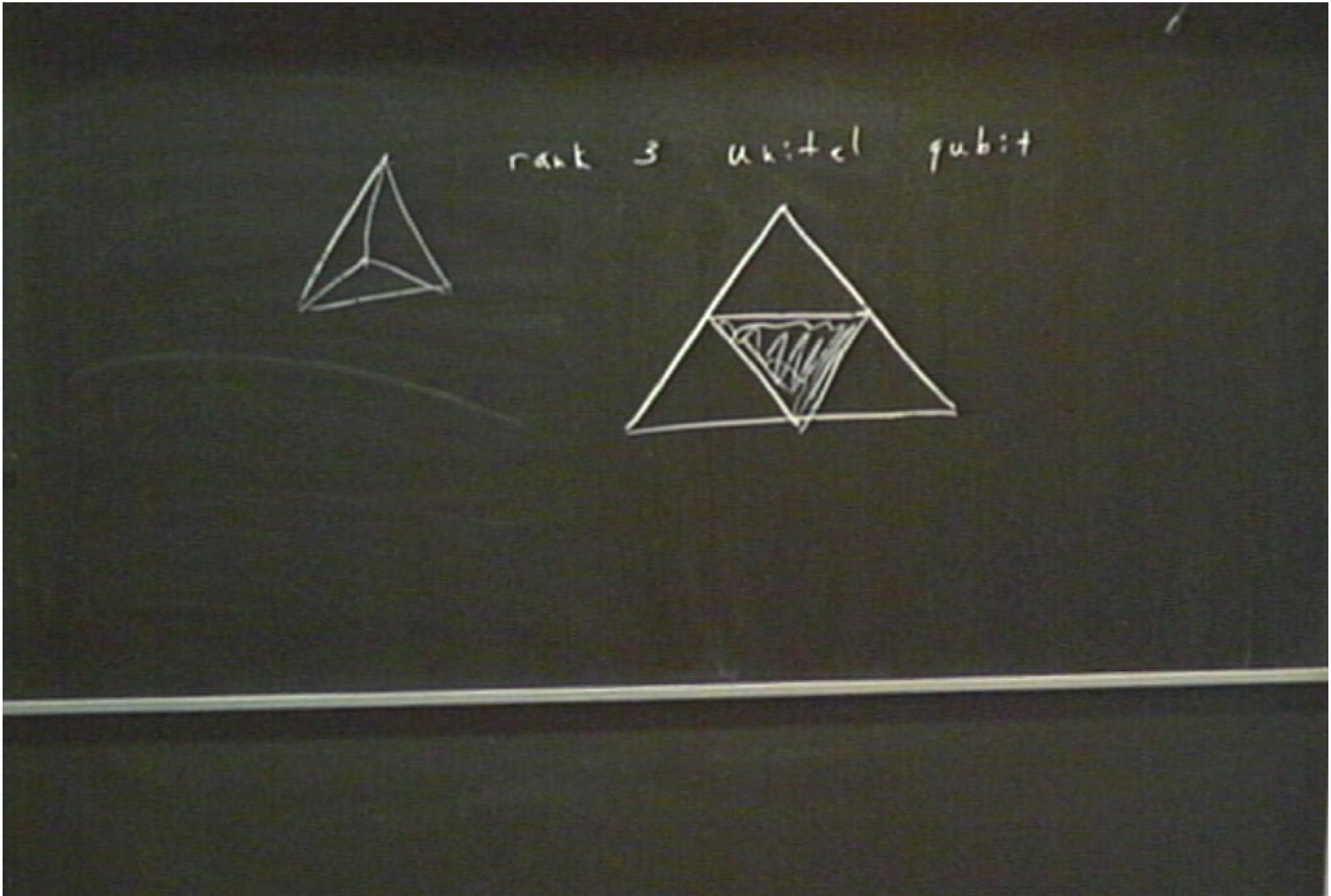
Example: unital qubit channels or rank 3. EB are $\frac{1}{4}$.


rank 3 unital qubit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