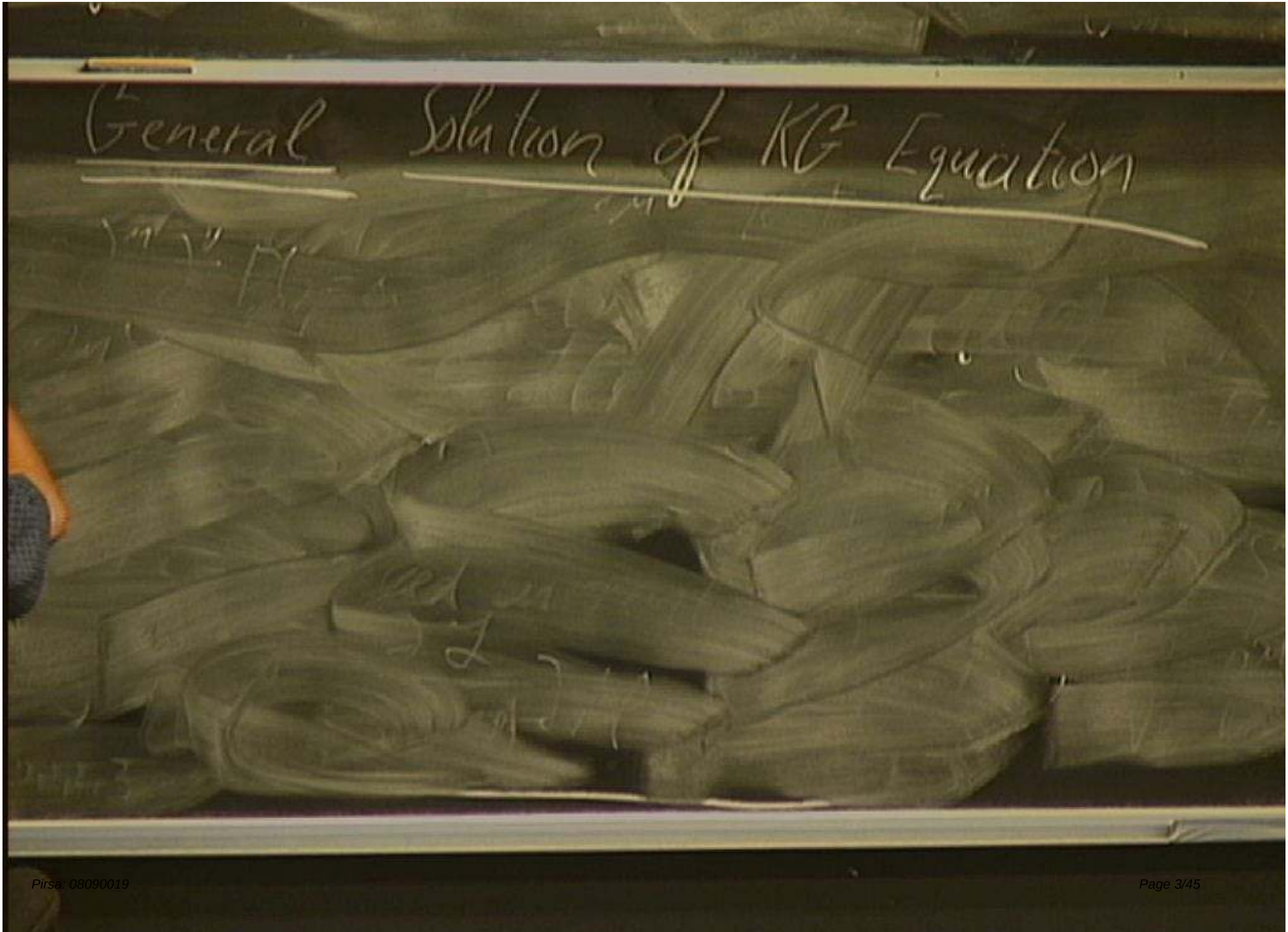
Title: Quantum Field Theory 1 - Lecture 3B

Date: Sep 24, 2008 03:30 PM

URL: http://pirsa.org/08090019

Abstract: Quantum Field Theory I course taught by Volodya Miransky of the University of Western Ontario

Topologically conserved currents. Take $F_{\mu\nu}$ ($F_{\mu\nu} = -F_{\nu\mu}$).
Introduce a current $J_\mu = \partial^\nu F_{\mu\nu}$, $\partial^\mu J_\mu = 0$ because
$\partial^\mu J_\mu = \partial^\mu \partial^\nu F_{\mu\nu} = 0$


General Solution of KG Equation

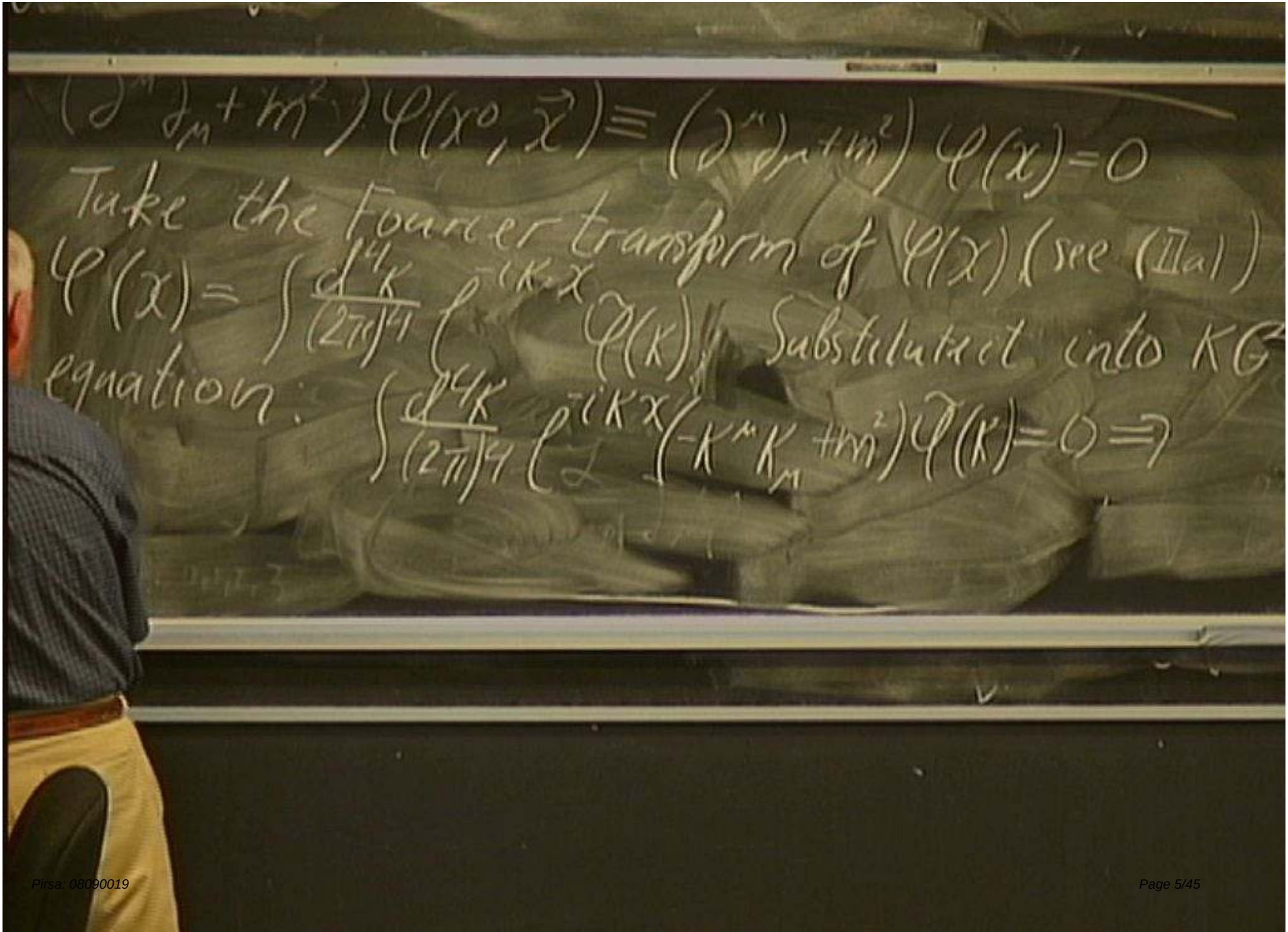
PDE

General Solution of KG Equation

$$(\partial^\mu \partial_\mu + m^2)\,\varphi(x^0, \vec{x}) \equiv (\partial^\mu \partial_\mu + m^2)\,\varphi(x) = 0$$

Take the Fourier transform of $\varphi(x)$.

$$\varphi(x) =$$


$(\partial^\mu \partial_\mu + m^2)\varphi(x^0, \vec{x}) \equiv (\partial^\mu \partial_\mu + m^2)\varphi(x) = 0$
Take the Fourier transform of $\varphi(x)$ (see IIa1)
$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\varphi}(k)$ Substitute it into KG equation:
$\int \frac{d^4k}{(2\pi)^4} e^{-ikx} (-k^\mu k_\mu + m^2)\tilde{\varphi}(k) = 0 \Rightarrow$

$(\partial^\mu \partial_\mu + m^2)\varphi(x^0, \vec{x}) \equiv (\partial^\mu \partial_\mu + m^2)\varphi(x) = 0$

Take the Fourier transform of $\varphi(x)$ (see (II.a))

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{\varphi}(k)$$

Substitute into KG equation:

$$\int \frac{d^4k}{(2\pi)^4} e^{-ikx} (-k^\mu k_\mu + m^2)\tilde{\varphi}(k) = 0 \Rightarrow$$

$$(-k^\mu k_\mu + m^2)\tilde{\varphi}(k) = 0$$

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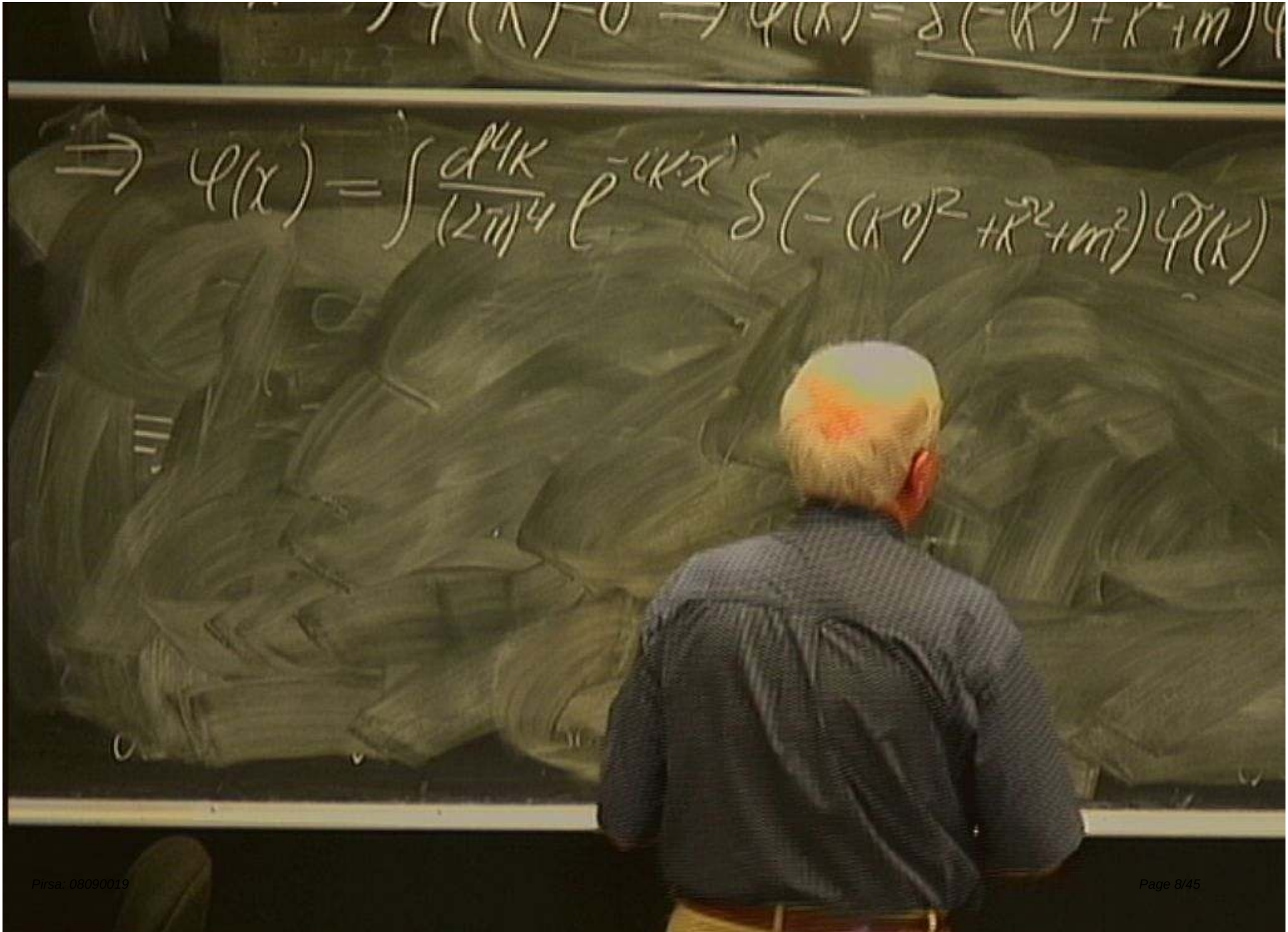
Take the Fourier transform of $\varphi(x)$ (see (II.a.1))

$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\varphi}(k)$! Substitute it into KG equation:

$\int \frac{d^4k}{(2\pi)^4} e^{ikx} (-k^\mu k_\mu + m^2)\tilde{\varphi}(k) = 0 \Rightarrow$ arbitrary

$\Rightarrow (-k^\mu k_\mu + m^2)\tilde{\varphi}(k) = 0 \Rightarrow \tilde{\varphi}(k) = \delta(-(k^0)^2 + \vec{k}^2 + m^2)\tilde{\varphi}(k)$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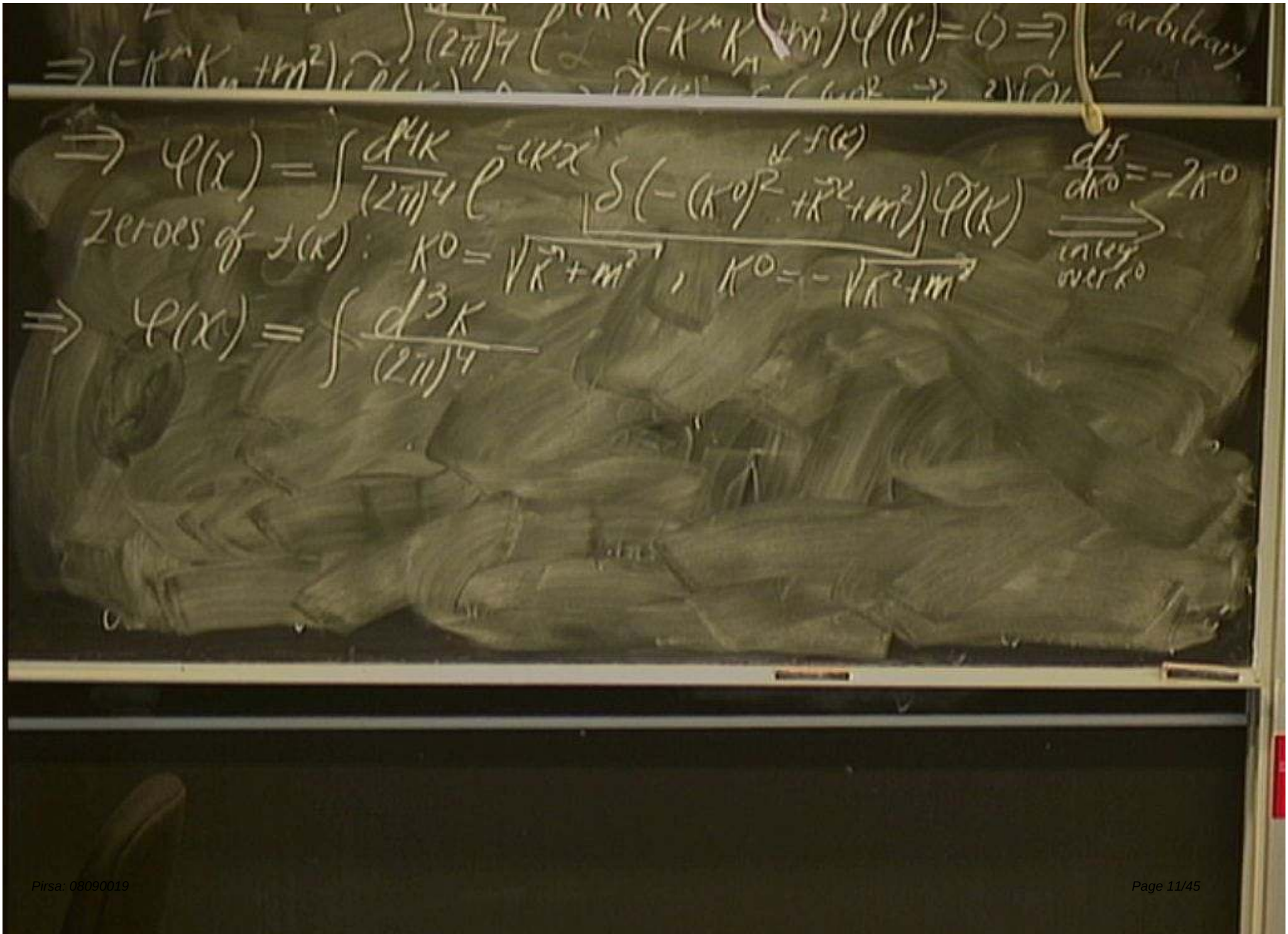
$\varphi(x) = 0 \Rightarrow \varphi(k) = \delta(-k^0 + k + m)$

$$\Rightarrow \varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \underbrace{\delta\left(-(k^0)^2 + \vec{k}^2 + m^2\right)}_{\text{"}f(k)\text{"}} \varphi(k)$$

Zeroes of $f(k)$: $k^0 = \sqrt{\vec{k}^2 + m^2}$, $k^0 = -\sqrt{\vec{k}^2 + m^2}$


$\Rightarrow (-K^\mu K_\mu + m^2)\,\tilde{\varphi}(K) = 0 \Rightarrow$ arbitrary
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$f(K)$
$\frac{df}{dK^0} = -2K^0$
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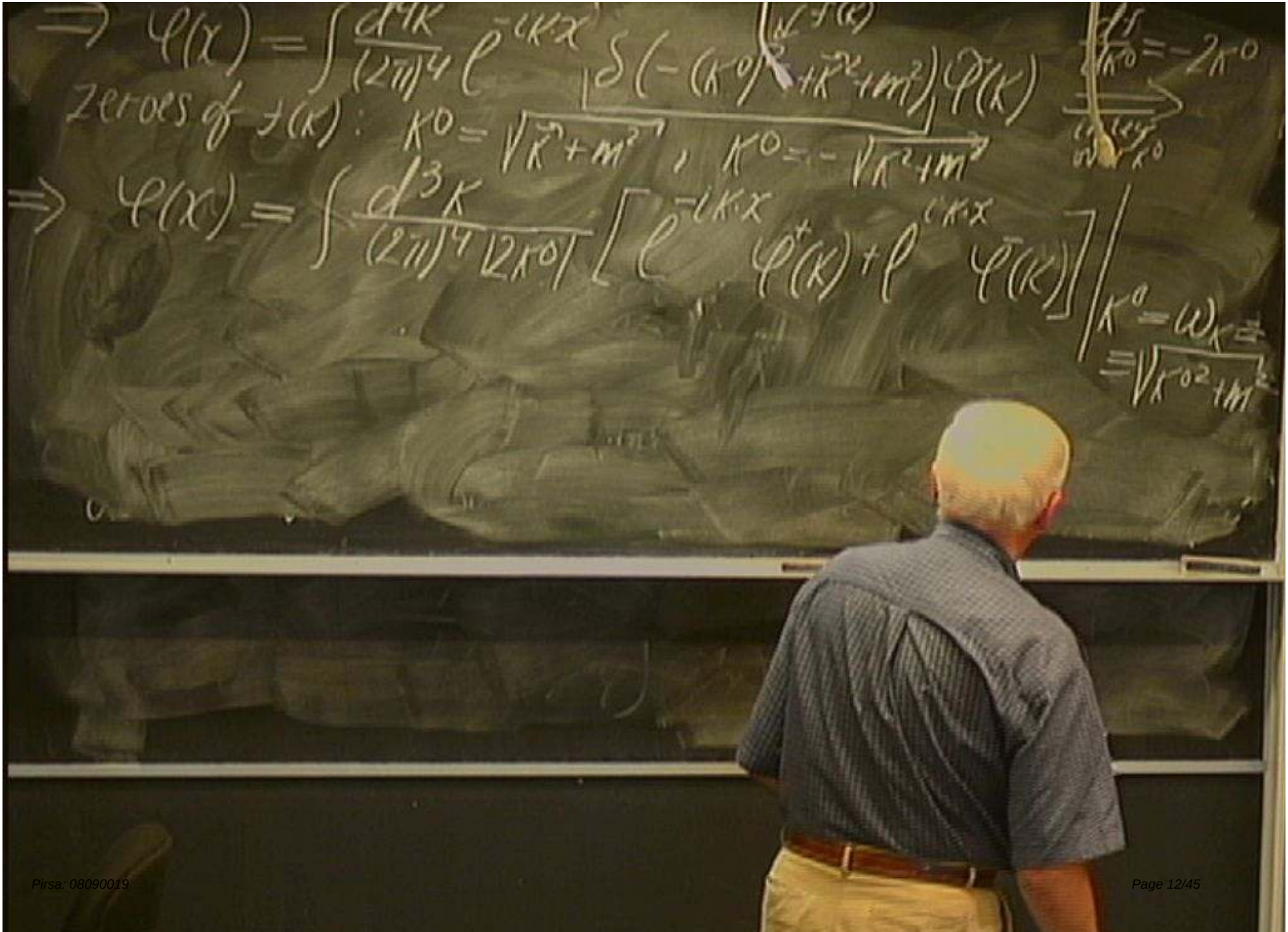
integ. over K^0
$\Rightarrow \varphi(x) = \int \frac{d^3K}{(2\pi)^4}$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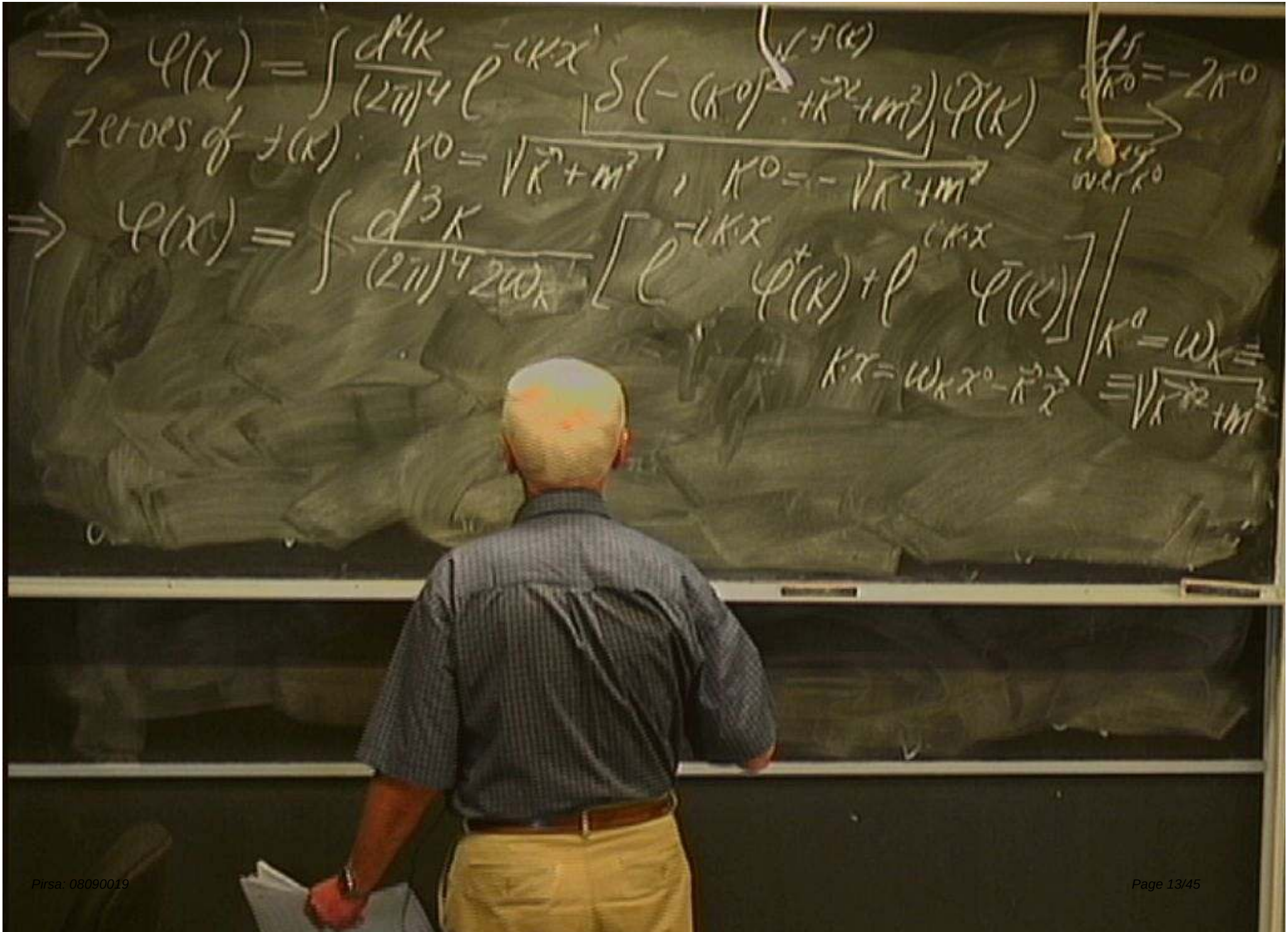
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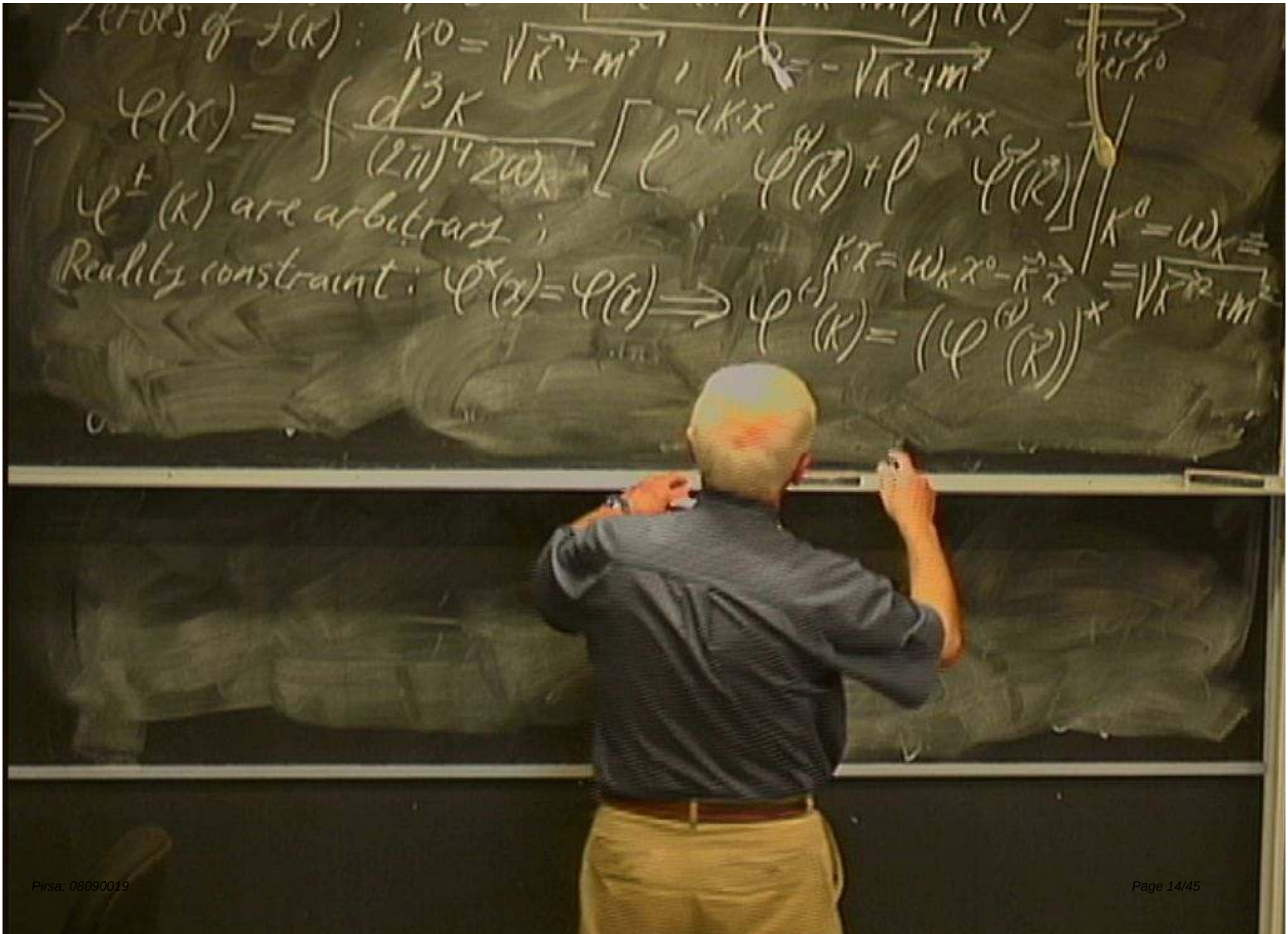
over k^0
$\Rightarrow \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ikx} \varphi^+(k) + e^{ikx} \varphi^-(k) \right]$
$k^0 = \omega_k =$
$= \sqrt{\vec{k}^2 + m^2}$
$kx = \omega_k x^0 - \vec{k}\vec{x}$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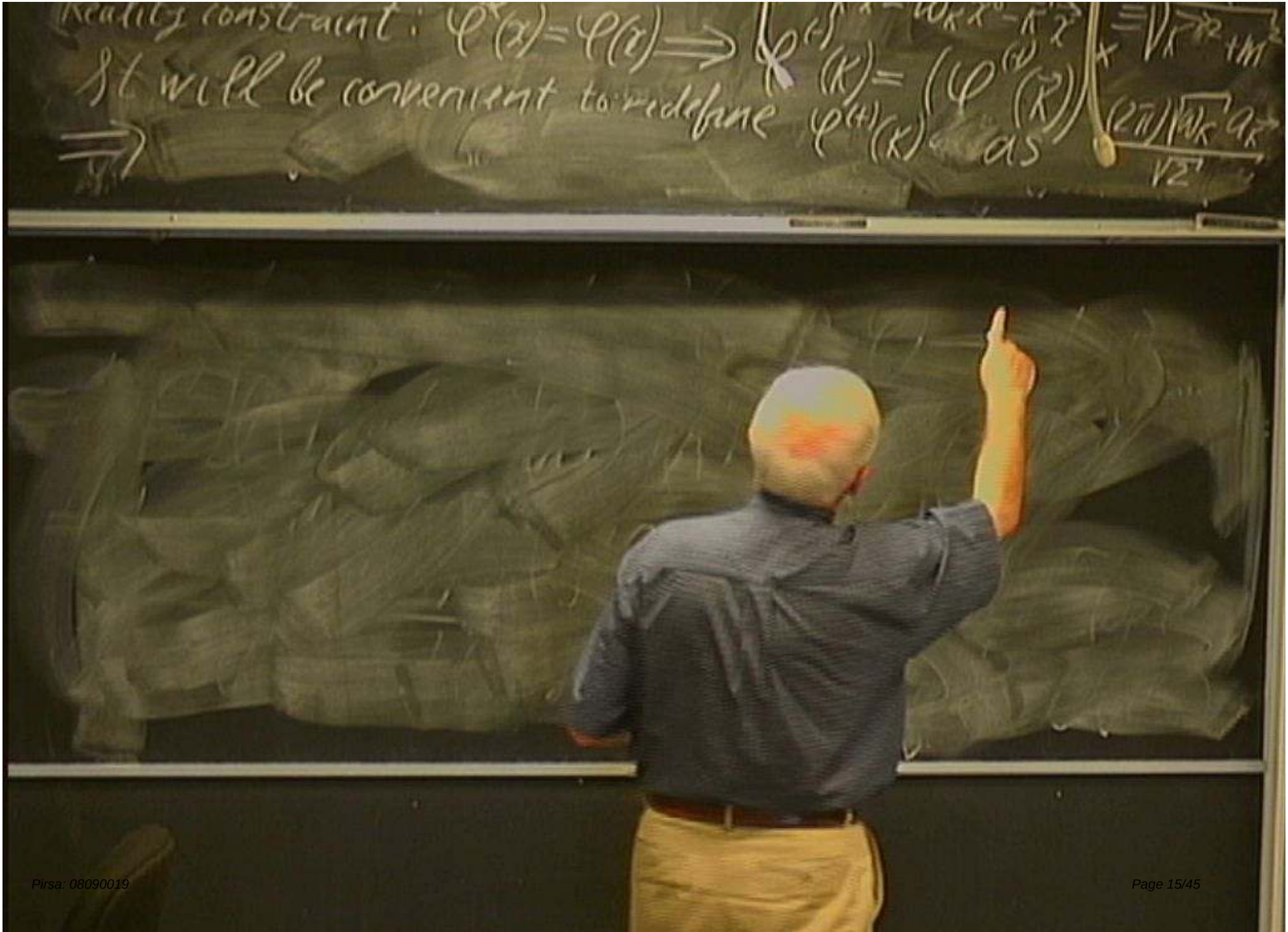
$\Rightarrow \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik\cdot x} \varphi^+(k) + e^{ik\cdot x} \varphi^-(k) \right]$
$\varphi^\pm(k)$ are arbitrary ;
Reality constraint: $\varphi^*(x) = \varphi(x) \Rightarrow \varphi^-(k) = (\varphi^+(k))^*$
$k^0 = \omega_k = \sqrt{\vec{k}^2 + m^2}$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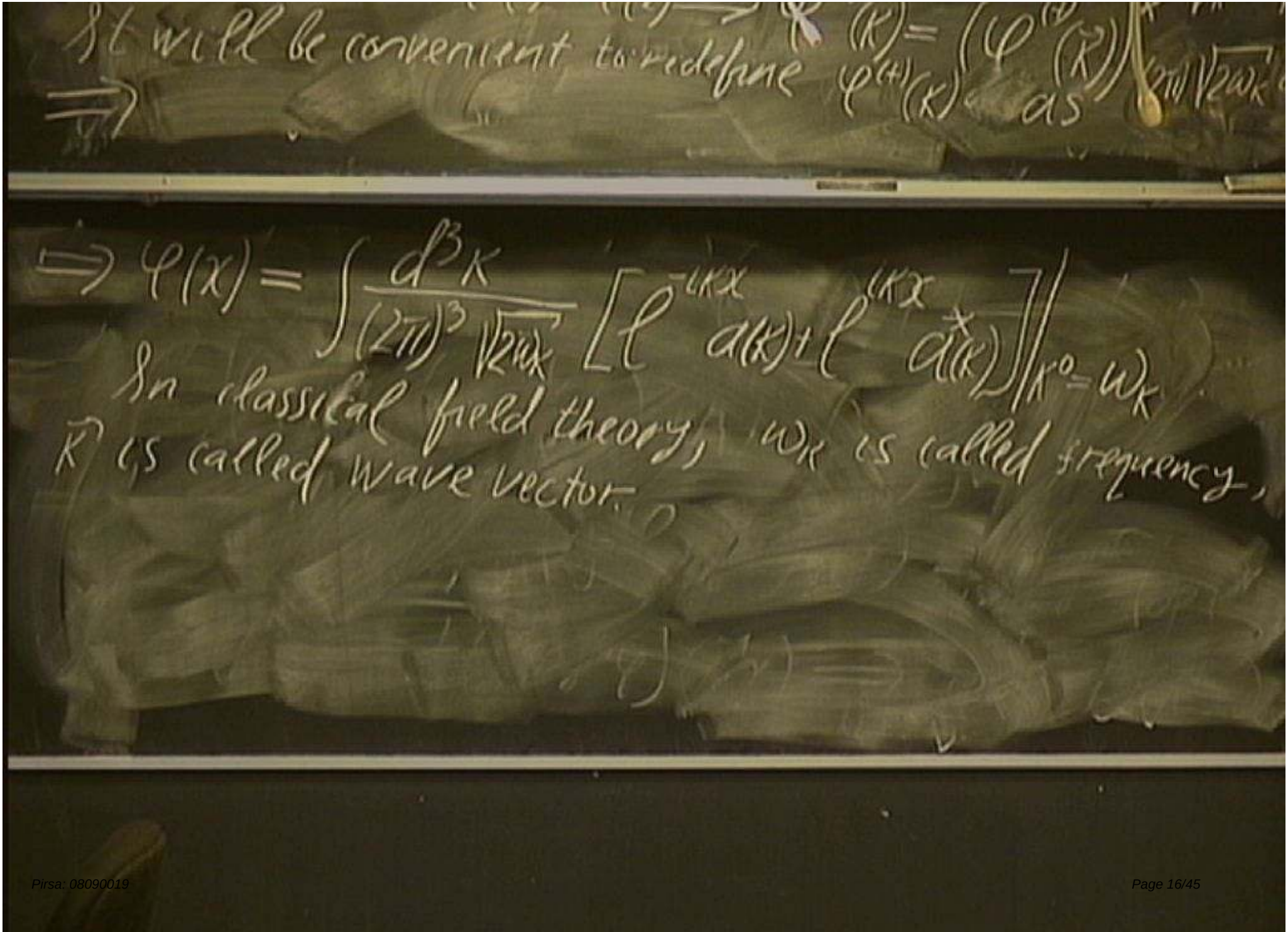
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$\Rightarrow \varphi(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} \left[e^{-ikx} a(k) + e^{ikx} a^*(k) \right]_{k^0 = \omega_k}$
In classical field theory, ω_k is called frequency,
$\vec{k}$ is called wave vector.

It will be convenient to redefine $\varphi^{(+)}(k)$ as $\varphi^{(+)}(k) = \left(\varphi^{(+)}(k)\right) \frac{1}{(2\pi)^{3/2}\sqrt{2\omega_k}}$

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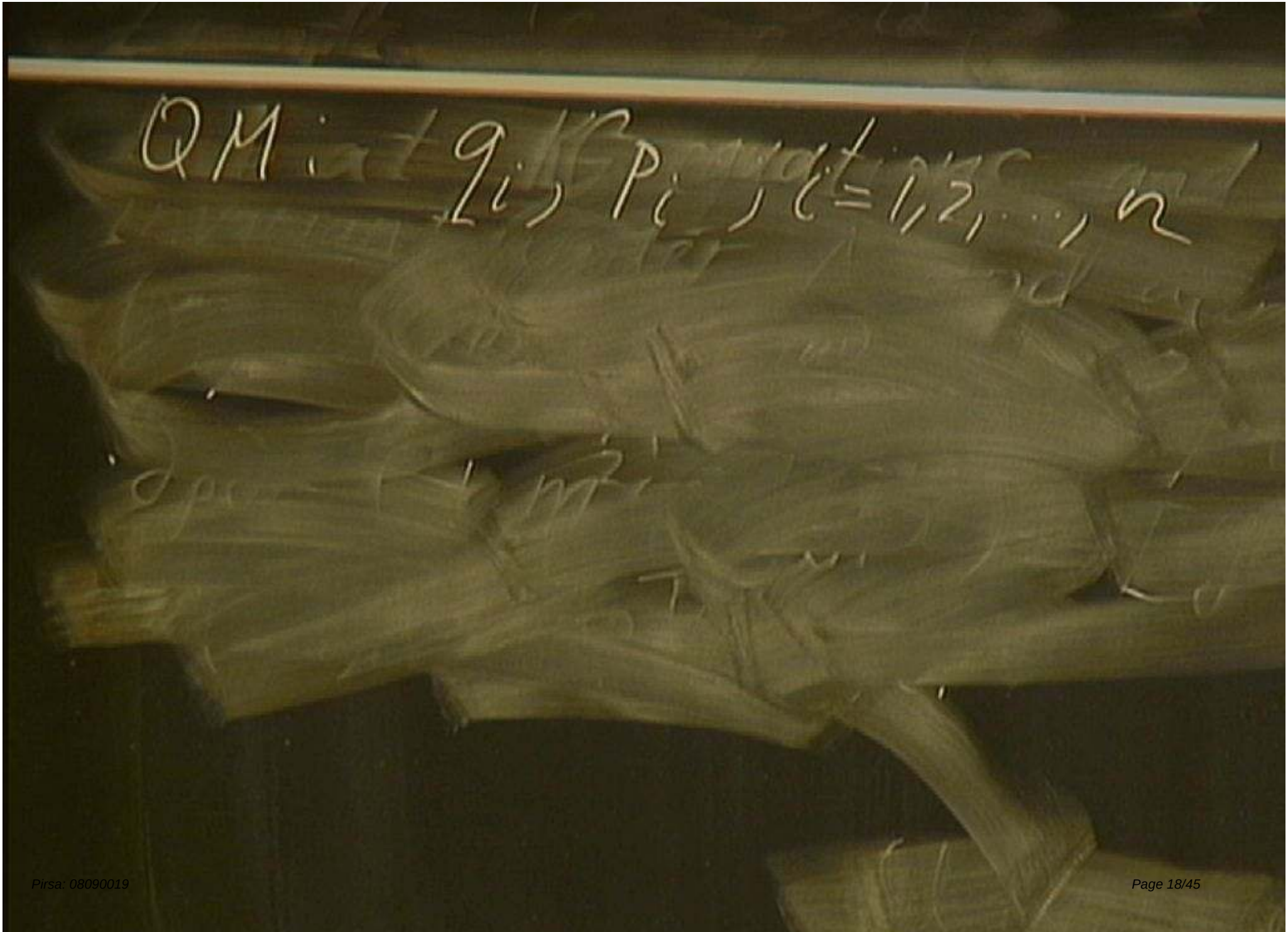
Quantization of KG field

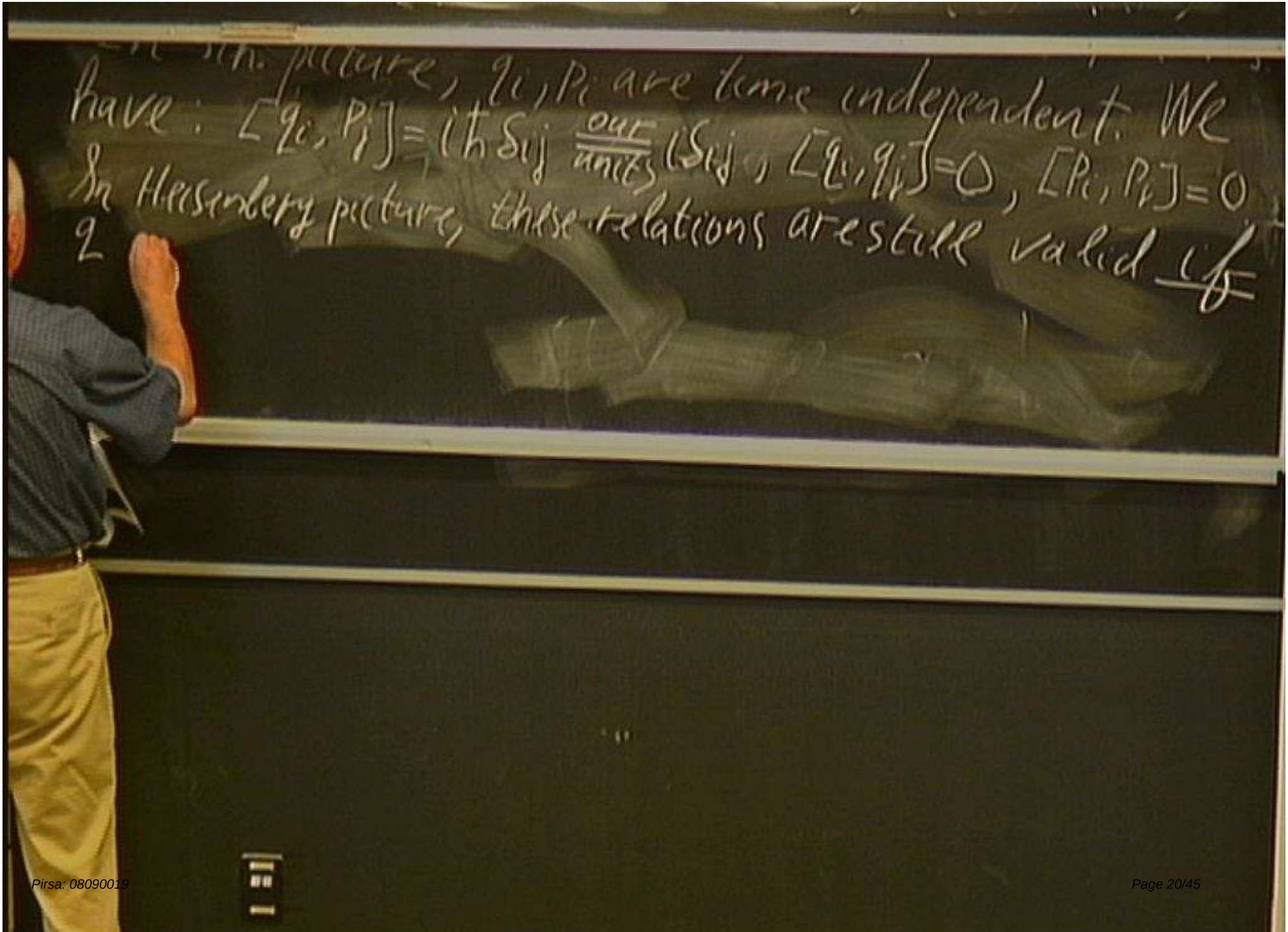
Review of Quantum Mechanics


QM: q_i, p_i $i = 1, 2, \ldots, n$

QM: $q_i, p_i, i=1,2,\ldots,n$ become operators.

In QM, there Schrödinger and Heisenbery pictures.

In Sch. picture, q_i, p_i are time independent. We have: $[q_i, p_j] = i\hbar\delta_{ij}$ $\underline{\underline{\text{our units}}}$ δ_{ij}, $[q_i, q_j]=0$, $[p_i, p_j]=0$.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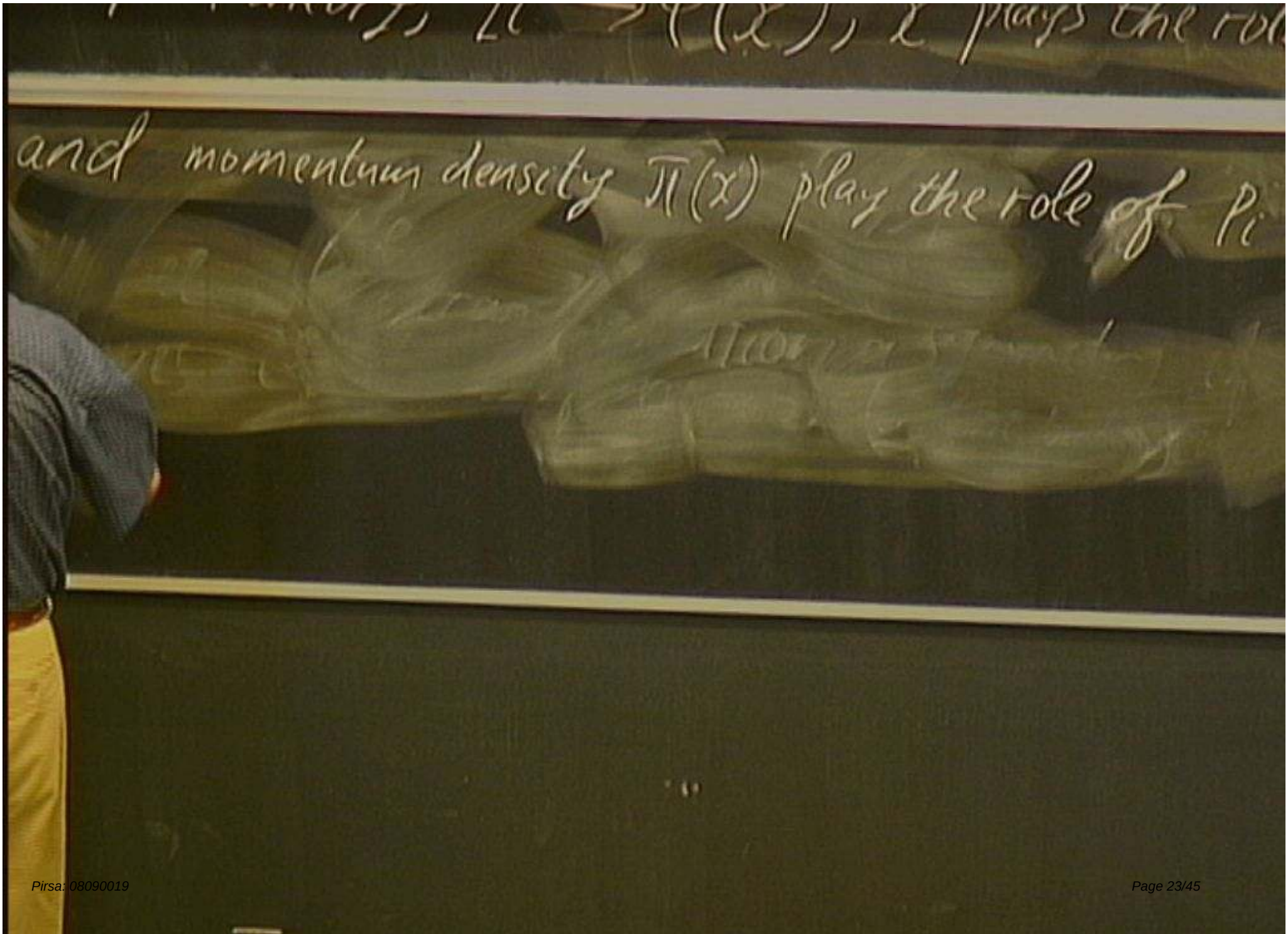
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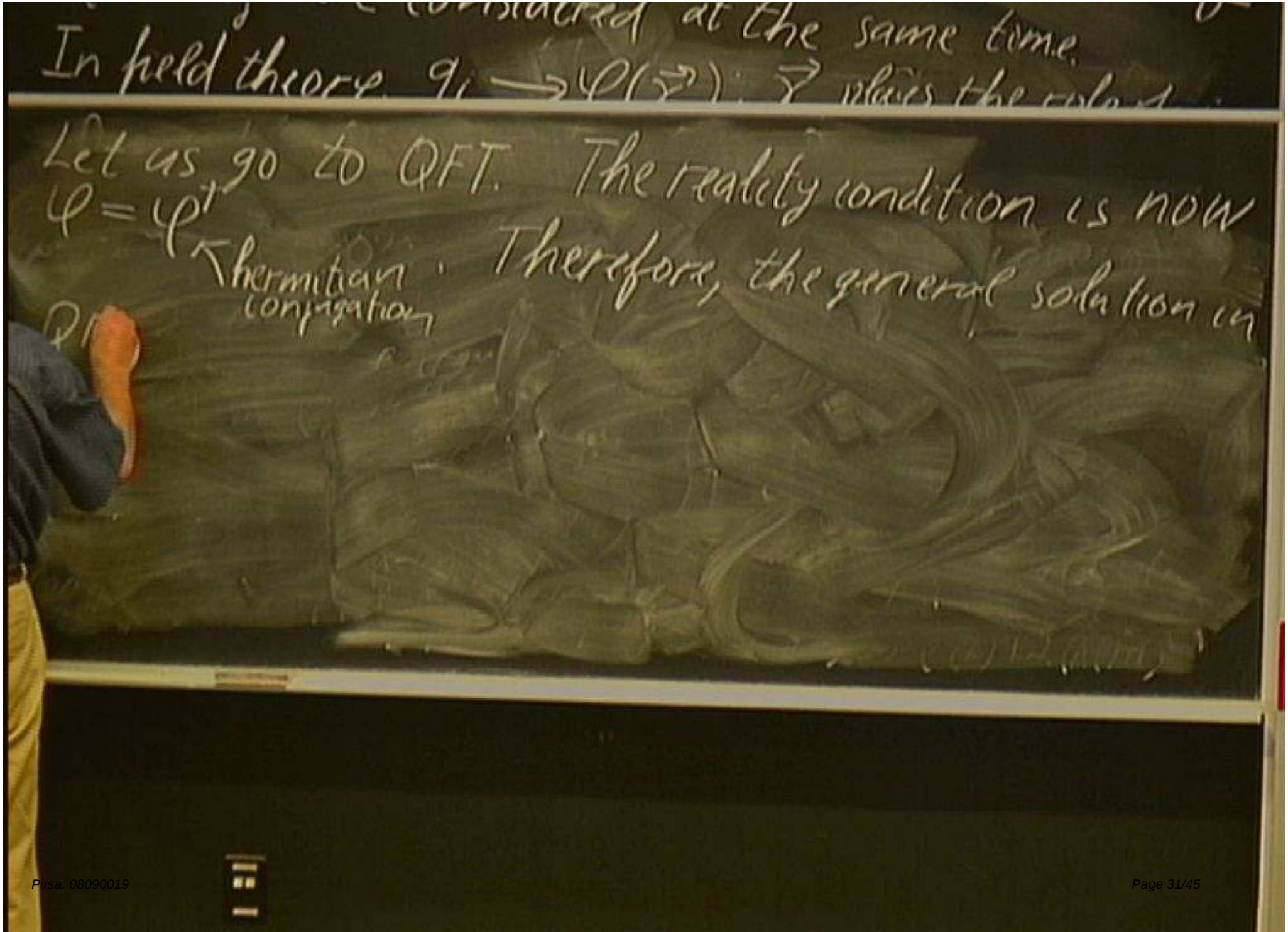
↑ hermitian conjugation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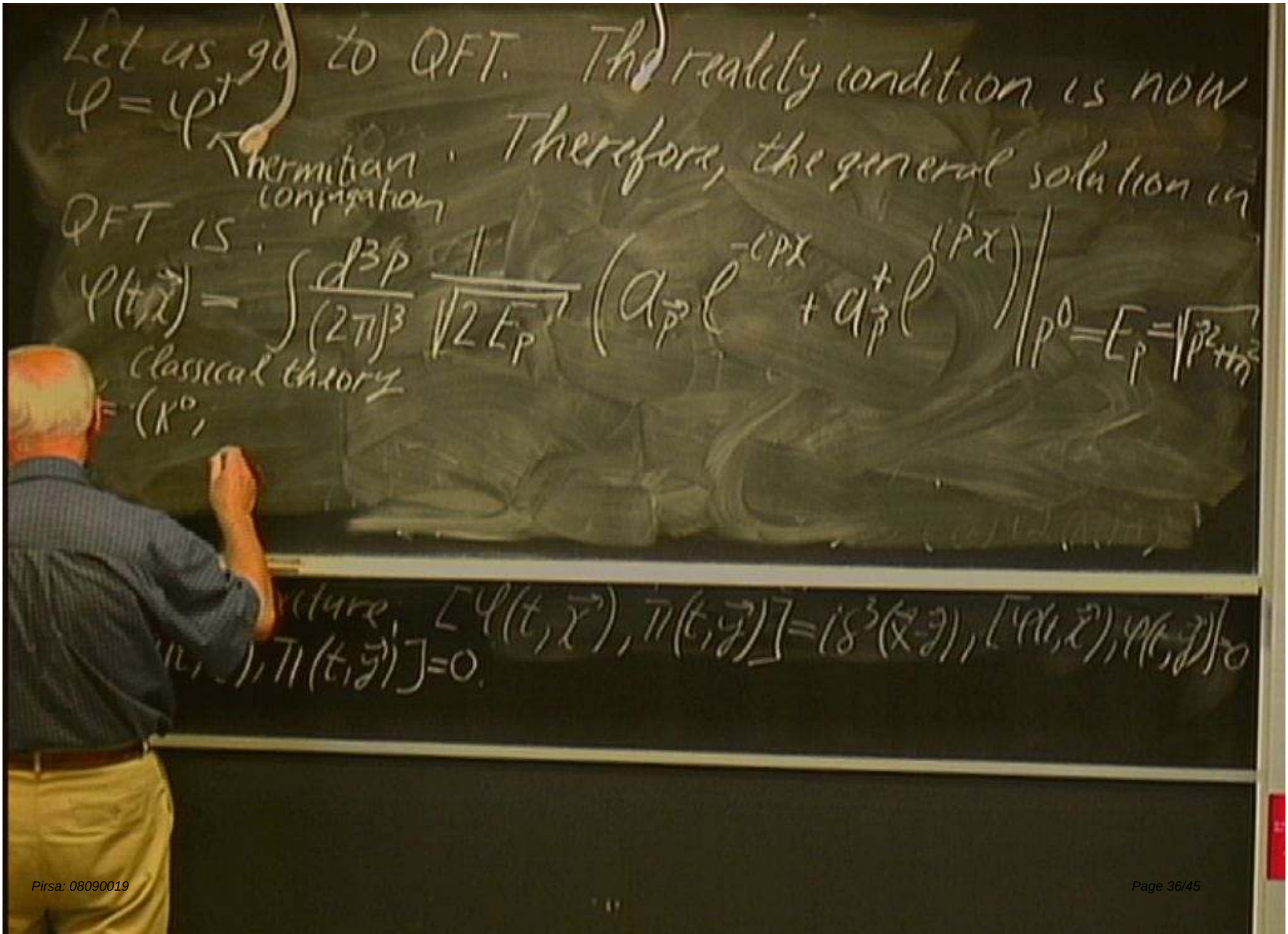
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Classical theory

$K = (K^0, \vec{K}) \longrightarrow (p^0 = \sqrt{\vec{p}^2 + m^2},\ \vec{p})$

K^0: frequency; $\vec{K}$: wave vector; p^0: energy; $\vec{p}$: momentum

$$\boxed{\begin{aligned} p^0 &= \hbar K^0 \\ \vec{p} &= \hbar \vec{K} \end{aligned}} \qquad \hbar = 1 \text{ in our units}$$

Heisenberg picture: $[\varphi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})$, $[\pi(t,\vec{x}), \varphi(t,\vec{y})] = 0$

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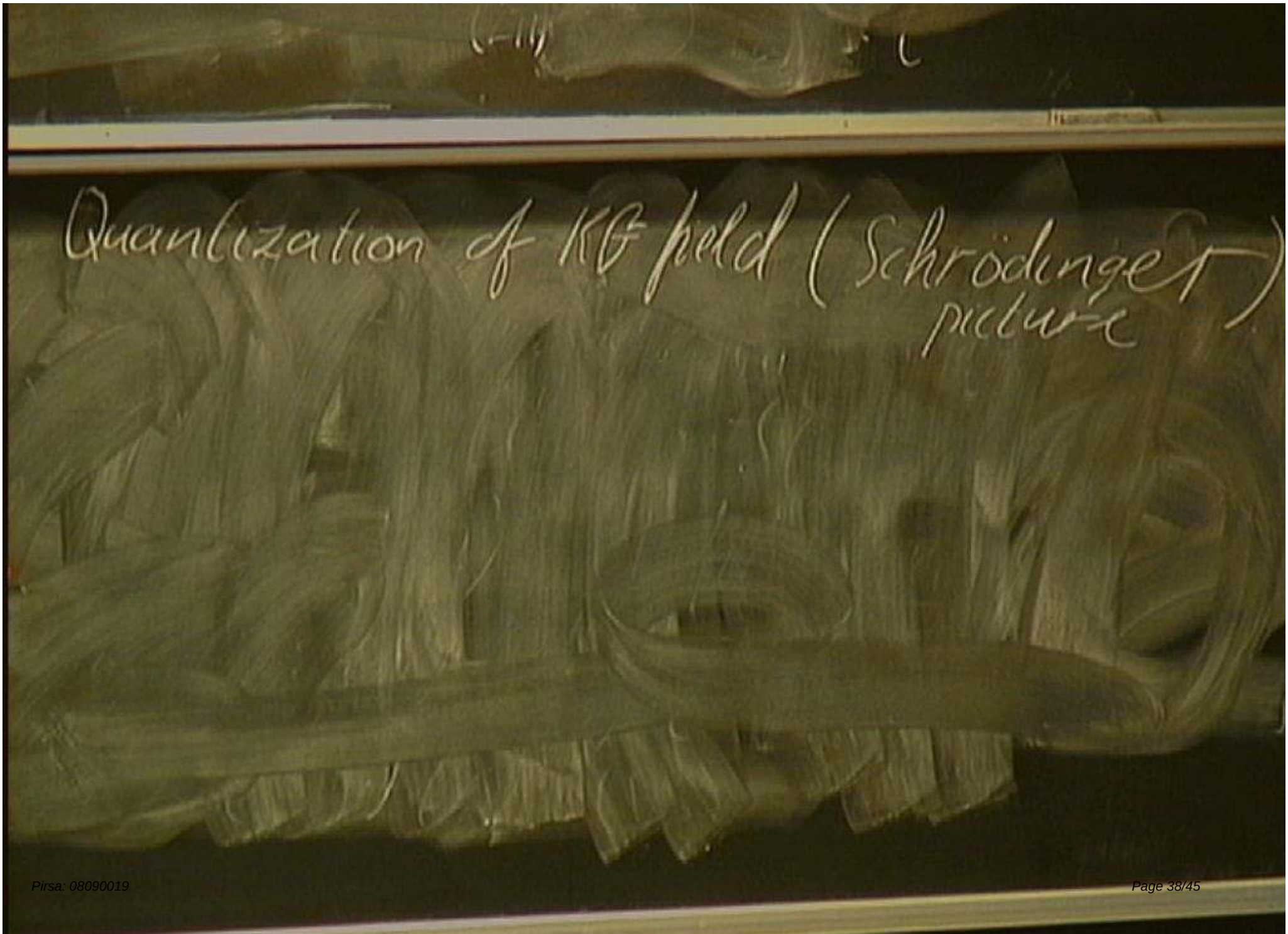
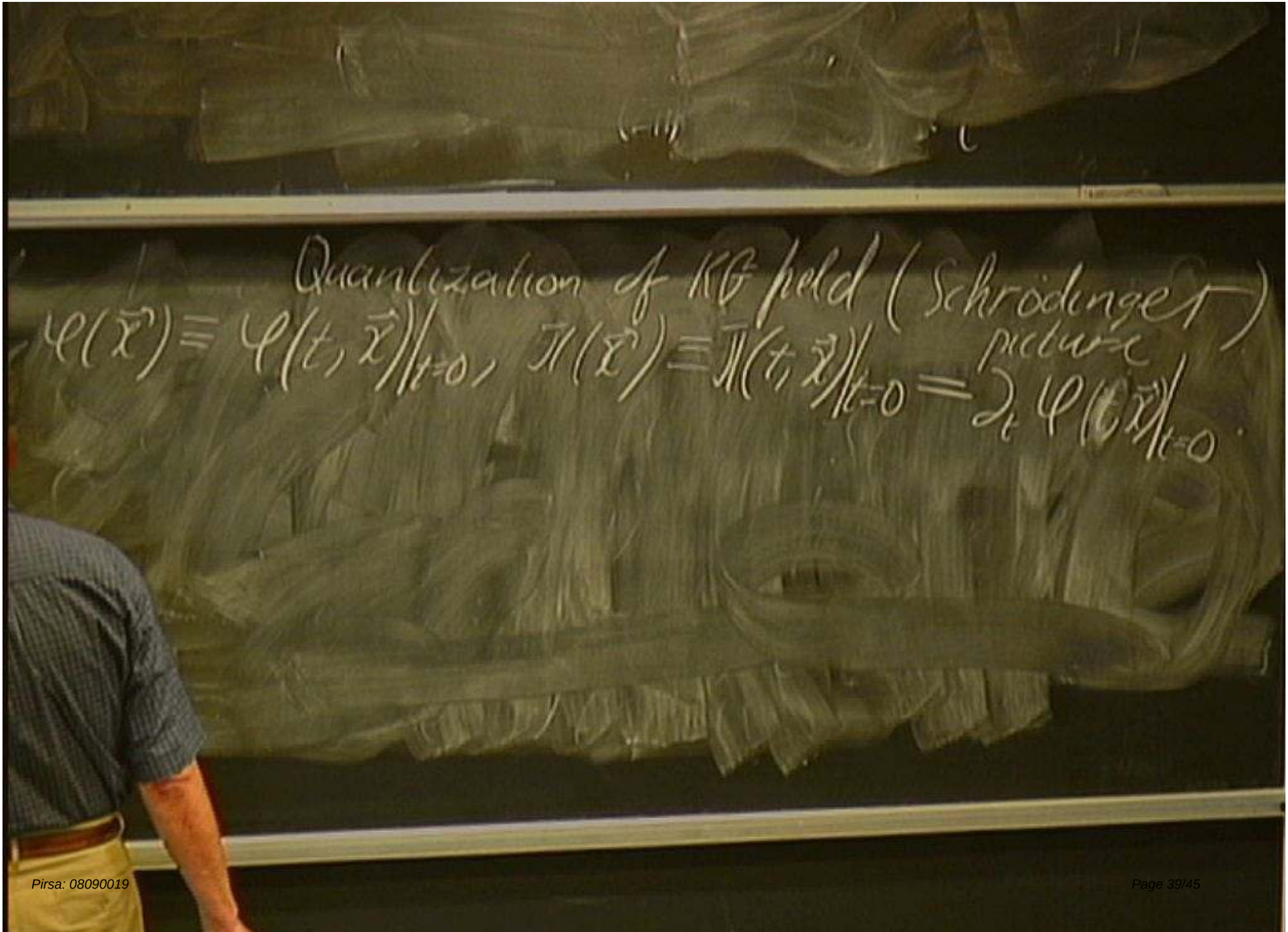

Quantization of KG field (Schrödinger picture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$\varphi(\vec{x}) \equiv \varphi(t,\vec{x})|_{t=0}$, $\pi(\vec{x}) \equiv \dot{\varphi}(t,\vec{x})|_{t=0} = \partial_t \varphi(t,\vec{x})|_{t=0}$.

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Commutation relations (1).

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$\varphi(\vec{x})$ is in Eq. (2).

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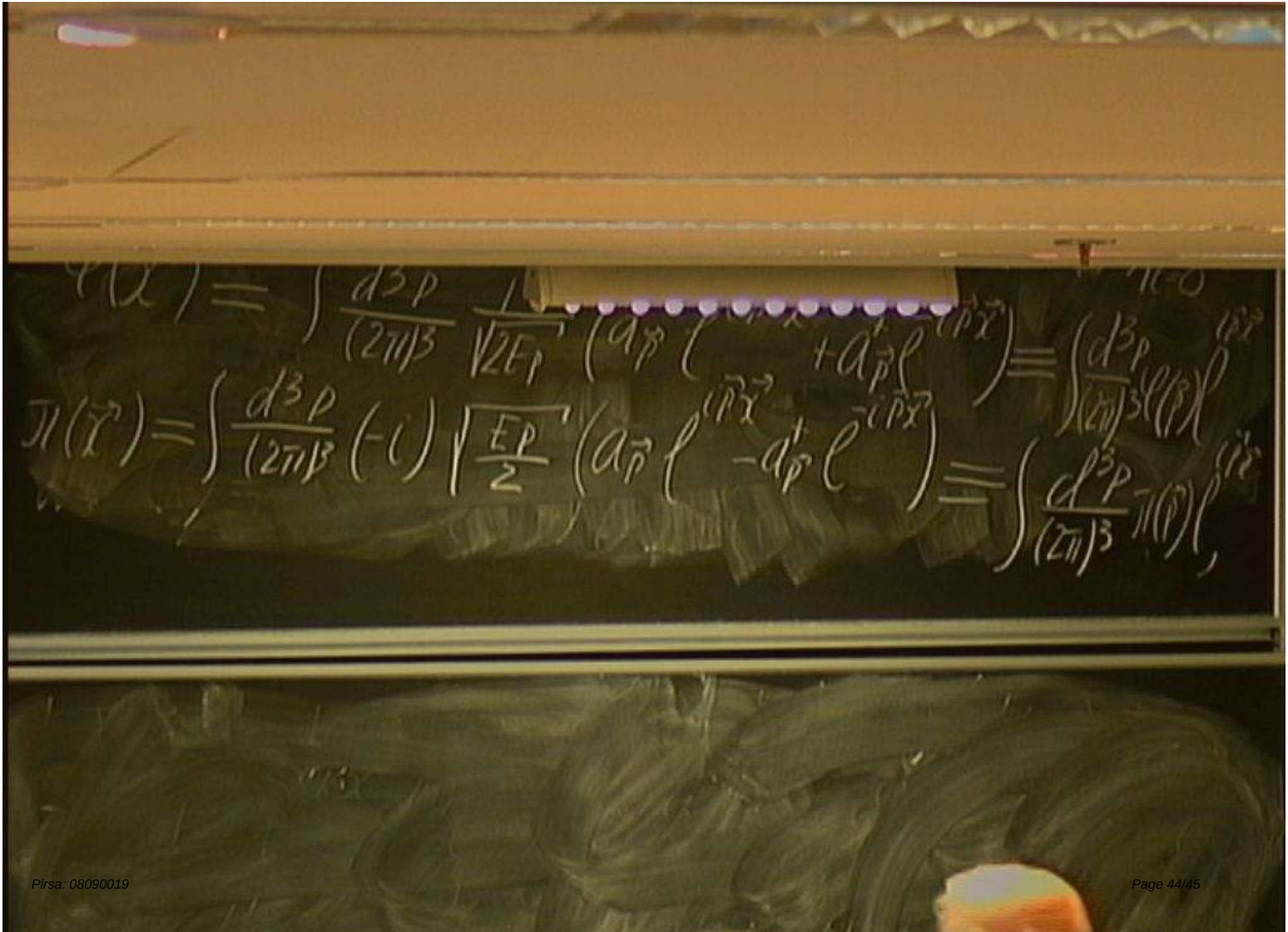
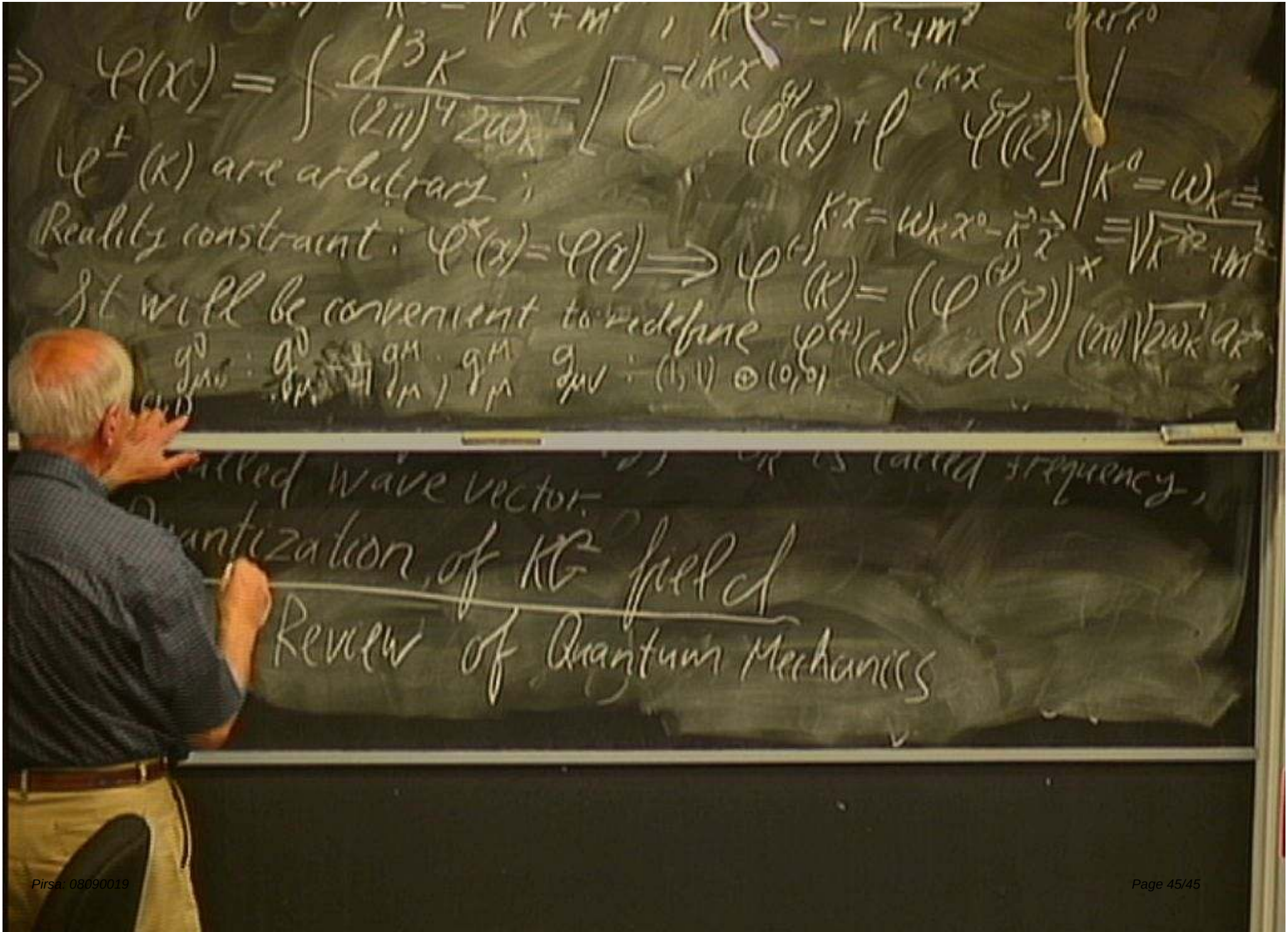
$\varphi(\vec{x})$ is in Eq. (2), at $t=0$


$\varphi(x) \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_{\vec{p}}\, e^{-ipx} + a^\dagger_{\vec{p}}\, e^{ipx}) \equiv \int \frac{d^3p}{(2\pi)^3} \varphi(p) e^{ipx}$
$\Pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} (a_{\vec{p}}\, e^{-ipx} - a^\dagger_{\vec{p}}\, e^{ipx}) \equiv \int \frac{d^3p}{(2\pi)^3} \Pi(p) e^{ipx}$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Review of Quantum Mechanics