Title: Noncommutative Geometry Ã la Connes

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Abstract: During the last two decades Alain Connes developed Noncommutative Geometry, which allows to unify two of the basic theories of modern physics: General Relativity and the Standard Model of Particle Physics. In the noncommutative framework the Higgs boson, which had previously to be put in by hand, and many of the ad hoc features of the standard model, appear in a natural way. The aim of my talk is to motivate this unification from basic physical principles and to give a flavour of its derivation. I will give an overview of the basic tools such as almost-commutative spectral triples and the spectral action principle. The latter allows to derive the Einstein-Hilbert Lagrangian and the Standard Model Lagrangian together with a set of relations among the Standard Model parameters.

Noncommutative Geometry à la Connes

Christoph A. Stephan
Institut für Mathematik
Universität Potsdam

Perimeter Institute
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Overview

1. Quick and Dirty
2. Geometry
3. Physics
4. The Standard Model
5. Conlusions

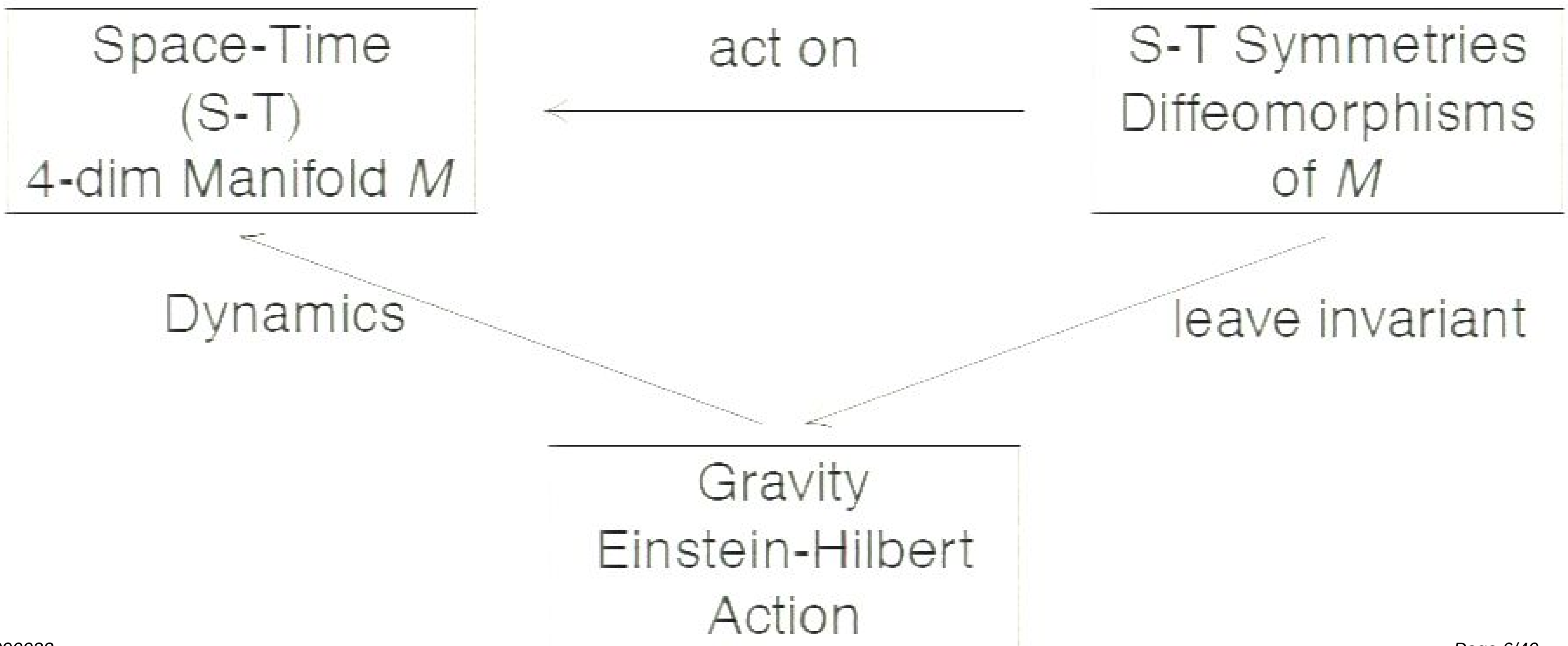
Overview

1. Quick and Dirty

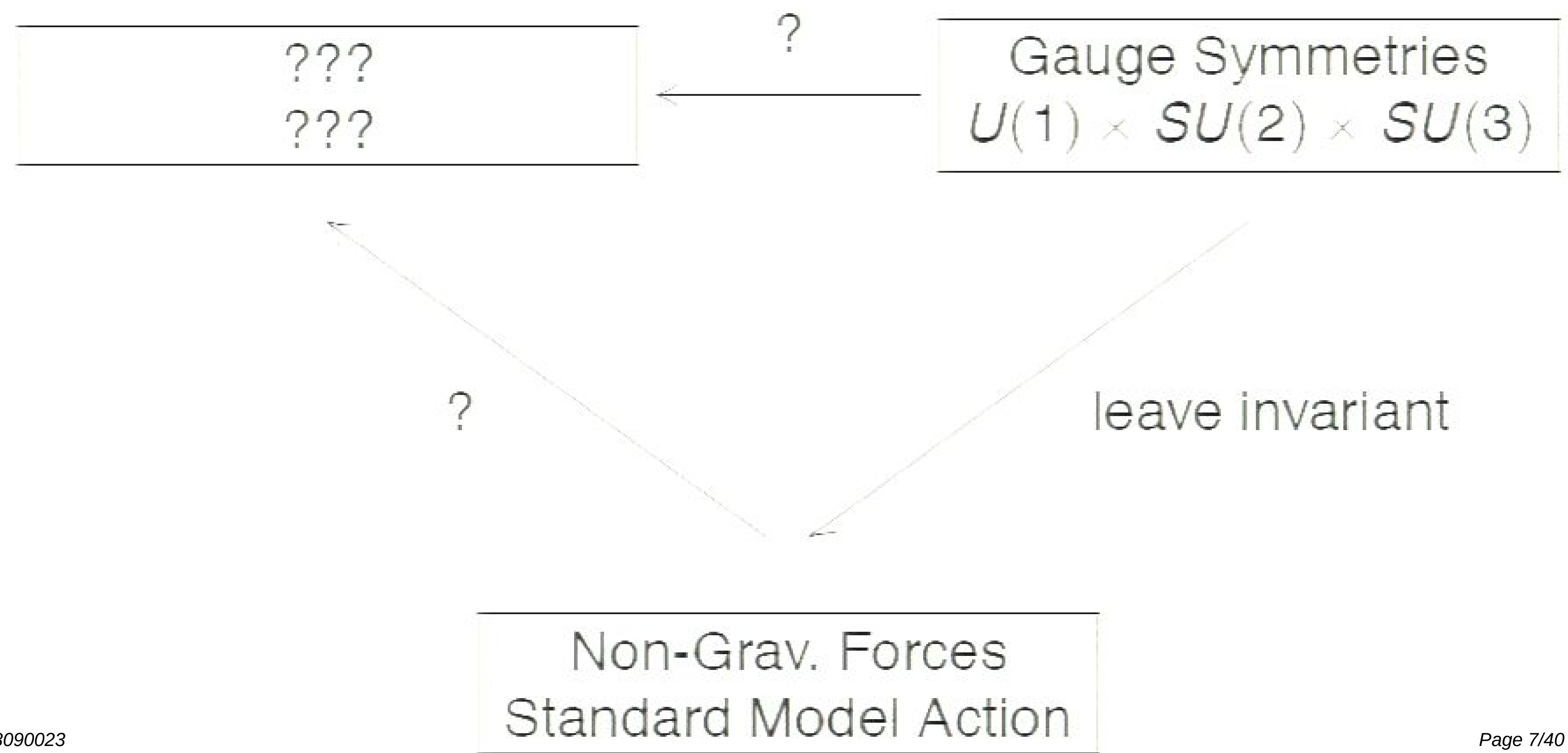
The aim:

To unify general relativity (GR) and the standard model of particle physics (SM) on the same geometrical level.
This means to describe gravity and the electro-weak and strong forces as gravitational forces of a unified space-time.

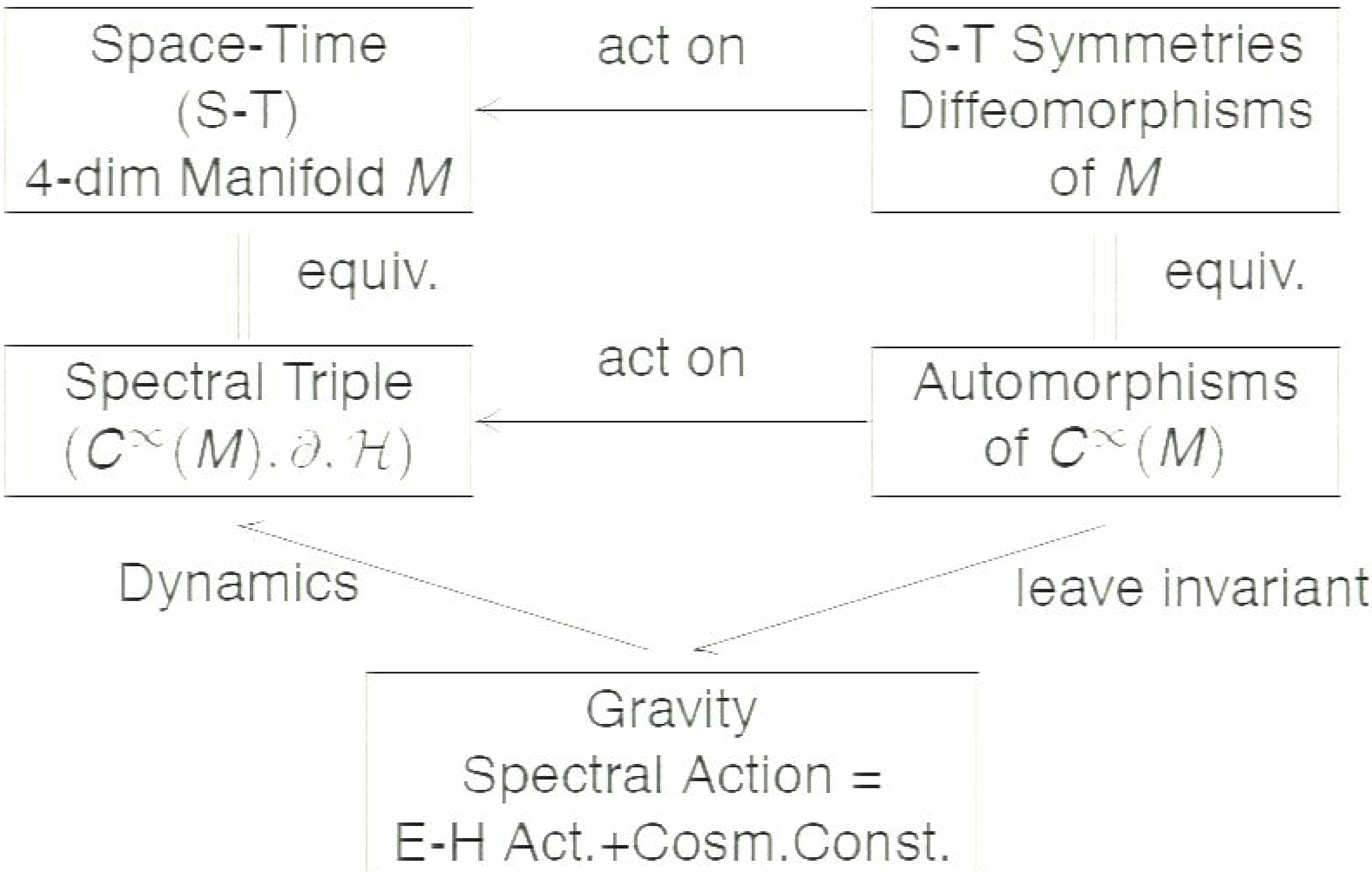
Schematic Structure of GR:
Gravity emerges as a pseudo-force associated to the space-time symmetries, i.e. the diffeomorphisms of the manifold M.



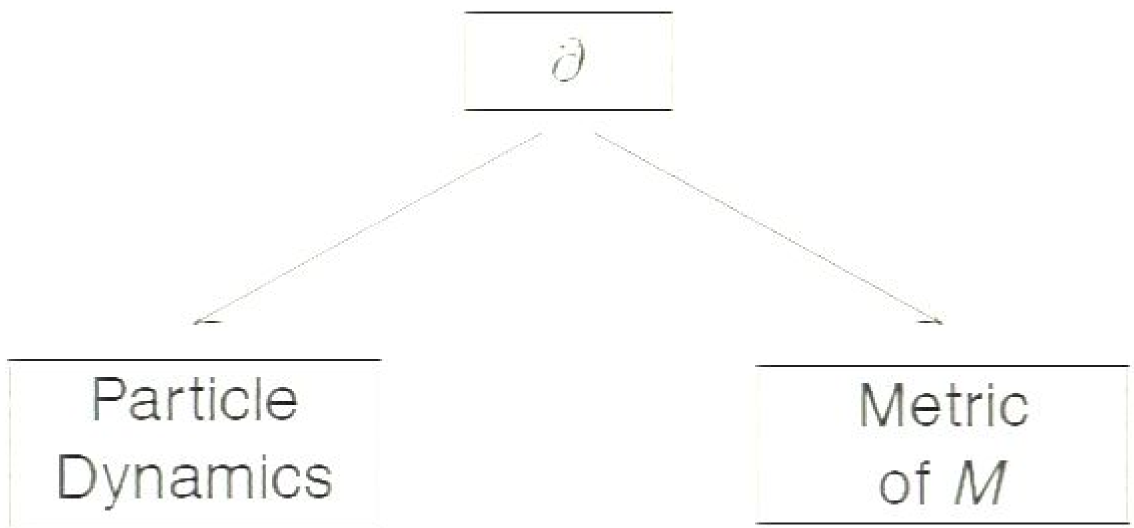
If one tries to put the SM into the same scheme, one cannot find an underlying geometric structure, which is equivalent to space time:



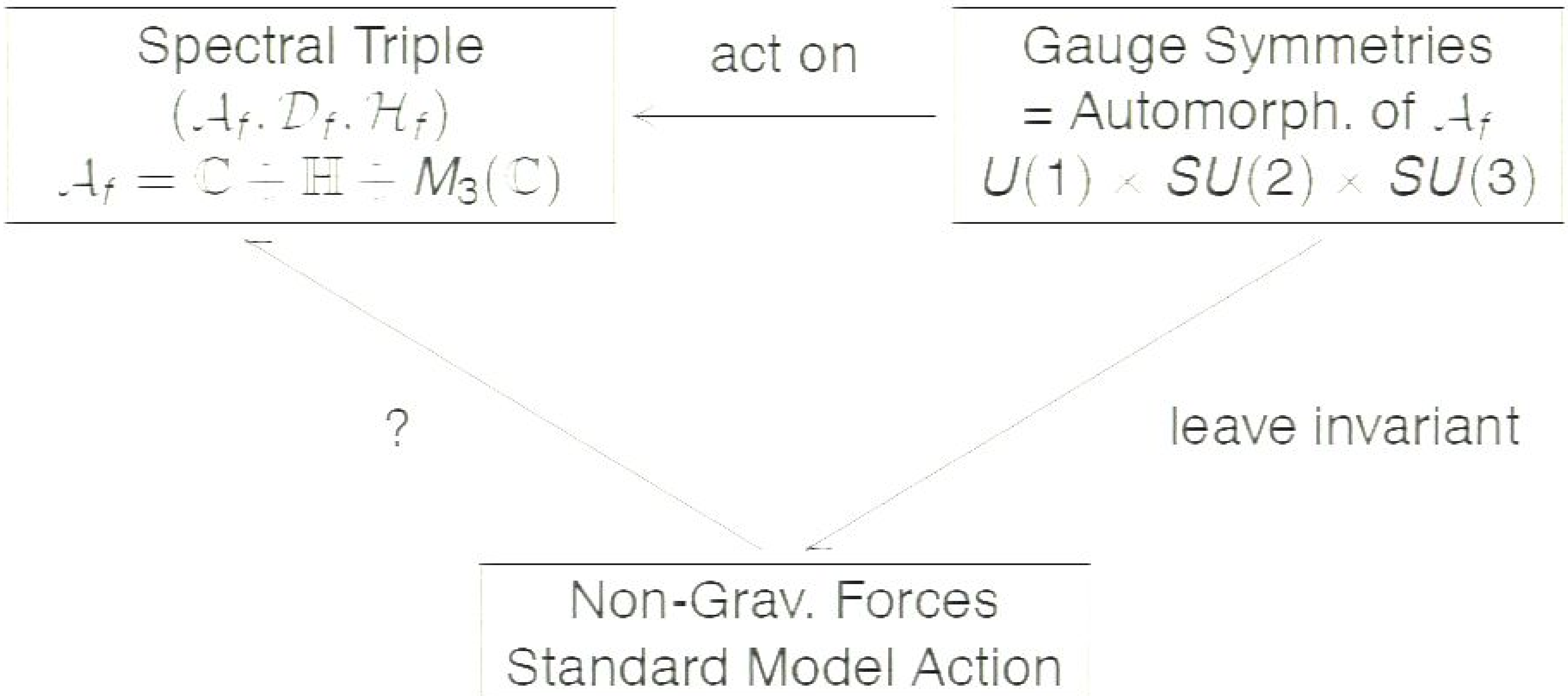
Euclidean space-time!



The Dirac operator plays a double role:



The Connes-Lott Model:



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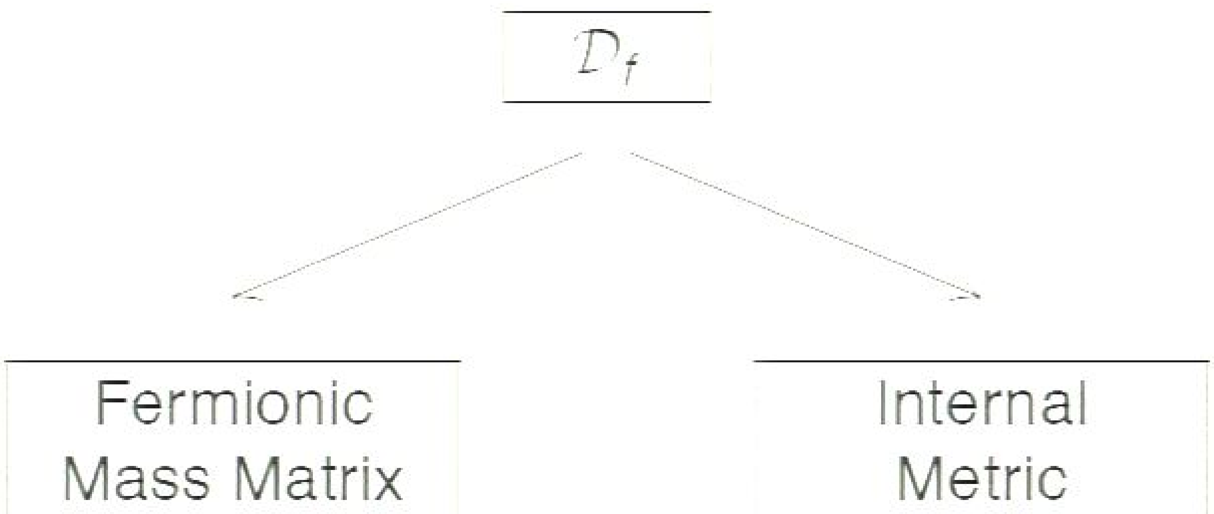


The internal Hilbert space $\mathcal{H}_f$ is given by the Fermion multiplets of the SM.

Almost-Commutative Standard Model (A.Chamseddine, A.Connes):

Almost-Commutative Spectral Triple
$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f$

act on

Int.+Ext. Symmetries
$= \mathrm{Diff}(M) \rtimes U(1) \times SU(2) \times SU(3)$

Dynamics

leave invariant

Spectral Action =
E-H Act.+Cosm.Const.
+ Stand. Model Action

The almost-commutative standard model automatically produces:

- The combined Einstein-Hilbert and standard model action
- A cosmological constant
- The Higgs boson with the correct quartic Higgs potential

The Dirac operator plays a multiple role:

$$\partial \otimes 1 + \gamma^5 \otimes D_f$$

Higgs & Gauge Bosons	Particle Dynamics, Ferm. Mass Matrix	Metric of M, Internal Metric

Overview

2 Geometry

An even, real spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$**:**
the ingredients (A. Connes):

- A real, associative, unital pre-C^*-algebra $\mathcal{A}$
- A Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ is faithfully represented via a representation ρ
- A self-adjoint operator $\mathcal{D}$ with compact resolvent, the Dirac operator
- An anti-unitary operator J on $\mathcal{H}$, the real structure
- A unitary operator γ on $\mathcal{H}$, the chirality

The spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has two notions of dimension:

- The metric dimension n associated with the Dirac operator
- The algebraic KO-dimension p associated with the real structure J and the chirality γ

Both dimensions are taken to be even but in general $n = p$.
In the case of $n = p = 0$ the following item will be required

- A unitary operator ϵ on $\mathcal{H}$ with $\epsilon^2 = 1$, the S^0-real structure. It obeys to the following commutation relations: $[\epsilon, \mathcal{D}] = [\epsilon, \gamma] = \{\epsilon, J\} = 0$ and $[\epsilon, \rho(a)] = 0$ for all $a \in \mathcal{A}$.

The 'classical" conditions or axioms of noncommutative geometry (A. Connes 1996):

Condition 1: Classical Dimension n (we assume n even)

Condition 2: Regularity

Condition 3: Finiteness

Condition 4: First Order of the Dirac Operator

Condition 5: Poincaré Duality

Condition 6: Orientability

Condition 7: (Reality)
The anti-unitary operator J obeys the following commutation relations in KO-dim. p (even):

$$[\rho(a), J\rho(a')J^{-1}] = 0 \text{ for all } a, a' \in \mathcal{A}$$

p mod 8	0	2	4	6
$J^2 = \pm 1$	+	-	-	+
$J\mathcal{D}J^{-1} = \pm\mathcal{D}$	+	+	+	+
$J\gamma J^{-1} = \pm\gamma$	+	-	+	-

Connes' Reconstruction "Theorem" (sloppy version):

Compact Riemannian spin manifolds are equivalent to spectral triples with $\mathcal{A}$ commutative.

One can therefore replace a compact 4-dim. Riemannian space-time $\mathcal{M}$ by the spectral triple $(C^\infty(\mathcal{M}), \mathcal{H}, \partial\!\!\!/)$.

Finite, $n = 0$, spectral triples:

- $\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \ldots$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$
- $\mathrm{Aut}(M_n(\mathbb{C})) = U(n)$, $\mathrm{Aut}(\mathbb{H}) = SU(2)$
- $\mathcal{H}_f \simeq \mathbb{C}^N$, N is the total number of particles (i.e. left-/right-handed particles/antiparticles are counted separately)
- $\mathcal{D}_f \in M_N(\mathbb{C})$, $\mathcal{D}_f$ is the fermionic mass matrix.

Commutation relations of the Dirac operator, the real-structure, the chirality, (the S^0-real structure) => Restrictions for the Dirac operator and the Hilbert space

The finite Hilbert space & Dirac operator:

The commutation relations of the chirality and the real structure allow to split the Hilbert space (and the representation) into left- and right-handed particle and antiparticle subspaces:

$$\mathcal{H}_f = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c$$

The Dirac operator takes the form

$$\mathcal{D}_f = \begin{pmatrix} \Delta & \Gamma \\ \Gamma^* & \bar{\Delta} \end{pmatrix} \text{ with with } \Delta = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} \text{ and } \Gamma = \Gamma^T.$$

Almost-commutative geometry:

An almost-commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is defined as the tensor product of a spectral triple $(\mathcal{A}_M = C^\infty(\mathcal{M}), \mathcal{H}_M, \mathcal{D}_M = \not\partial)$ with dimensions $n_M = p_M > 0$ (for space-time $n_M = 4$) and a finite spectral triple $(\mathcal{A}_f, \mathcal{H}_f, \mathcal{D}_f)$ with metric dimension $n_f = 0$.

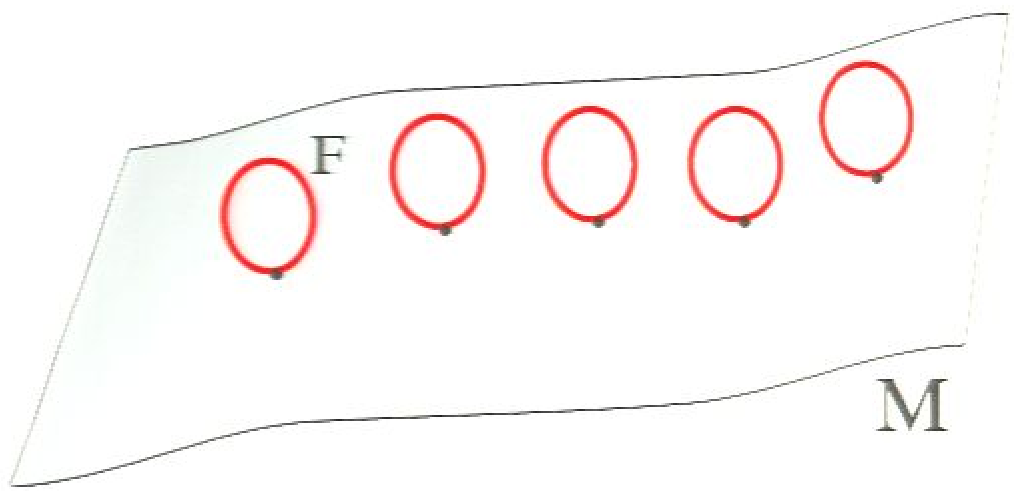
$$\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_f, \quad \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_f,$$

$$J = J_M \otimes J_f, \quad \gamma = \gamma_M \otimes \gamma_f,$$

$$\mathcal{D} = \mathcal{D}_M \otimes 1_f + \gamma_M \otimes \mathcal{D}_f$$

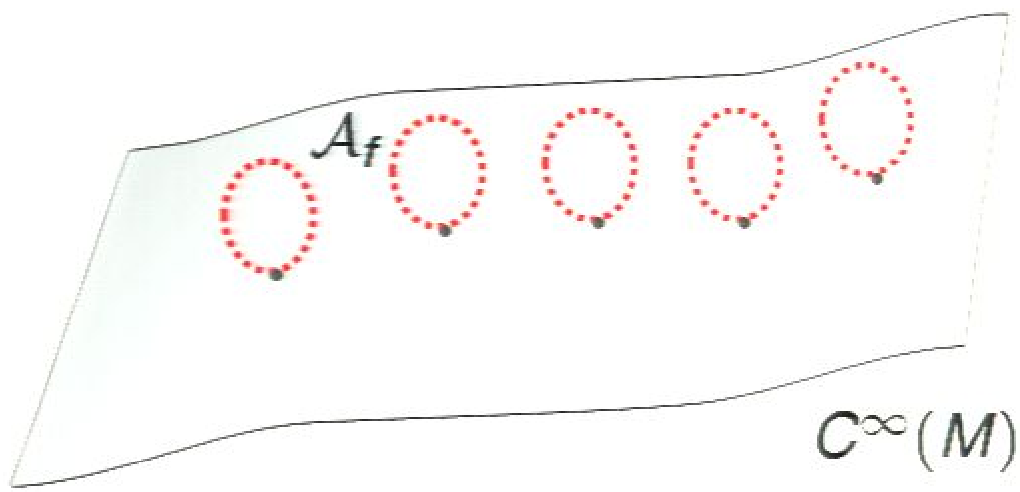
$$\text{Aut}(\mathcal{A}_M \otimes \mathcal{A}_f) = \text{Diff}(M) \ltimes \text{Aut}(\mathcal{A}_f)$$

Analogy: Almost-comm. geometry — Kaluza-Klein space



Idea: M — $C^\infty(M)$,
F — some "finite space",
differential geometry — spectral triple

Almost-commutative Geometry



"finite space" — $\mathcal{A}_f = M_1(\mathbb{K}) \oplus M_2(\mathbb{K}) \oplus \dots$

Kaluza-Klein space — almost-com. geometry. $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_f$

Overview

3 Physics

Fluctuating the Dirac operator $\mathcal{D}$:

The Dirac operator $\mathcal{D}$ does not contain gauge bosons or a Higgs boson.

Fluctuating $\mathcal{D}$ with the lifted automorphisms of $\mathcal{A}$
=> gauge bosons, Higgs bosons and curvature

$$\mathcal{D} \to {}^f\mathcal{D} = L(\sigma)\mathcal{D}L^{-1}(\sigma) = \mathcal{D} + A + JAJ^{-1}$$

$L(\sigma) = \rho(\sigma)J\rho(\sigma)J^{-1}$ = lift of the automorphisms $\sigma \in$Aut($\mathcal{A}$) to the Hilbert space. A = gauge potential (1-form).

Curvature: Promote A to an arbitrary 1-form.

Fluctuating the almost-comm. Dirac operator:

$\mathcal{D} = \partial \otimes 1_f + \gamma_5 \otimes \mathcal{D}_f, \quad \sigma \in \mathrm{Aut}((\mathcal{A}_M \otimes)\mathcal{A}_f)$

${}^f\mathcal{D} = L(\sigma)(\partial \otimes 1_f + \gamma_5 \otimes \mathcal{D}_f)L^{-1}(\sigma)$

$L(\sigma)(\partial \otimes 1_f)L^{-1}(\sigma)$ — (curvature), gauge bosons

$L(\sigma)(\gamma_5 \otimes \mathcal{D}_f)L^{-1}(\sigma)$ — Higgs boson

The spectral action (A. Connes & A. Chamseddine 1996):

The spectral action is defined to be the number of eigenvalues of the fluctuated Dirac operator up to a cut-off Λ.

$$S_{sp.} = \mathrm{Tr}(f(\frac{{}^f\mathcal{D}}{\Lambda}))$$

f is a positive test function. only its momenta play a role. The effective action is obtained by a heat-kernel expansion of $S_{sp.}$. It is the **bosonic** action of the theory.

The fermionic action:

The fermionic action is given by the scalar product $< \cdot | \cdot \cdot >$ of the Hilbert space and the fluctuated Dirac operator:

$$S_{ferm.} = < \xi | {}^{f}\mathcal{D}\xi >, \xi = \psi \otimes \phi \text{ with } \psi \in \mathcal{H}_M, \phi \in \mathcal{H}_f$$

It gives the Yukawa couplings for the fermions and the couplings of between gauge bosons and fermions.

Fermion quadrupling: the fermionic degrees of freedom are four-fold over counted.

$$\xi = (\psi_L \oplus \psi_R \oplus \psi_L^c \oplus \psi_R^c) \otimes (\phi_L \oplus \phi_R \oplus \phi_L^c \oplus \phi_R^c).$$

They have to be projected out!

Overview

4. The Standard Model

The standard model (A. Chamseddine, A. Connes 1996 & J.-H. Jureit, T. Schücker, C.S. 2005):

- $\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})(\oplus\mathbb{C})$
- $\mathrm{Aut}(\mathcal{A}_f) = \mathcal{U}^{nc}(\mathcal{A}_f) = SU(2) \times U(3)$
- $\mathcal{H}_f = \mathcal{H}_{SM}$ Hilbert space of minimal standard model fermion multiplets
- $\mathcal{D}_f$: Fermionic mass matrix with CKM matrix and PMNS matrix
- Majorana masses and SeeSaw mechanism for right-handed neutrinos (J. Barrett & A. Connes 2006)

The spectral action of the SM:

$$
\begin{aligned}
S_{sp.} = \; & \frac{48}{\pi^2} f_4 \Lambda^4 \int dV + \frac{4}{\pi^2} f_2 \Lambda^2 \int R \, dV \\
& - \frac{2a}{\pi^2} f_2 \Lambda^2 \int |\varphi|^2 \, dV + \frac{b}{2\pi^2} f_0 \int |\varphi|^4 \, dV \\
& + \frac{a}{2\pi^2} f_0 \int |D_\mu \varphi|^2 \, dV - \frac{a}{12\pi^2} f_0 \int R \, |\varphi|^2 \, dV \\
& + \frac{f_0}{2\pi^2} \int (\rho_2 B_{\mu\nu} B^{\mu\nu} + \rho_1 F^{\alpha}_{\mu\nu} F^{\alpha\mu\nu} + \rho_3 G^{i}_{\mu\nu} G^{i\mu\nu}) \, dV \\
& + \frac{f_0}{10\pi^2} \int \left(\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}\right) dV
\end{aligned}
$$

f_i moments of the test-function f; a, b functions of $\mathrm{tr}(\mathcal{D}_f)^2$;
ρ_i numerical factors depending on rep. ρ.

Extracting the constraints at the cut-off Λ:

$$\varphi \quad \longrightarrow \quad \tilde{\varphi}$$

$$\frac{a}{2\pi^2} f_0 |D_\mu \varphi|^2 \quad \longrightarrow \quad \frac{1}{2} |D_\mu \tilde{\varphi}|^2$$

$$\frac{b}{2\pi^2} f_0 |\varphi|^4 \quad \longrightarrow \quad \frac{\lambda}{24} |\tilde{\varphi}|^4$$

$$\frac{f_0}{2\pi^2} \rho_2 B_{\mu\nu} B^{\mu\nu} \quad \longrightarrow \quad \frac{1}{g_2^2} B_{\mu\nu} B^{\mu\nu}$$

$$\frac{f_0}{2\pi^2} \rho_3 G^i_{\mu\nu} G^{i\mu\nu} \quad \longrightarrow \quad \frac{1}{g_3^2} G^i_{\mu\nu} G^{i\mu\nu}$$

$$\sum_{fermions} (\psi, \varphi \psi) \quad \longrightarrow \quad \sum_{fermions} (\psi, \tilde{\varphi} \psi)$$

Constraints on the SM parameters at the cut-off Λ:

$$N_{SM}\, g_2^2 = N_{SM}\, g_3^2 = 3\, \frac{Y_2^2}{H}\, \frac{\lambda}{24} = \frac{3}{4}\, Y_2$$

- g_2, g_3: $SU_w(2), SU_c(3)$ gauge couplings
- λ: quartic Higgs coupling
- Y_2: sum of all Yukawa couplings squared
- H: sum of all Yukawa couplings to the fourth power
- N_{SM}: number of standard model generations

Consequences from the SM constraints:

Input:

- Big Desert
- $g_2(m_Z) = 0.6514, \; g_3(m_Z) = 1.221$
- $m_{top} = 170.9 \pm 2.6$ GeV
- heaviest left-handed neutrino: $0.05 \text{ eV} < m_\nu < 0.3 \text{ eV}$

Output:

- $g_2^2(\Lambda) = g_3^2(\Lambda)$ at $\Lambda = 1.1 \times 10^{17}$ GeV
- $m_{Higgs} = 168.3 \pm 2.5$ GeV
- neutrino: $y_\nu \sim 1.3 \times y_{top}$, $M_\nu \sim 10^{13}$ GeV
  right-handed neutrino: $m_\nu \sim 10^{13}$ GeV

Overview

5 Conlusions

To-do-List:

- Models beyond the Standard Model (LHC signature and/or dark matter?)
- Mechanisms for Neutrino masses (Dirac/Majorana masses or something different?)
- Renorm. group flow for all couplings in the spectral action (Exact Renorm Groups in the spirit of M. Reuter et al. ?)
- Understand fluctuations with Aut$(C^\infty(M))$ =Diff(M)
- Spectral triples with Lorentzian signature (A. Rennie, M. Paschke, R. Verch,...)
- Action principle in Lorentzian signature (Local index formula, Wodzicki residual? J. Tolksdorf et al.)

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