Title: Quantum Mechanics as a Real-Vector-Space Theory with a Universal Auxiliary Rebit

Date: Aug 11, 2009 02:30 PM

URL: http://pirsa.org/08080100

Abstract: In a 1960 paper, E. C. G. Stueckelberg showed how one can obtain the familiar complex-vector-space structure of quantum mechanics by starting with a real-vector-space theory and imposing a superselection rule. In this talk I interpret Stueckelberg’s construction in terms of a single auxiliary real-vector-space binary object—a universal rebit, or “ubit." The superselection rule appears as a limitation on our ability to measure the ubit or to use it in state transformations. This interpretation raises the following questions: (i) What is the ubit? (ii) Could the superselection rule emerge naturally as a result of decoherence? (iii) If so, could one hope to see experimentally any effects of imperfect decoherence?

Background reading:

E. C. G. Stueckelberg, "Quantum Theory in Real Hilbert Space," Helv. Phys. Acta 33, 727 (1960).

P. Goyal, "From Information Geometry to Quantum Theory," arXiv:0805.2770.

C. M. Caves, C. A. Fuchs, and P. Rungta, "Entanglement of Formation of an Arbitrary State of Two Rebits," arXiv:quant-ph/0009063.

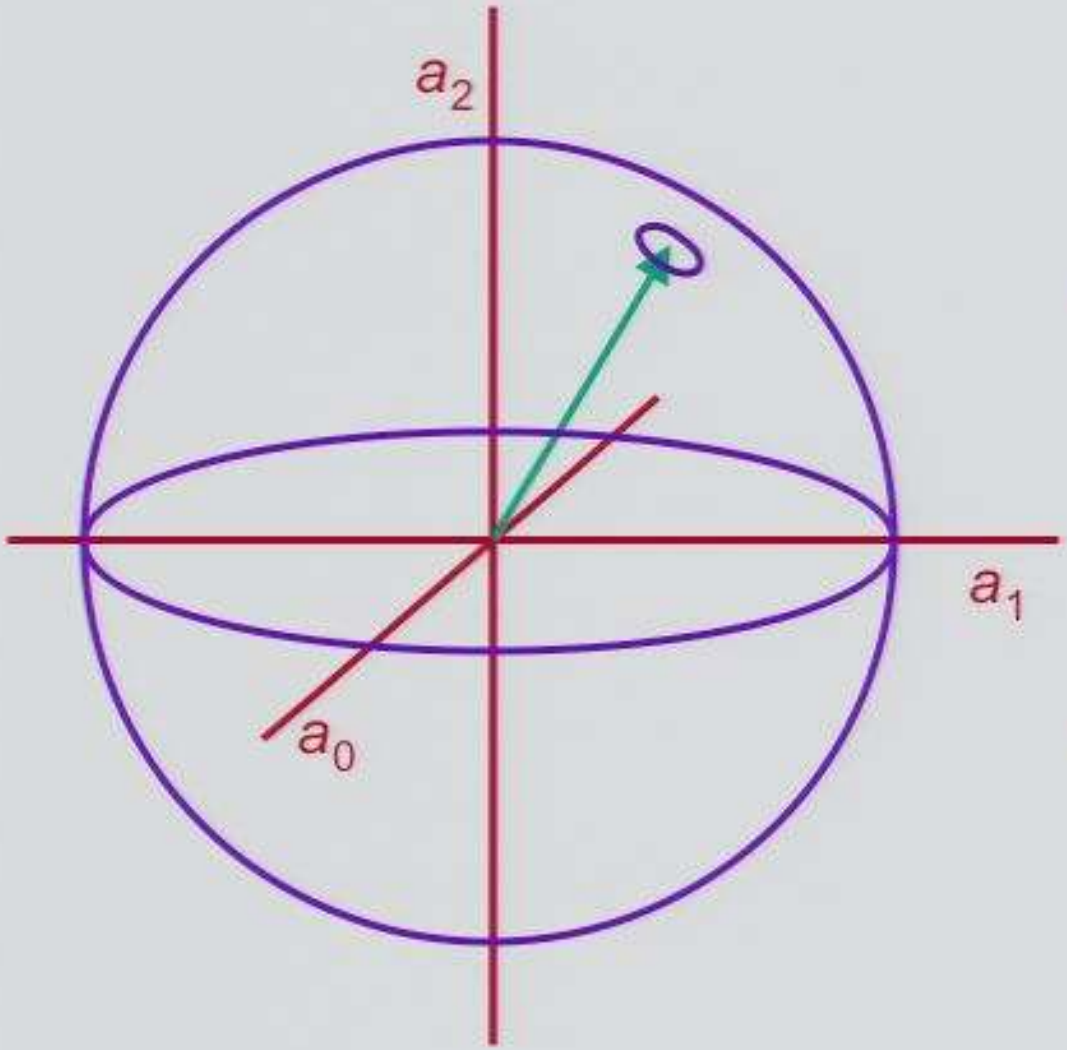
Quantum Mechanics as a Real-Vector-Space Theory with a Universal Auxiliary Rebit*

William K. Wootters

Williams College

*or, what do you do if your reconstruction produces *real-vector-space* quantum mechanics?

Real-Vector-Space Quantum Mechanics



- As many real parameters as independent probabilities.
- Statistical error after many trials is isotropic, with a spread that is independent of the location of the state.

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle + a_2|2\rangle$$

Stueckelberg's superselection rule (for finite dimensions) (1960)

To describe a system with N orthogonal states:

- Start with a $2N$-dimensional real vector space.
- States are symmetric $2N \times 2N$ density matrices.
- A transformation is an orthogonal matrix with determinant 1.

Rule: All observables and evolutions must commute with J.

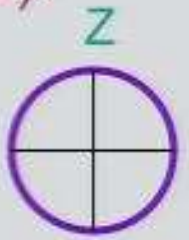
Note that $J^2 = -I$.

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Restating Stueckelberg's rule

To describe a system with *N* orthogonal states:

- Start with an *N*-dimensional real vector space.
- Assume an additional binary object, the universal rebit, or "ubit" (Z).



Restrictions:

- Observables and transformations must commute with rotations of the ubit; e.g., a transformation can't be conditioned on the state of the ubit.
- In particular, the ubit itself cannot be measured.

States differing by a rotation of the ubit can't be distinguished.

So, represent a pure state as a rank-two projection operator (divided by 2) obtained by "twirling" the ubit.

Stueckelberg's superselection rule (for finite dimensions) (1960)

To describe a system with N orthogonal states:

- Start with a $2N$-dimensional real vector space.
- States are symmetric $2N \times 2N$ density matrices.
- A transformation is an orthogonal matrix with determinant 1.

Rule: All observables and evolutions must commute with J.

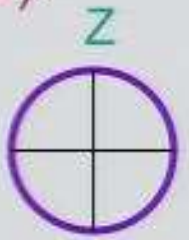
Note that $J^2 = -I$.

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Restating Stueckelberg's rule

To describe a system with *N* orthogonal states:

- Start with an *N*-dimensional real vector space.
- Assume an additional binary object, the universal rebit, or "ubit" (Z).



Restrictions:

- Observables and transformations must commute with rotations of the ubit; e.g., a transformation can't be conditioned on the state of the ubit.
- In particular, the ubit itself cannot be measured.

States differing by a rotation of the ubit can't be distinguished.

So, represent a pure state as a rank-two projection operator (divided by 2) obtained by "twirling" the ubit.

Conversion from the complex to the real representation

In every square matrix: $a + ib \longrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Or, given a pure state $|\psi\rangle = (a + ib)|0\rangle + (c + id)|1\rangle$;

$$i|\psi\rangle = (-b + ia)|0\rangle + (-d + ic)|1\rangle :$$

$$|\psi_1\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

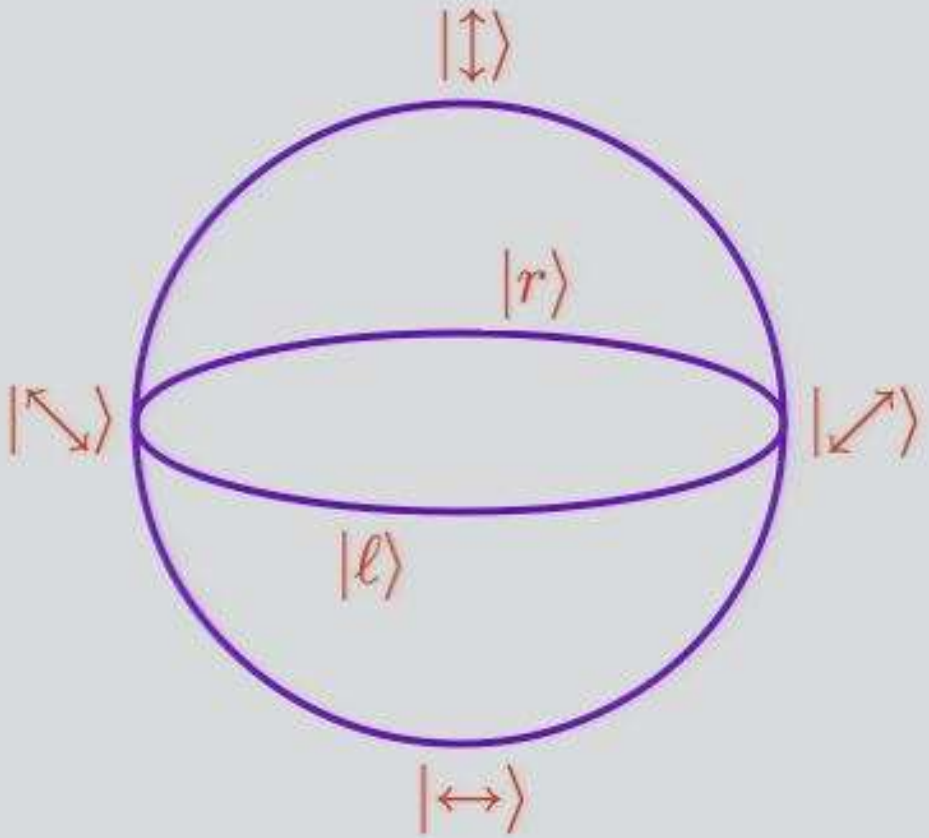
$$|\psi_2\rangle = -b|00\rangle + a|01\rangle - d|10\rangle + c|11\rangle$$

ubit Z

Then $\rho = (1/2)(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)$.

Example: Photon Polarization

Standard complex representation:



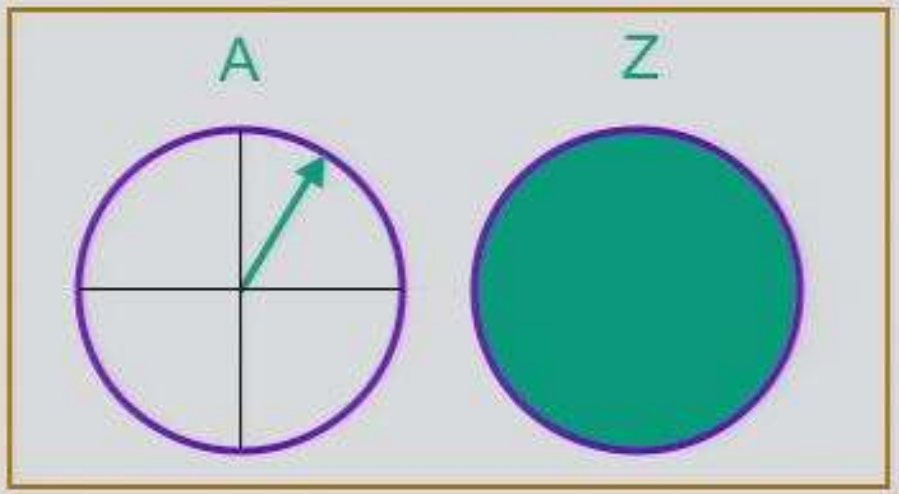
$$|\nearrow\rangle = \left(|\updownarrow\rangle + |\leftrightarrow\rangle\right)/\sqrt{2}$$

$$|\nwarrow\rangle = \left(|\updownarrow\rangle - |\leftrightarrow\rangle\right)/\sqrt{2}$$

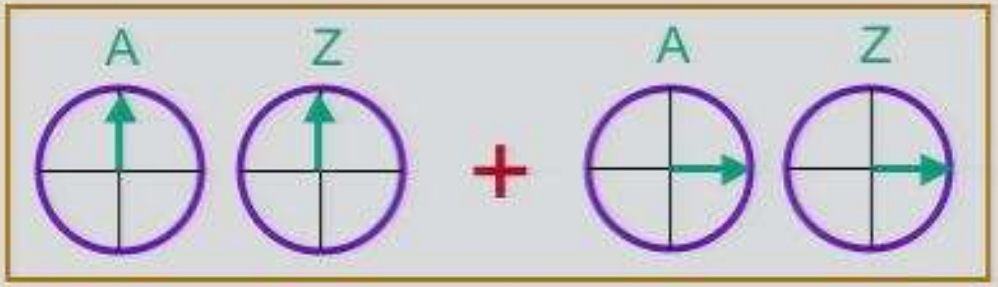
$$|r\rangle = \left(|\updownarrow\rangle + i\,|\leftrightarrow\rangle\right)/\sqrt{2}$$

$$|\ell\rangle = \left(|\updownarrow\rangle - i\,|\leftrightarrow\rangle\right)/\sqrt{2}$$

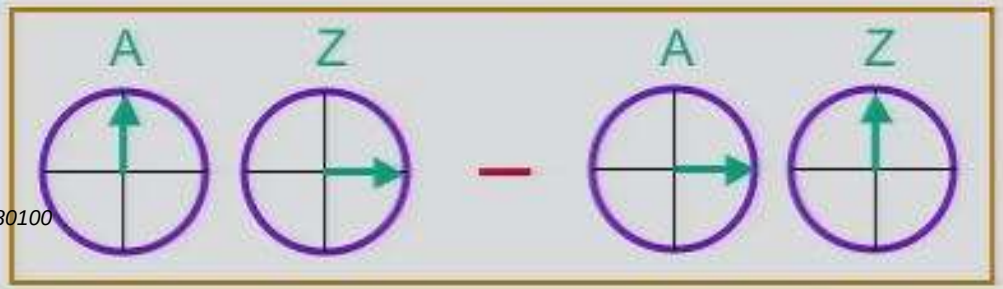
Photon Polarization--Real Representation with the Ubit



linear polarization



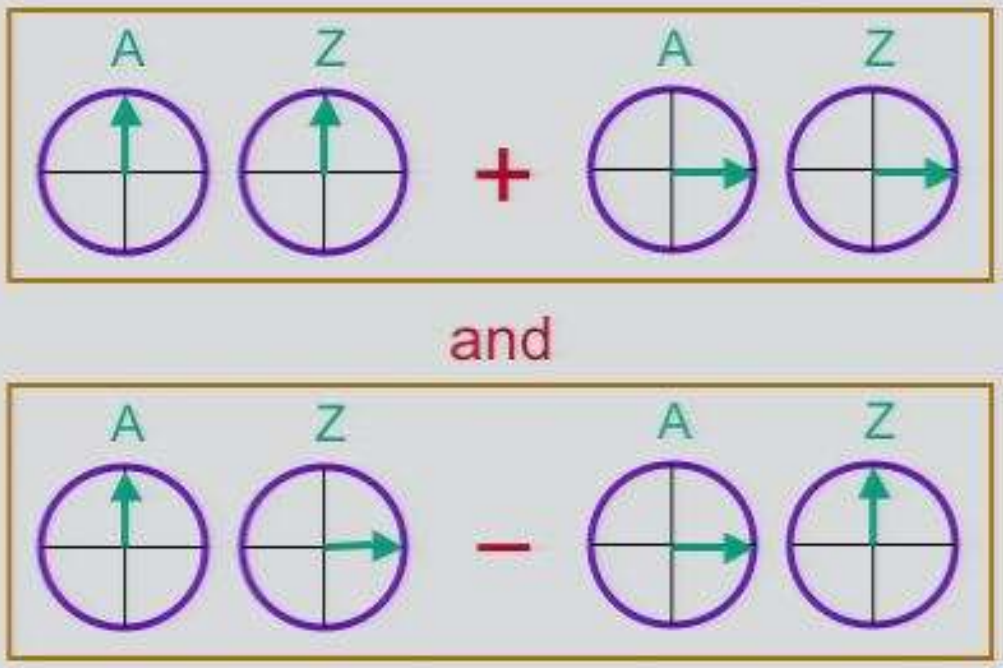
and



right circular polarization

This ***mixed*** state is maximally entangled (but not correlated!).

Surprising features of the state representing right-circular polarization

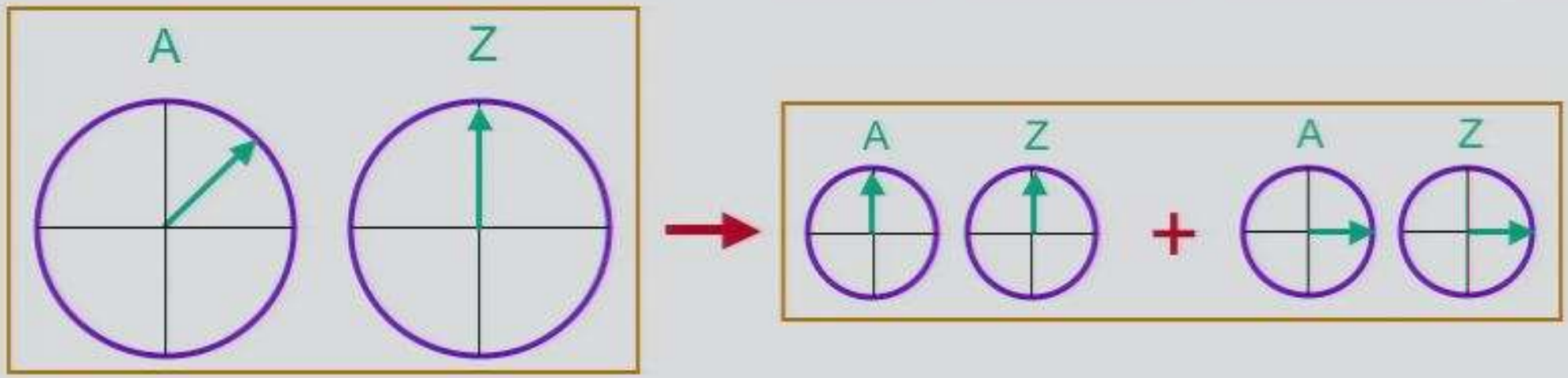


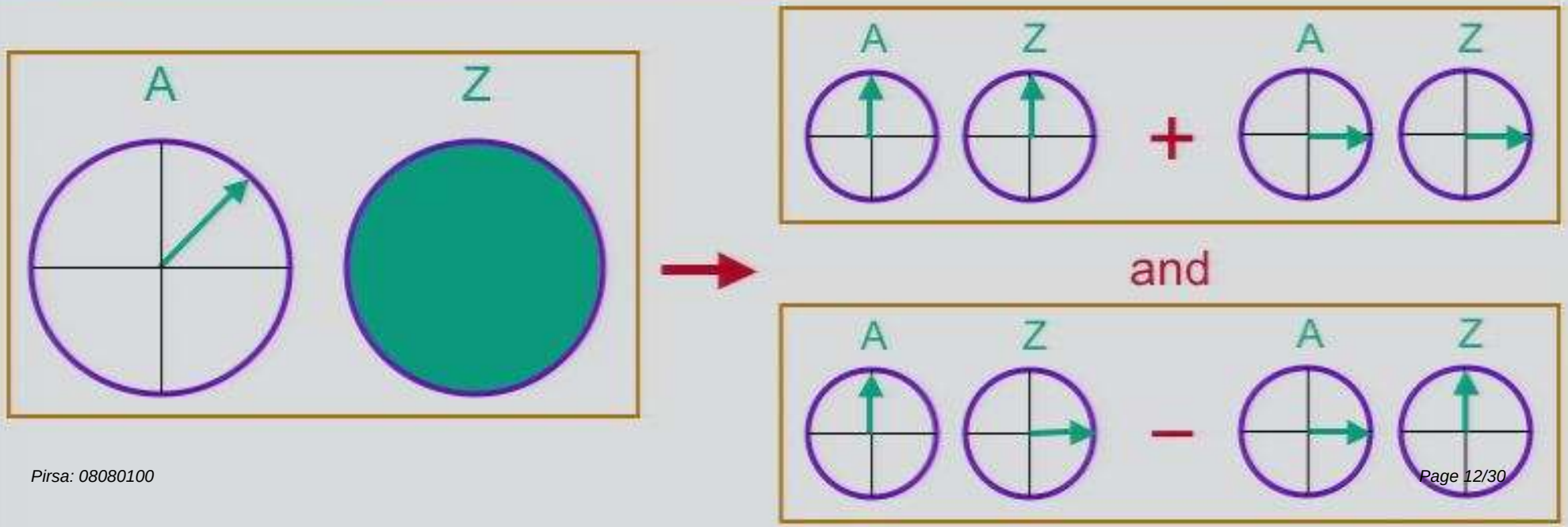
- It is mixed but has ***maximal entanglement***: in any decomposition of the density matrix, every pure state is maximally entangled. (Batle *et al*, 2002)

- There is ***no correlation*** between A and Z. If there were, a measurement of A could reveal something about the universal rebit, which is not allowed.

Quarter-wave plate: C-ROT$_{AZ}$

Rotate Z by 90° if A is horizontal.

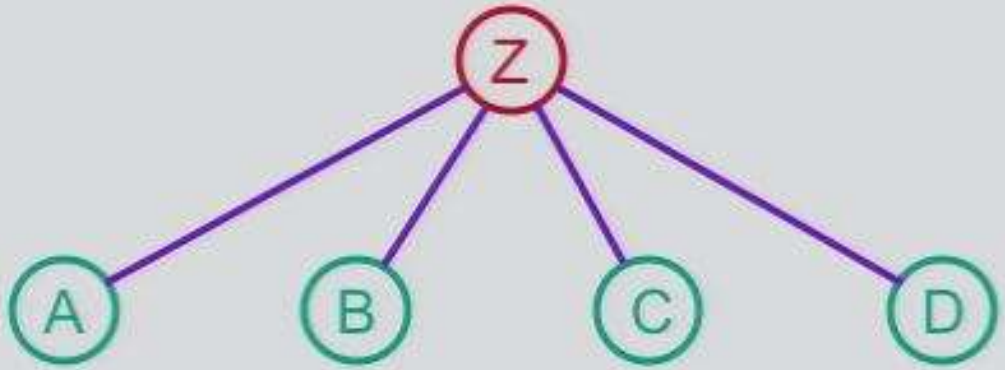




The real representation of several photons' polarization

Q: Suppose they're all right-circular. The "protophotons" can't ***all*** be ***maximally*** entangled with the same rebit, can they?

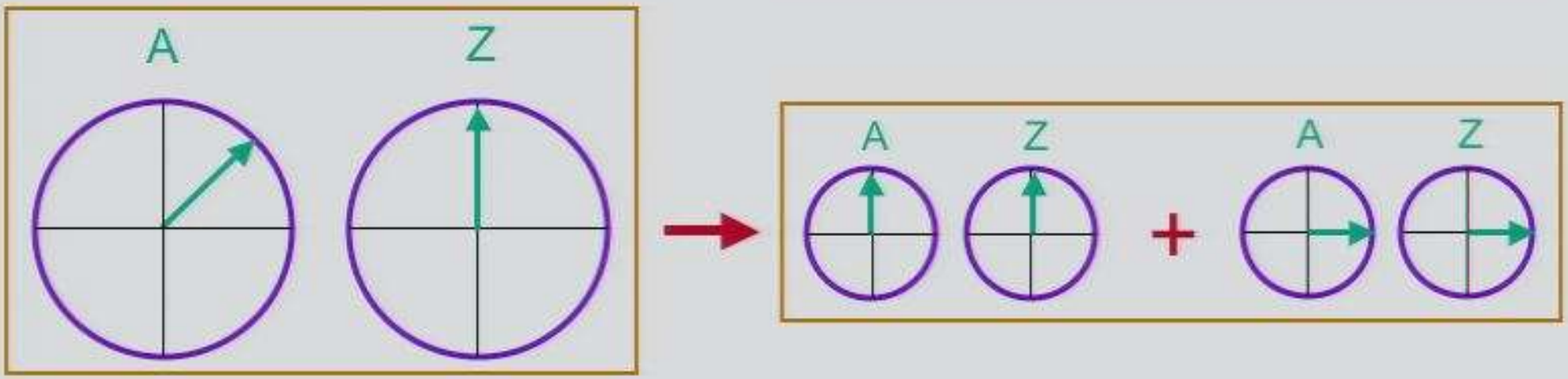
A: Yes they can. There's no entanglement monogamy in real-vector-space quantum mechanics.

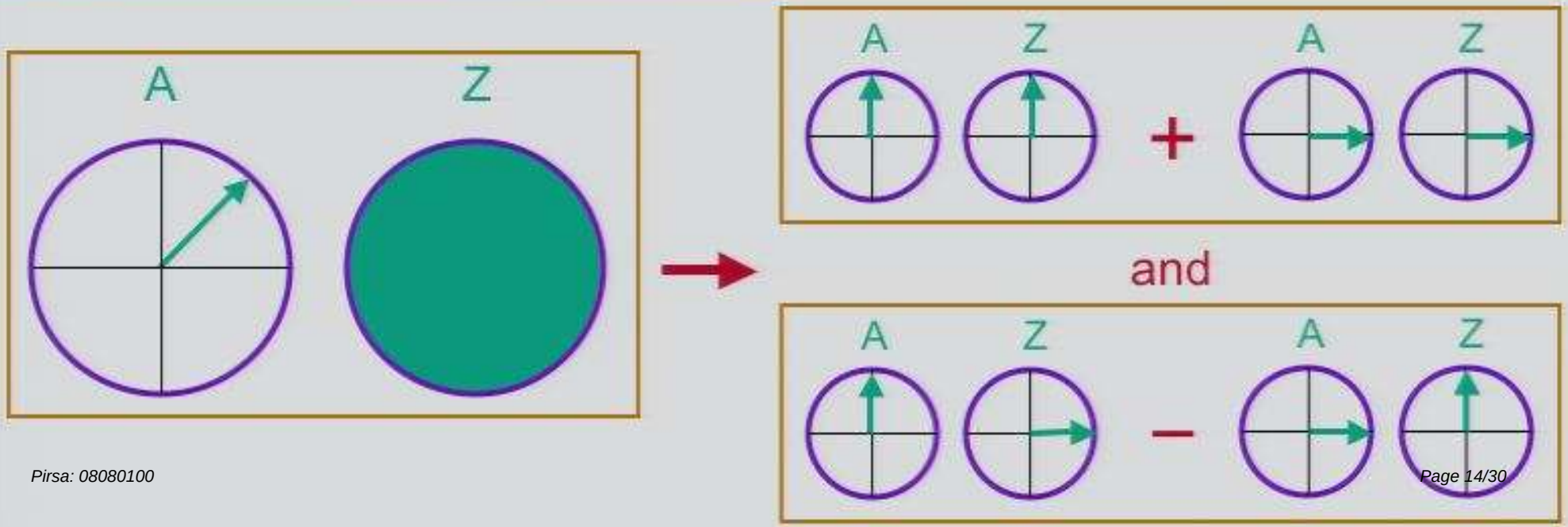


So a ***single*** extra rebit can turn a circle of states into a sphere of states, for ***arbitrarily many*** particles.
(*cf* Fernandez and Schneeberger, 2004)

Quarter-wave plate: C-ROT$_{AZ}$

Rotate Z by 90° if A is horizontal.

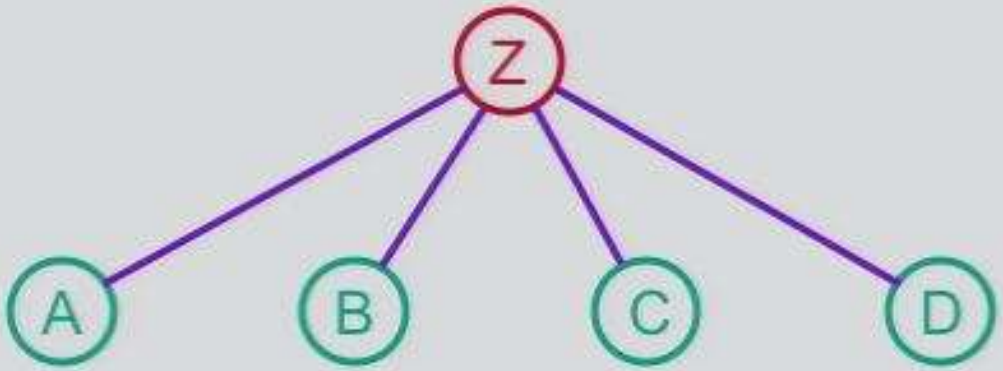




The real representation of several photons' polarization

Q: Suppose they're all right-circular. The "protophotons" can't ***all*** be ***maximally*** entangled with the same rebit, can they?

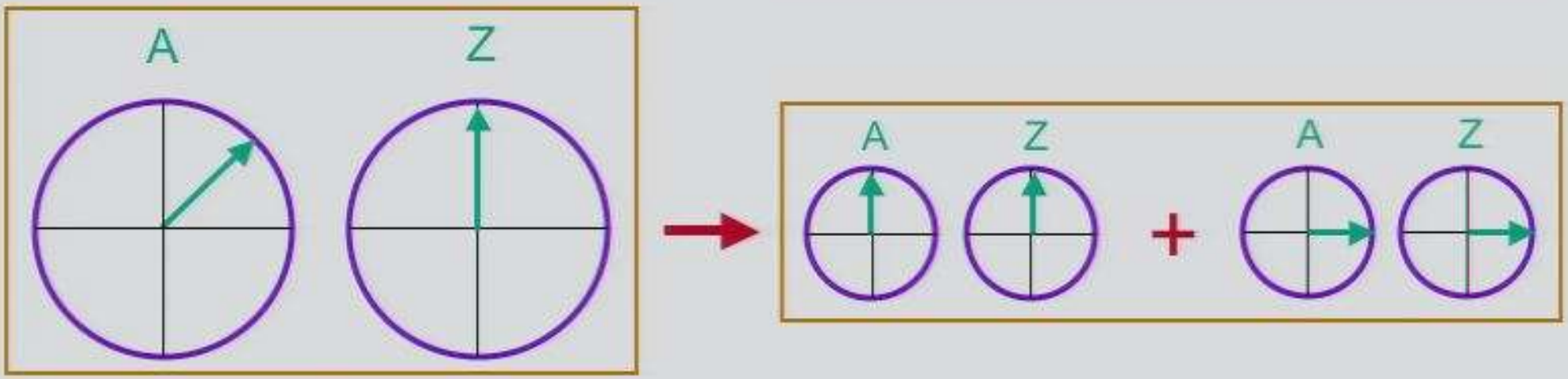
A: Yes they can. There's no entanglement monogamy in real-vector-space quantum mechanics.

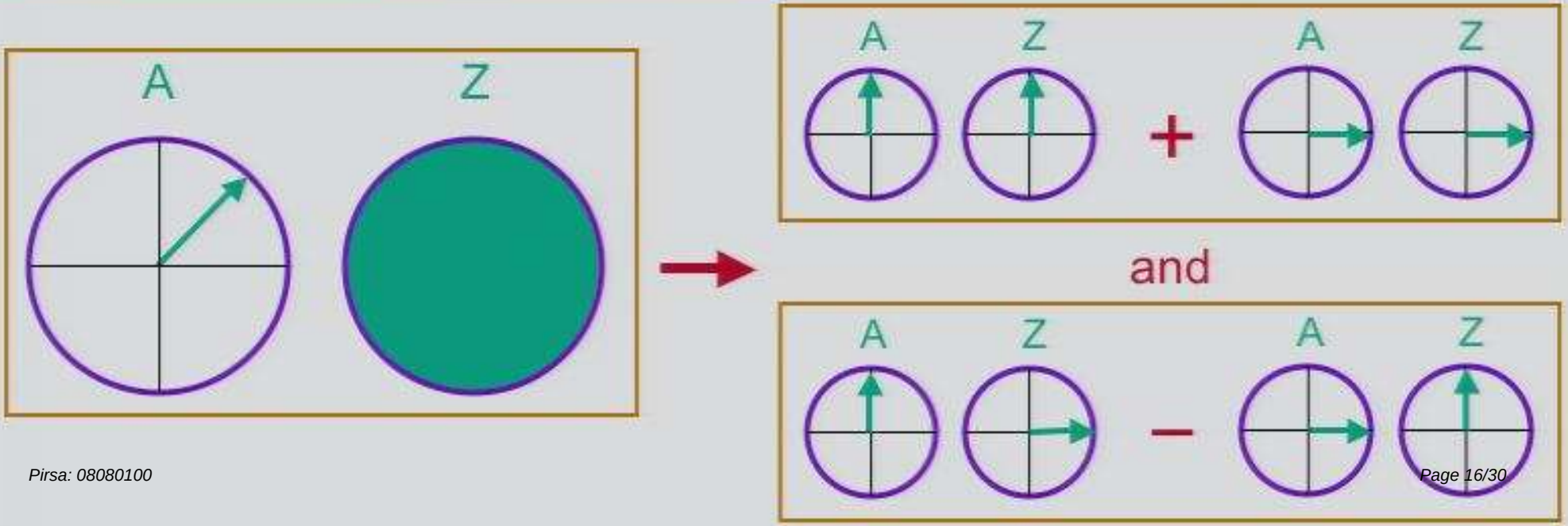


So a ***single*** extra rebit can turn a circle of states into a sphere of states, for ***arbitrarily many*** particles.
(*cf* Fernandez and Schneeberger, 2004)

Quarter-wave plate: C-ROT$_{AZ}$

Rotate Z by 90° if A is horizontal.

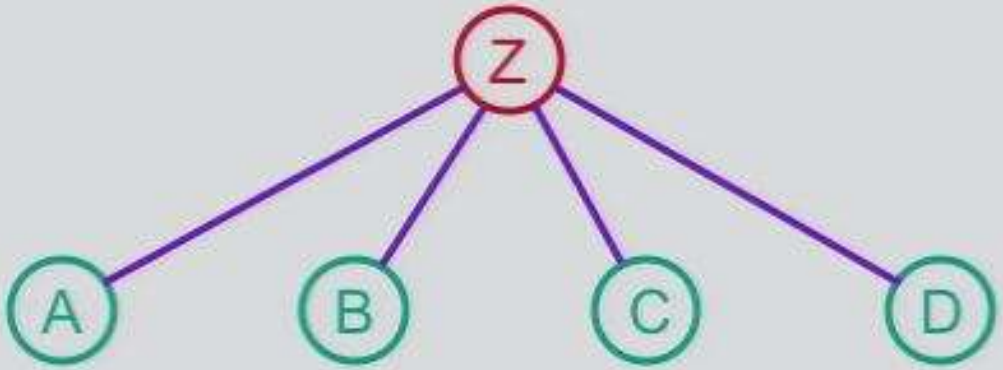




The real representation of several photons' polarization

Q: Suppose they're all right-circular. The "protophotons" can't ***all*** be ***maximally*** entangled with the same rebit, can they?

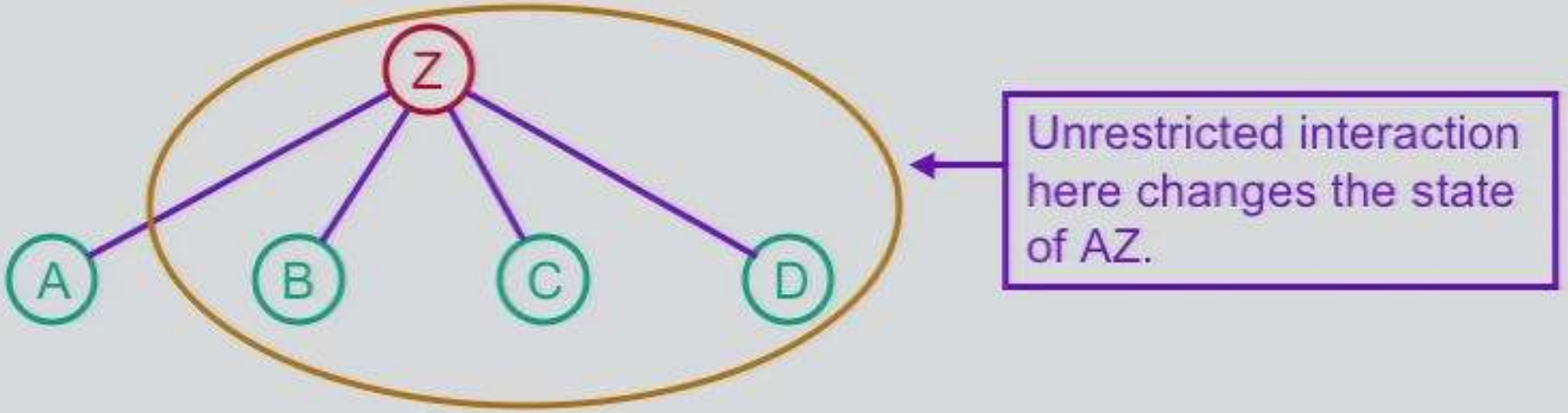
A: Yes they can. There's no entanglement monogamy in real-vector-space quantum mechanics.



So a ***single*** extra rebit can turn a circle of states into a sphere of states, for ***arbitrarily many*** particles.
(*cf* Fernandez and Schneeberger, 2004)

Could the superselection rule emerge naturally?

Because the ubit is ***everywhere***, interacting with many objects, would decoherence quickly bring all states and transformations to the standard form?



What follows are the results of numerical simulations, with a randomly chosen "Hamiltonian" (antisymmetric matrix) for the ubit and the rest of the universe.

The good news: A large universe makes states of A and Z commute with J.

$$\rho = \left(\frac{\rho - J\rho J}{2}\right) + \left(\frac{\rho + J\rho J}{2}\right)$$

ρ_{good} ρ_{bad}

$$\text{“badness”} = 2\,\mathrm{Tr}(\rho_{bad}^2).$$

The badness rapidly goes to a value inversely proportional to the dimension of the rest of the universe, and it remains at that level.

The bad news: The effect is too strong.

$$\rho_{AZ} = \begin{pmatrix} a & b & c & d \\ b & a' & d' & c' \\ c & d' & e & f \\ d & c' & f & e' \end{pmatrix} \longrightarrow (1/2) \begin{pmatrix} a+a' & 0 & c+c' & 0 \\ 0 & a+a' & 0 & c+c' \\ c+c' & 0 & e+e' & 0 \\ 0 & c+c' & 0 & e+e' \end{pmatrix}$$

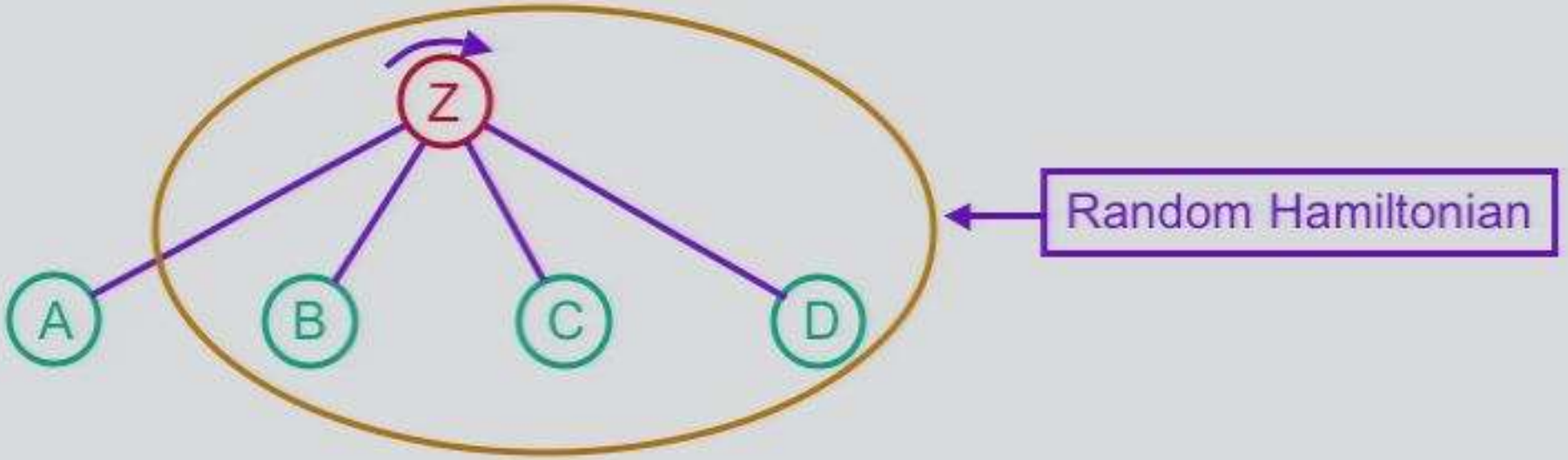
This kind of decoherence goes too far: the ubit factors out.

E.g., we would have only linear polarizations.

We wanted

$$\rho_{AZ} = \begin{pmatrix} a & b & c & d \\ b & a' & d' & c' \\ c & d' & e & f \\ d & c' & f & e' \end{pmatrix} \longrightarrow (1/2) \begin{pmatrix} a+a' & 0 & c+c' & d-d' \\ 0 & a+a' & d'-d & c+c' \\ c+c' & d'-d & e+e' & 0 \\ d-d' & c+c' & 0 & e+e' \end{pmatrix}$$

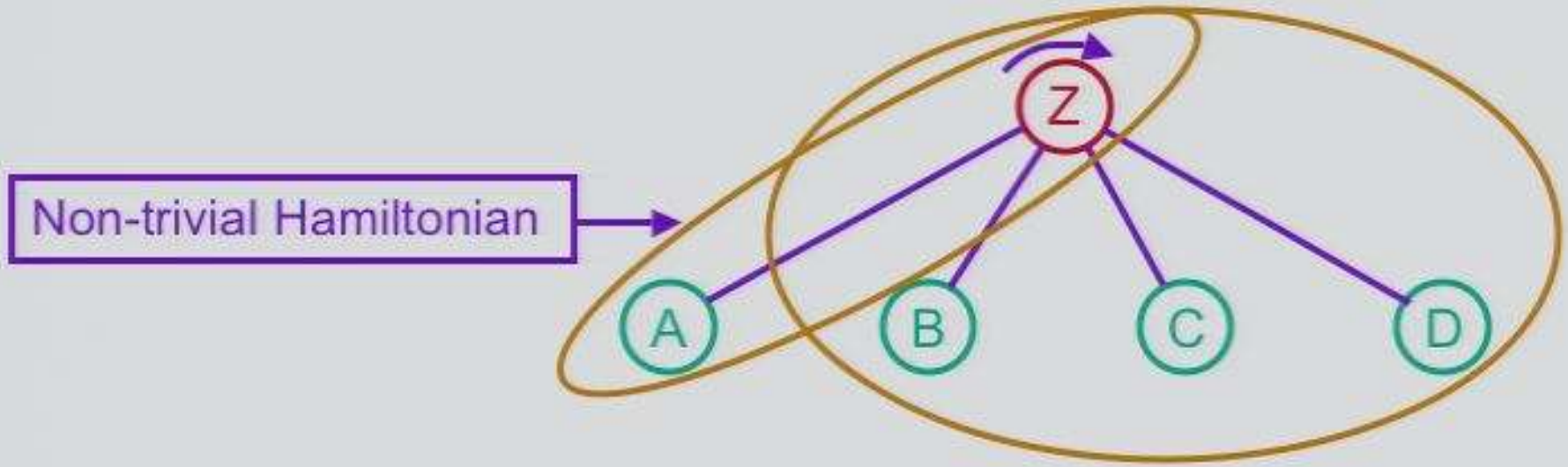
Suppose the ubit naturally rotates very fast



With increasing rotation rate, the result approaches the desired form:

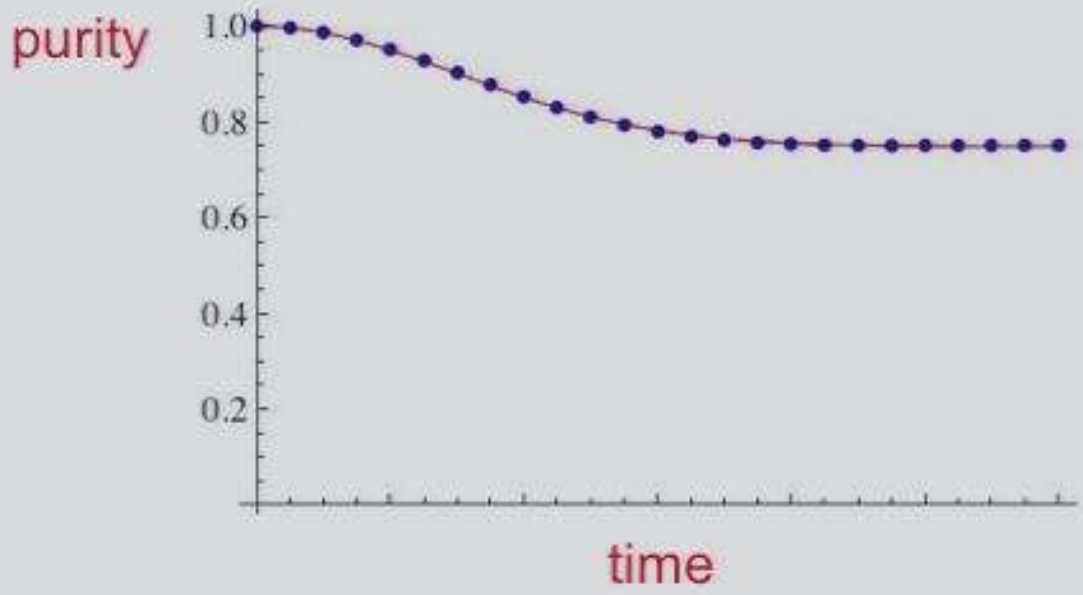
$$\rho_{AZ} = \begin{pmatrix} a & b & c & d \\ b & a' & d' & c' \\ c & d' & e & f \\ d & c' & f & e' \end{pmatrix} \longrightarrow (1/2) \begin{pmatrix} a+a' & 0 & c+c' & d-d' \\ 0 & a+a' & d'-d & c+c' \\ c+c' & d'-d & e+e' & 0 \\ d-d' & c+c' & 0 & e+e' \end{pmatrix}$$

Now add a Hamiltonian for A and Z



- Does the state of AZ eventually commute with *J*?
- Does AZ act like a qubit, or does Z factor out, leaving A as a simple rebit?

Case (i): Suppose H_{AZ} commutes with J



$$\text{purity} = 2\,\mathrm{Tr}(\rho_{bad}^2)$$

- The purity decays as a Gaussian, with decay time scaling like

$$\text{decay time} \propto \omega_Z{}^2/[\beta_{AZ}\,\beta^2{}_{Z(\text{rest of universe})}]$$

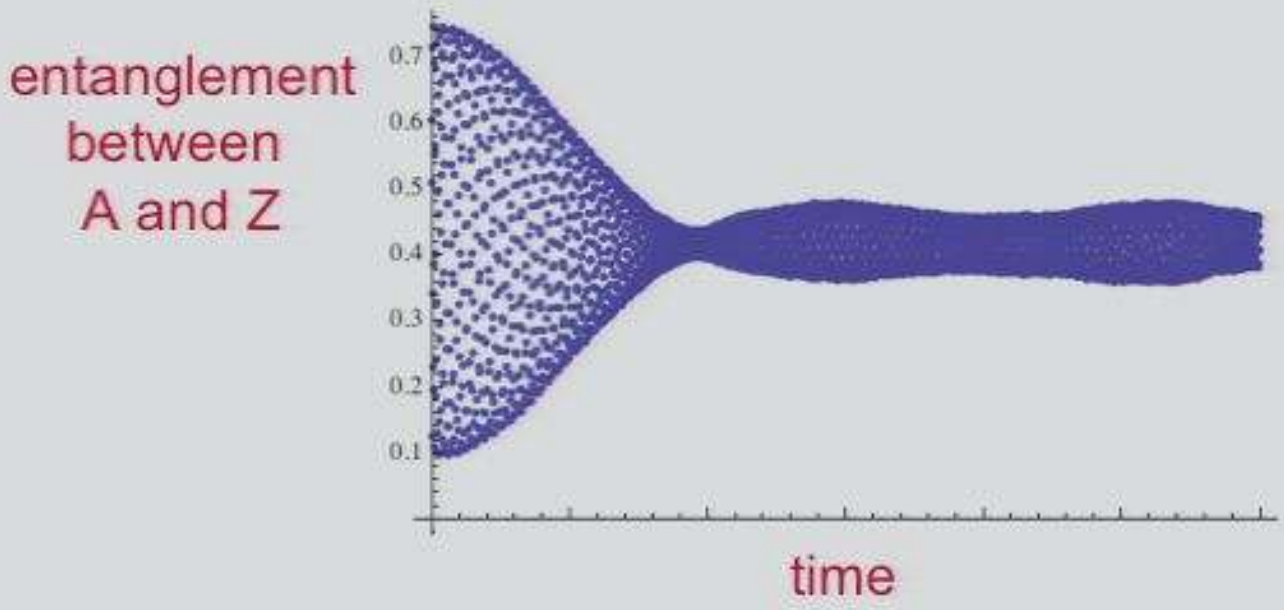
- Eventually Z approximately factors out, leaving A as a rebit. The dynamics effects a slow projection onto the rebit slice of the Bloch sphere.

Case (ii): Typical H_{AZ} not commuting with J

- The purity decays as a Gaussian, with decay time scaling like

  decay time $\propto \omega_Z/[\beta_{AZ}\,\beta_{Z(\text{rest of universe})}]$

- Some qubit-like evolution of AZ persists even after the decay.



Tentative Conclusion

We might be able to recover an arbitrarily good approximation to ordinary quantum mechanics if the ubit is spinning sufficiently rapidly.

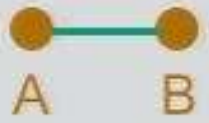
But if we wait long enough, we will see an otherwise unexplained increase in entropy.

Entanglement as measured by the tangle

Pure two-qubit state $|\psi\rangle = c_0|00\rangle + c_1|11\rangle$: Tangle $\tau = 4|c_0 c_1|^2$; $0 \le \tau \le 1$.

Mixed state ρ: Tangle $\tau = \min \sum p_j \tau(\psi_j)$, where $\rho = \sum p_j |\psi_j\rangle\langle\psi_j|$.

Same definitions for real states (but only real decompositions allowed).
(Caves, Fuchs, and Rungta, 2000.)

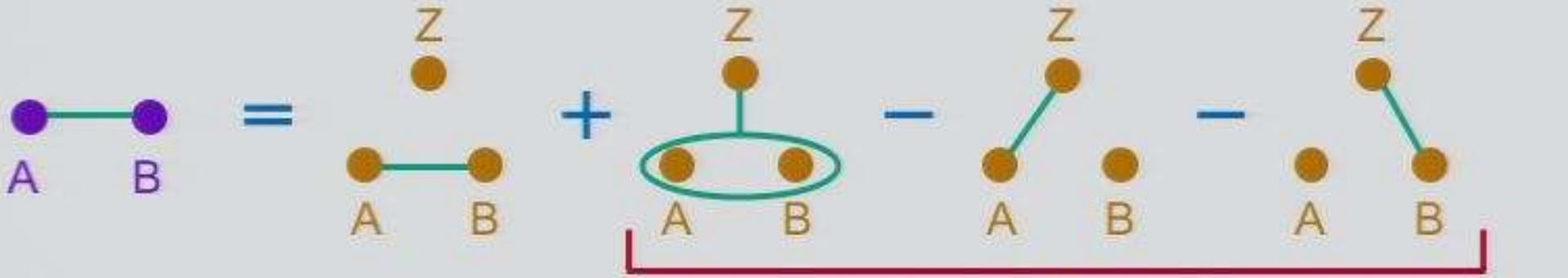


two rebits

$$\tau = (\mathrm{Tr}[\rho\ (\sigma_y \otimes \sigma_y)])^2$$

Entanglement of two qubits, as seen in the real representation

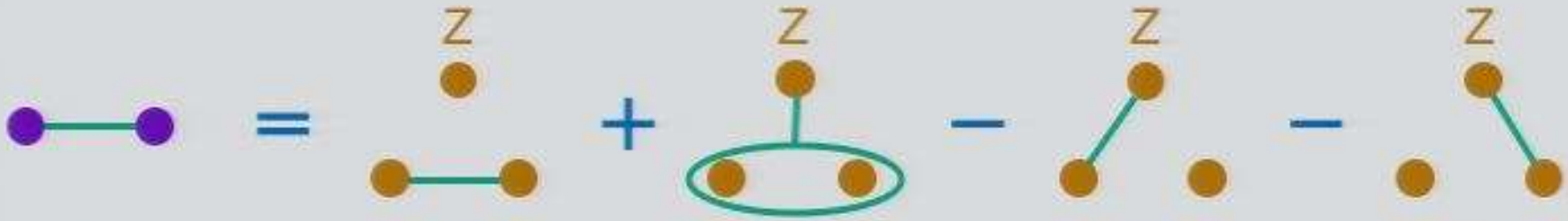
Ordinary two-qubit tangle can be written in terms of two-rebit tangles (for pure or mixed states).



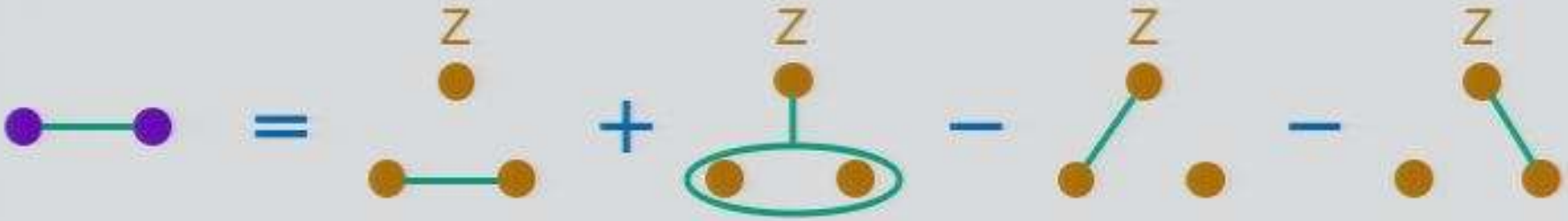
This part would be non-negative for complex states but it can be negative for real states.

Examples of entanglement

1. The entangled state $(1/\sqrt{2})(|\updownarrow\updownarrow\rangle + |\leftrightarrow\leftrightarrow\rangle)$.



2. Two right-circular photons.



Problems with the ubit picture

1. ***Arbitrariness of the real/imaginary split.***
   If the system is isolated and its spectrum has no degeneracy, we can take the energy eigenstates to be the special basis for the real vector space.
   But we still have to choose a special phase for each eigenstate.

2. ***Instantaneousness.*** It would seem that one has to choose a special foliation of spacetime into spacelike surfaces. But maybe there is a more subtle way of interpreting the "sharing" of the ubit.

Conclusions and Questions

- We can obtain ordinary quantum mechanics from a real-vector-space theory, if we add a universal rebit whose state cannot be accessed either by direct measurement or by coupling to other objects.
- It is conceivable that under the right circumstances, such a restriction (superselection rule) might occur naturally.
- Ordinary two-qubit entanglement can be expressed in terms of rebit entanglement.
- If the superselection rule is due to decoherence, how could one experimentally test for *incomplete* decoherence?
- What might the ubit be? Why would it spin rapidly?