Title: Quantum Mechanics 15 - The Mathematics of Electron Spin

Date: Aug 10, 2008 02:00 PM

URL: http://pirsa.org/08080090

Abstract: Development of a successful mathematical model of spin.
Learning Outcomes:
â€¢ A review of the mathematics of vectors.
â€¢ Applying the experimental results of QM-14 to construct a mathematical model of an electron spinning in any direction as a certain superposition of the spin up and spin down states.
â€¢ Discovering that this mathematical model predicts that an electron does not return to its original state when rotated once (through 360 degrees) â€“ it must be rotated twice (through 720 degrees). A discussion of experimental tests of this remarkable prediction.

The Mathematics of Spin

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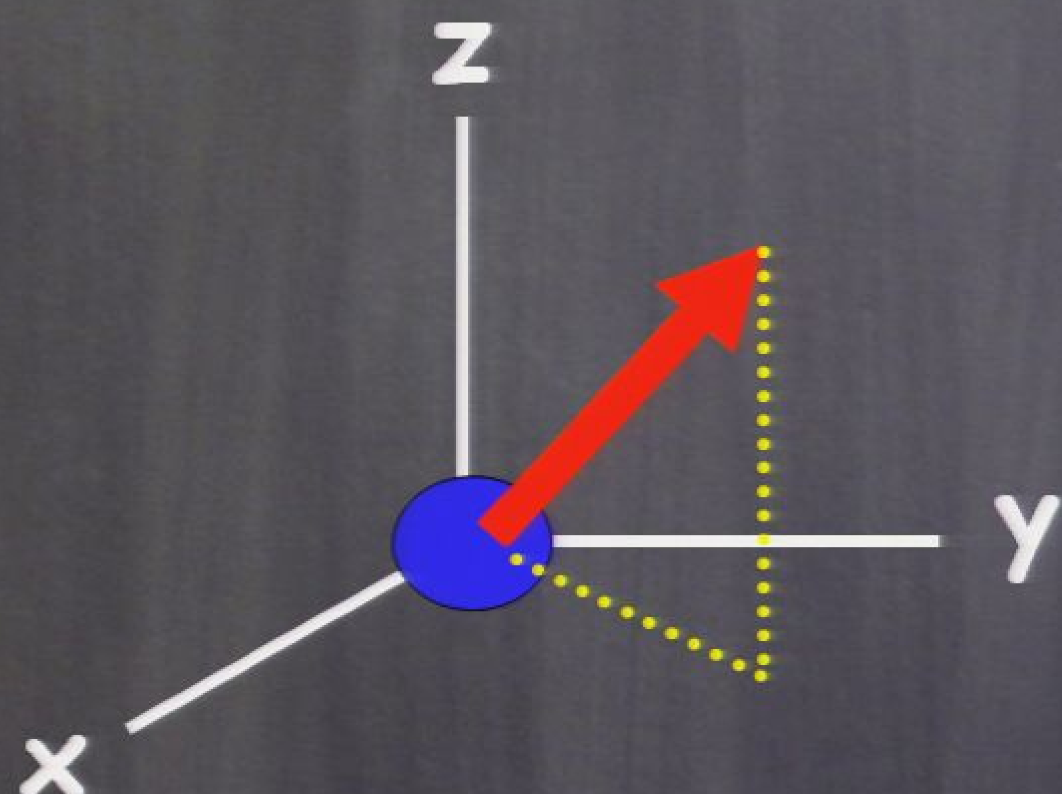

What do we know about VECTORS?

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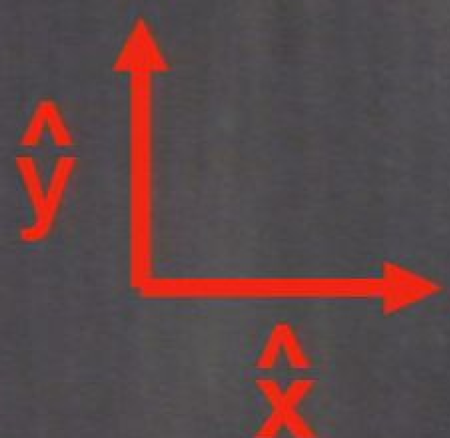
Unit basis vectors: $\hat{x}$, $\hat{y}$



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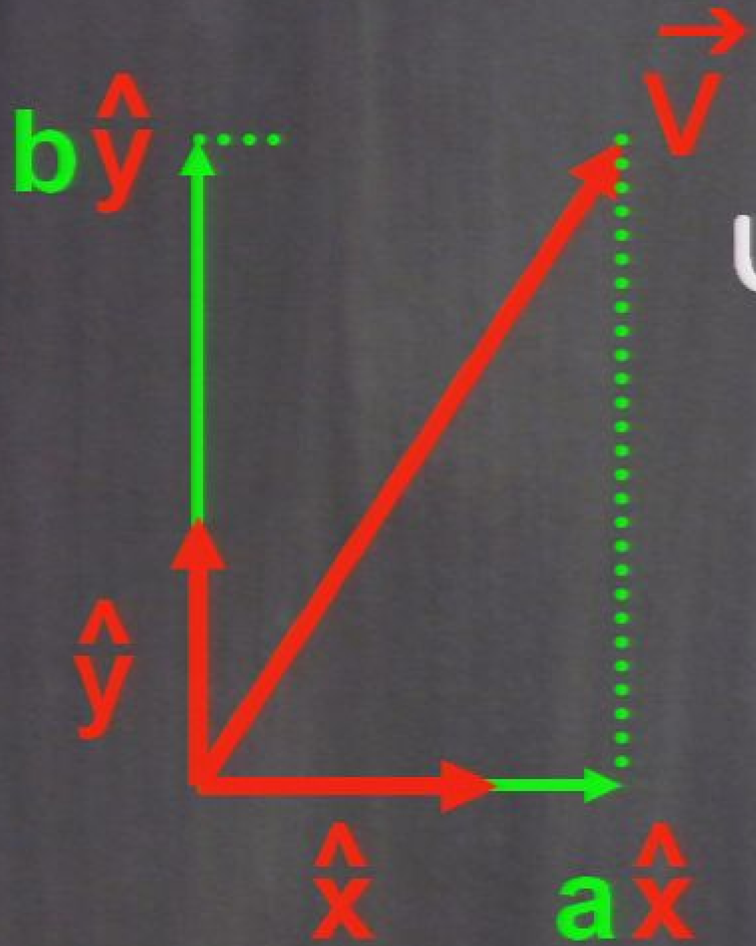

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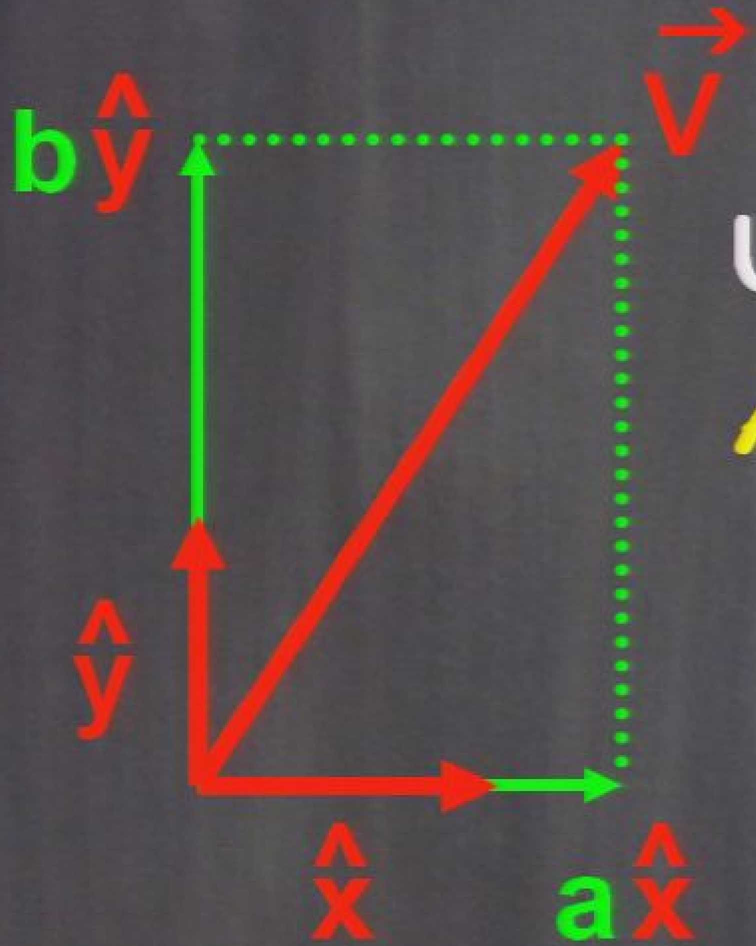

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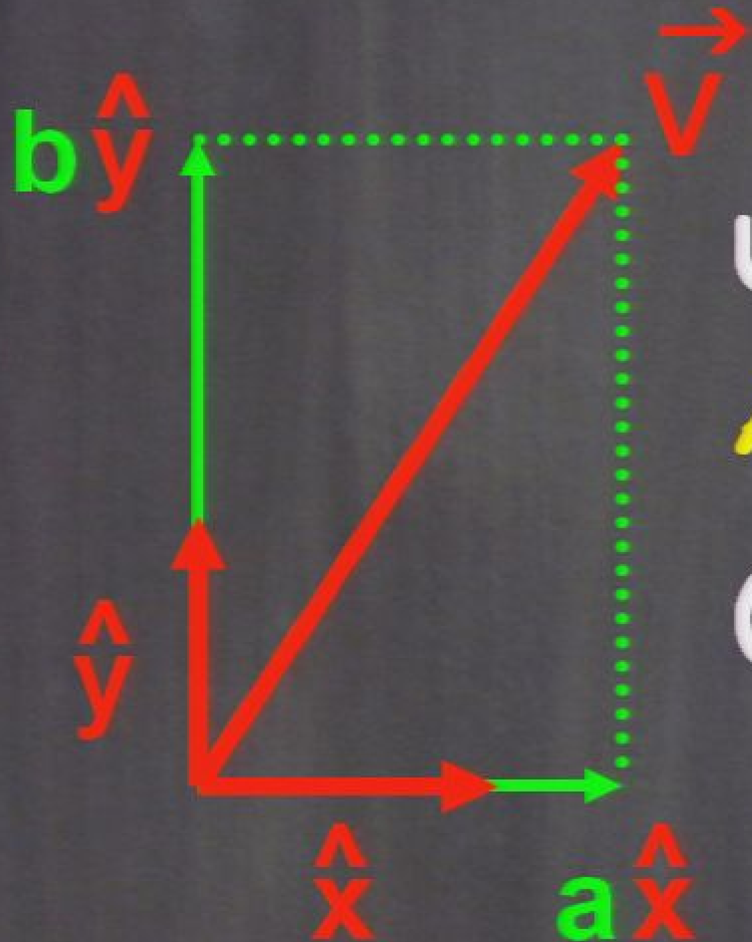

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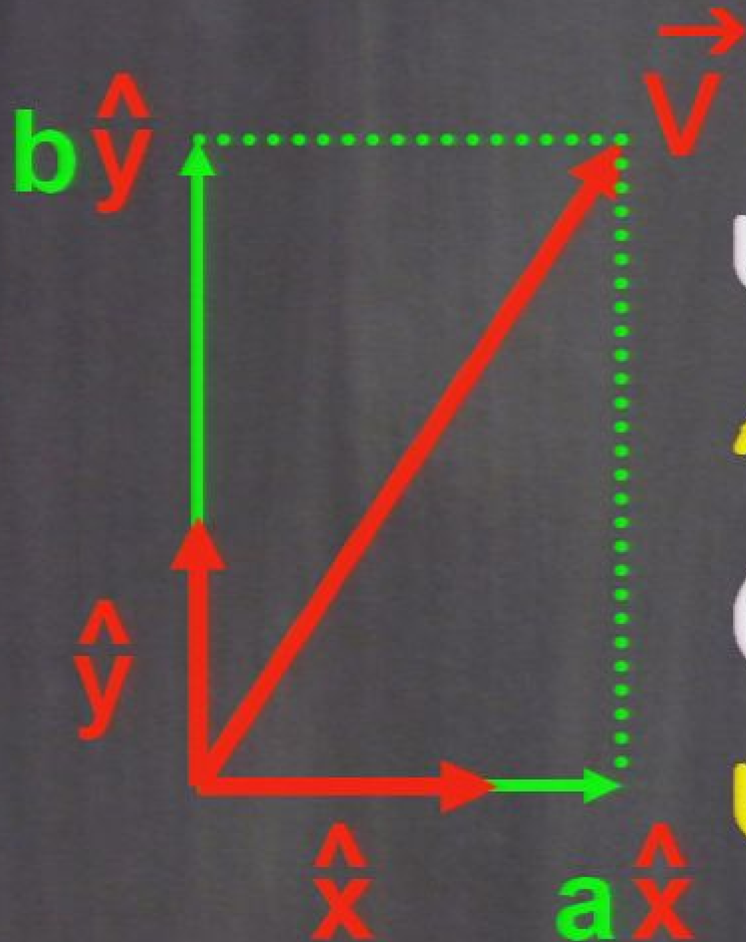

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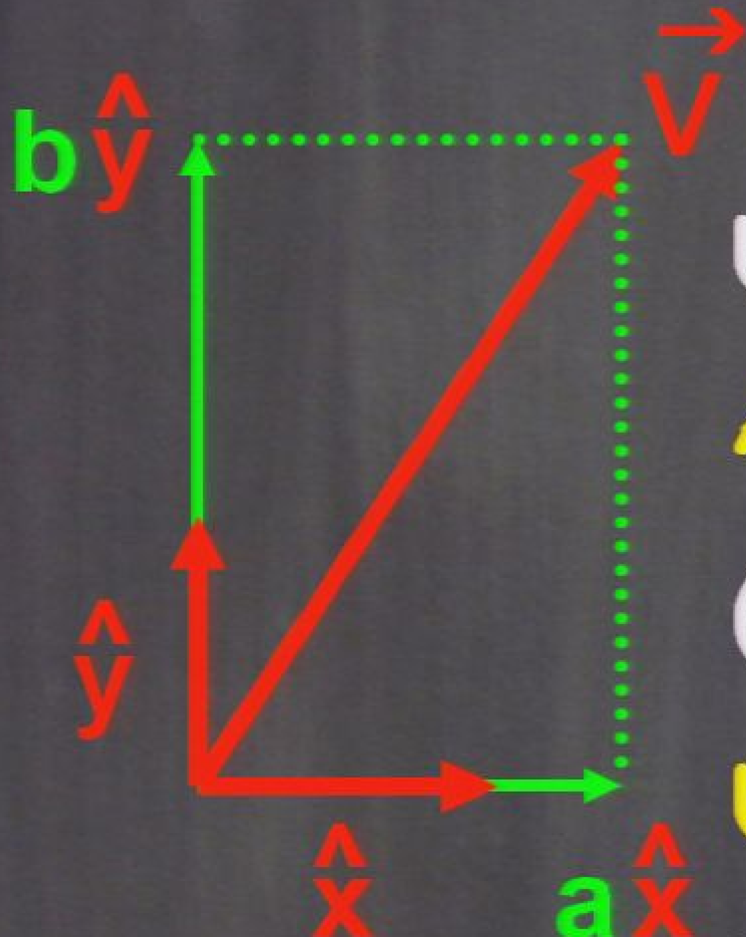

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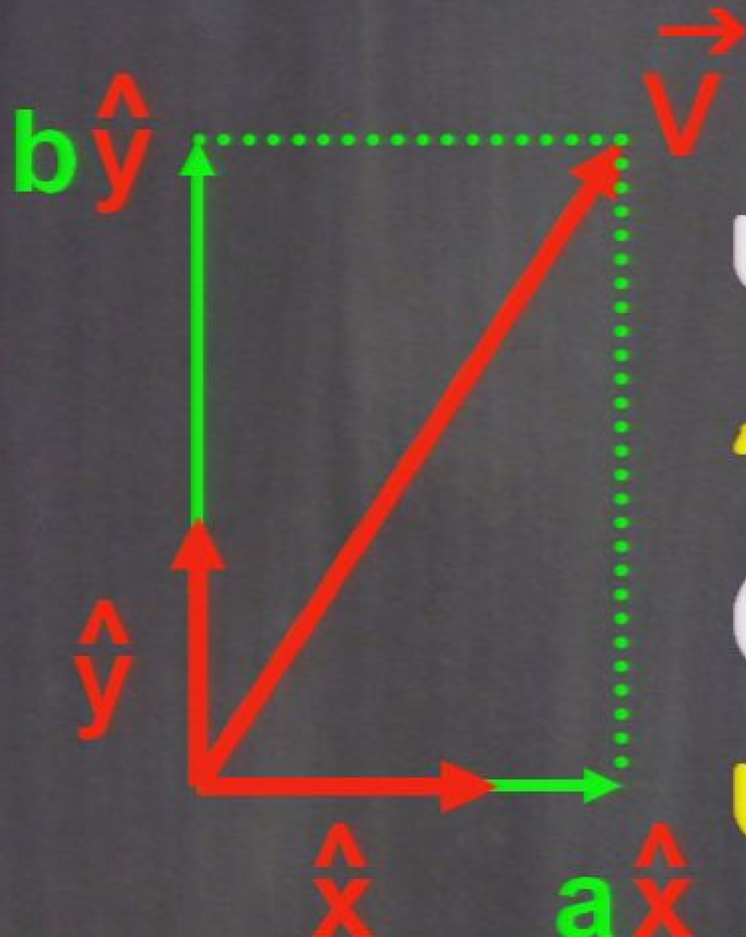

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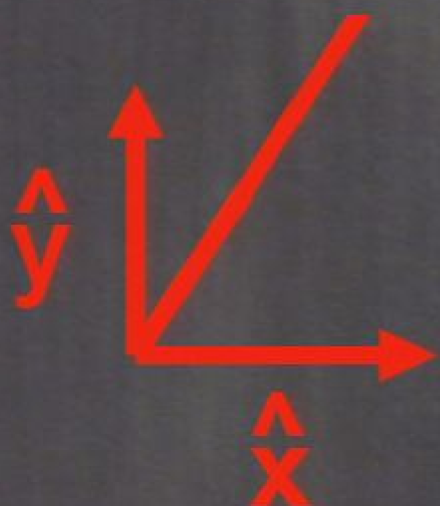

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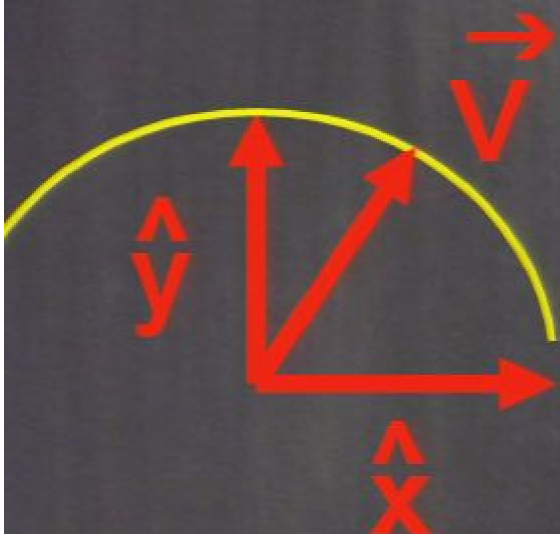

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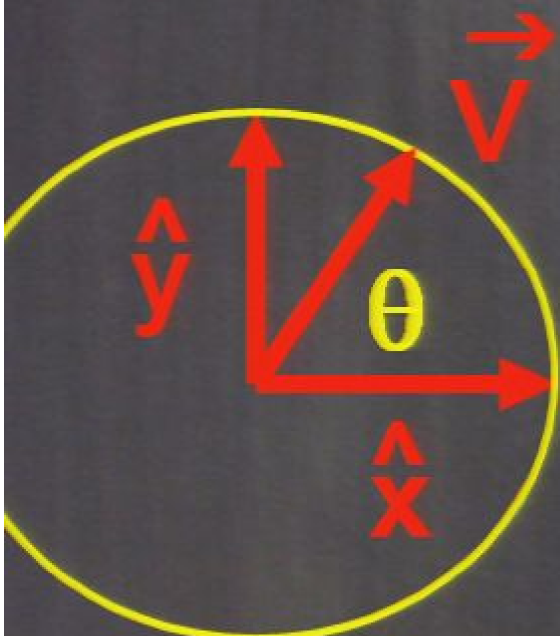

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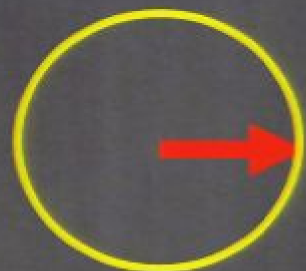
$\theta = 0$:



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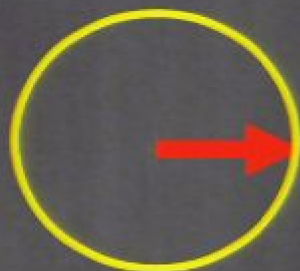
$\theta = 0$:  $\vec{V} = \hat{x}$

$\theta = 90$:  $\vec{V} = \hat{y}$

$\theta = 45$: 

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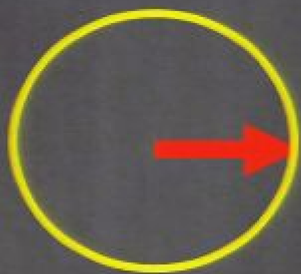
$\theta = 0$:  $\vec{V} = \hat{x}$

$\theta = 90$:  $\vec{V} = \hat{y}$

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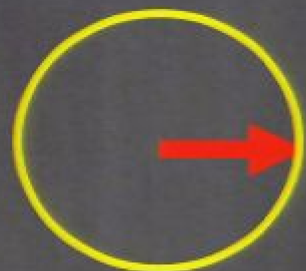
$\theta = 0$:  $\vec{V} = \hat{x}$

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$$a^2 + b^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$$

Application to Spin:

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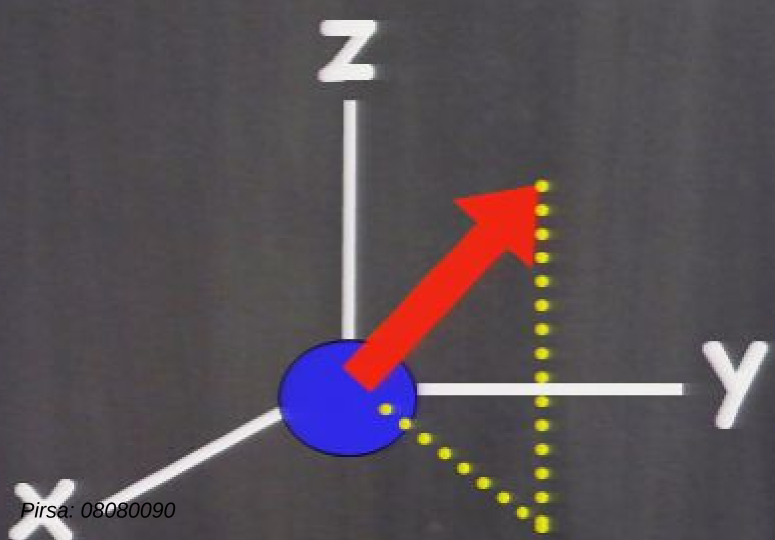
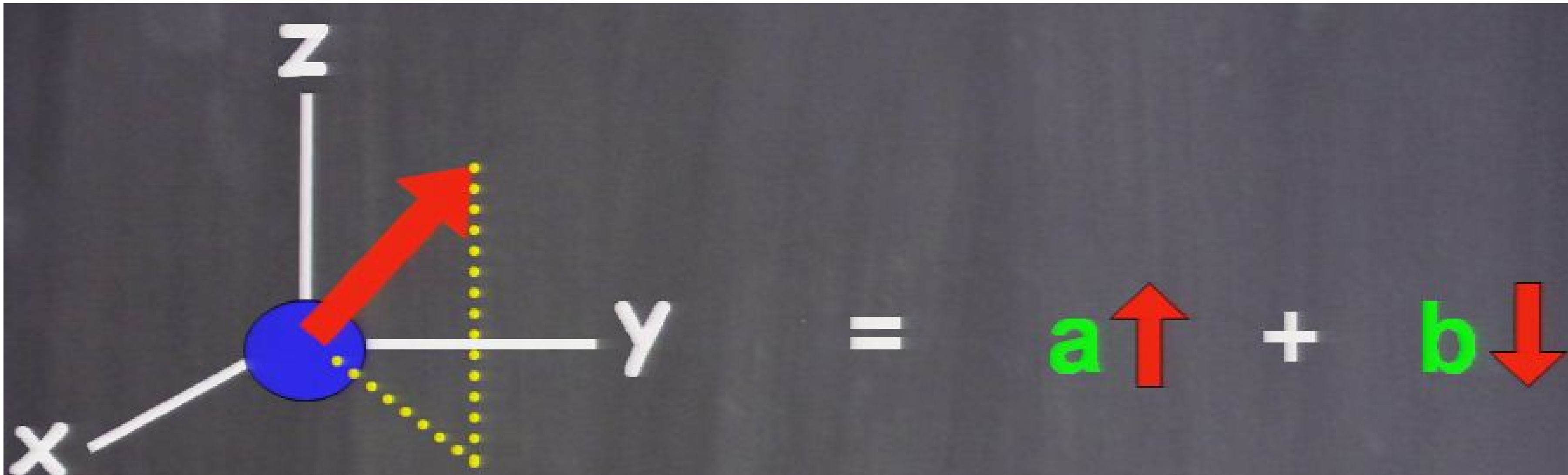
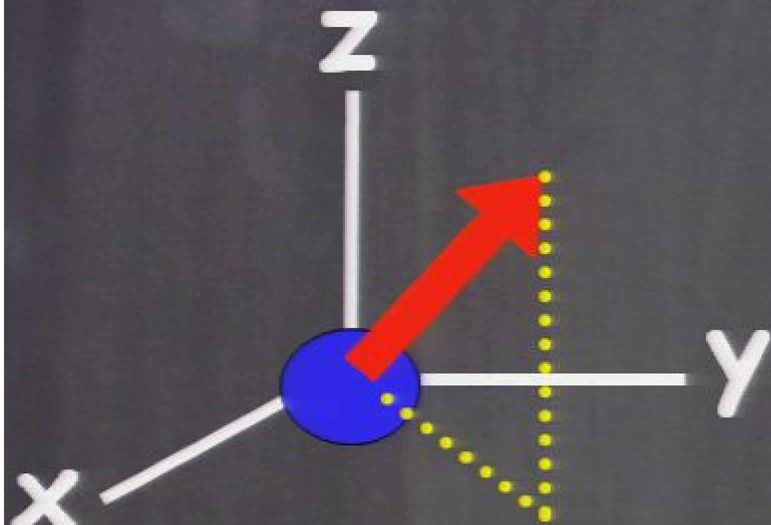


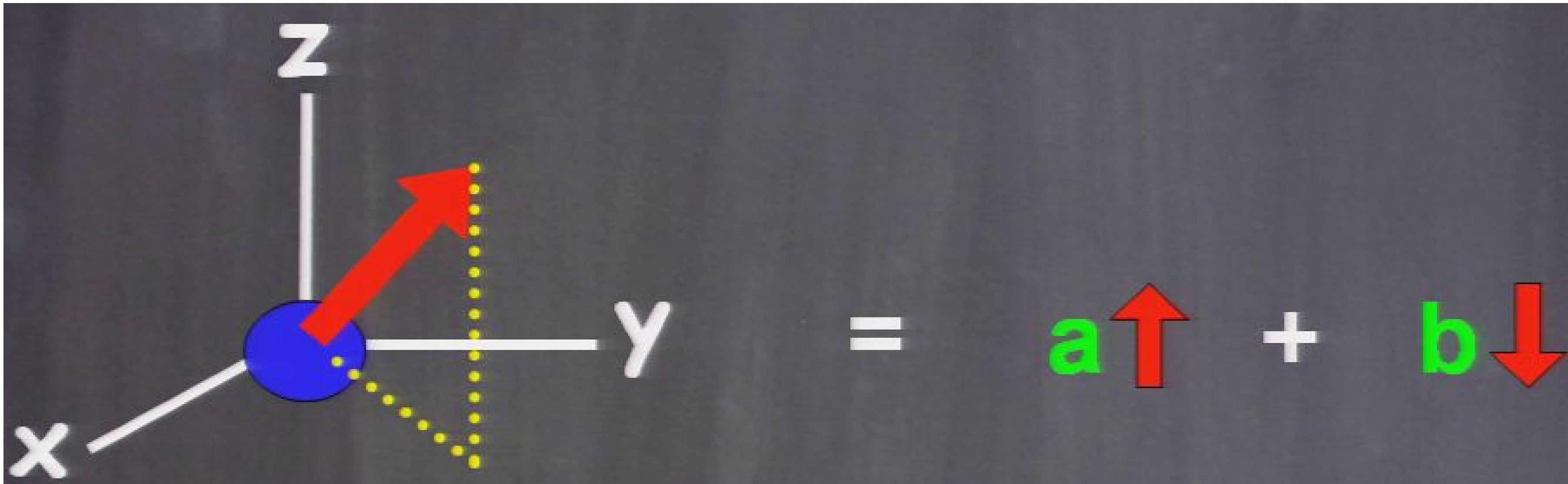
= a↑ + b↓ ??


z
y
x
=
a
+
b



$= a\uparrow + b\downarrow$

2: <u>Probability</u> suggests a natural interpretation for: $1 = a^2 + b^2$



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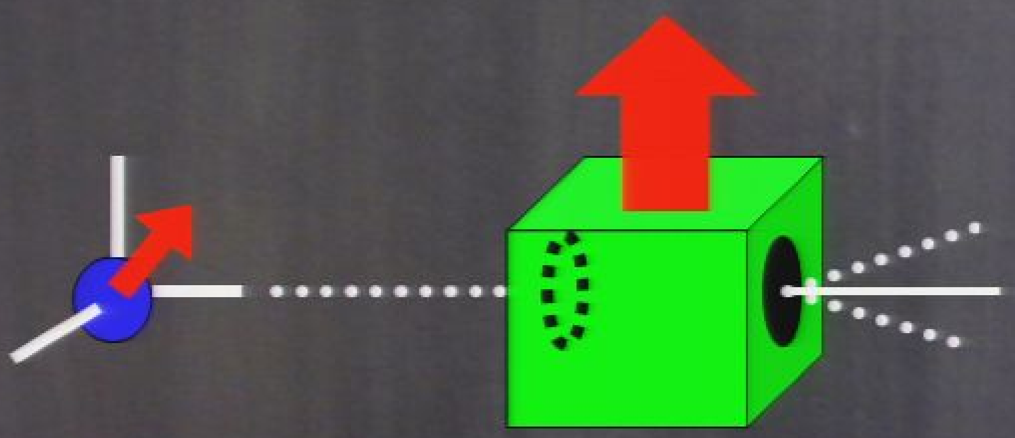

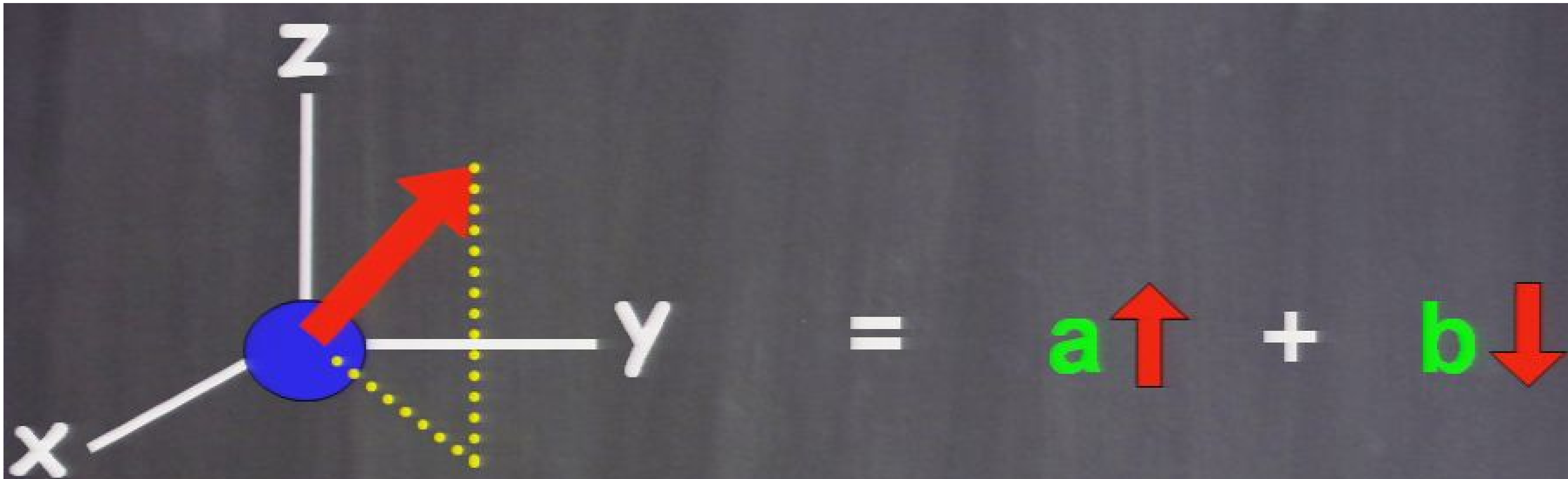

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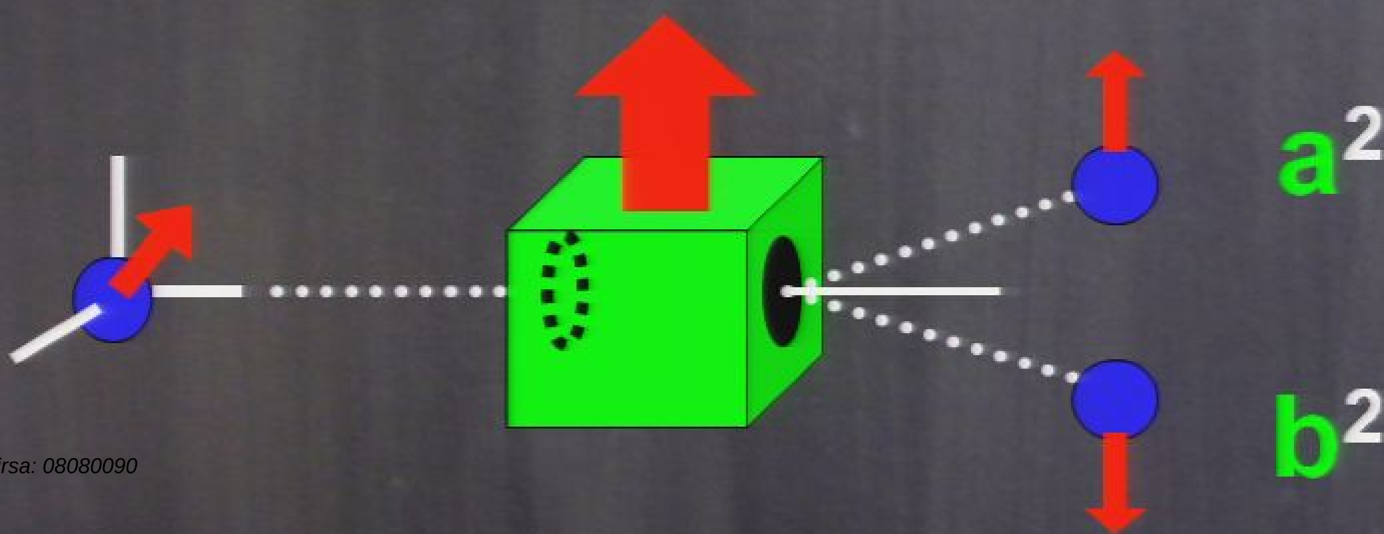
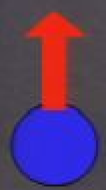
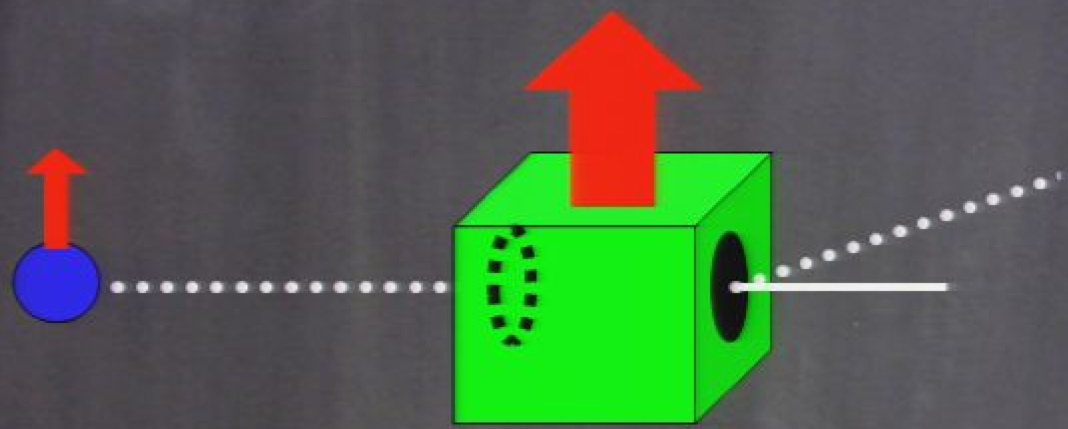
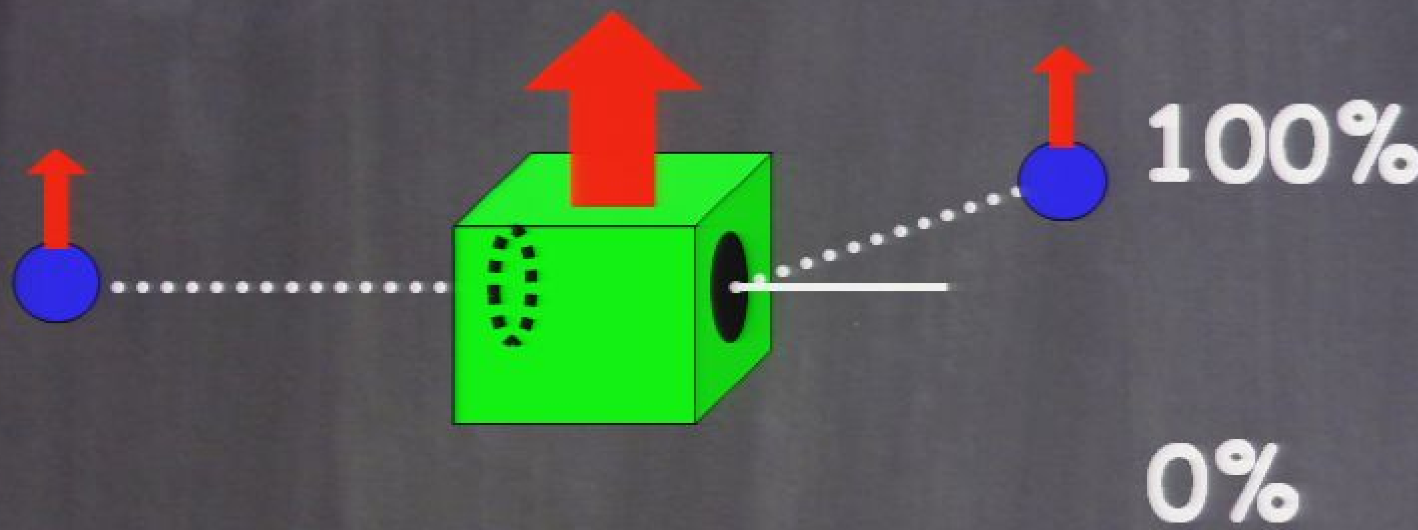
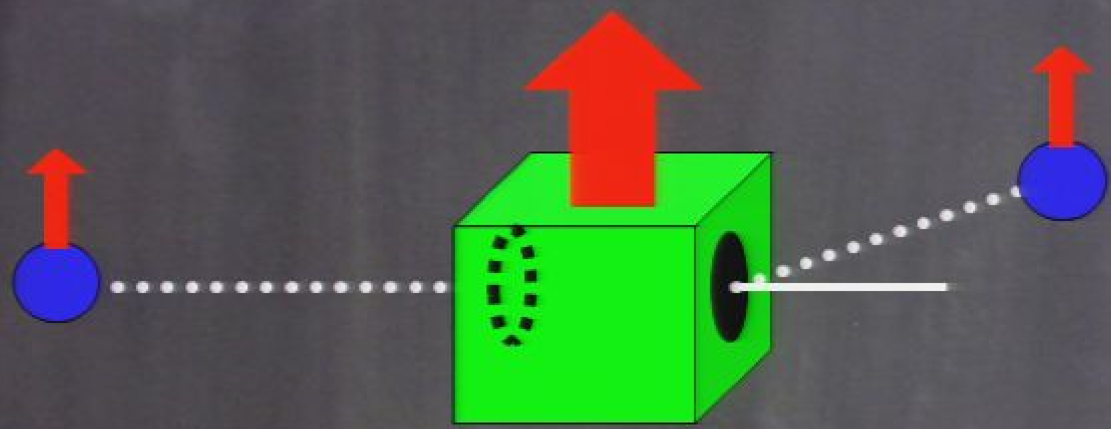




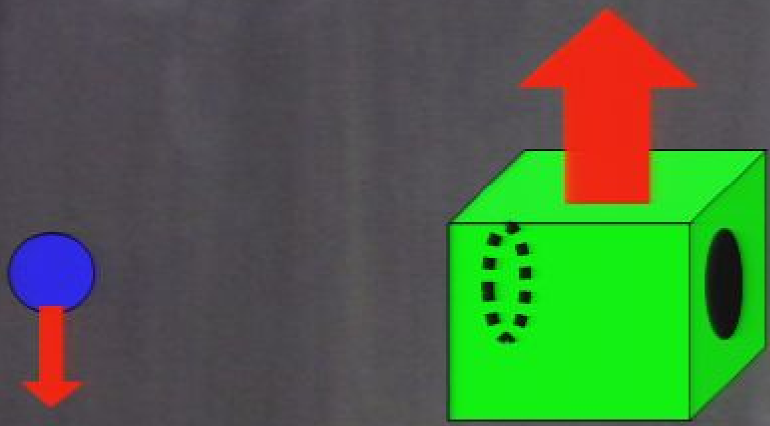


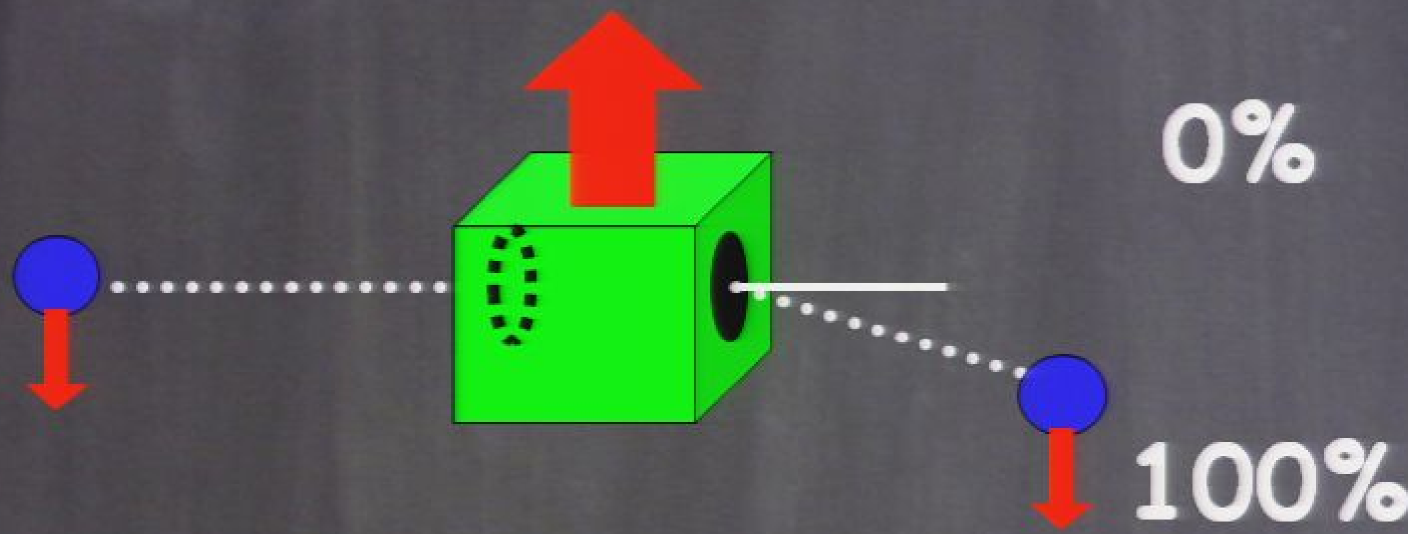

100%
0%



100% $a^2 = 1$

0% $b^2 = 0$

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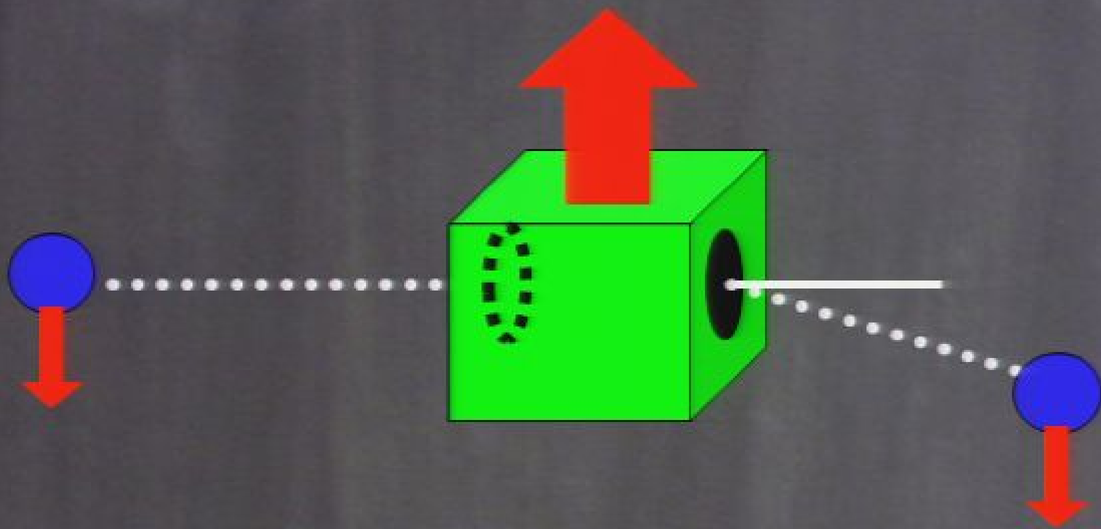

0% $a^2 = 0$

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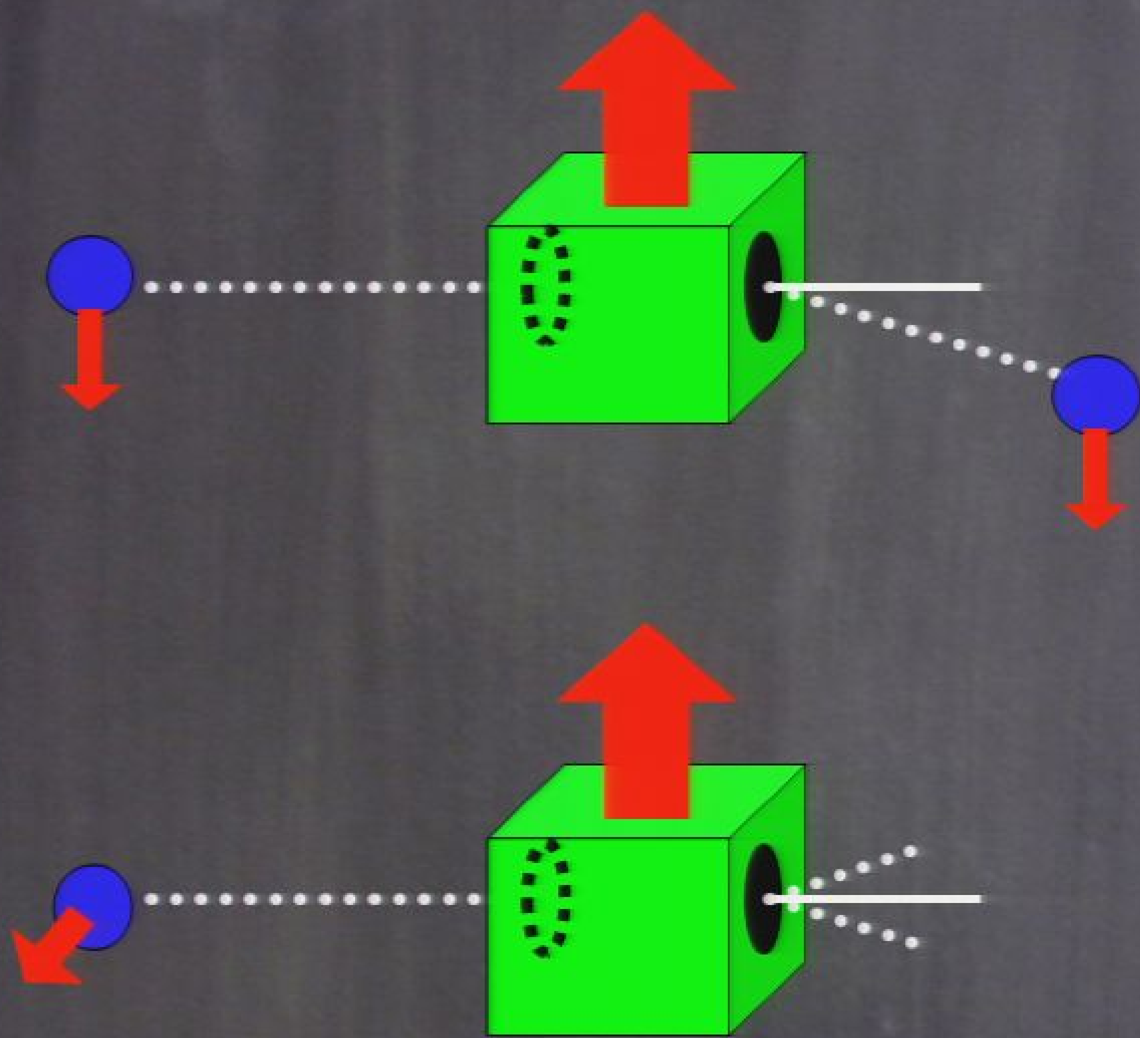
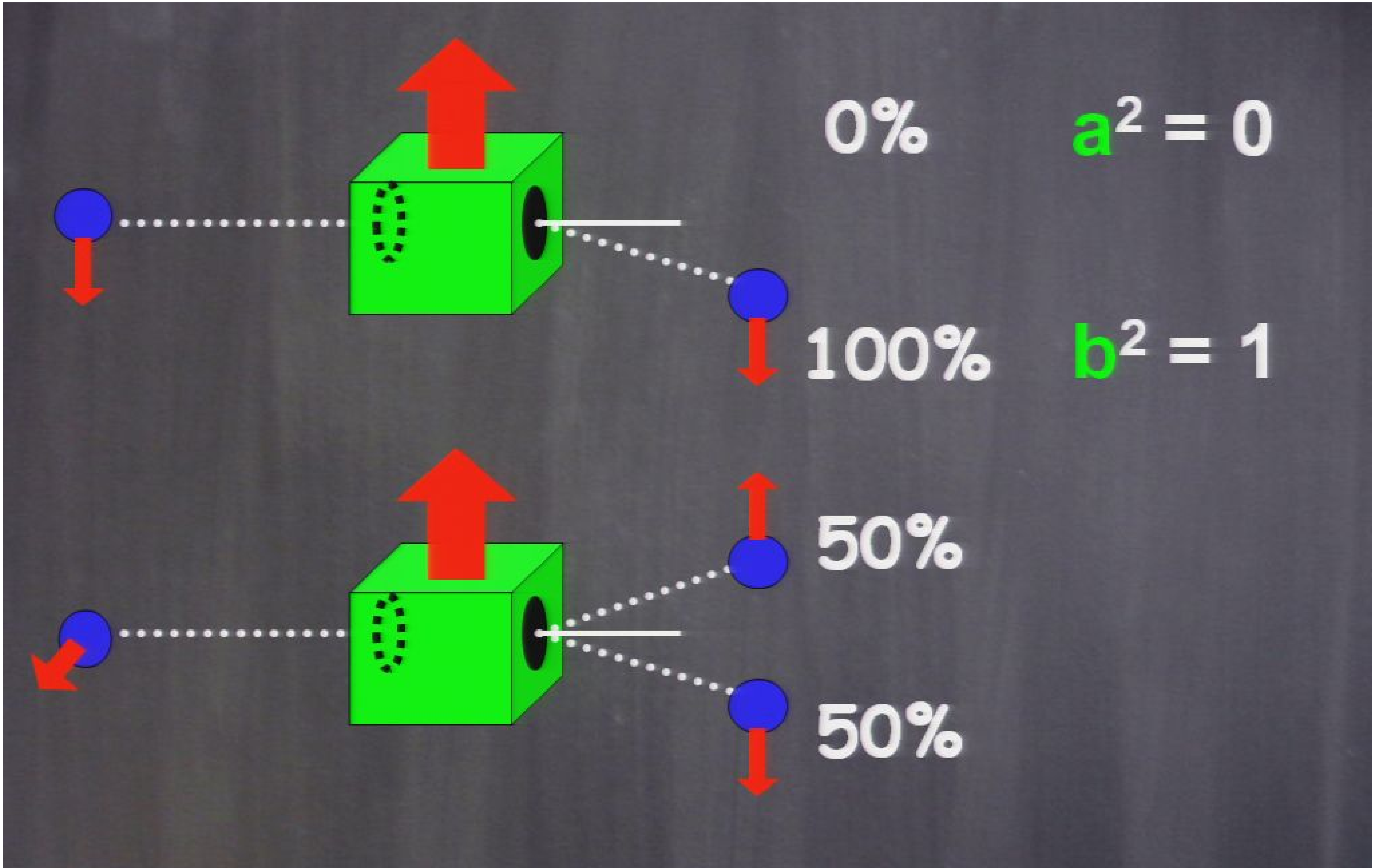
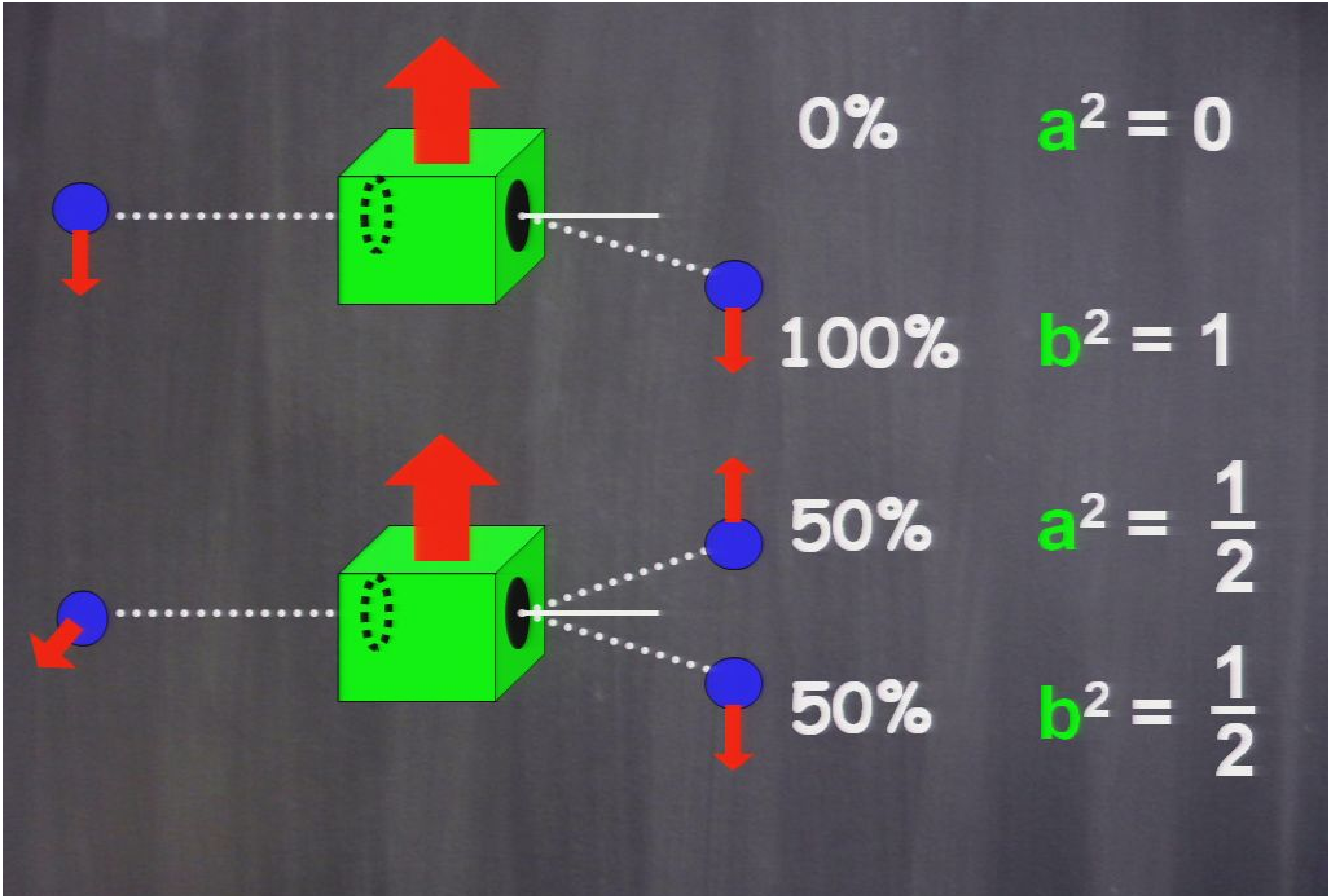
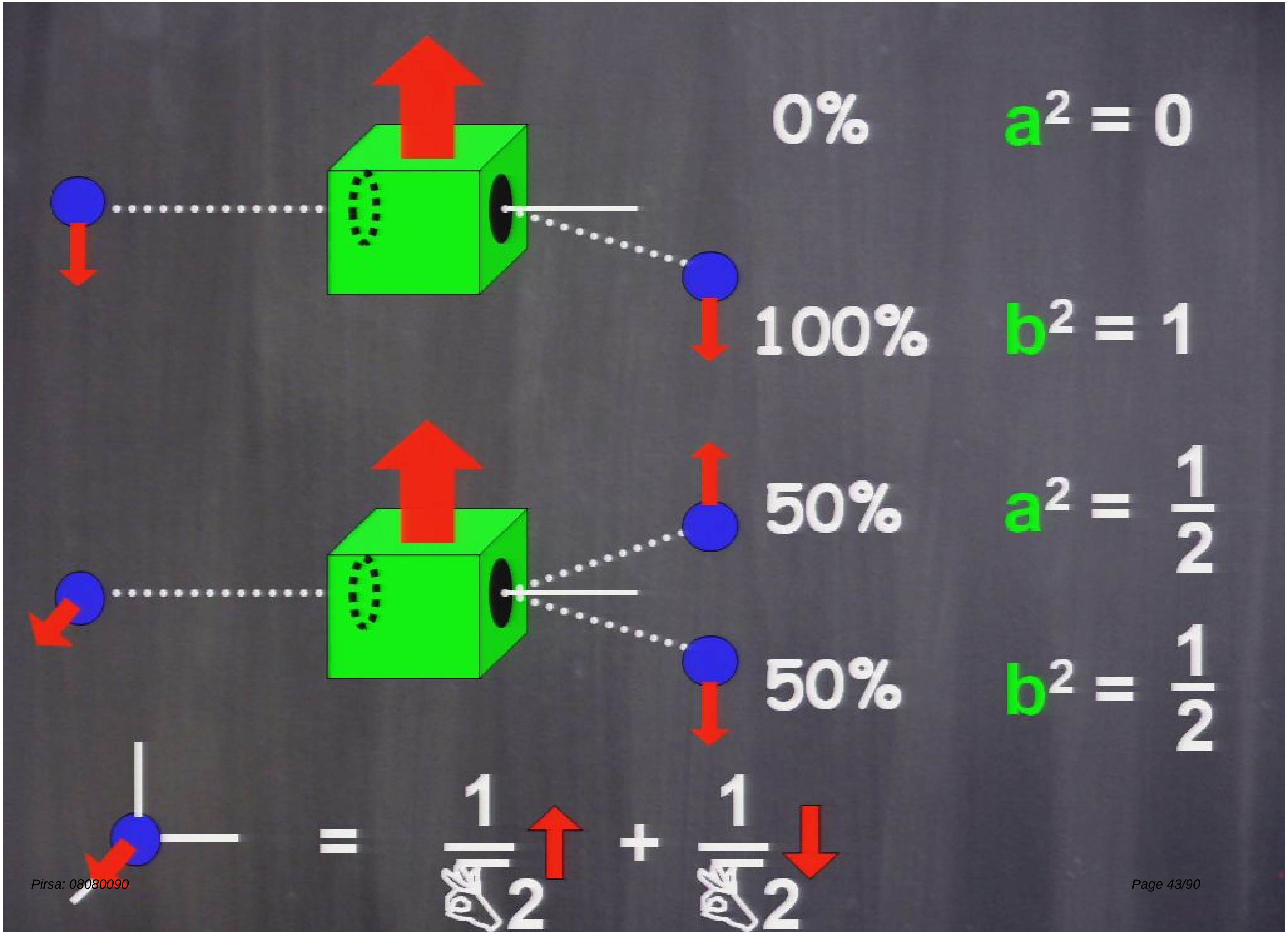


0% $a^2 = 0$

100% $b^2 = 1$


0%
$a^2 = 0$
100%
$b^2 = 1$
50%
50%


0%
$a^2 = 0$
100%
$b^2 = 1$
50%
$a^2 = \frac{1}{2}$
50%
$b^2 = \frac{1}{2}$


0%
a² = 0
100%
b² = 1
50%
a² = 1/2
50%
b² = 1/2
= 1/√2 + 1/√2

Summary:

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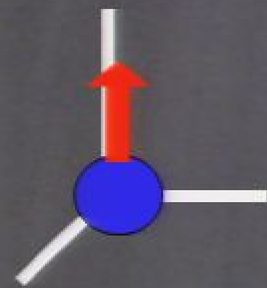

$$= 1 \uparrow + 0 \downarrow$$

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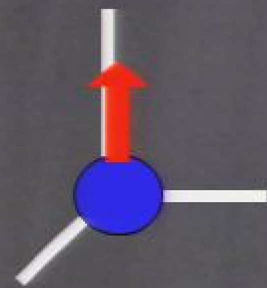

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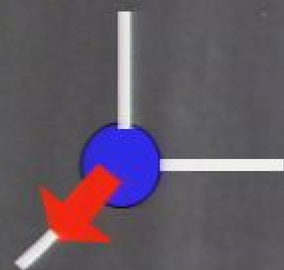
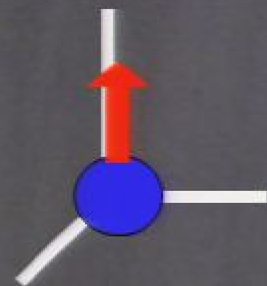
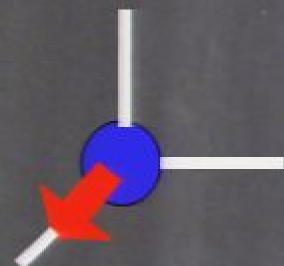


$$= \frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$

Summary:



$$= 1\uparrow + 0\downarrow$$



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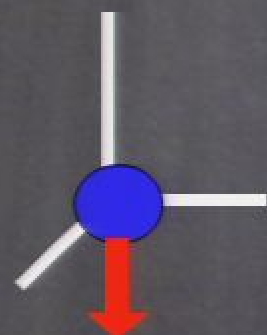

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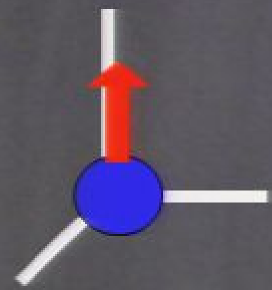

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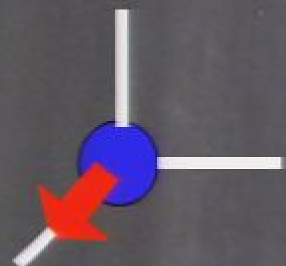

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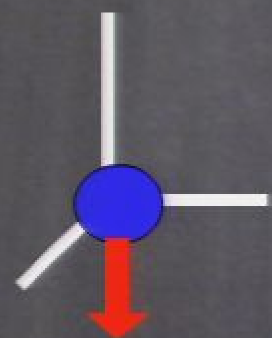
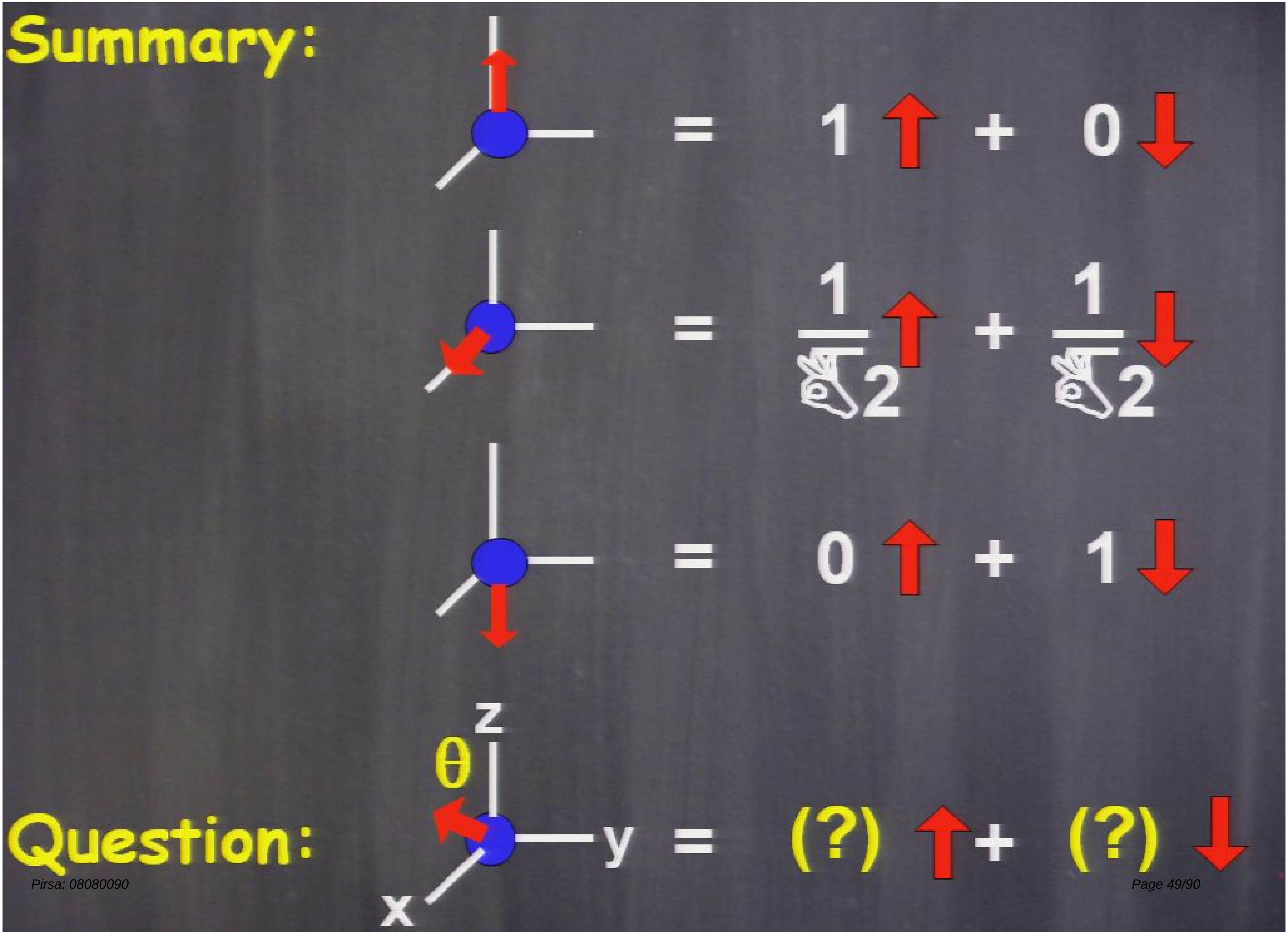
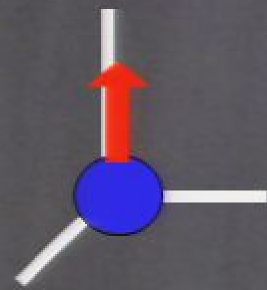


$$= 0\uparrow + 1\downarrow$$

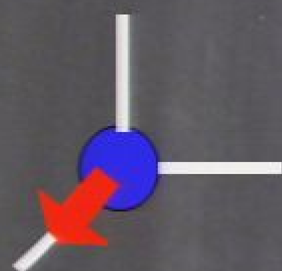
Question:


Summary:
= 1 + 0
= 1/√2 + 1/√2
= 0 + 1
z
θ
y
x
Question:
= (?) + (?)

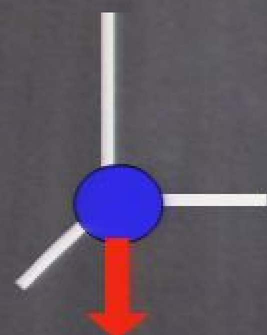
Summary:



$= 1\uparrow + 0\downarrow$



$= \frac{1}{\sqrt{2}}\uparrow + \frac{1}{\sqrt{2}}\downarrow$



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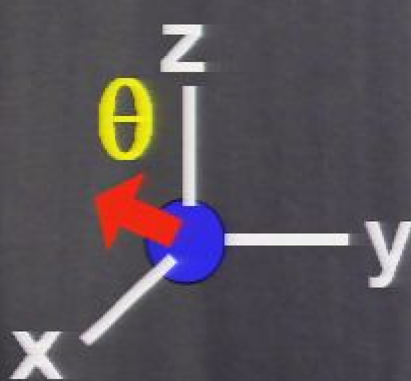
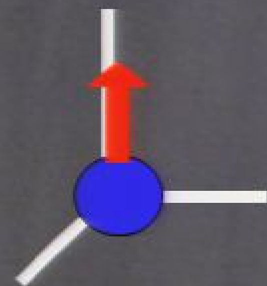
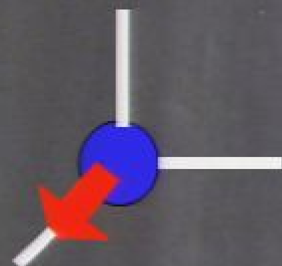


$= \cos\theta\uparrow + \sin\theta\downarrow$

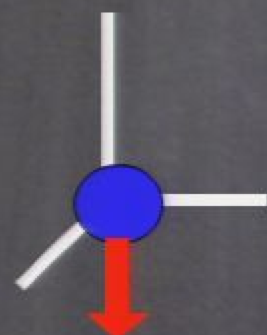
Summary:



$$= 1\uparrow + 0\downarrow$$



$$= \frac{1}{\sqrt{2}}\uparrow + \frac{1}{\sqrt{2}}\downarrow$$



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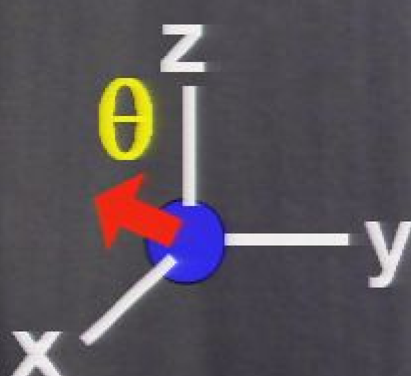
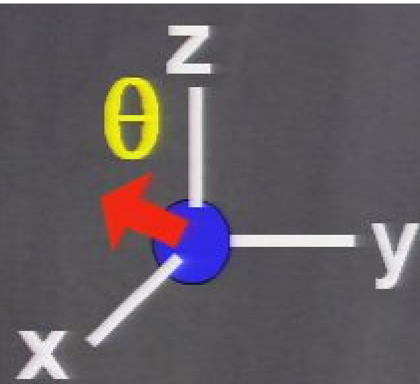
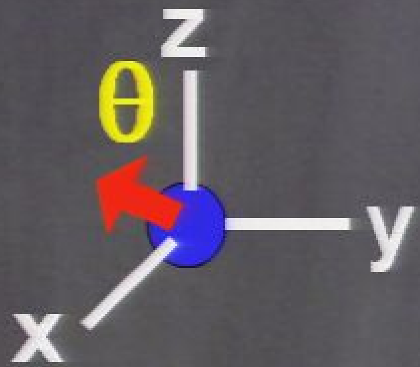


$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

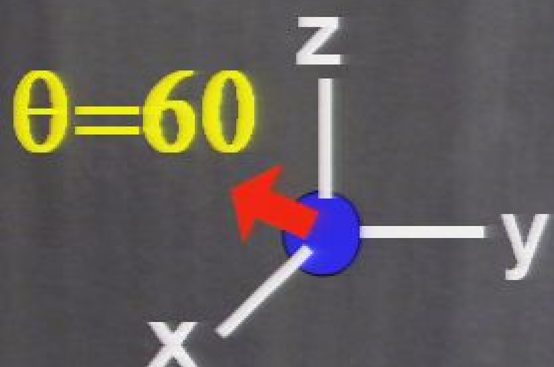
Result:



$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

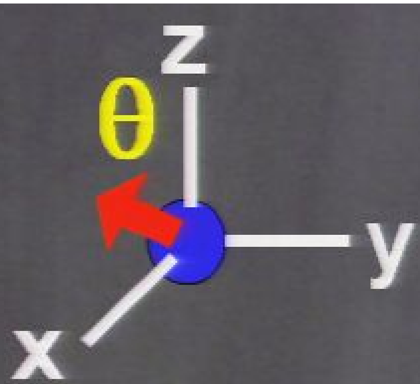
Result: 

$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

E.g.: 

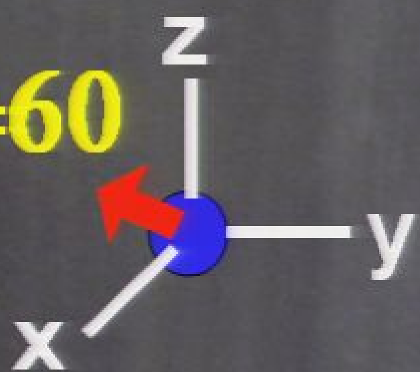
$$= \frac{\sqrt{3}}{2}\uparrow + \frac{1}{2}\downarrow$$

Result:



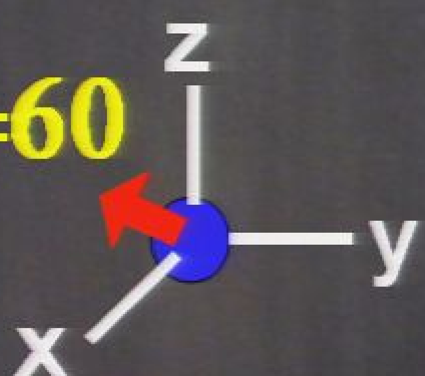
$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

E.g.: θ=60



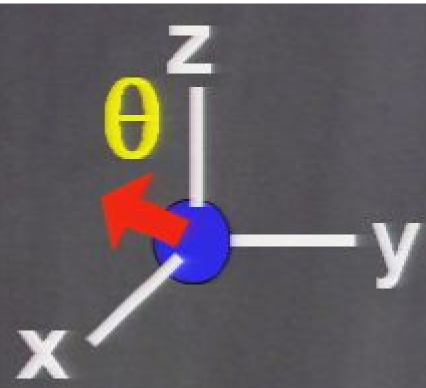
$$= \frac{\sqrt{3}}{2}\uparrow + \frac{1}{2}\downarrow$$

θ=60

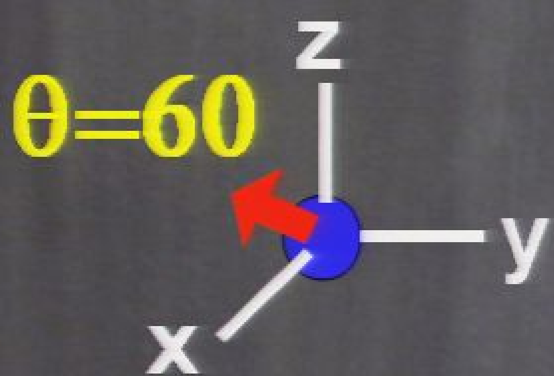


Result: (spin along direction θ) $= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$

E.g.: $\theta=60$ $= \frac{\sqrt{3}}{2}\uparrow + \frac{1}{2}\downarrow$

Result: 

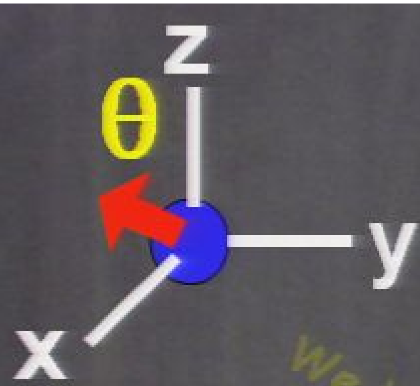
$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

E.g.: 

$$= \frac{\sqrt{3}}{2}\uparrow + \frac{1}{2}\downarrow$$

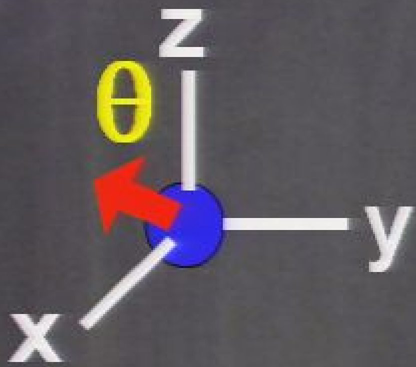
$$\left(\frac{\sqrt{3}}{2}\right)^2 = 75\%$$

$$\left(\frac{1}{2}\right)^2 = 25\%$$



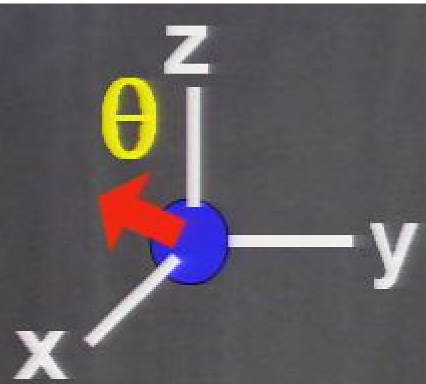
$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

We have constructed a successful "mathematical model" of spin!



$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

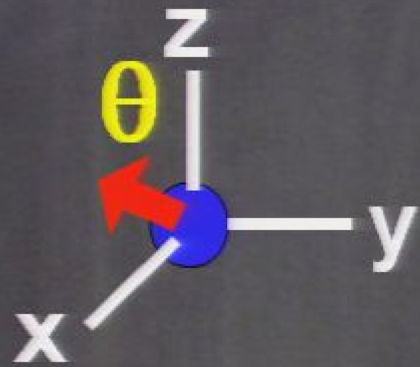
We have constructed a <u>successful</u> "mathematical model" of spin!



$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

We have constructed a <u>successful</u>
"mathematical model" of spin!

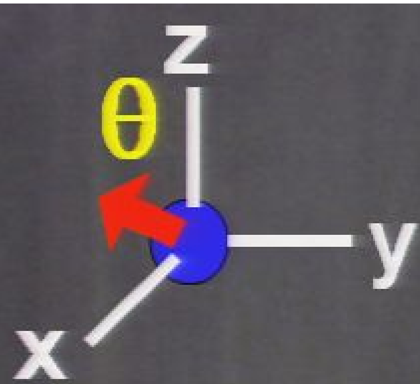
Does the mathematics predict anything
nteresting we h a v e n



$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

We have constructed a successful "mathematical model" of spin!

Does the mathematics predict anything nteresting we have not thought of yet?



$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

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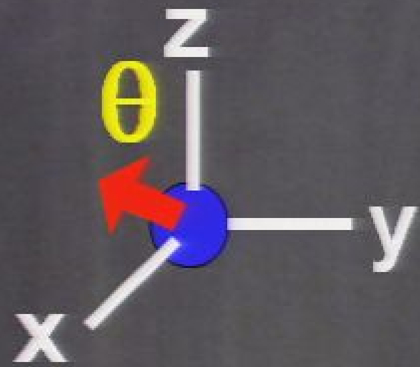
Does the mathematics predict anything nteresting we have not thought of yet?

The <u>power</u> of mathematics in the

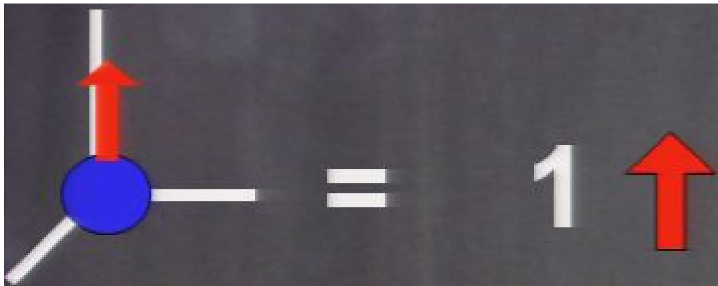


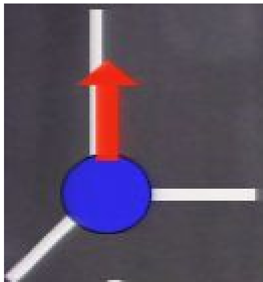
$$= \cos\frac{\theta}{2}\uparrow + \sin\frac{\theta}{2}\downarrow$$

We have constructed a <u>successful</u> "mathematical model" of spin!

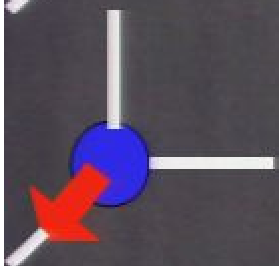
Does the mathematics predict anything nteresting we have not thought of yet?

The <u>power</u> of mathematics in the <u>process</u> of science

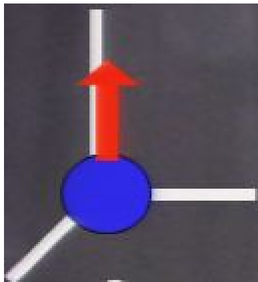

= 1



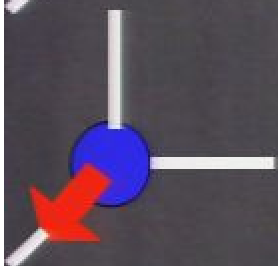
$= 1 \uparrow$



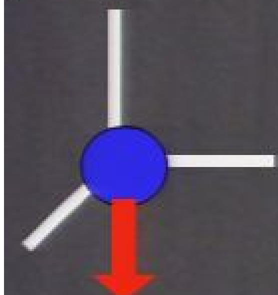
$= \frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$



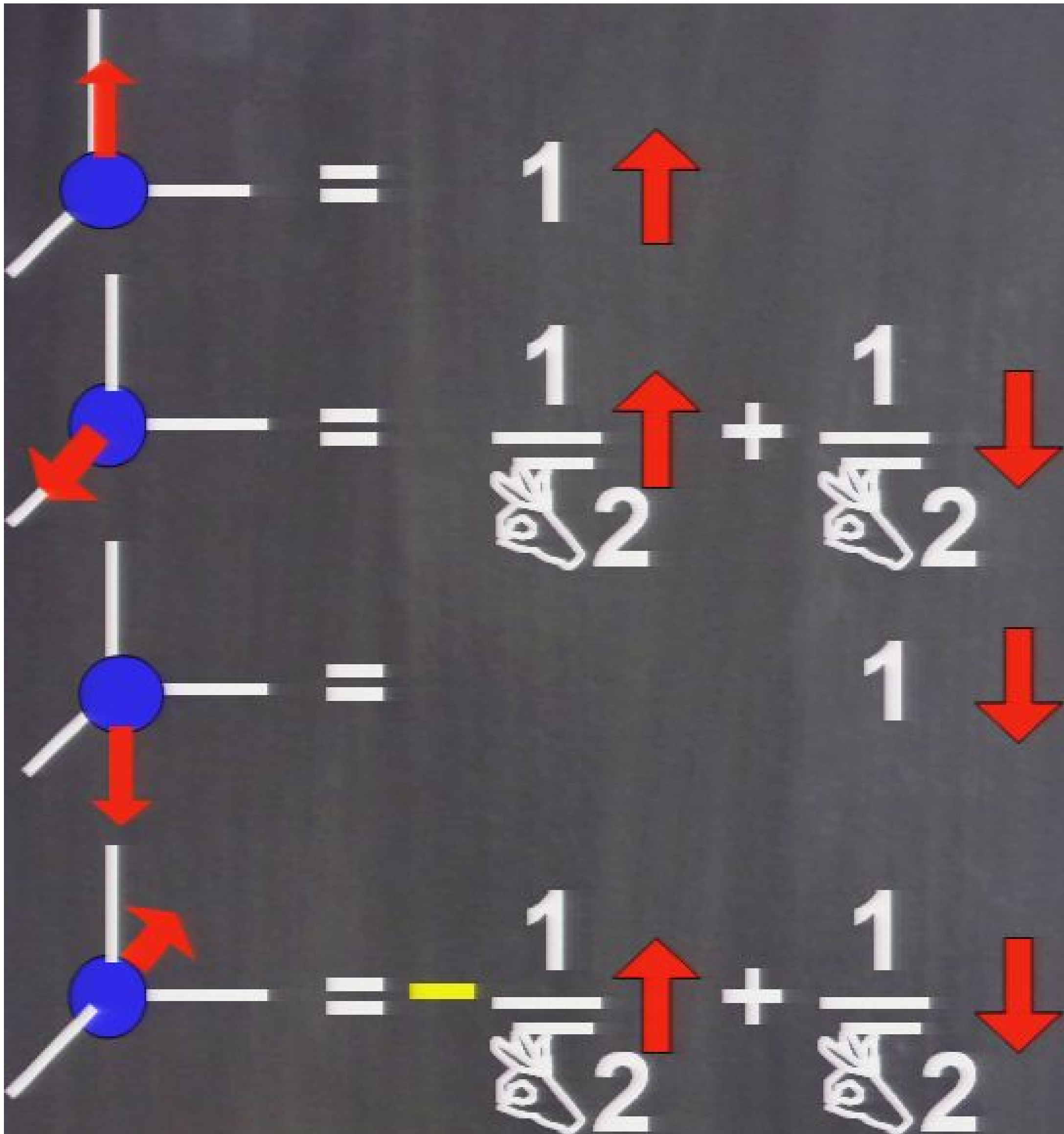
$= 1 \uparrow$

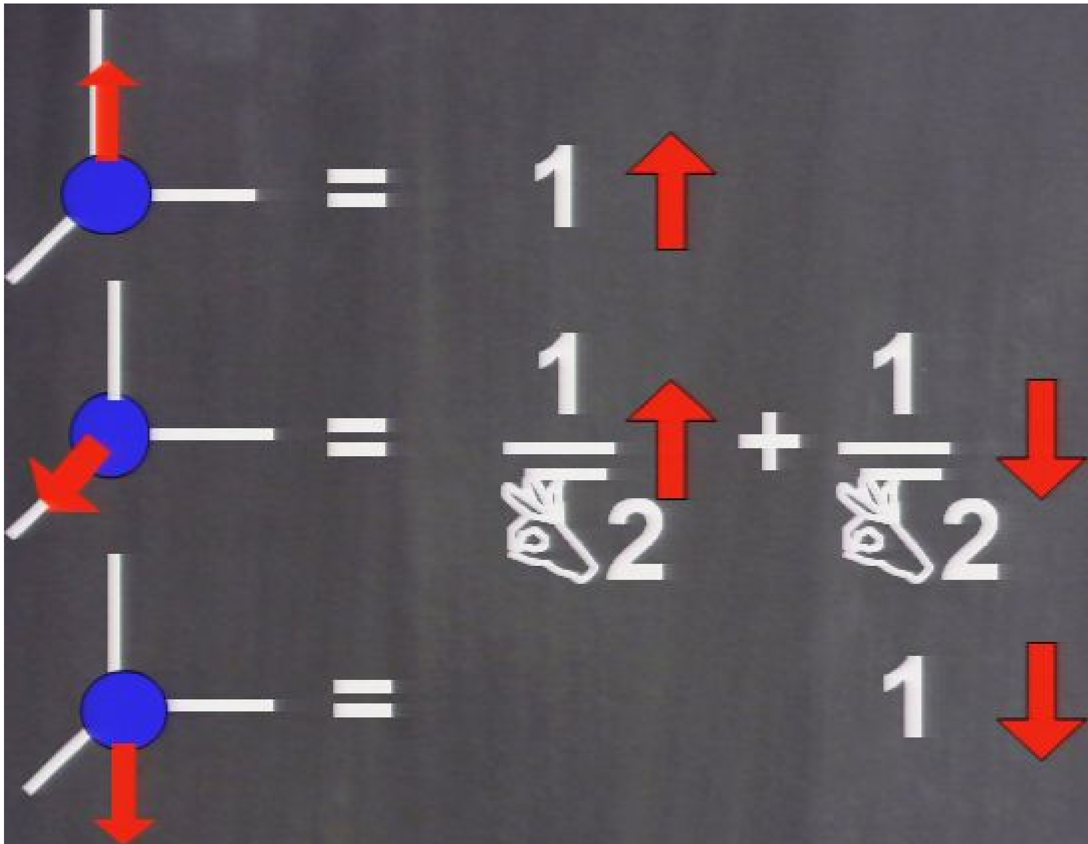


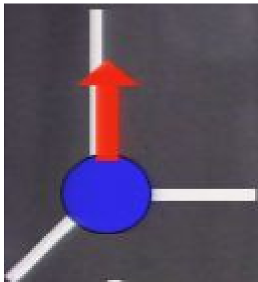
$= \frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$



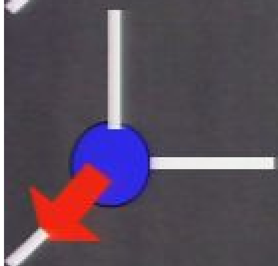
$= 1 \downarrow$



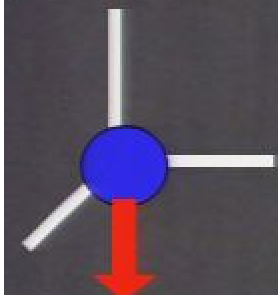

= 1
= 1 + 1
2 2
= 1



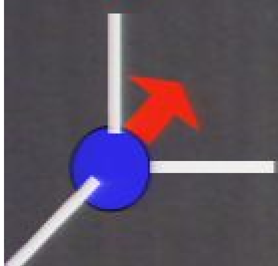
$$= 1 \uparrow$$



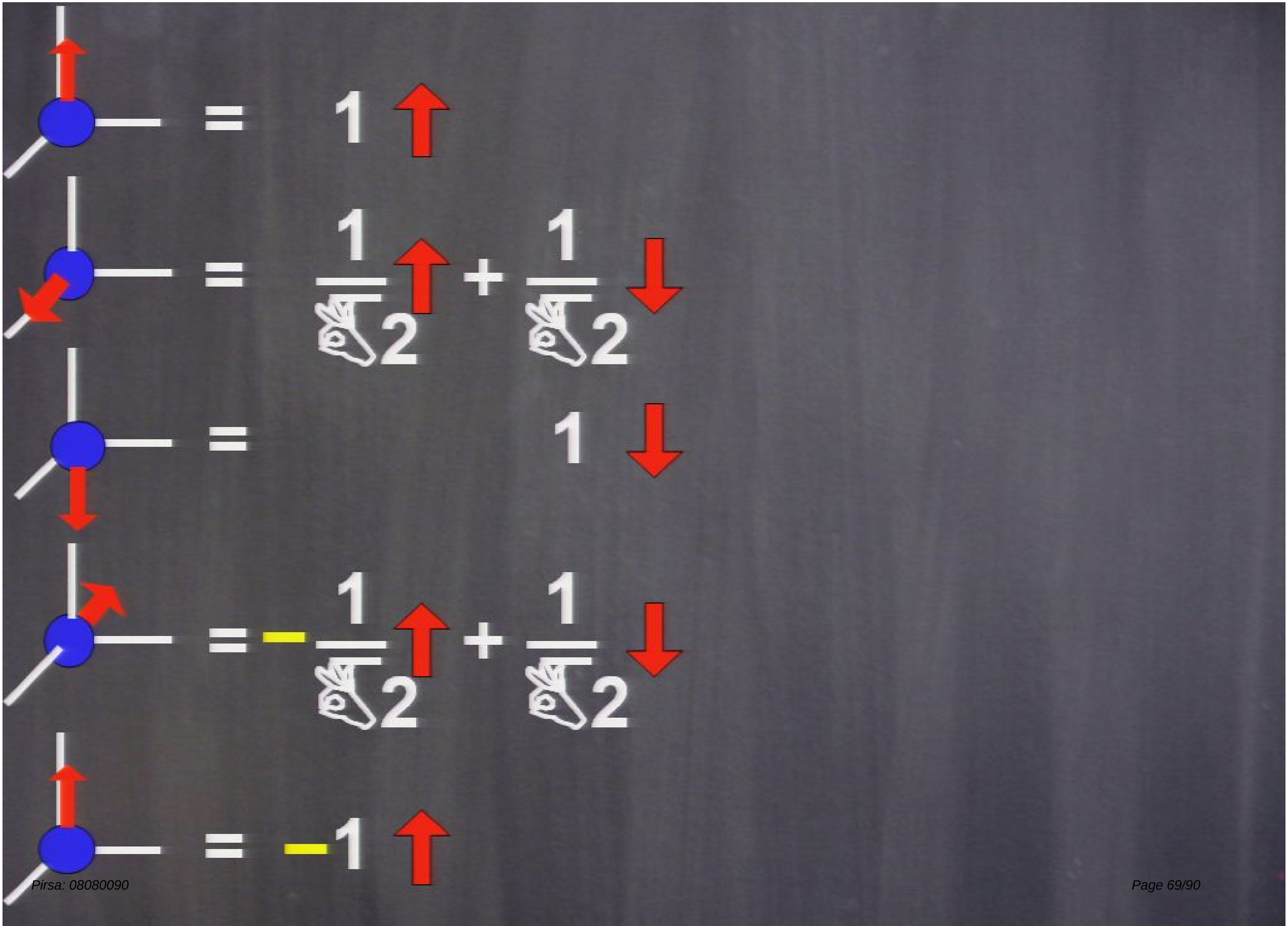
$$= \frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$

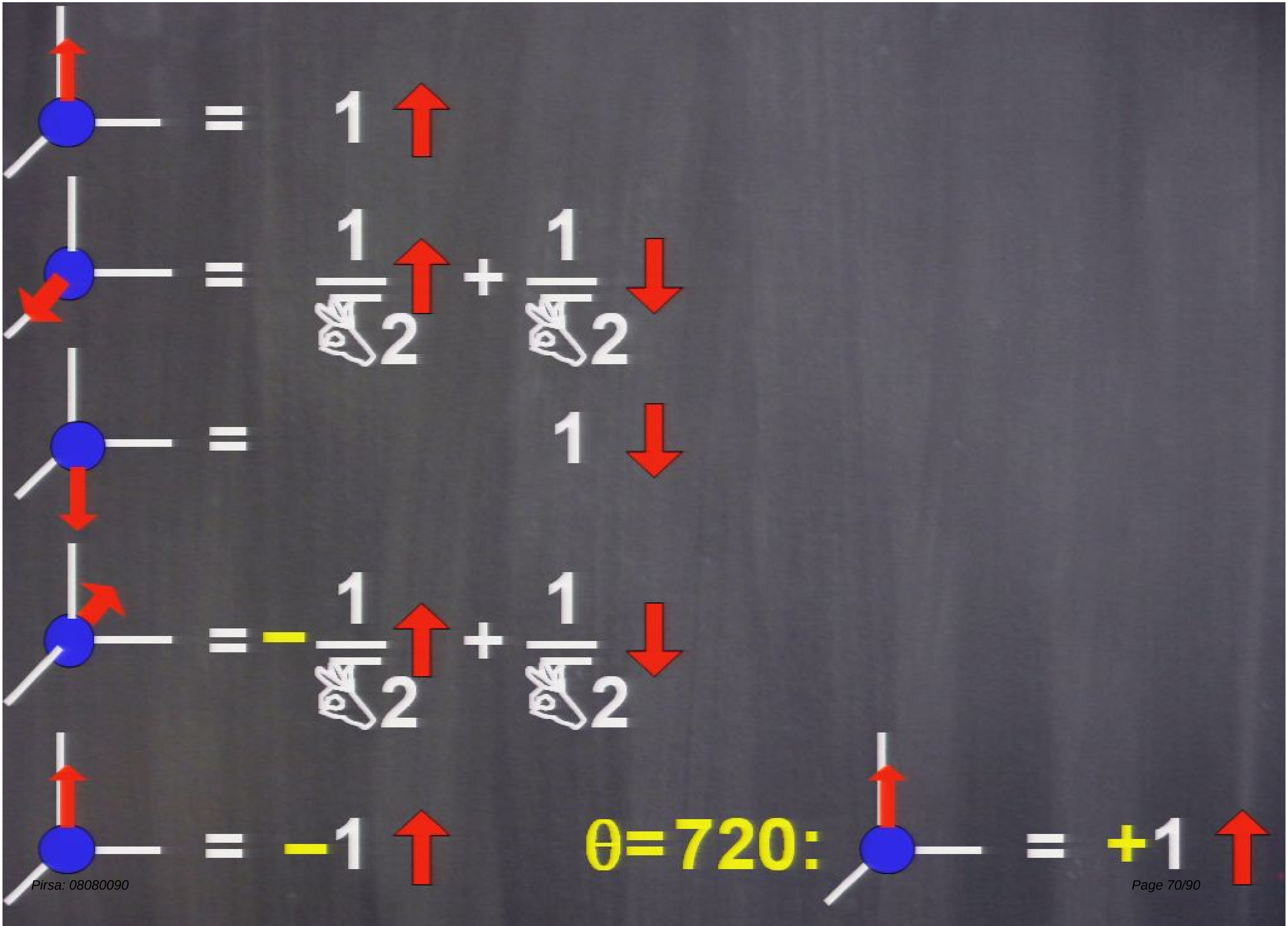


$$= 1 \downarrow$$



$$= -\frac{1}{\sqrt{2}} \uparrow + \frac{1}{\sqrt{2}} \downarrow$$


= 1
= 1/√2 + 1/√2
= 1
= − 1/√2 + 1/√2
= −1


= 1
= 1/√2 + 1/√2
= 1
= − 1/√2 + 1/√2
= −1
θ=720:
= +1

The Myst

The Mysteri

The Mysterious Min u s

The Mysterious Minus Sign:

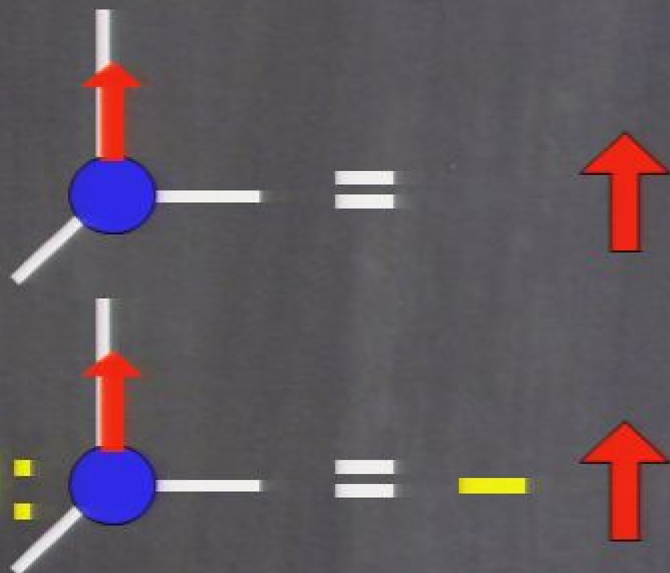
The Mysterious Minus Sign:

$\theta = 0$:

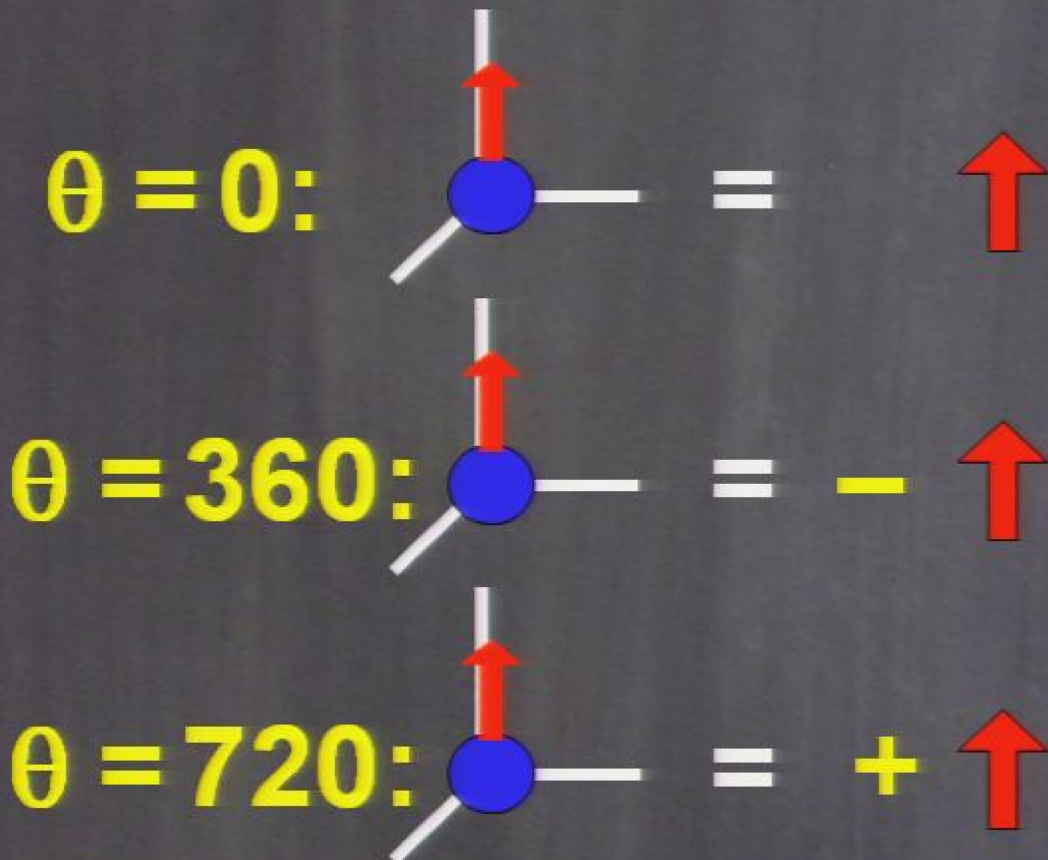
The Mysterious Minus Sign:

$\theta = 0$:

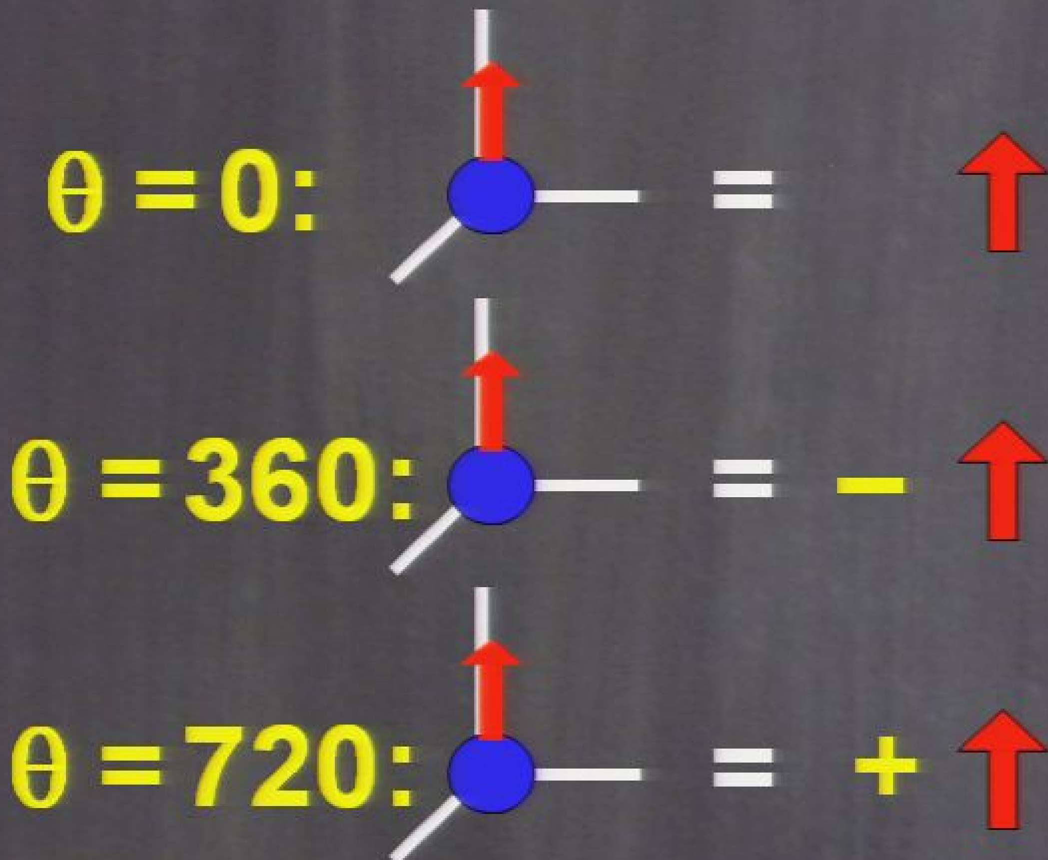
$\theta = 360$:



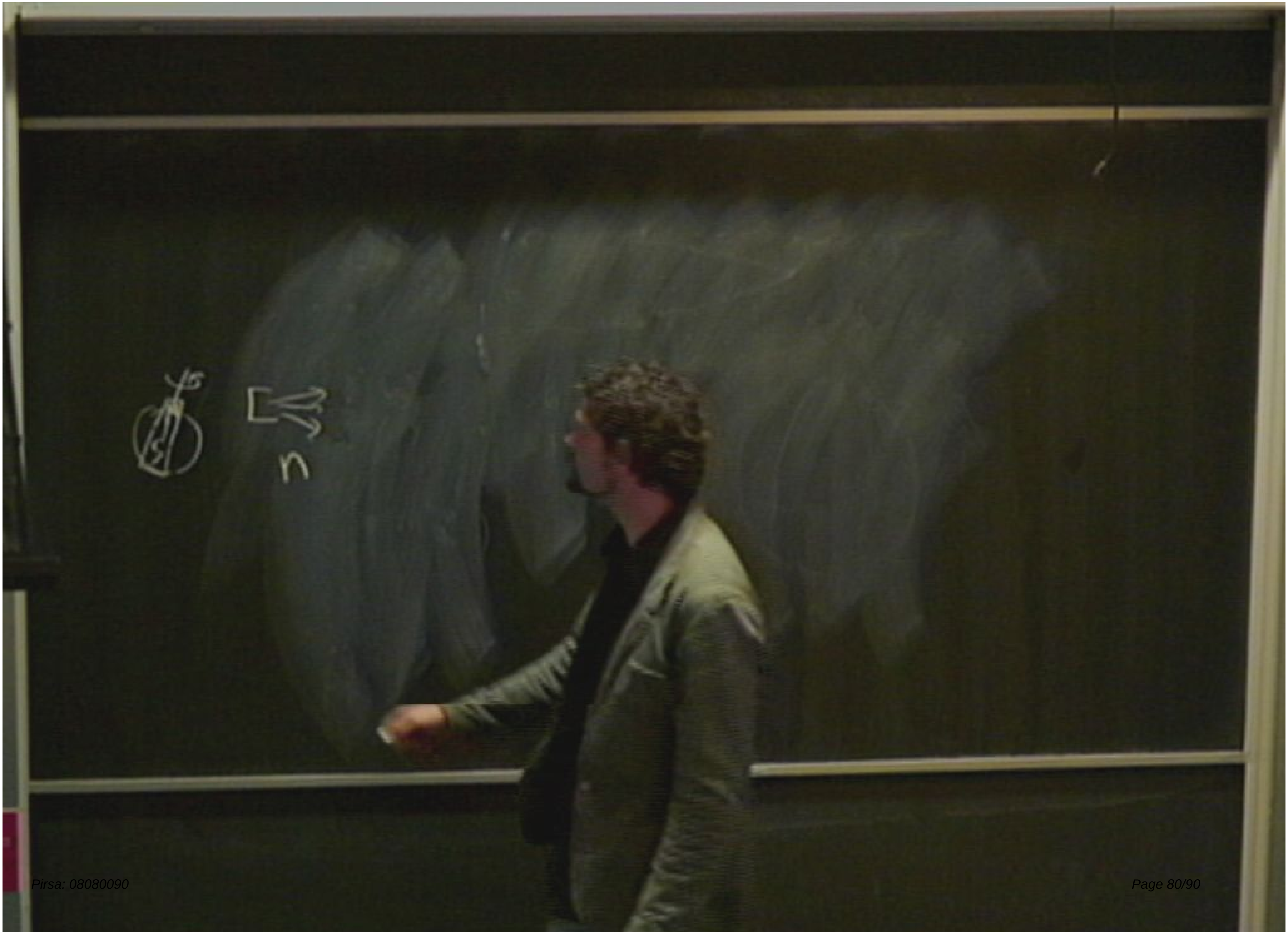
The Mysterious Minus Sign:

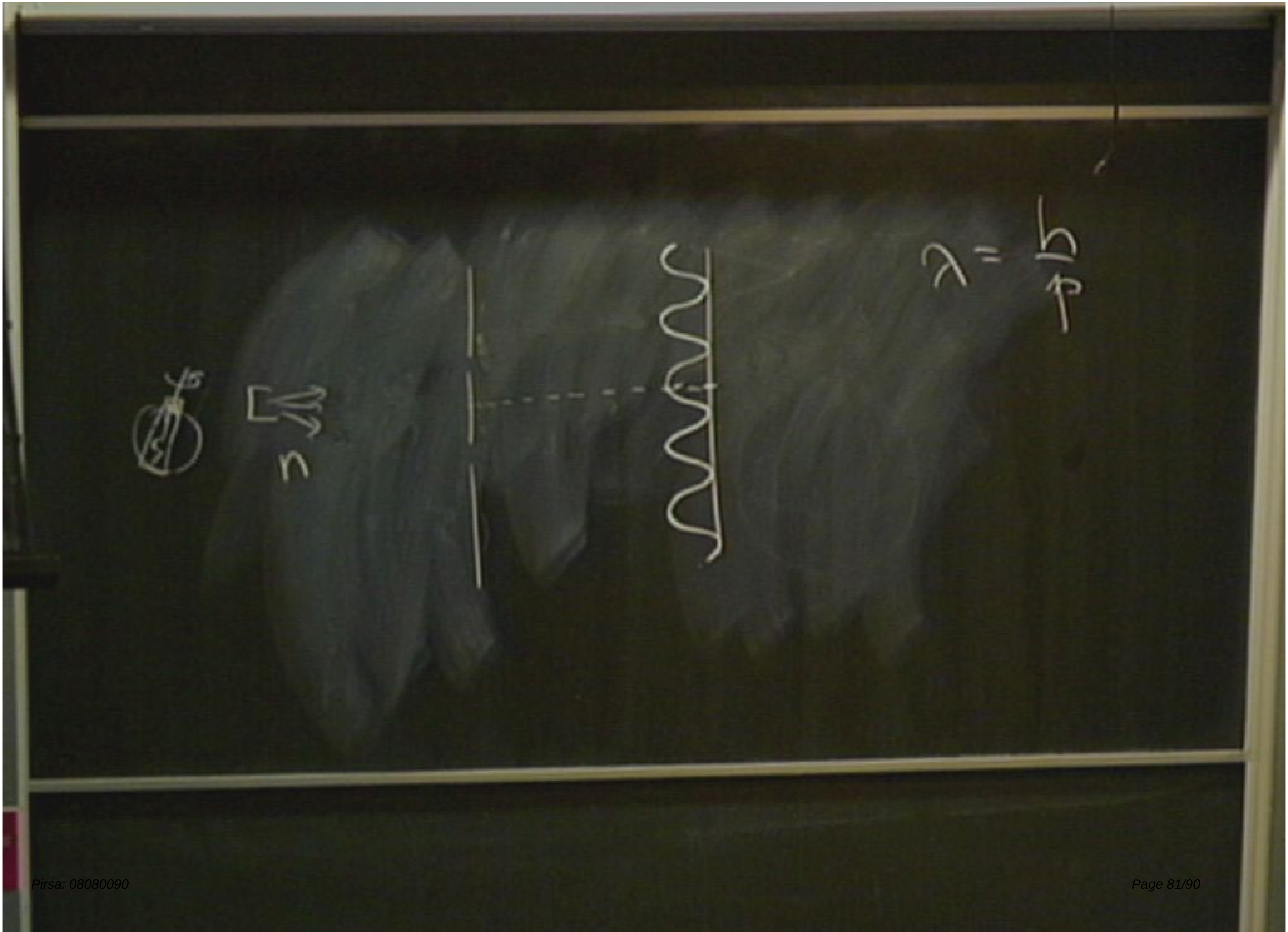


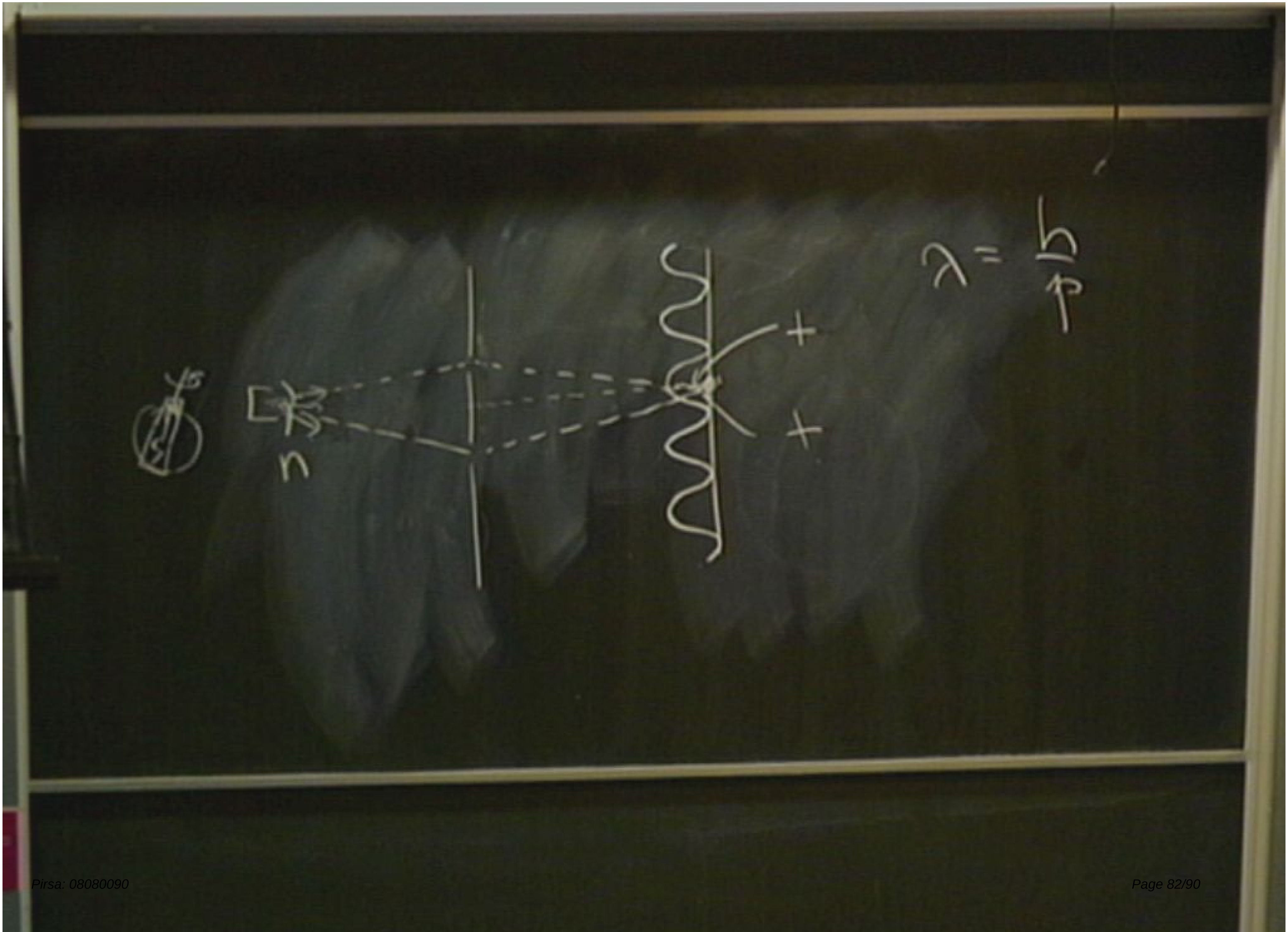
The Mysterious Minus Sign:

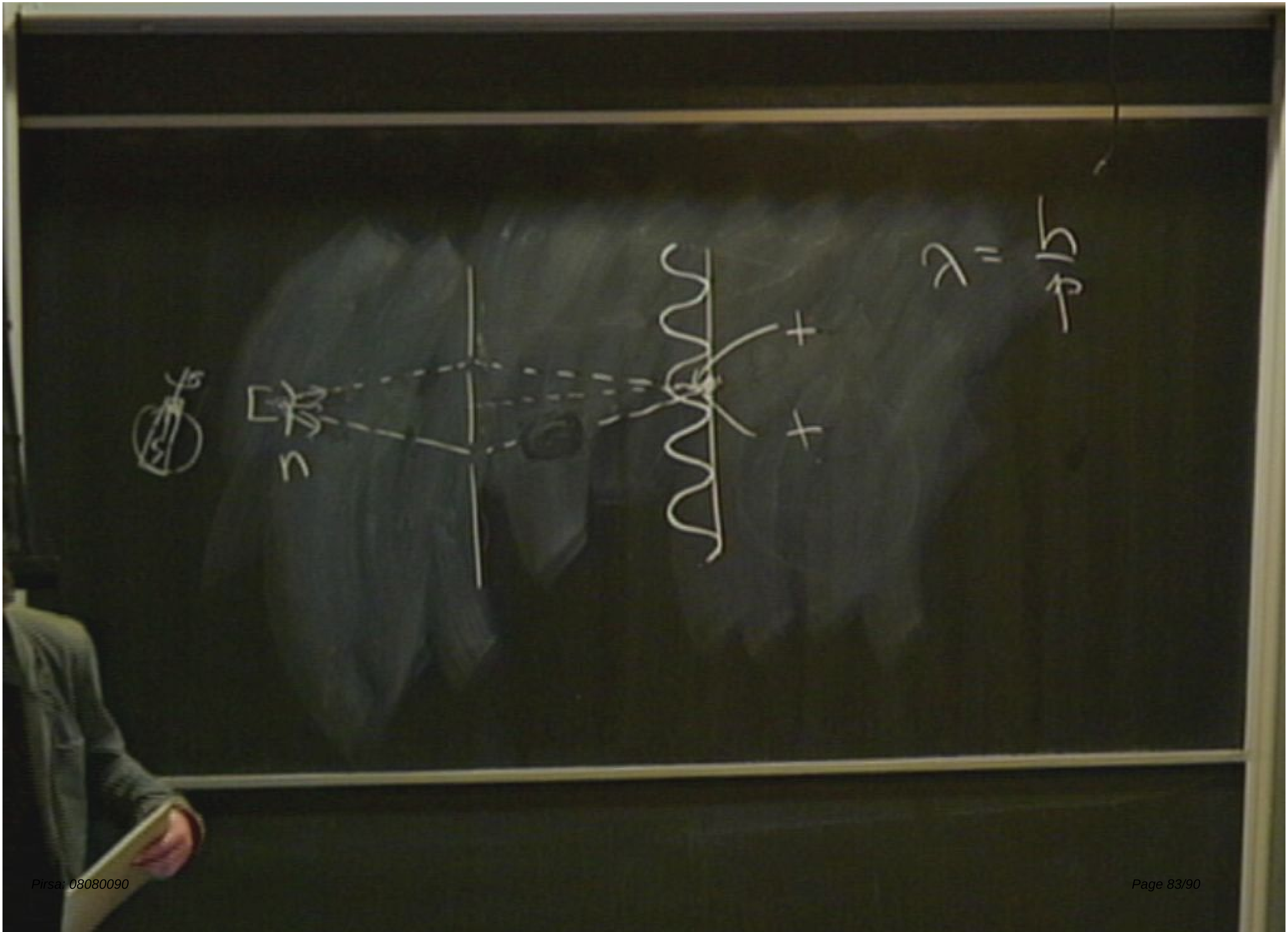


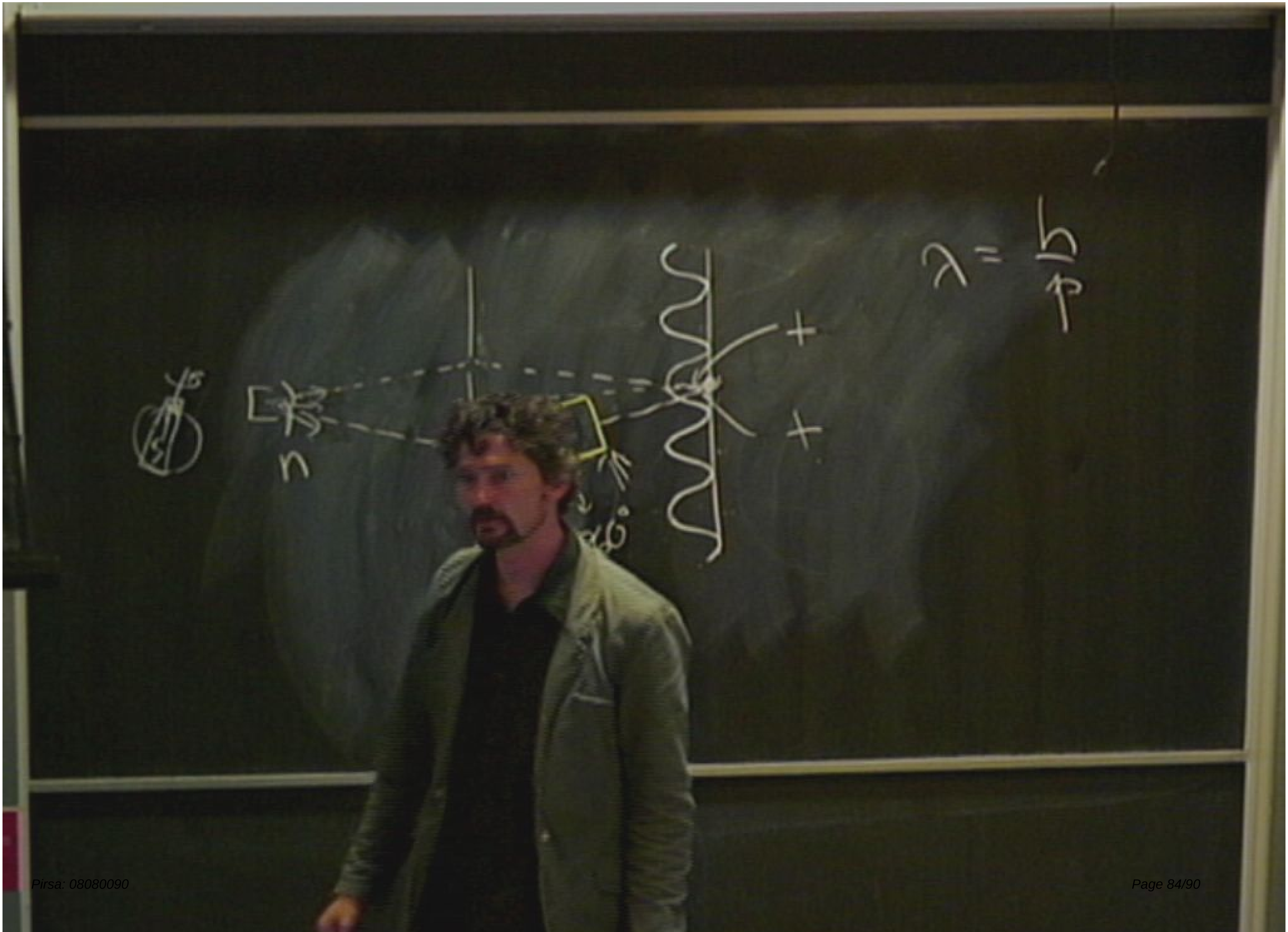
: Can be observed experimentally

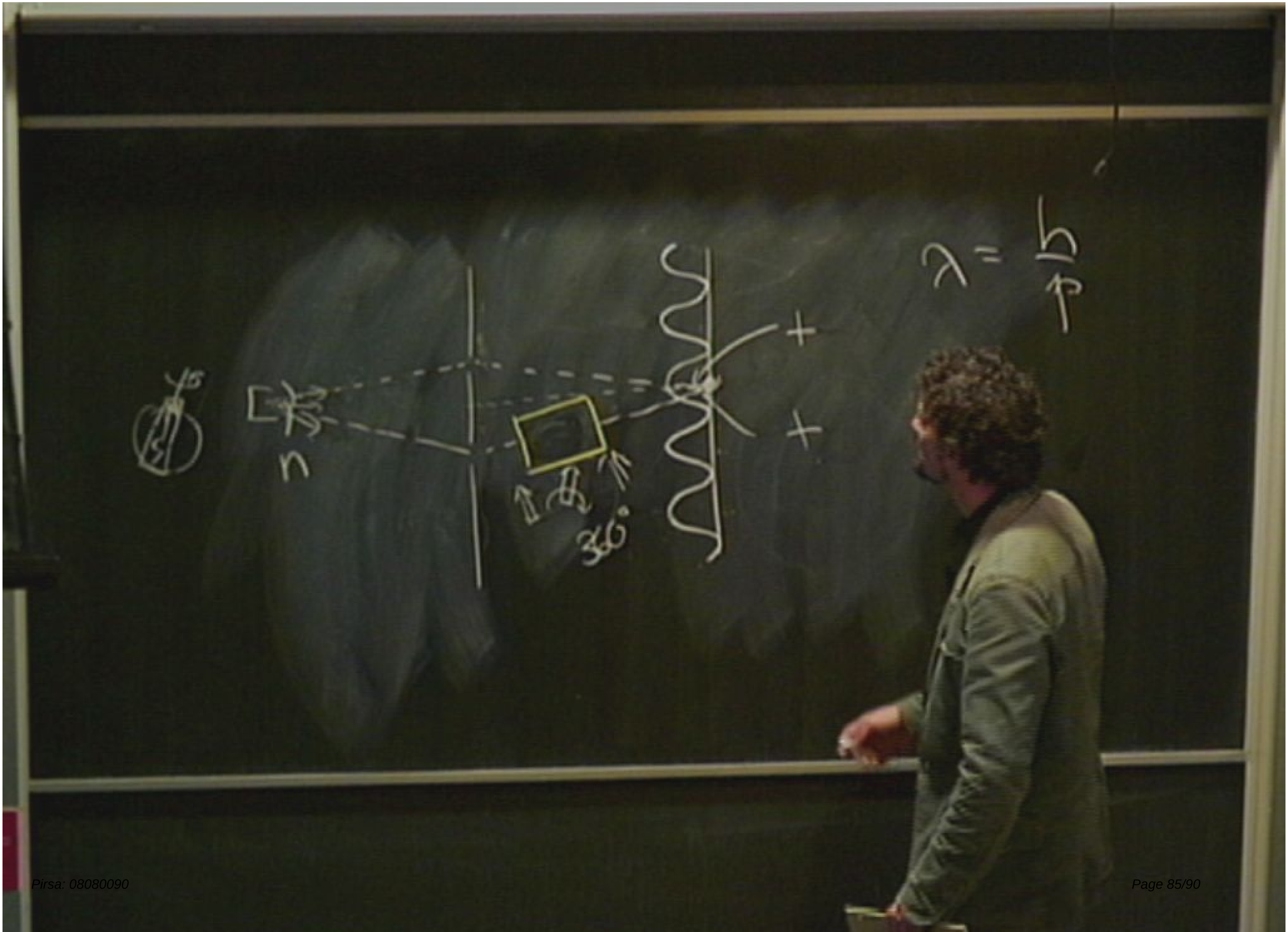



n
λ = h/p

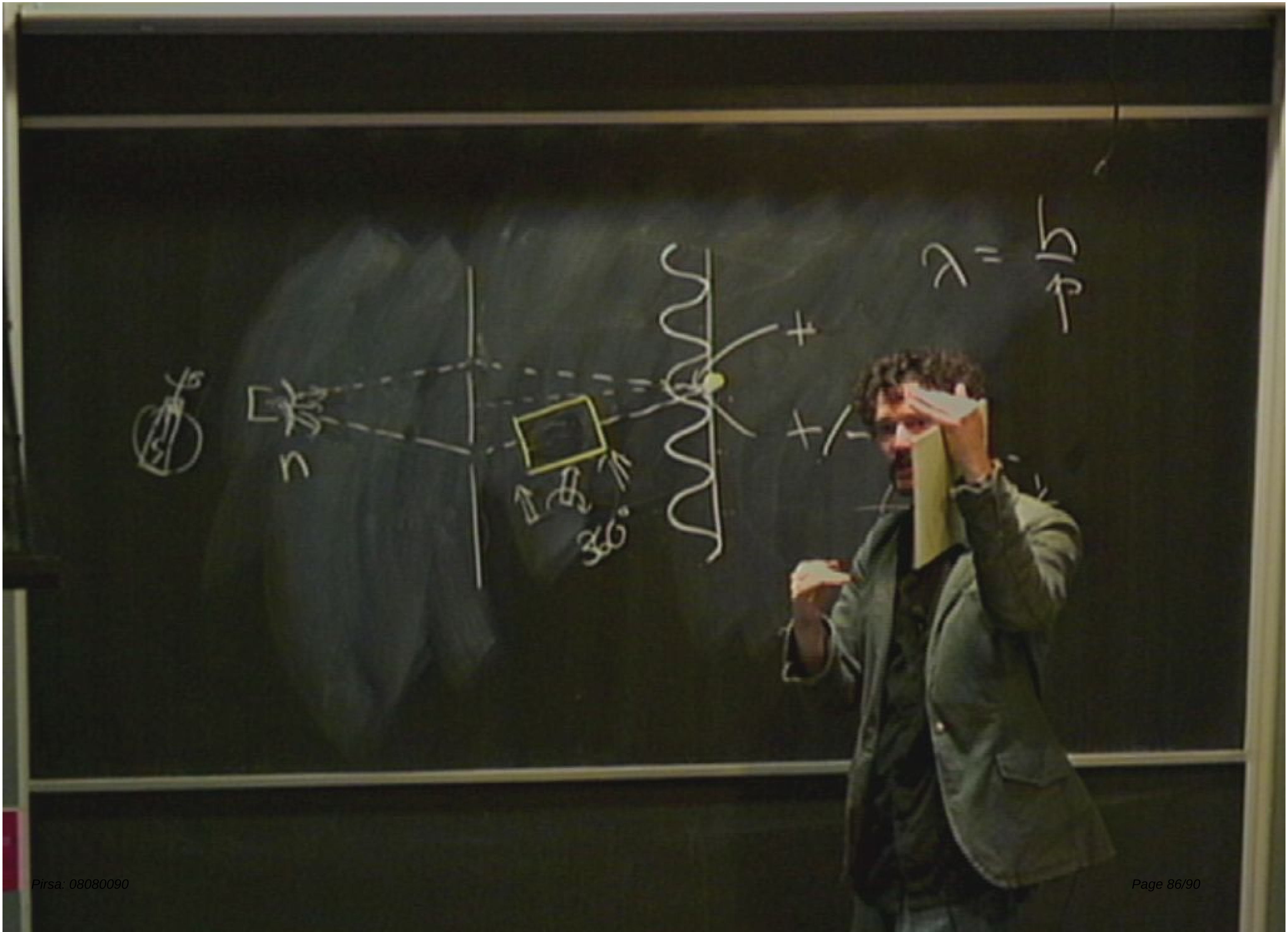

n
$\lambda = \frac{h}{p}$

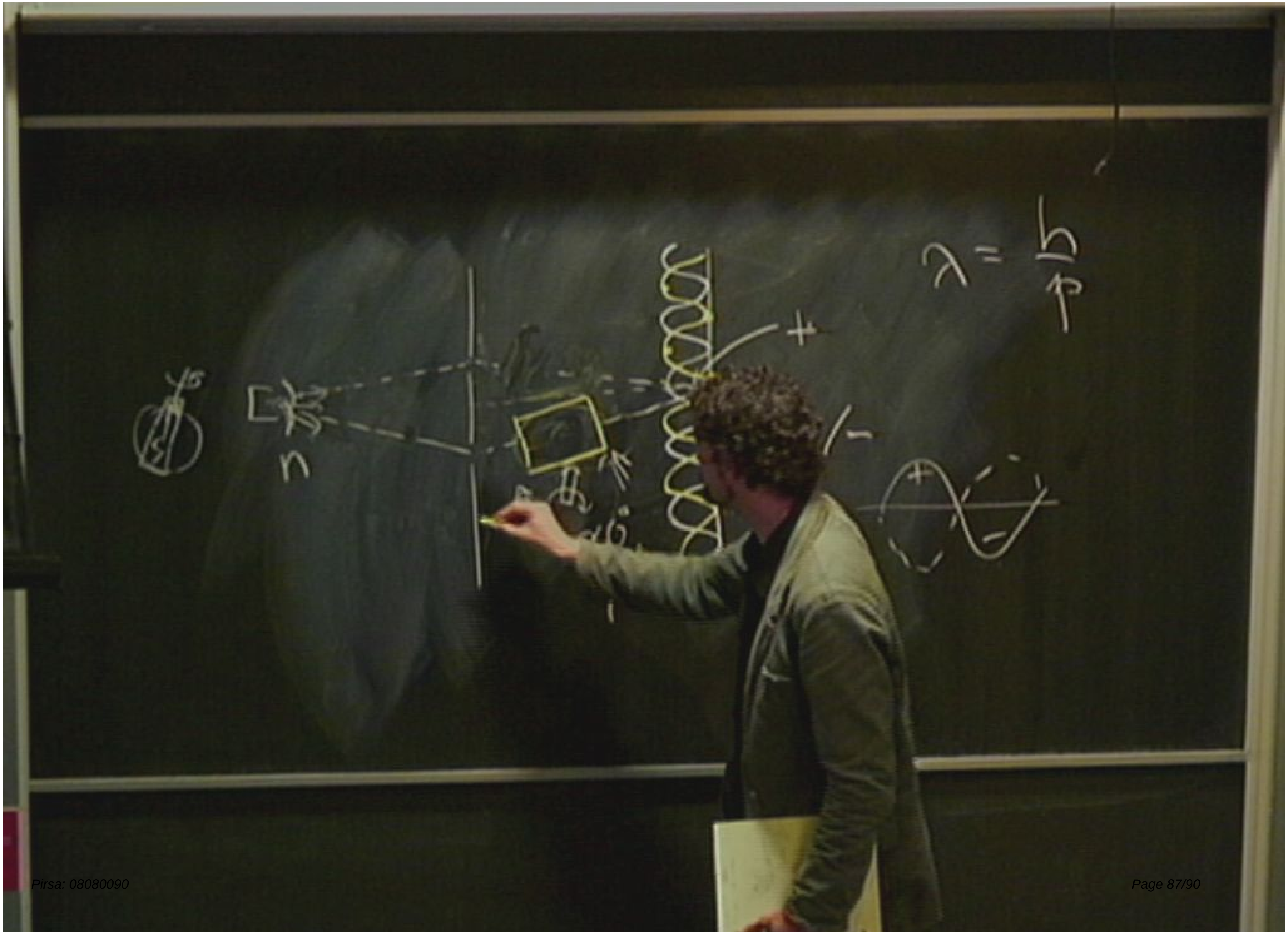

λ = h/p
n

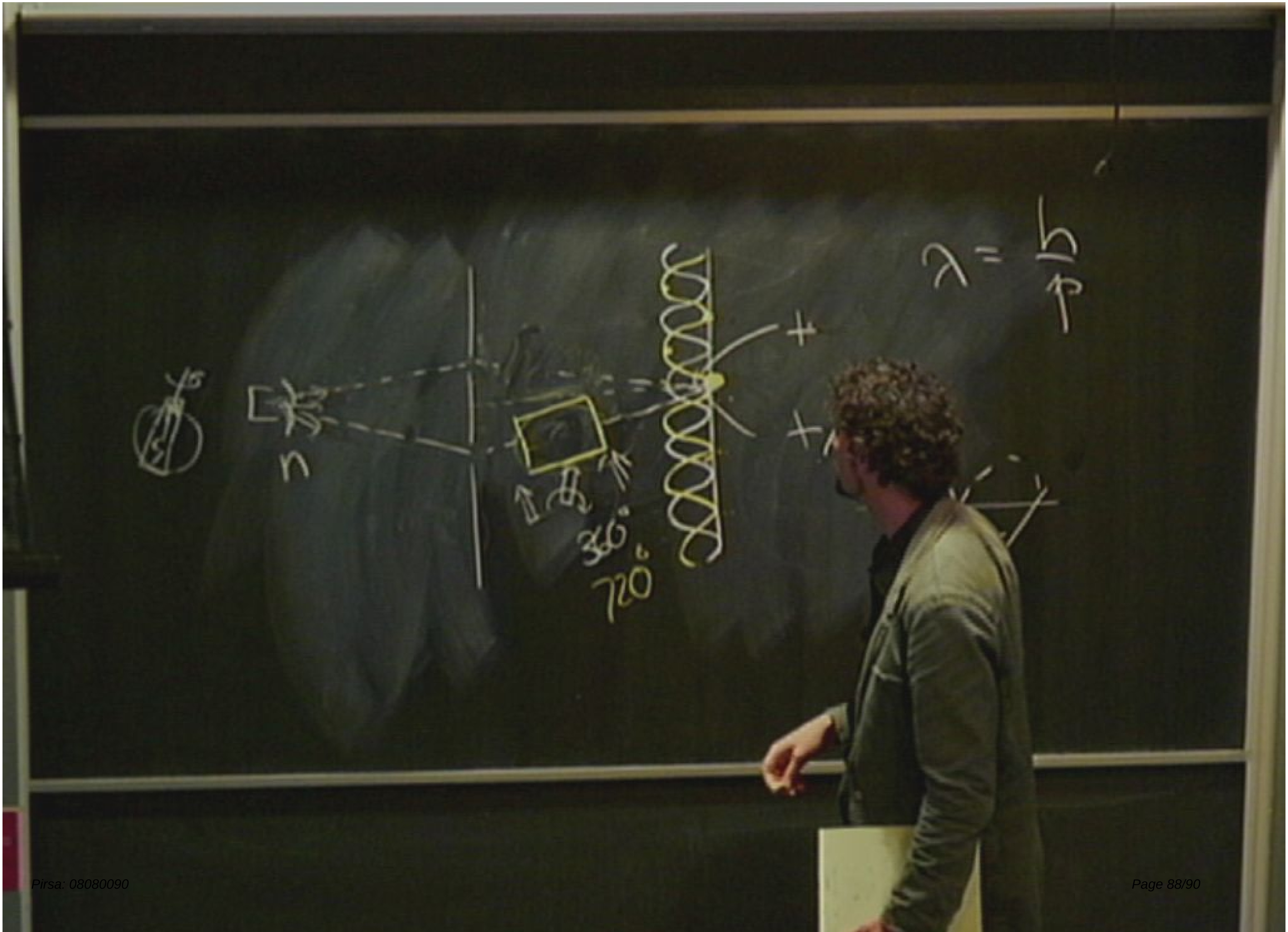

λ = h/p
n

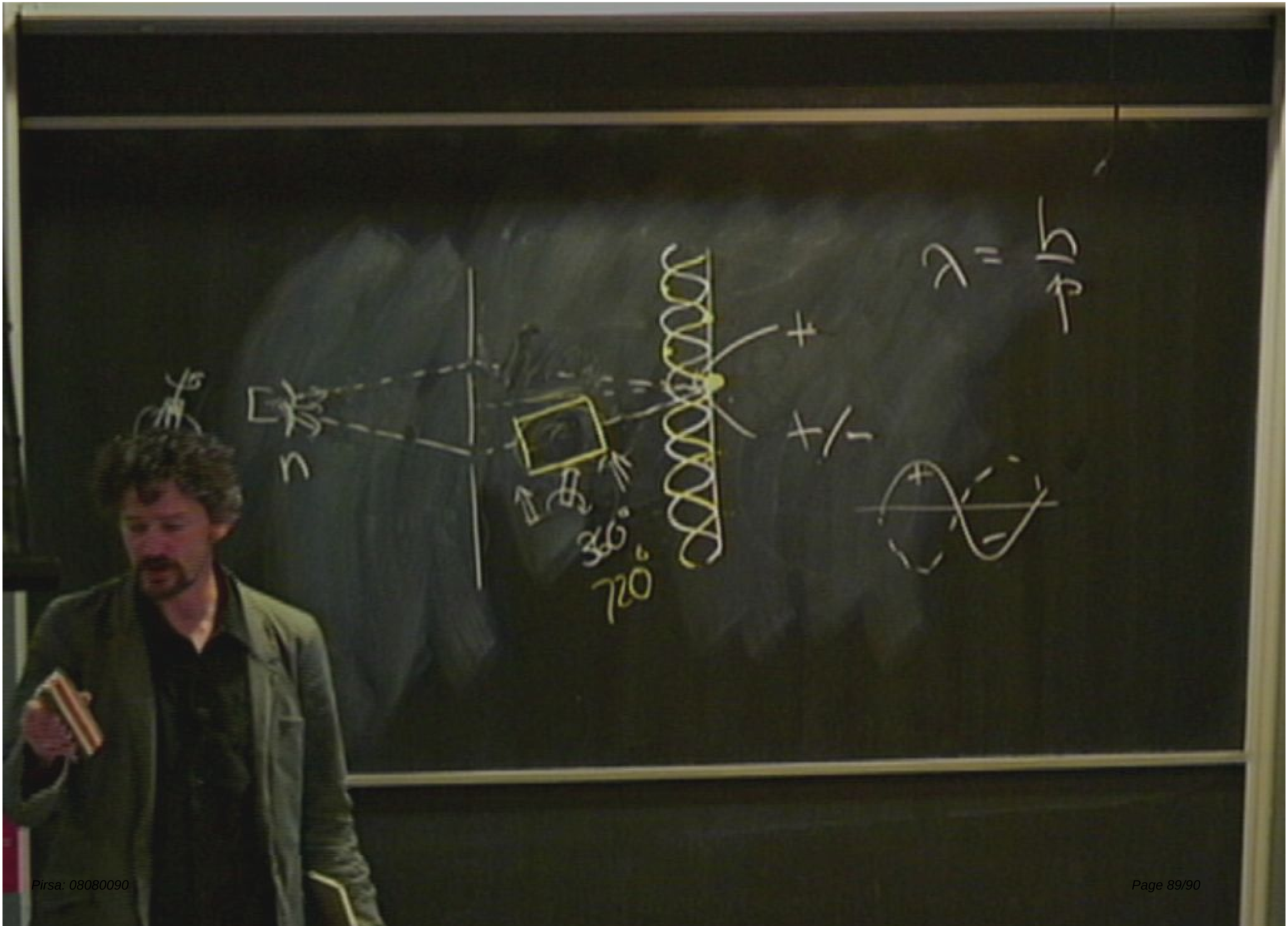


n
360°
λ = h/p

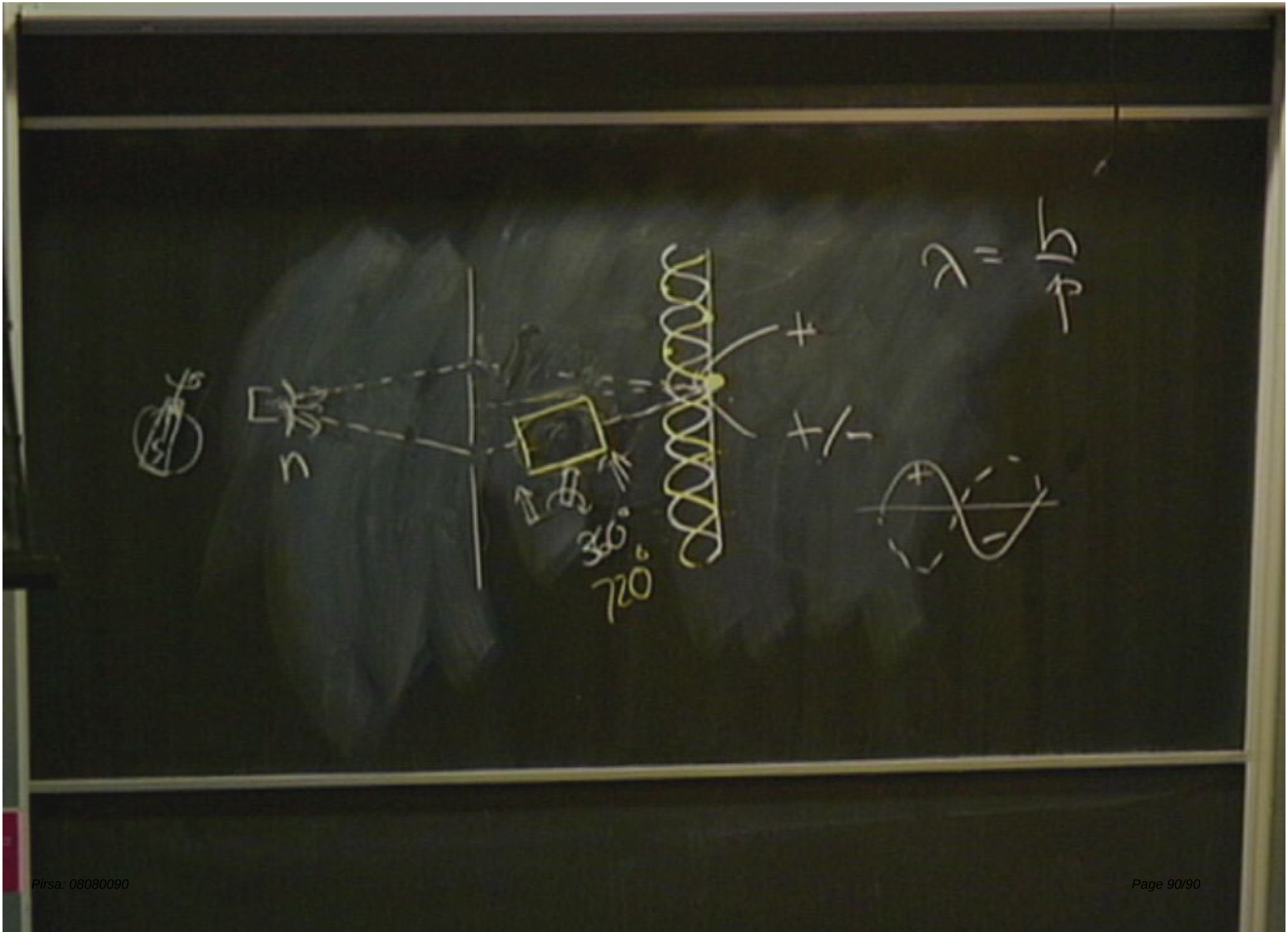

λ = h/p
+
+/-
n
360°


λ = h/p
n
+
-


λ = h/p
n
360°
720°



λ = h/p
+
+/-
n
360°
720°


λ = h/p
+
+/-
+
360°
720°
n