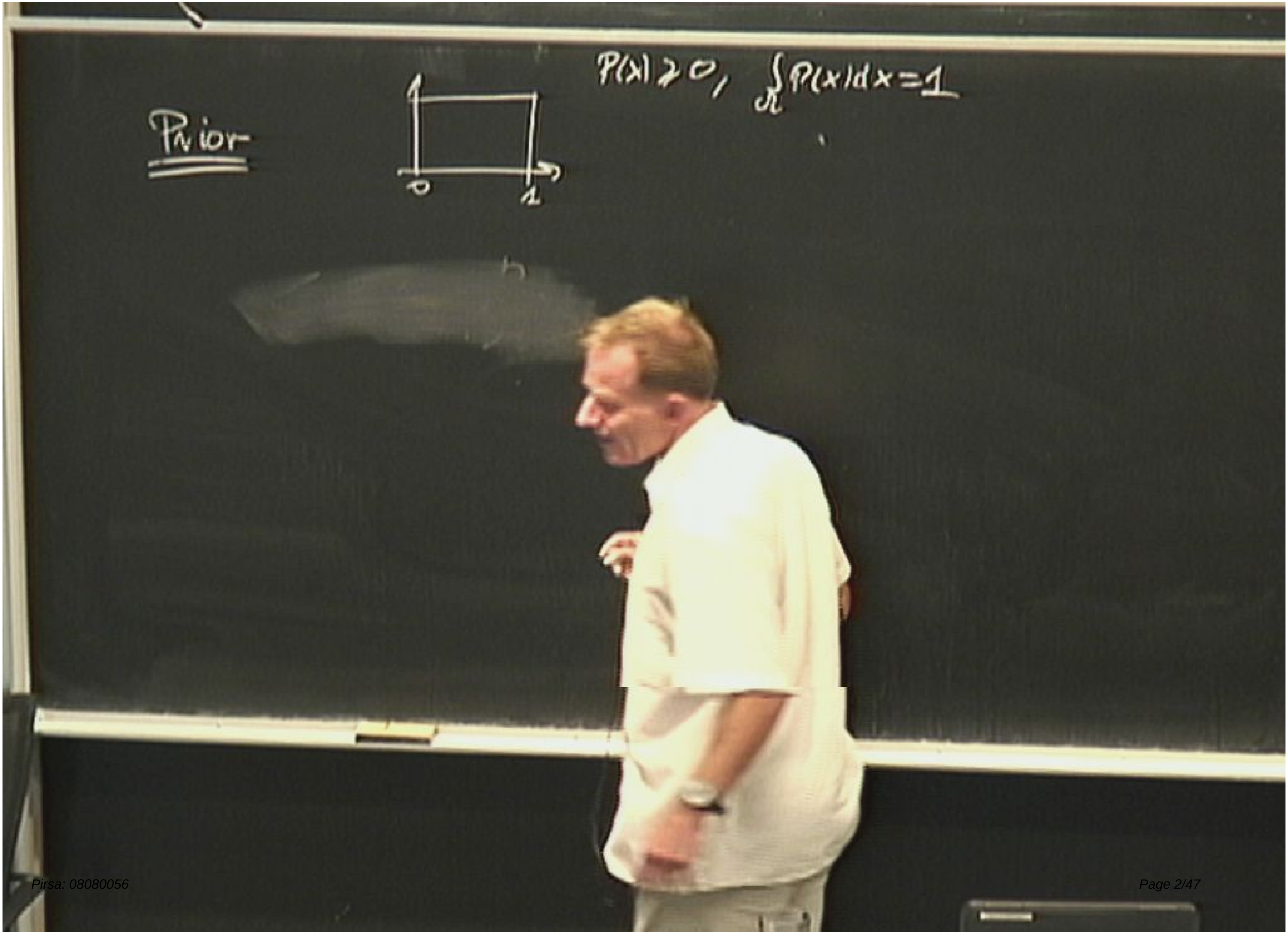
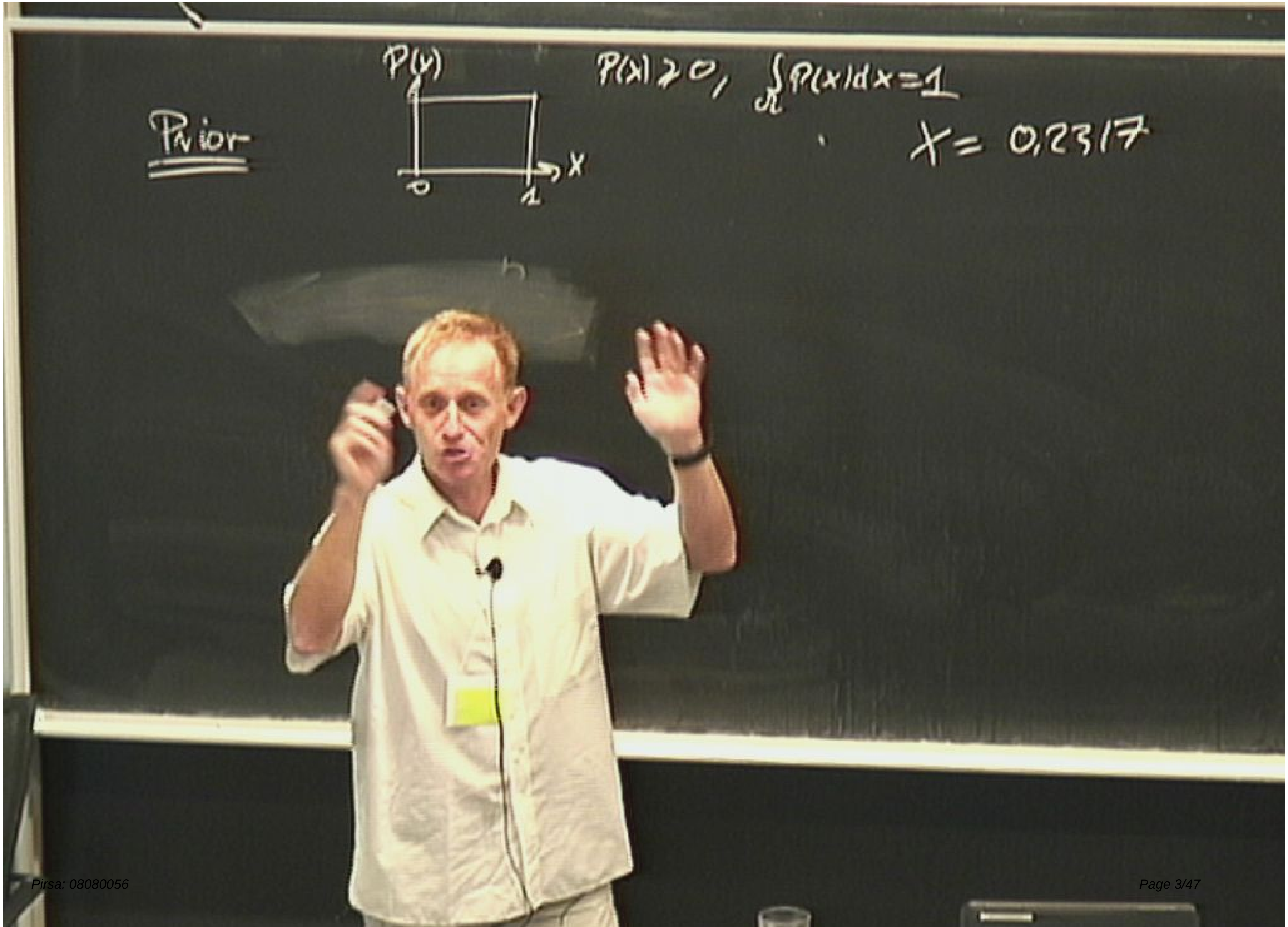
Title: Priors for quantum states & processes

Date: Aug 29, 2008 11:00 AM

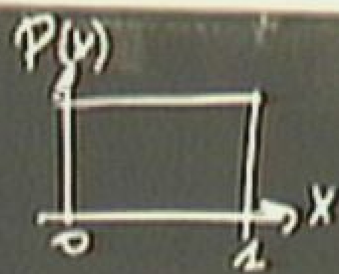
URL: http://pirsa.org/08080056

Abstract: As became apparent during Koenraad\'s talk, there are some important subleties to concepts like \'flat prior\' and \'uniform distribution\'... especially over probability simplices and quantum state spaces. This is a key problem for Bayesian approaches. Perhaps we\'re more interested in Jeffreys priors, Bures priors, or even something induced by the Chernoff bound! I\'d like to start a discussion of the known useful distributions over quantum states & processes, and I nominate Karol Zyckowski to lead it off.


Prior
P(x) ≥ 0, ∫_Ω P(x) dx = 1
0
1


Prior
P(y)
0
1
x
P(x) ≥ 0, ∫_Ω P(x) dx = 1
X = 0,2317

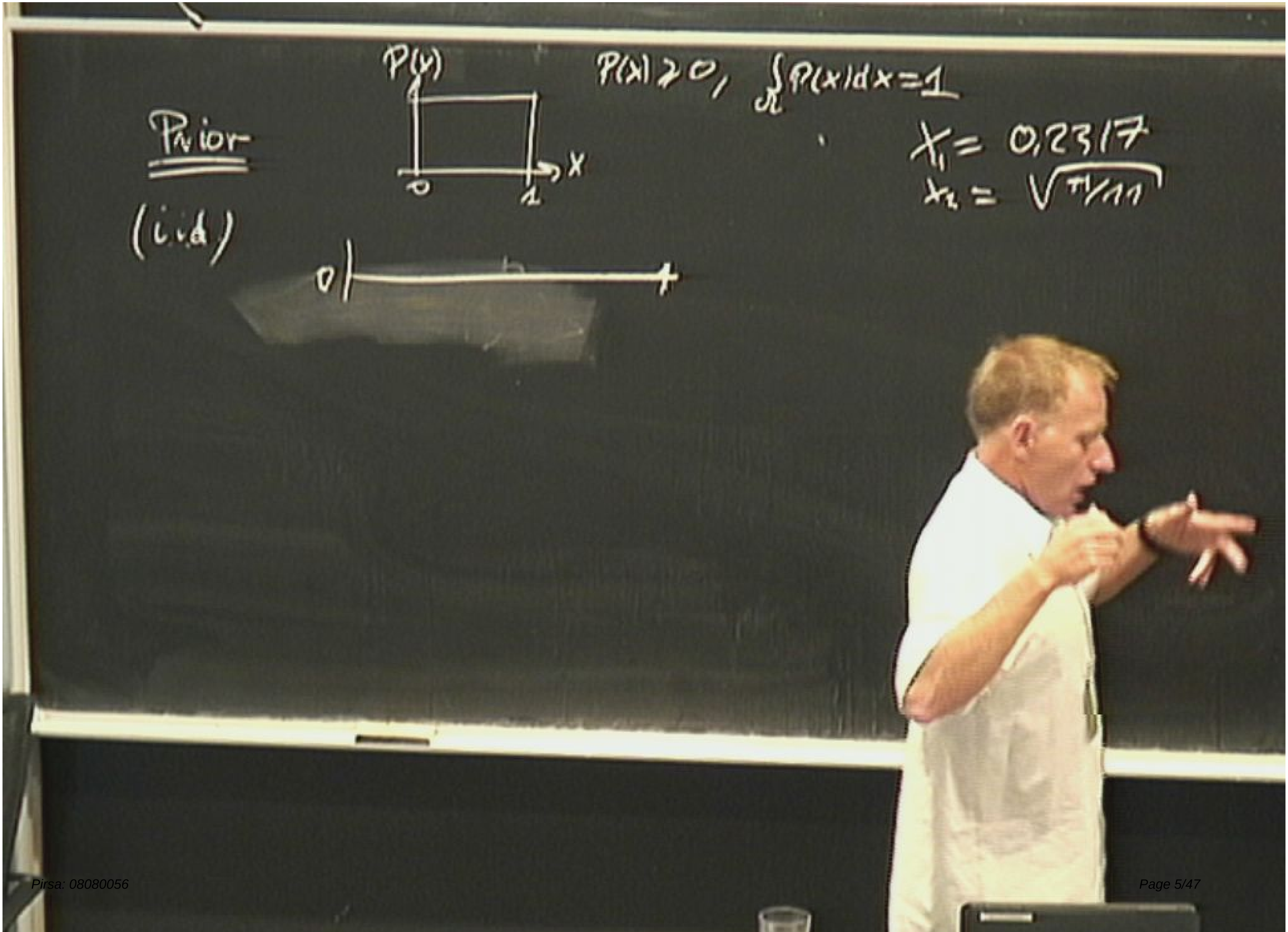
**Prior**

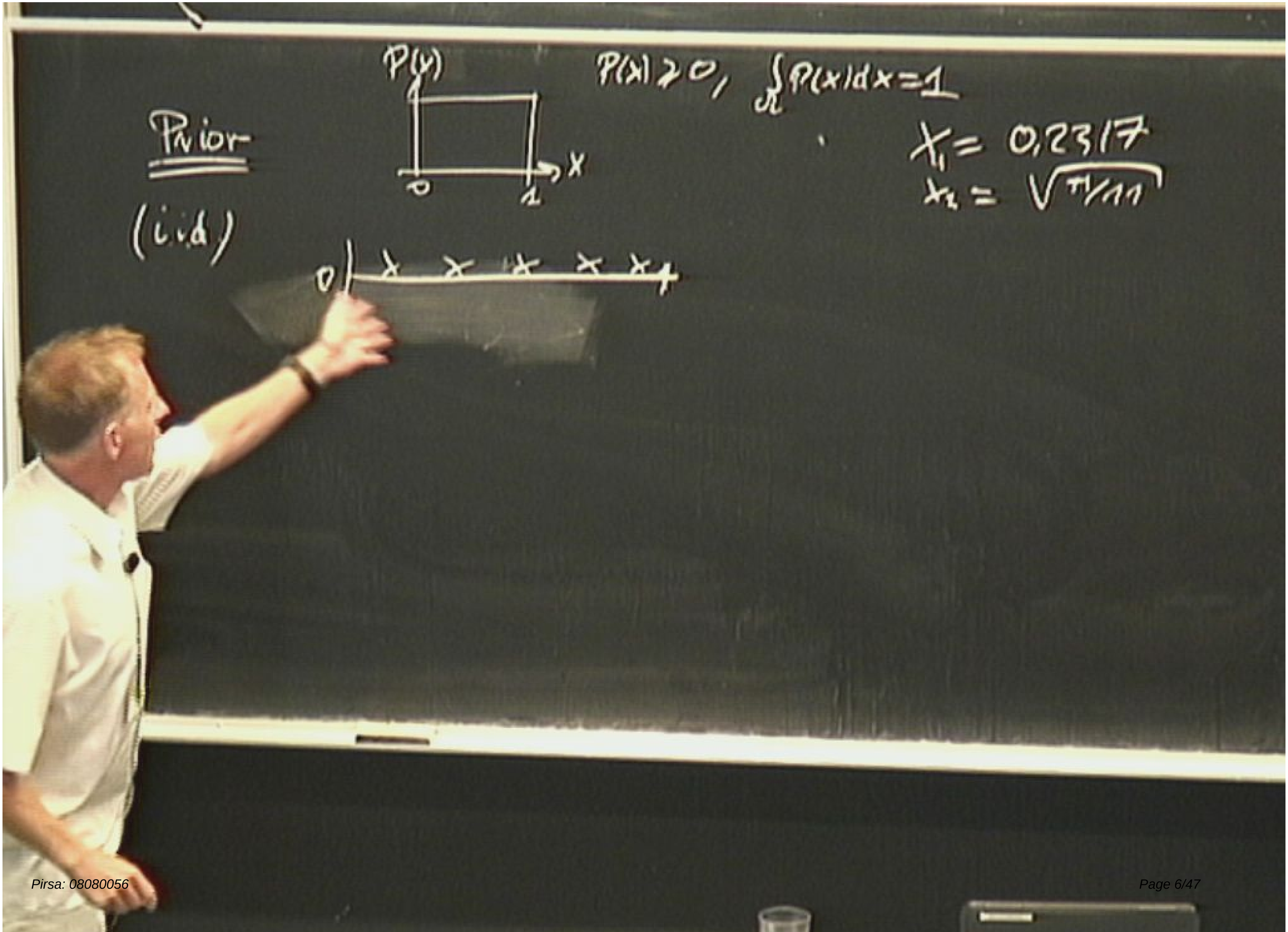



$P(x) \geq 0, \quad \int_{\Omega} P(x)\,dx = 1$

$x_1 = 0.2317$

$x_2 = \sqrt{\pi/11}$

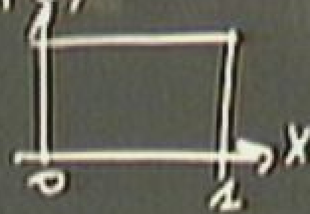

Prior
(i.i.d)
P(x)
X
0
1
P(x) ≥ 0, ∫_Ω P(x) dx = 1
x_1 = 0,2317
x_2 = √(π/11)
0
1


Prior
(i.i.d)
P(x)
x
0
1
P(x) ≥ 0, ∫Ω P(x) dx = 1
x1 = 0,2317
x2 = √π/11

Prior

(i.i.d.)

$P(x)$




$P(x) \geq 0, \quad \int_{\Omega} P(x)\,dx = 1$

$x_1 = 0.2317$

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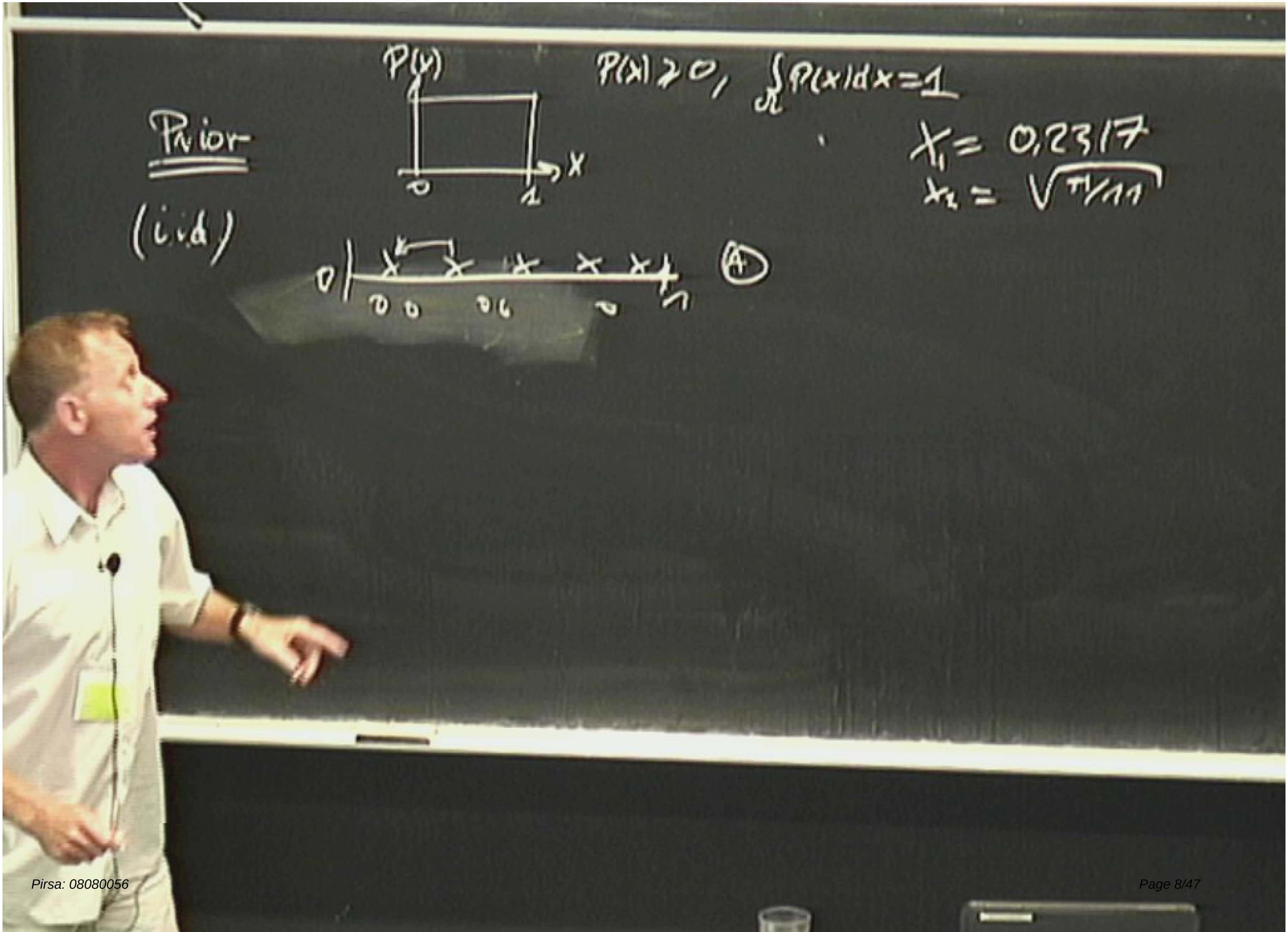
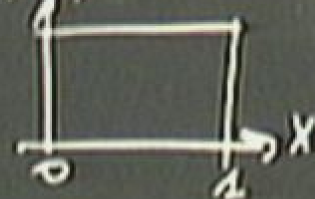

Prior
(i.i.d)
P(x)
P(x) ≥ 0, ∫_Ω P(x) dx = 1
x_1 = 0,2317
x_2 = √(π/11)

Prior

(i.i.d.)

$P(x)$



$P(x) \geq 0, \quad \int_{\Omega} P(x)\,dx = 1$

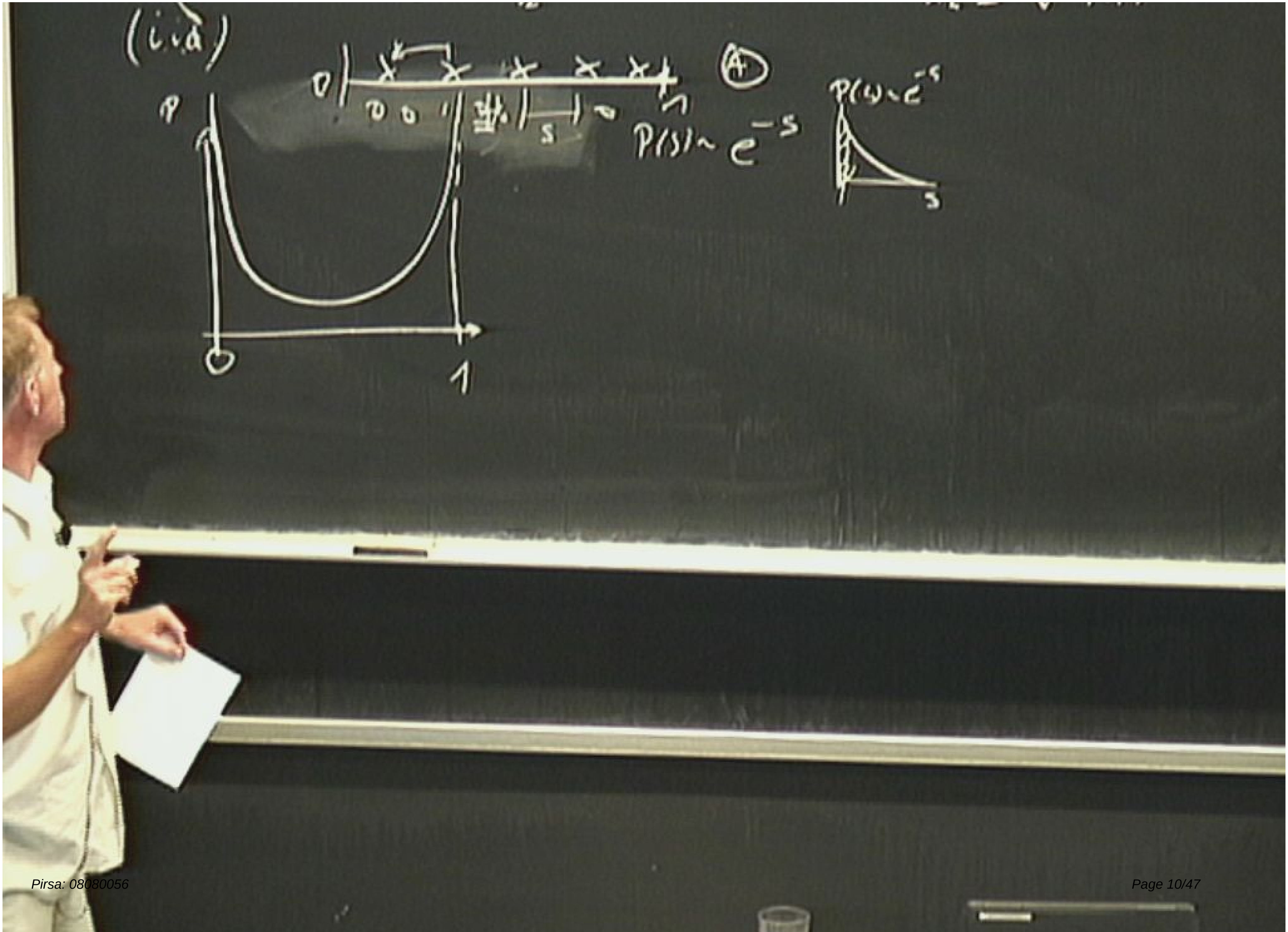
$X_1 = 0.2317$

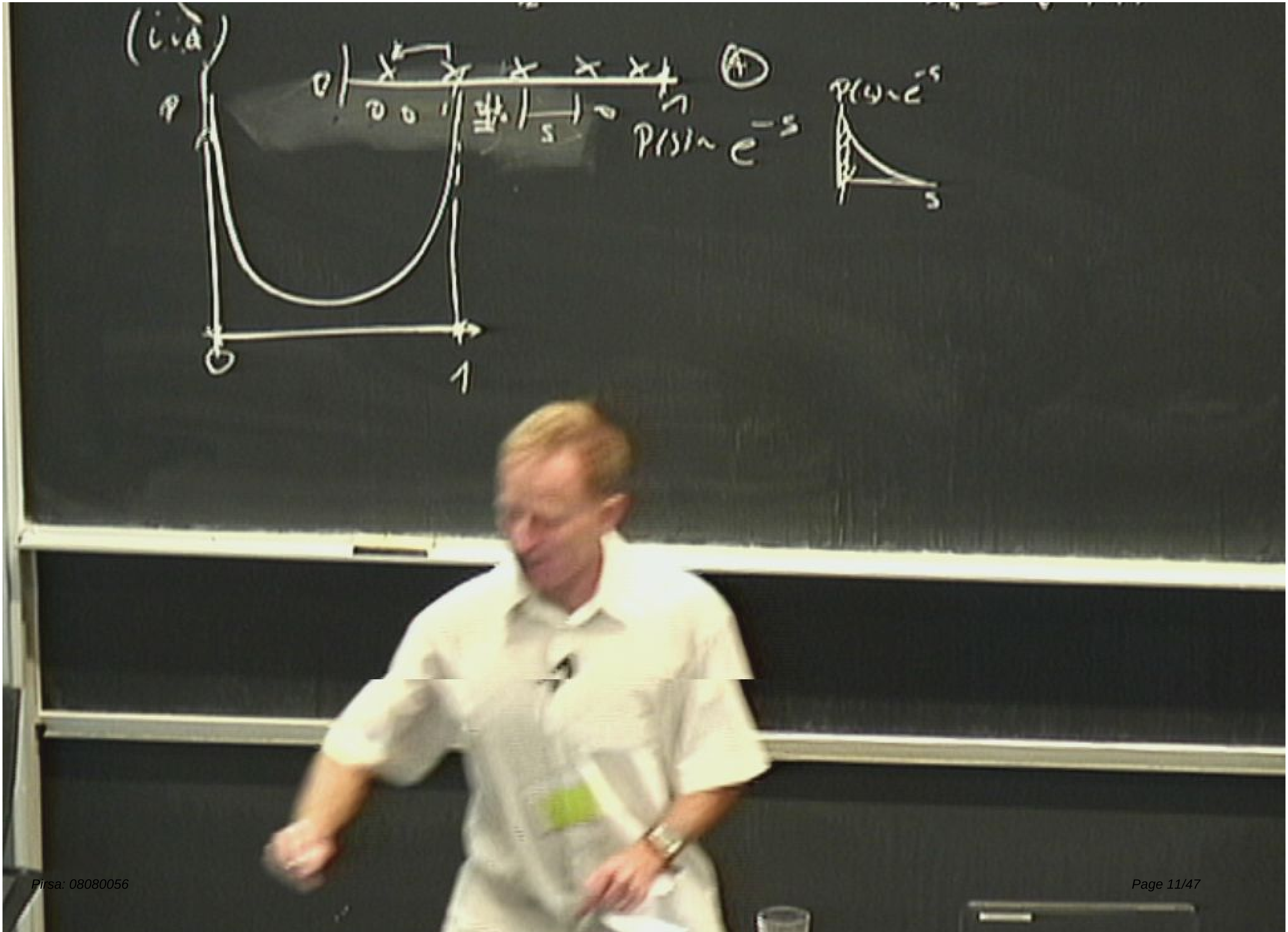
$x_2 = \sqrt{\pi/11}$

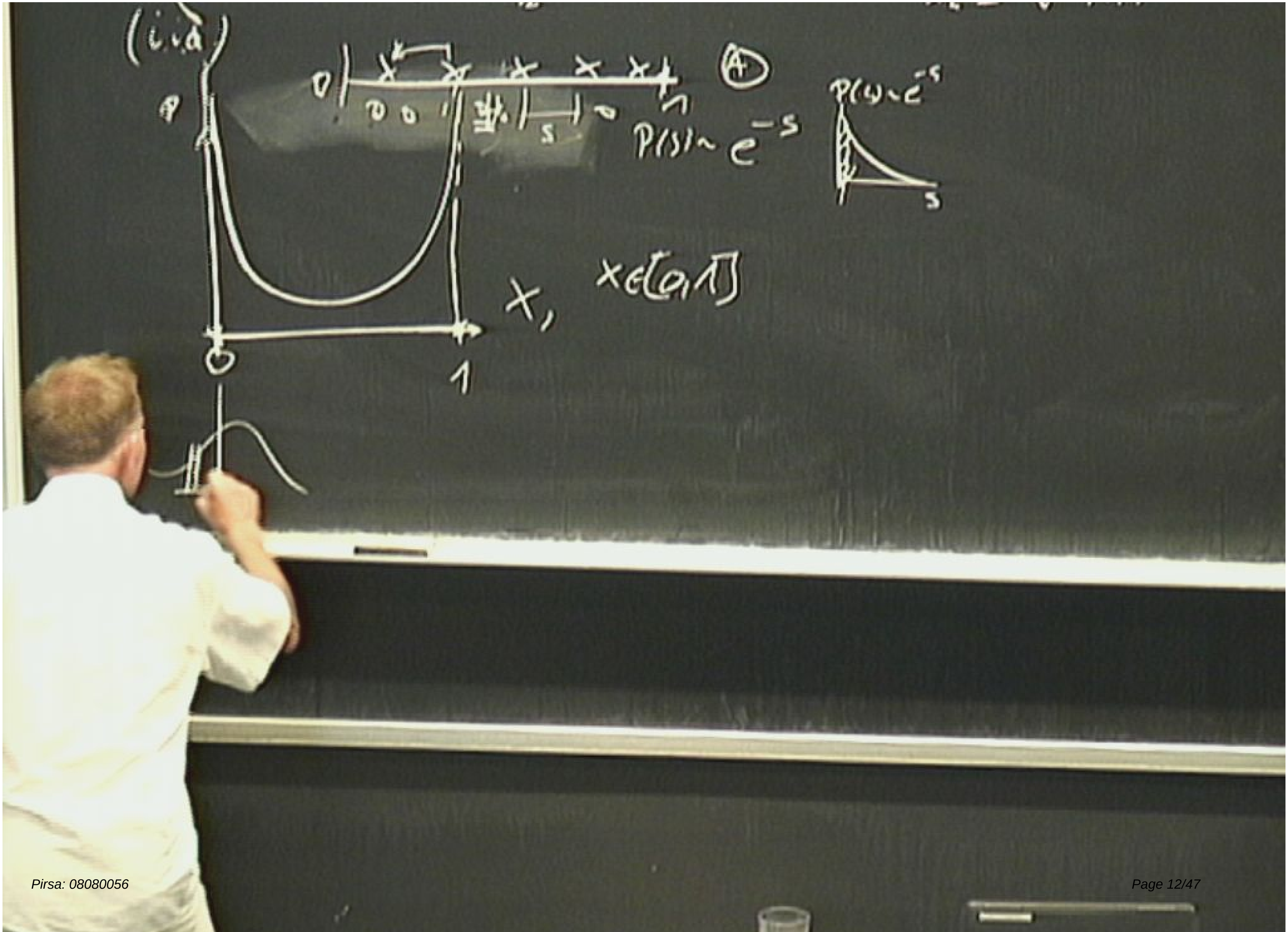
Ⓐ

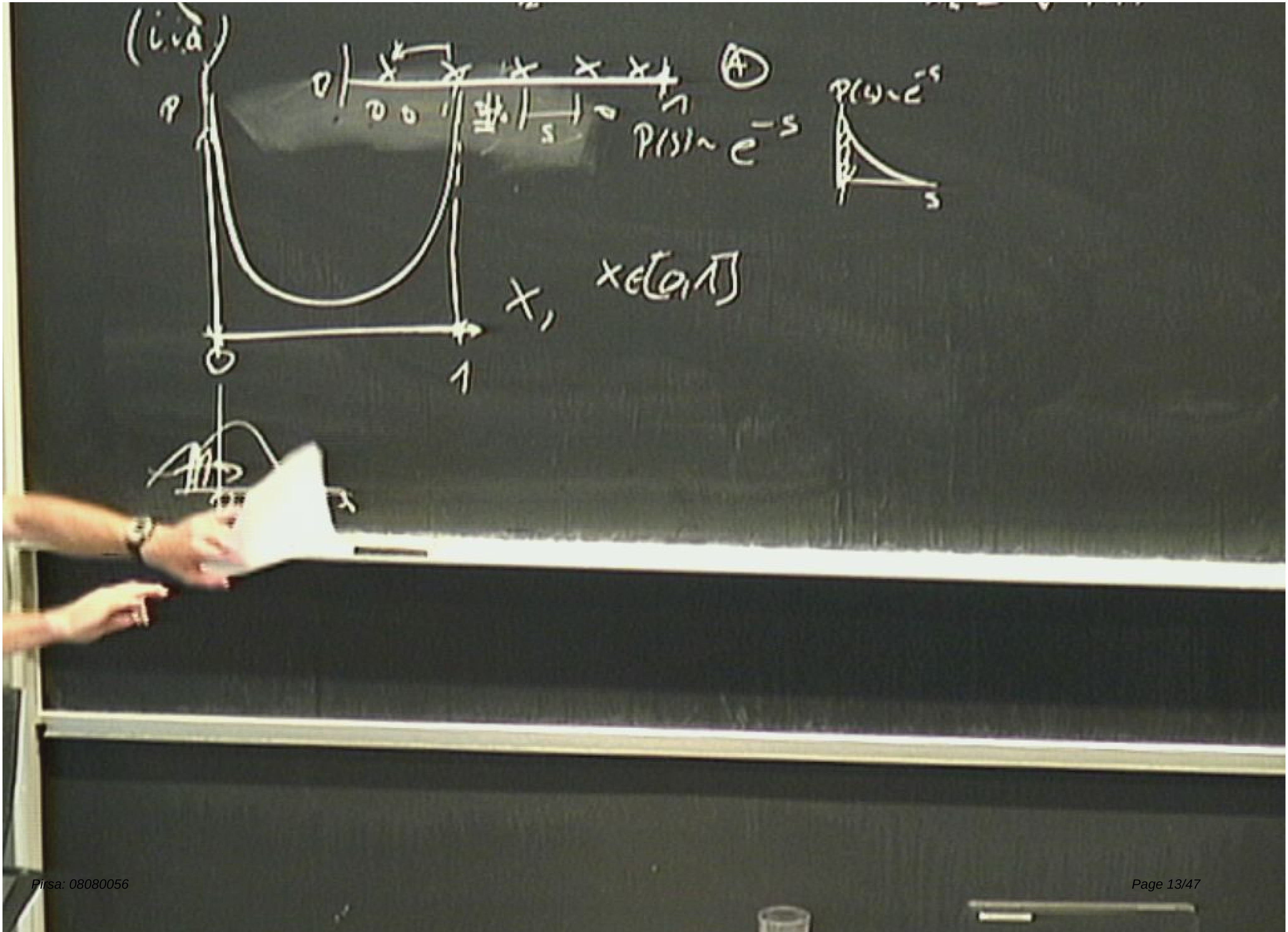
$P(s) \sim e^{-s}$

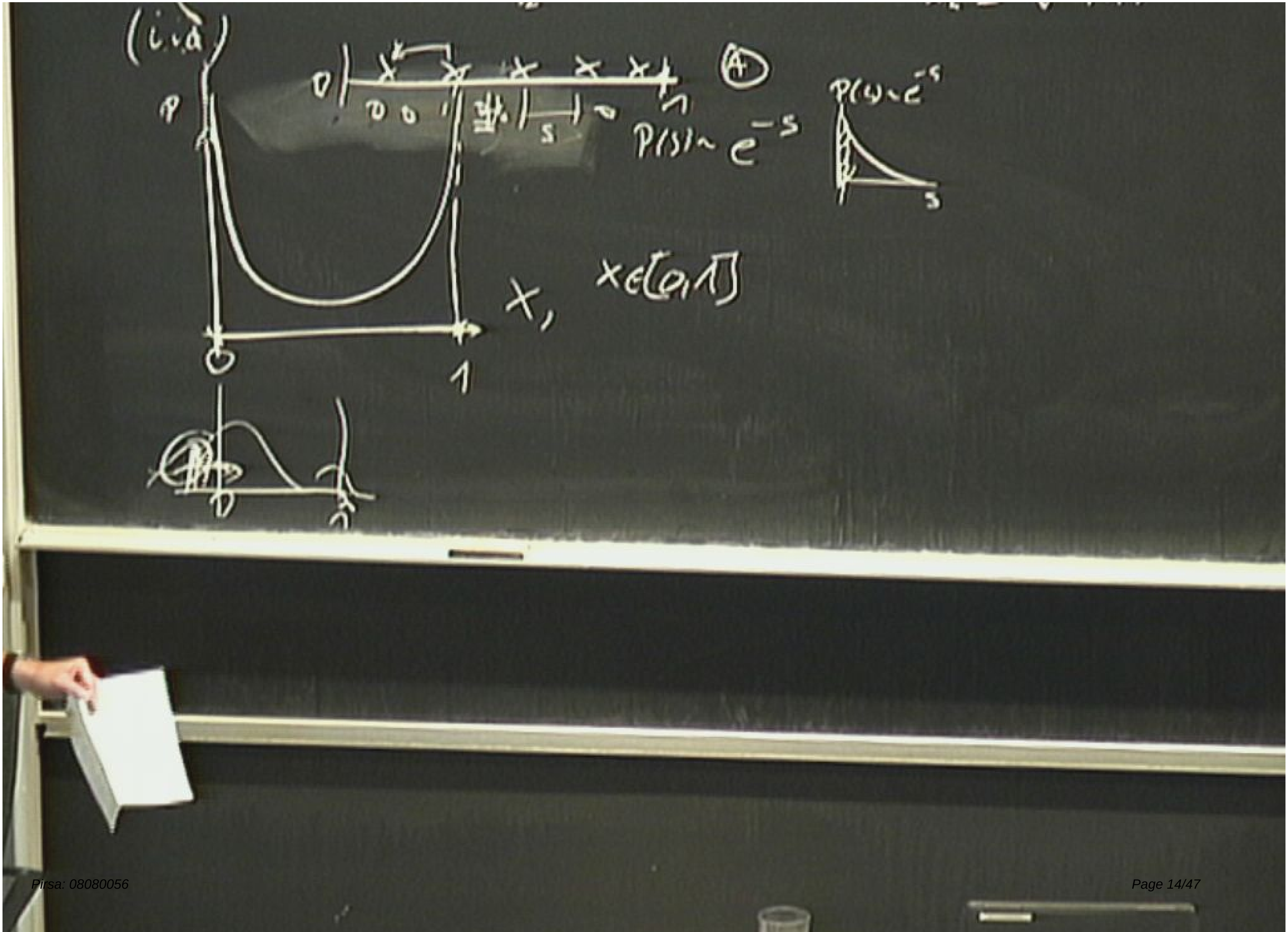
$P(s) \sim e^{-s}$

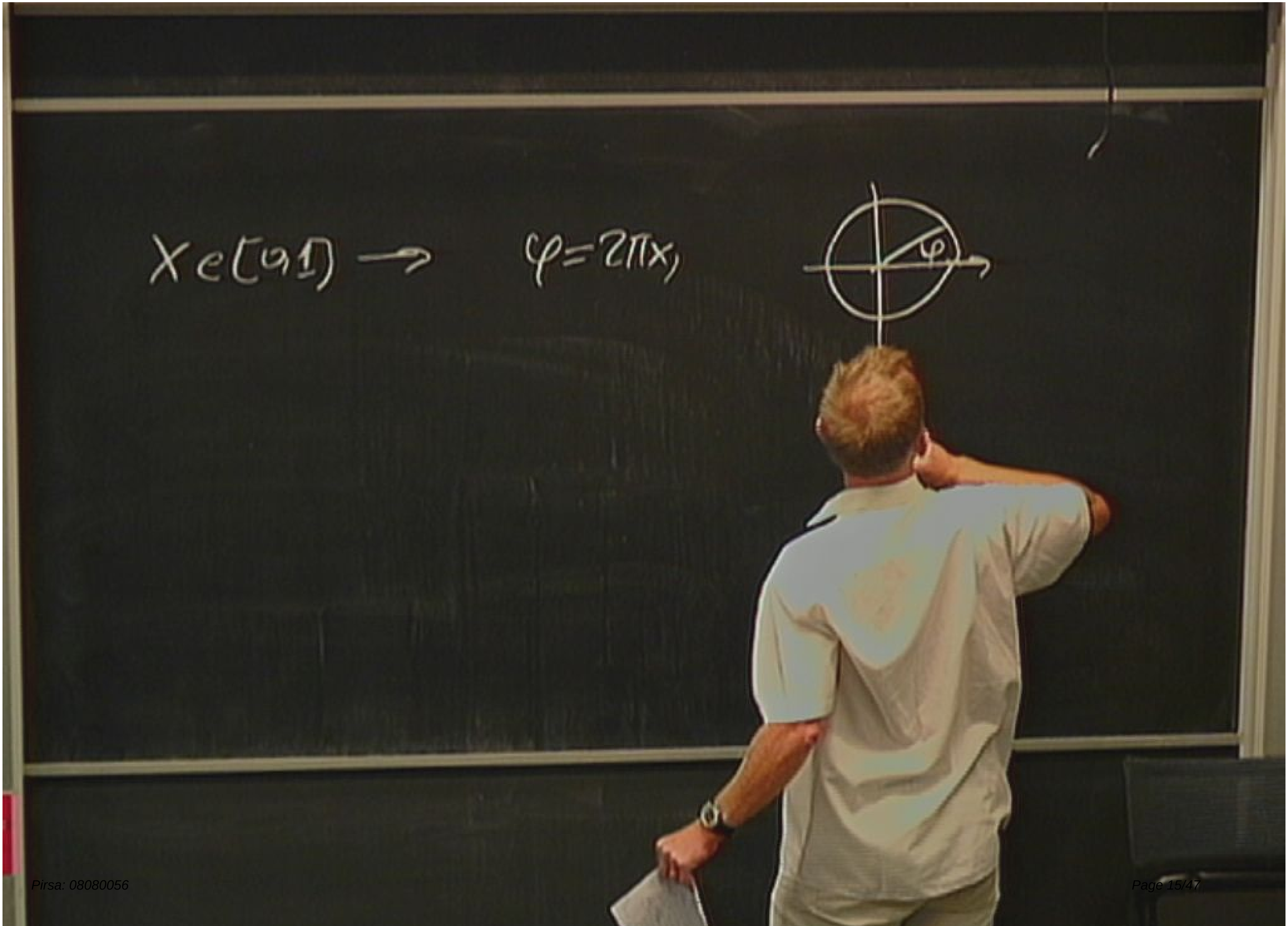

(i.i.d)
P(s) ~ e^-s
P(s) ~ e^-s
s
0
1

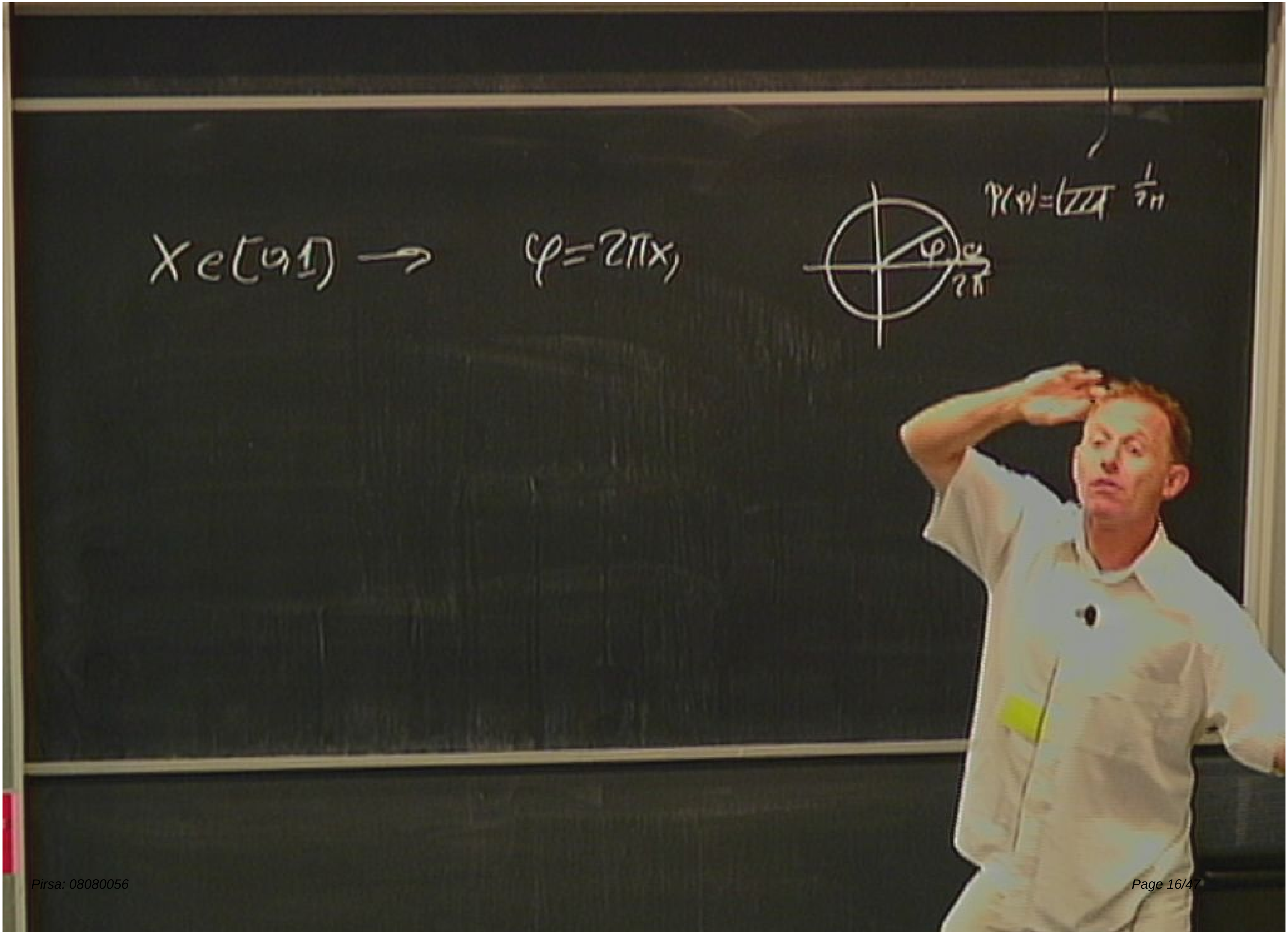

(i.a)
P(s) ~ e^-s
P(s) ~ e^-s

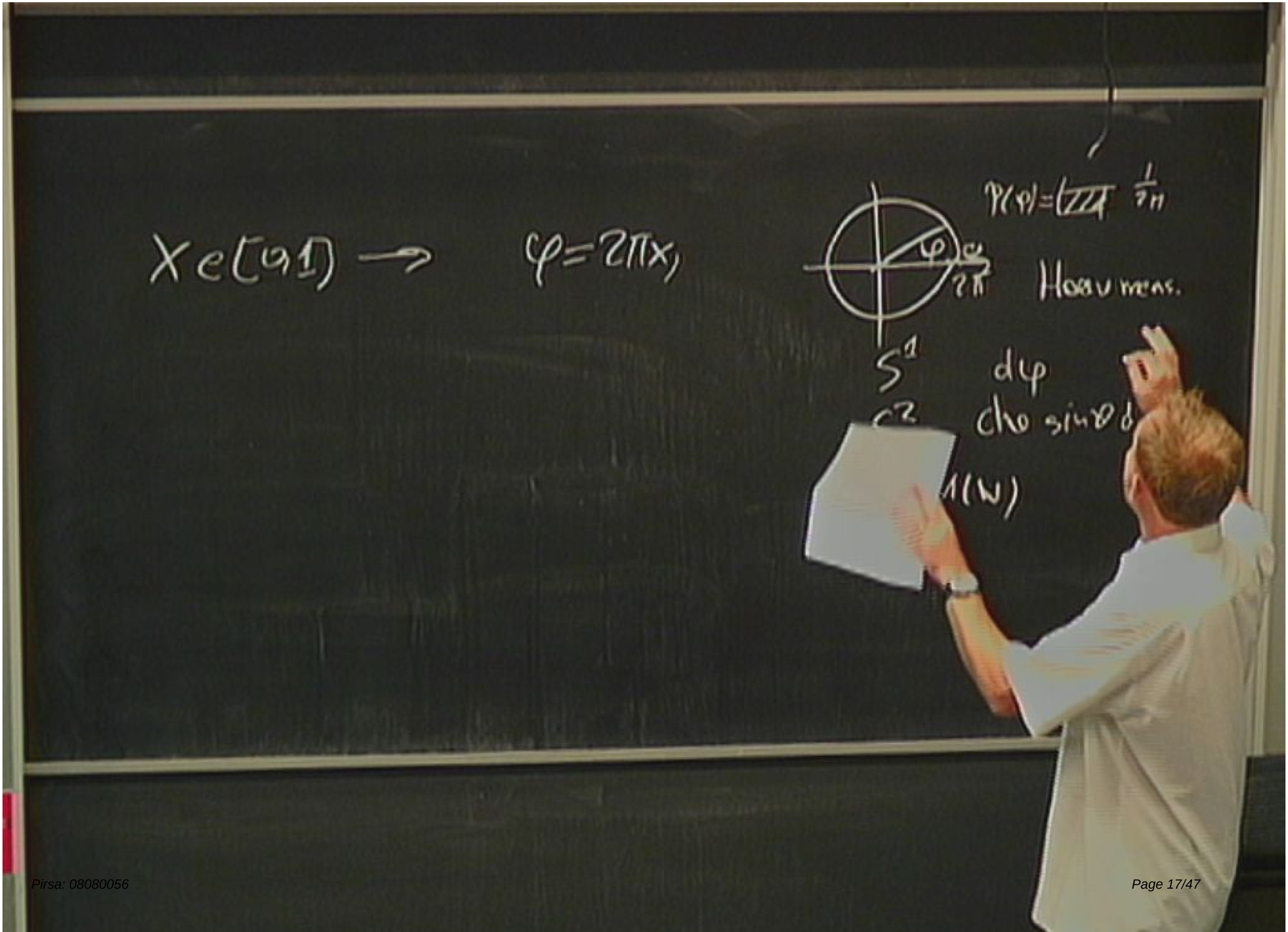

(i.i.d.)
P(s) ~ e^{-s}
x ∈ [0,1]


(i.i.d.)
P
0
1
P(s) ~ e^{-s}
P(s) ~ e^{-s}
s
x,
x ∈ [0,1]
0
1

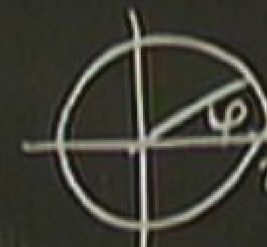

(i.i.d)
P(s) ~ e^{-s}
x∈[0,1]


X∈[0,1) →
φ=2πx,
φ


X ∈ [0,1] →
φ = 2πx,
φ/2π


X ∈ [0,1] →
φ = 2πx
φ
Haar meas.
S^1
dφ
(N)

$X \in [0,1] \rightarrow \quad \varphi = 2\pi x,$



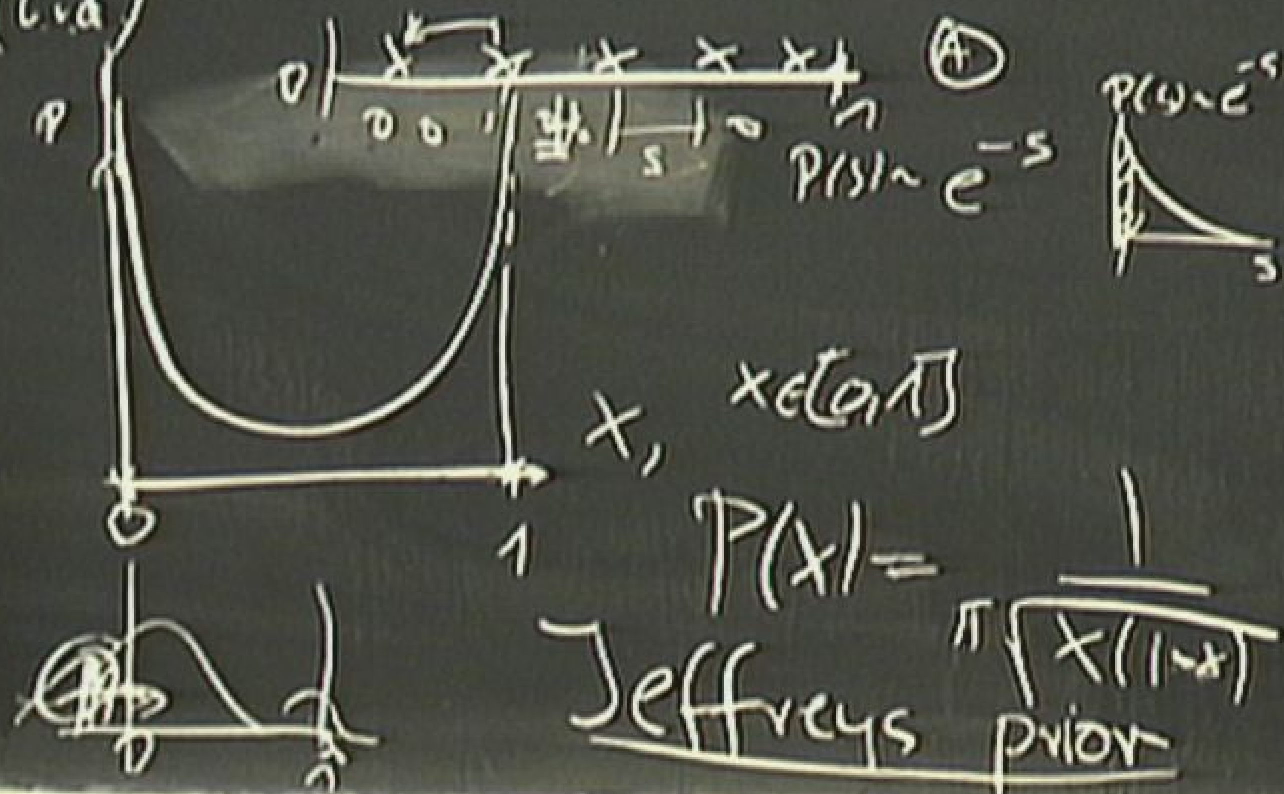
$P(\varphi) = \cancel{\ \ } \ \frac{1}{2\pi}$

$\frac{\varphi}{2\pi}$ Haar meas.

$S^1 \quad d\varphi$

$S^2 \quad d\theta \sin\theta\, d\varphi$

$\vdots \quad U(N)$



(i.i.d.)

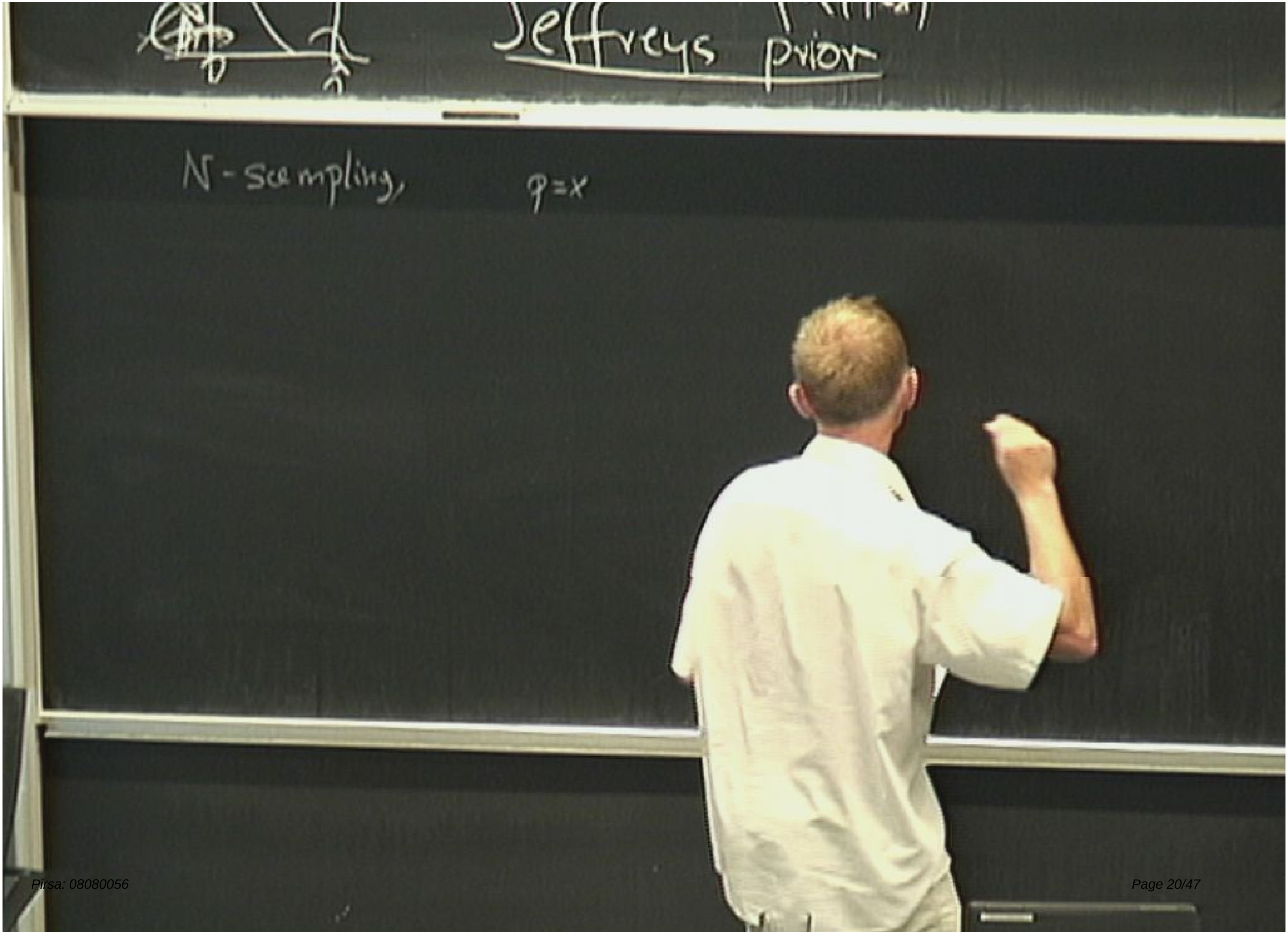
$P(s) \sim e^{-s}$

$x \in [0,1]$

$x,$

$$P(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

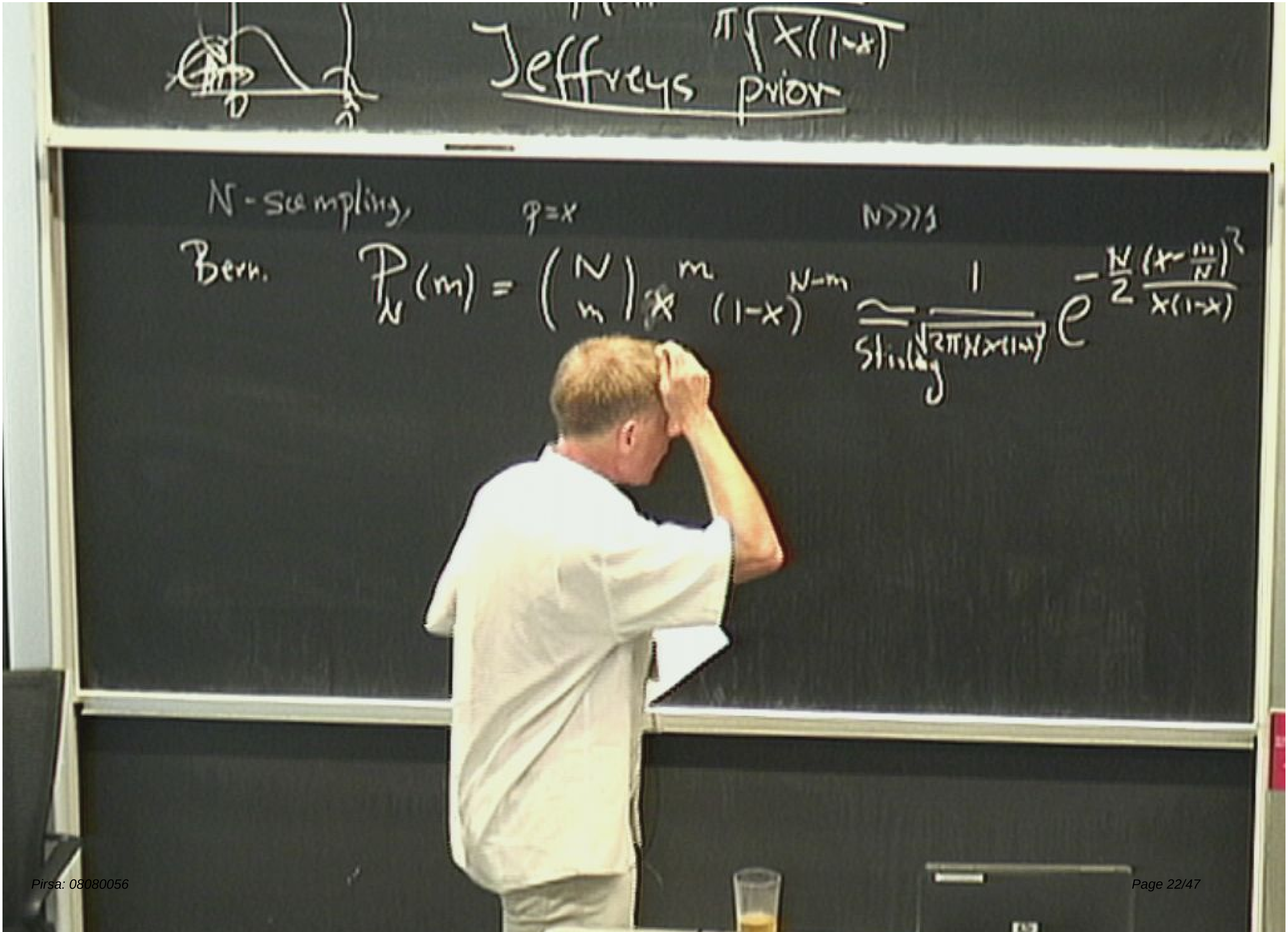
Jeffreys prior

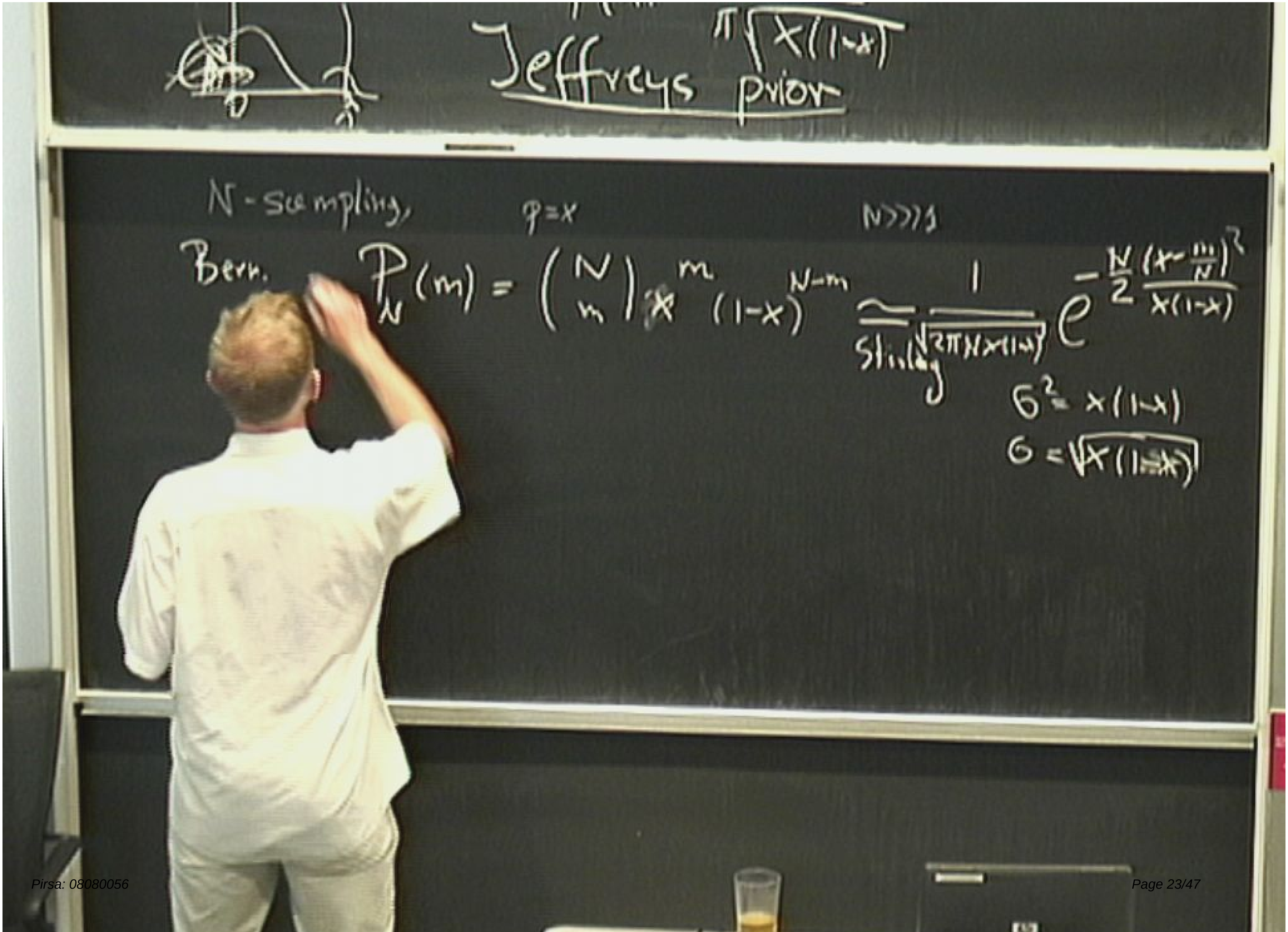

Jeffreys prior
N - sampling,
p=x

Jeffreys prior

N - sampling, $p = x$ $N \gg 1$

Bern. $P_N(m) = \binom{N}{m} x^m (1-x)^{N-m} \underset{\text{Stirling}}{\simeq}$


Jeffreys prior
N-sampling, $p = x$
$N \gg 1$
Bern. $P_N(m) = \binom{N}{m} x^m (1-x)^{N-m} \underset{\text{Stirling}}{\simeq} \frac{1}{\sqrt{2\pi N x(1-x)}} e^{-\frac{N}{2}\frac{(x-\frac{m}{N})^2}{x(1-x)}}$


Jeffreys prior
N-sampling,
$p = x$
$N \gg 1$
Bern.
$P_N(m) = \binom{N}{m} x^m (1-x)^{N-m} \simeq \frac{1}{\sqrt{2\pi N x(1-x)}} e^{-\frac{N}{2}\frac{(x-\frac{m}{N})^2}{x(1-x)}}$
Stirling
$\sigma^2 = x(1-x)$
$\sigma = \sqrt{x(1-x)}$

$\pi\sqrt{x(1-x)}$

Jeffreys prior

N - sampling, $p = x$ $N \gg 1$

Bern. $P_N(m) = \binom{N}{m} x^m (1-x)^{N-m} \underset{\text{Stirling}}{\simeq} \frac{1}{\sqrt{2\pi N x(1-x)}} e^{-\frac{N}{2}\frac{(x-\frac{m}{N})^2}{x(1-x)}}$

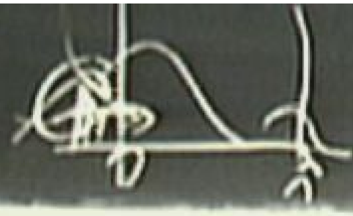
N istat

$\sigma^2 = x(1-x)$

$\sigma = \sqrt{x(1-x)}$

$x'$

$\dfrac{1}{\pi\sqrt{x(1-x)}}$



Jeffreys prior

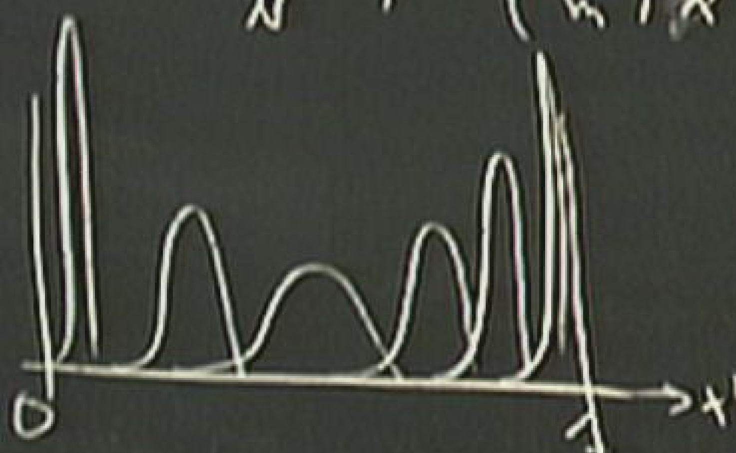
N - sampling, $\quad p = x \quad\quad N \gg 1$

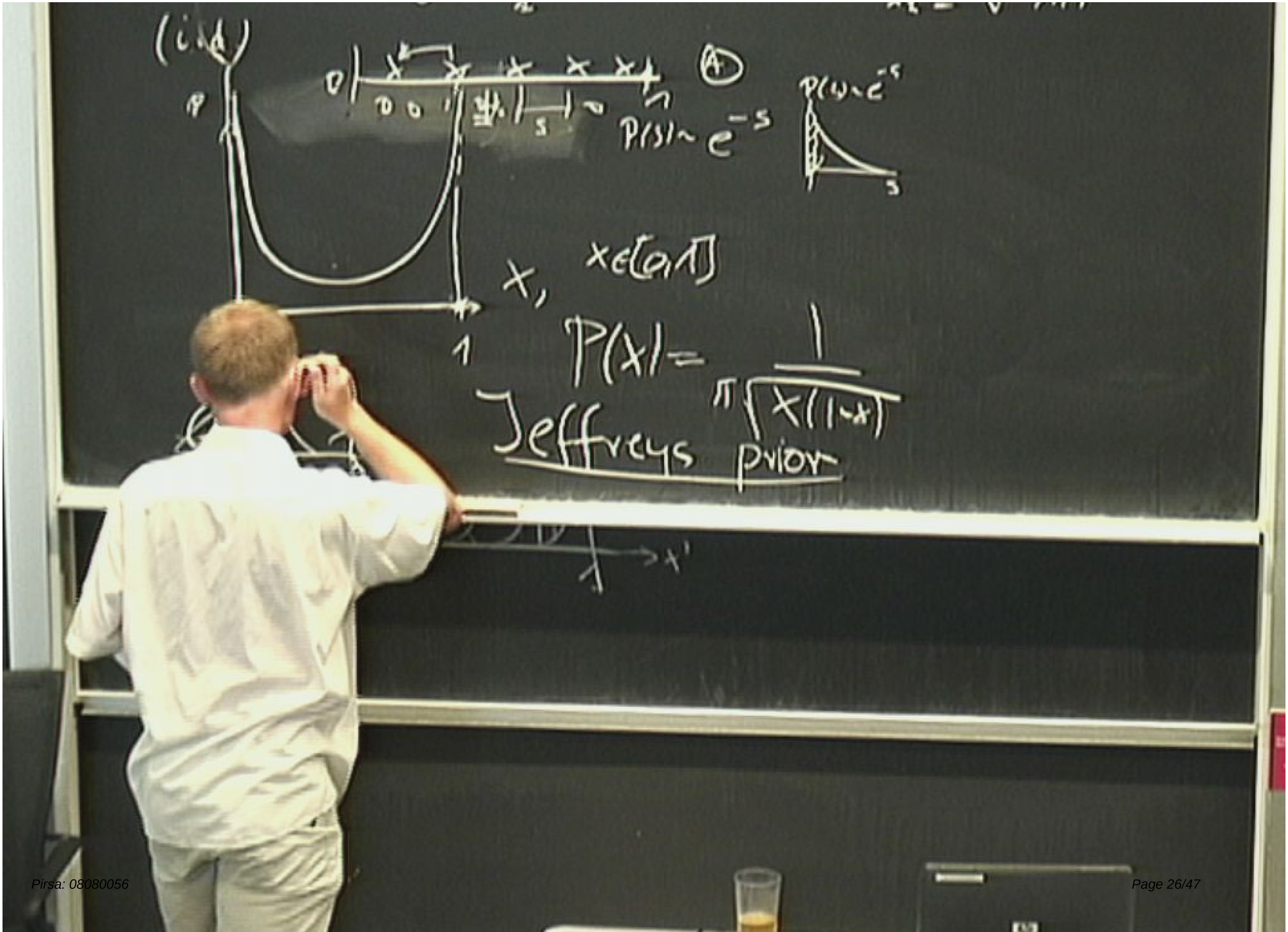
Bern. $\quad P_N(m) = \binom{N}{m} x^m (1-x)^{N-m} \underset{\text{Stirling}}{\simeq} \dfrac{1}{\sqrt{2\pi N x(1-x)}} e^{-\frac{N}{2}\frac{(x-\frac{m}{N})^2}{x(1-x)}}$

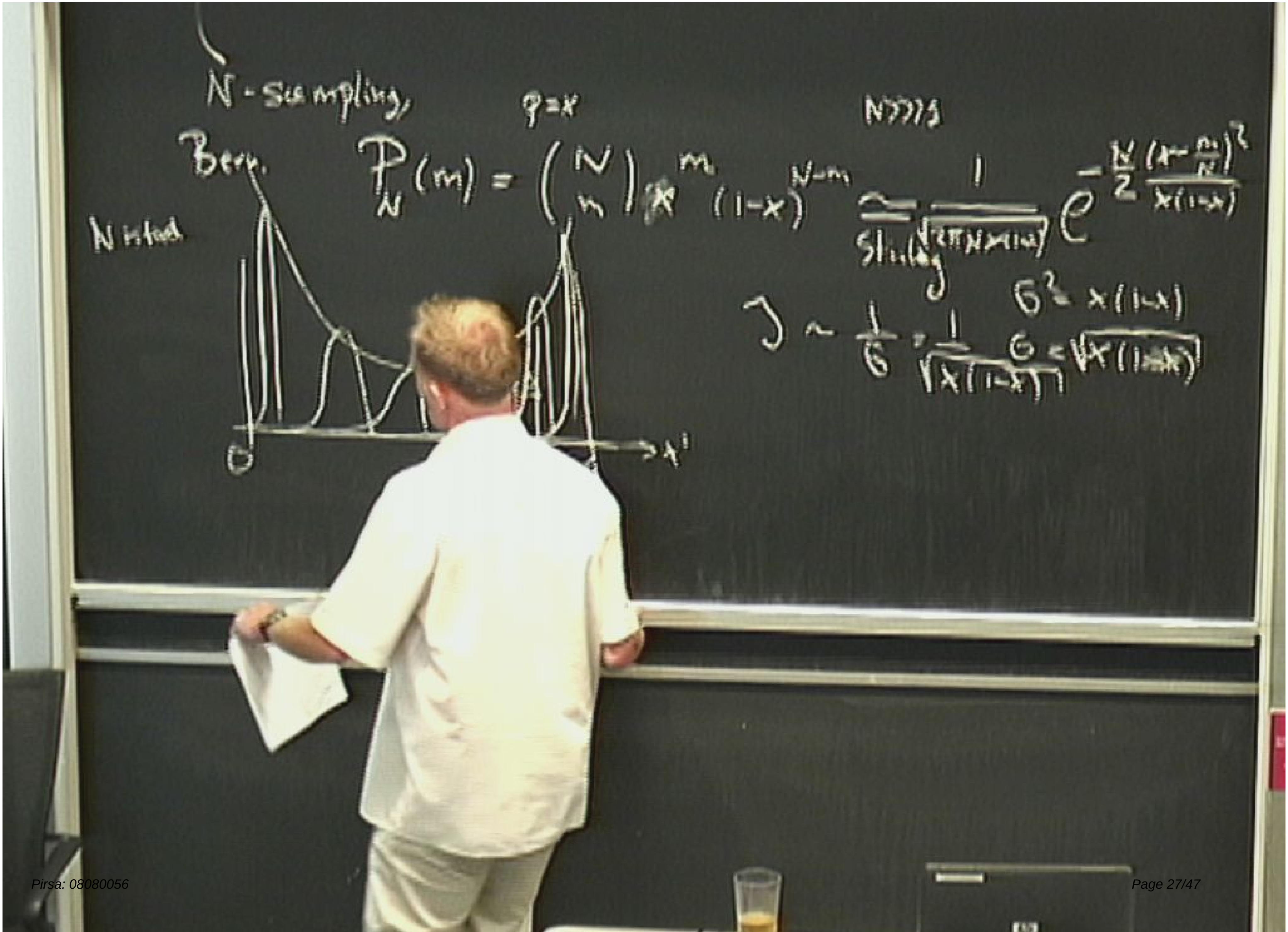
$\sigma^2 = x(1-x)$

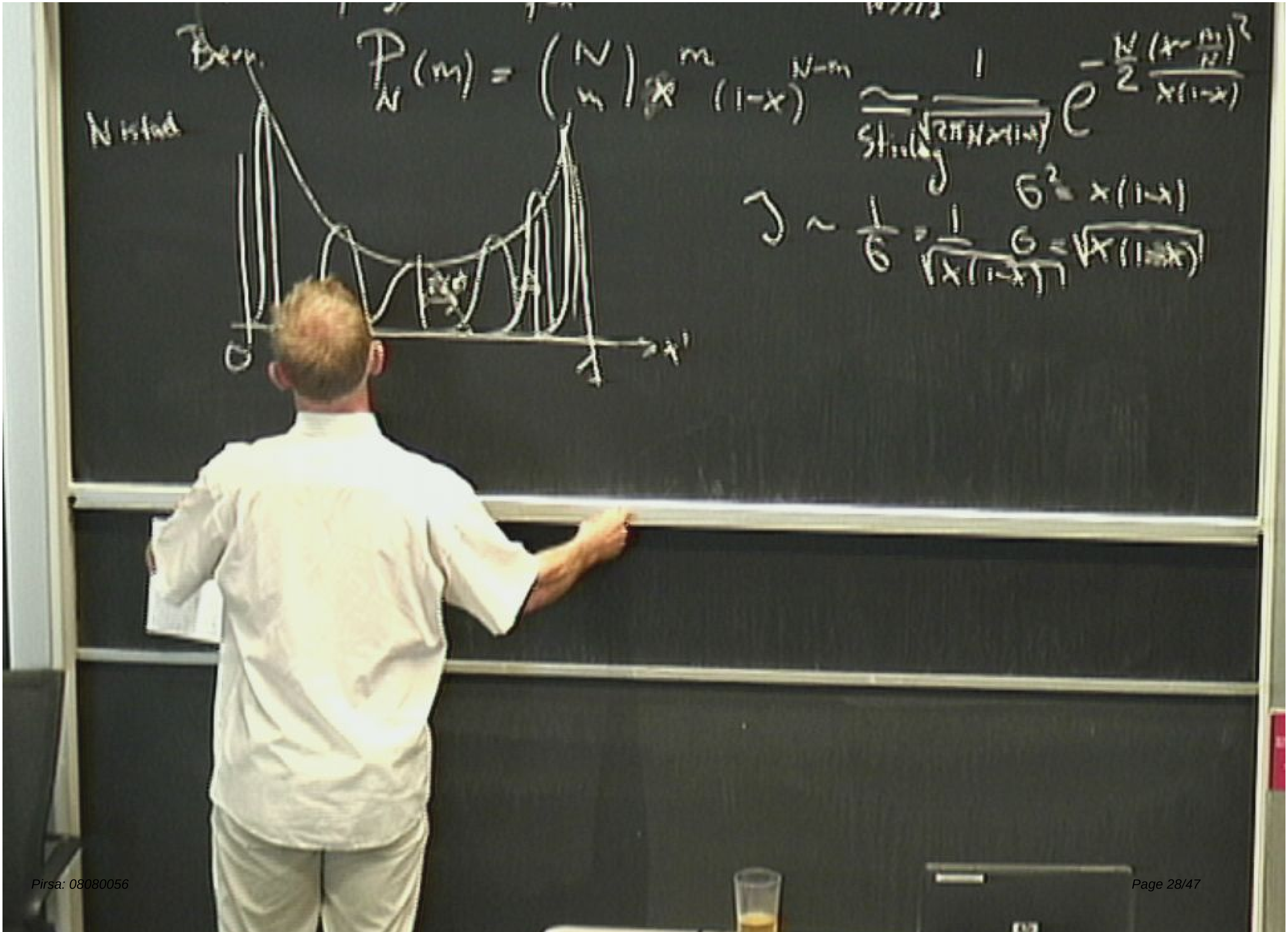
$\sigma = \sqrt{x(1-x)}$

N instead




(iid)
P(s) ~ e^{-s}
x ∈ [0,1]
P(x) = 1 / (π √(x(1-x)))
Jeffreys prior

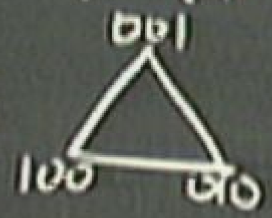

N-sampling,
$q = x$
$N \gg 1$
Bern.
$P_N(m) = \binom{N}{m} x^m (1-x)^{N-m} \simeq \frac{1}{\sqrt{2\pi N x(1-x)}} e^{-\frac{N}{2}\frac{(x-\frac{m}{N})^2}{x(1-x)}}$
Stirling
N fixed
$J \sim \frac{1}{\sigma} = \frac{1}{\sqrt{x(1-x)}}$
$\sigma^2 = x(1-x)$
$\sigma = \sqrt{x(1-x)}$
0
$x'$


Bern.
N fixed
$P_N(m) = \binom{N}{m} x^m (1-x)^{N-m} \simeq \frac{1}{\sqrt{2\pi N x(1-x)}} e^{-\frac{N}{2}\frac{(x-\frac{m}{N})^2}{x(1-x)}}$
Stirling
$\sigma^2 = x(1-x)$
$J \sim \frac{1}{\sigma} = \frac{1}{\sqrt{x(1-x)}}$
$\sigma = \sqrt{x(1-x)}$
0
1
x'

$M=2, \quad P(x_1, 1-x)$

$M, \quad \vec{X} = \{x_1 \cdots x_n\} \in \Delta_{M-1}$

$M=3$



$M=2$ $\quad P(x_1, 1-x)$

$M,$ $\quad \vec{x} = \{x_1 \ldots x_n\} \in \Delta_{M-1}$

$M=3$ $\quad$ 001 $\quad$ 100

Dirichlet $\quad D_\alpha(\vec{x}) = C_\alpha\, \delta\left(\sum_{i=1}^{M} x_i - 1\right) \prod_{i=1}^{M} x_i^{\alpha-1}$

$\alpha = 1 \to$ flat.

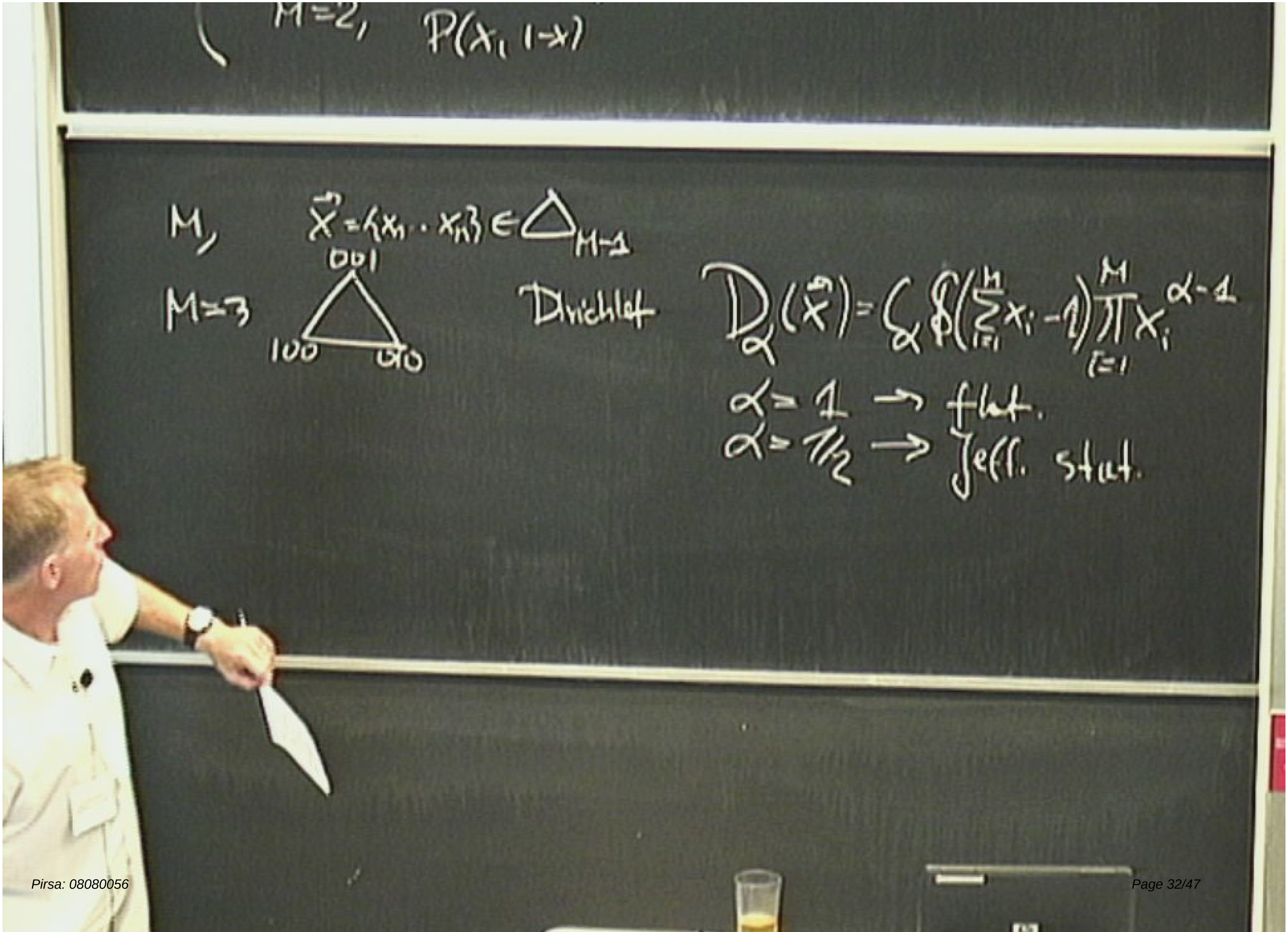
$M=2, \quad P(x, 1-x)$

$M, \quad \vec{x} = \{x_1 \cdots x_n\} \in \Delta_{M-1}$

$M=3$

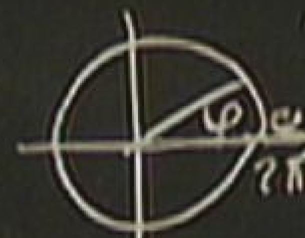

Dirichlet $$D_\alpha(\vec{x}) = C_\alpha\, \delta\Big(\sum_{i=1}^{M} x_i - 1\Big) \prod_{i=1}^{M} x_i^{\alpha-1}$$

$\alpha = 1 \rightarrow$ flat.


$M=2, \quad P(x, 1-x)$
$M, \quad \vec{X} = \{x_1 \cdot x_n\} \in \Delta_{M-1}$
001
$M=3$
100
010
Dirichlet
$D_\alpha(\vec{x}) = C_\alpha \delta\left(\sum_{i=1}^{M} x_i - 1\right) \prod_{i=1}^{M} x_i^{\alpha-1}$
$\alpha = 1 \rightarrow$ flat.
$\alpha = 1/2 \rightarrow$ Jeff. stat.

$X \in [0,1] \longrightarrow \quad \varphi = 2\pi x,$

fact $\; y \in S^{N-1}, \quad \sum_i y_i^2 = 1$




$P(\varphi) = \frac{1}{2\pi}$

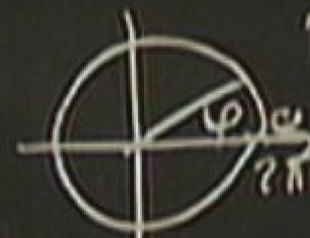
Haar meas.

$S^1 \quad d\varphi$

$S^2 \quad d\theta \sin\theta \, d\varphi$

$\vdots \quad U(N)$

$X \in [0,1] \longrightarrow \quad \varphi = 2\pi x_1$




$P(\varphi) = \cancel{\quad} \ \frac{1}{2\pi}$

Haar meas.

$S^1 \quad d\varphi$

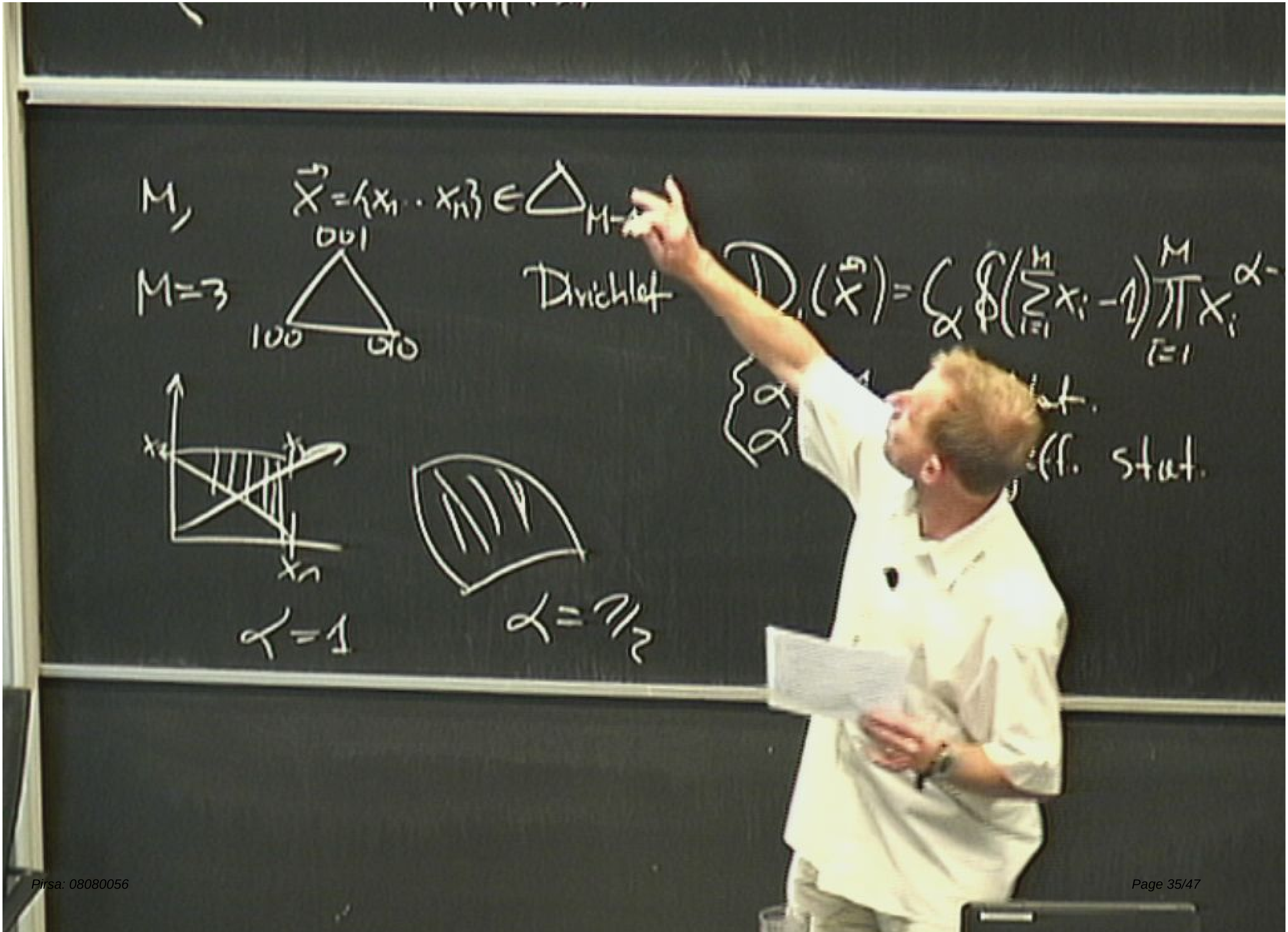
$S^2 \quad d\theta \sin\theta \, d\varphi$

$\vdots \quad U(N)$

fact $y \in S^{N-1}, \quad \sum_i y_i^2 = 1$

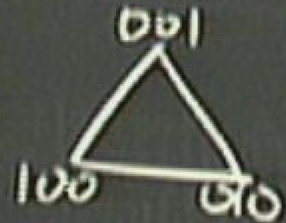
$x_i = y_i^2 \Rightarrow \vec{x} \in \Delta_{N-1}$

$\Downarrow \ \vec{x}$ is dist. stat. $\alpha = 1/2$


M=3
001
100
010
Dirichlet
$\alpha=1$
stat.

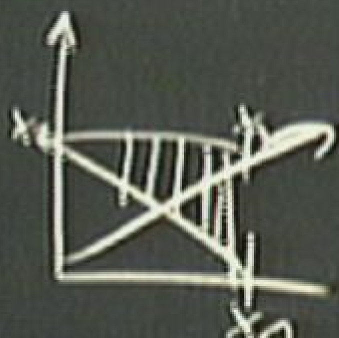
$M,\quad \vec{x} = \{x_1 \dots x_n\} \in \Delta_{M-1}$

$M=3$



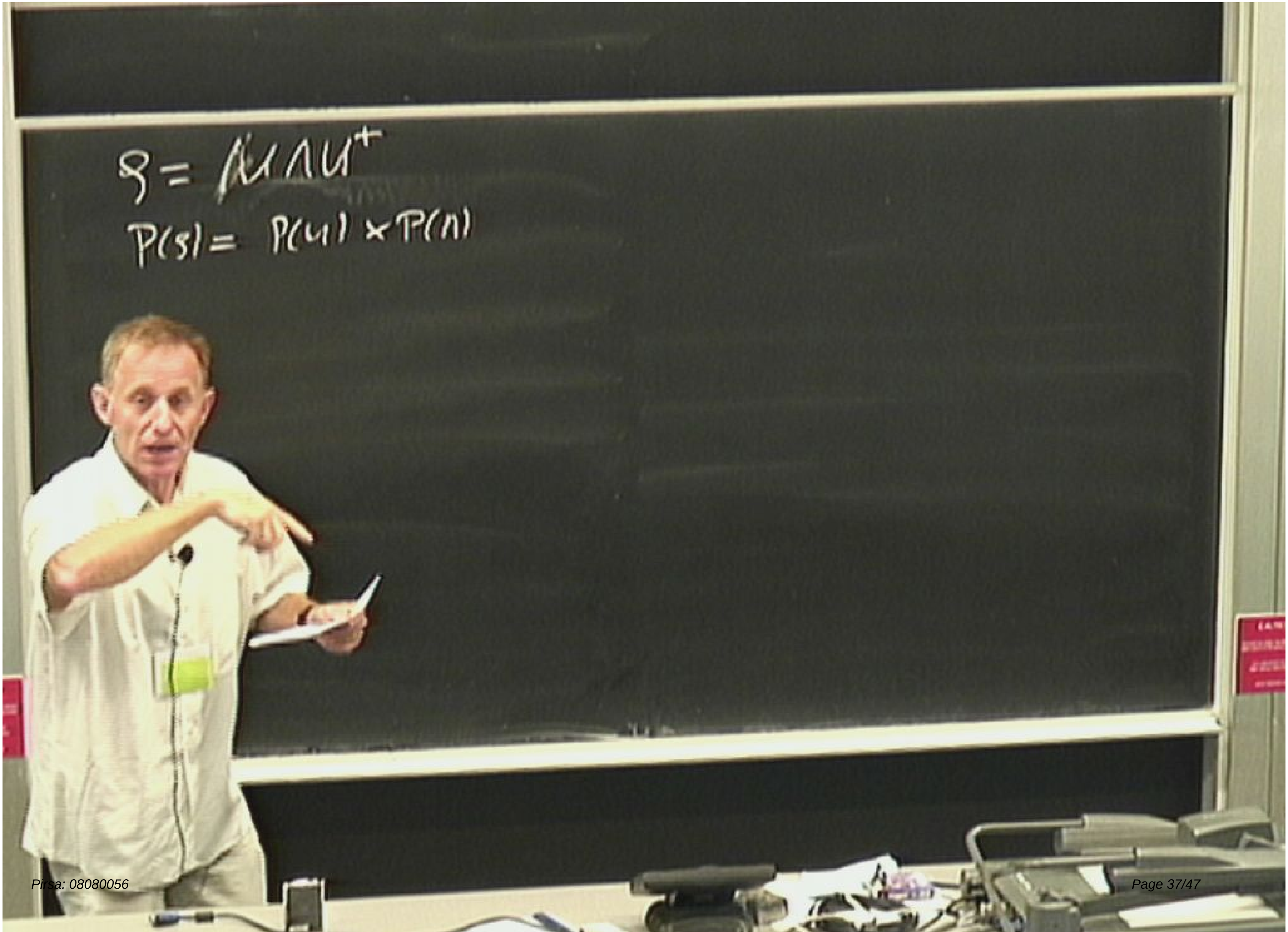

Dirichlet $\quad D_\alpha(\vec{x}) = C_\alpha\, \delta\left(\sum_{i=1}^{M} x_i - 1\right) \prod_{i=1}^{M} x_i^{\alpha-}$

$$\begin{cases} \alpha = 1 \rightarrow \text{flat.} \\ \alpha = 1/2 \rightarrow \text{Jeff. stat.} \end{cases}$$




$\alpha = 1$

$\alpha = 1/2$


ρ = UΛU⁺
P(ρ) = P(U) × P(Λ)

$$\rho = U \Lambda U^{\dagger}$$

$$P(\rho) = \frac{P(U) \times P(\Lambda)}{P} \quad , \quad P(U) = P_H$$

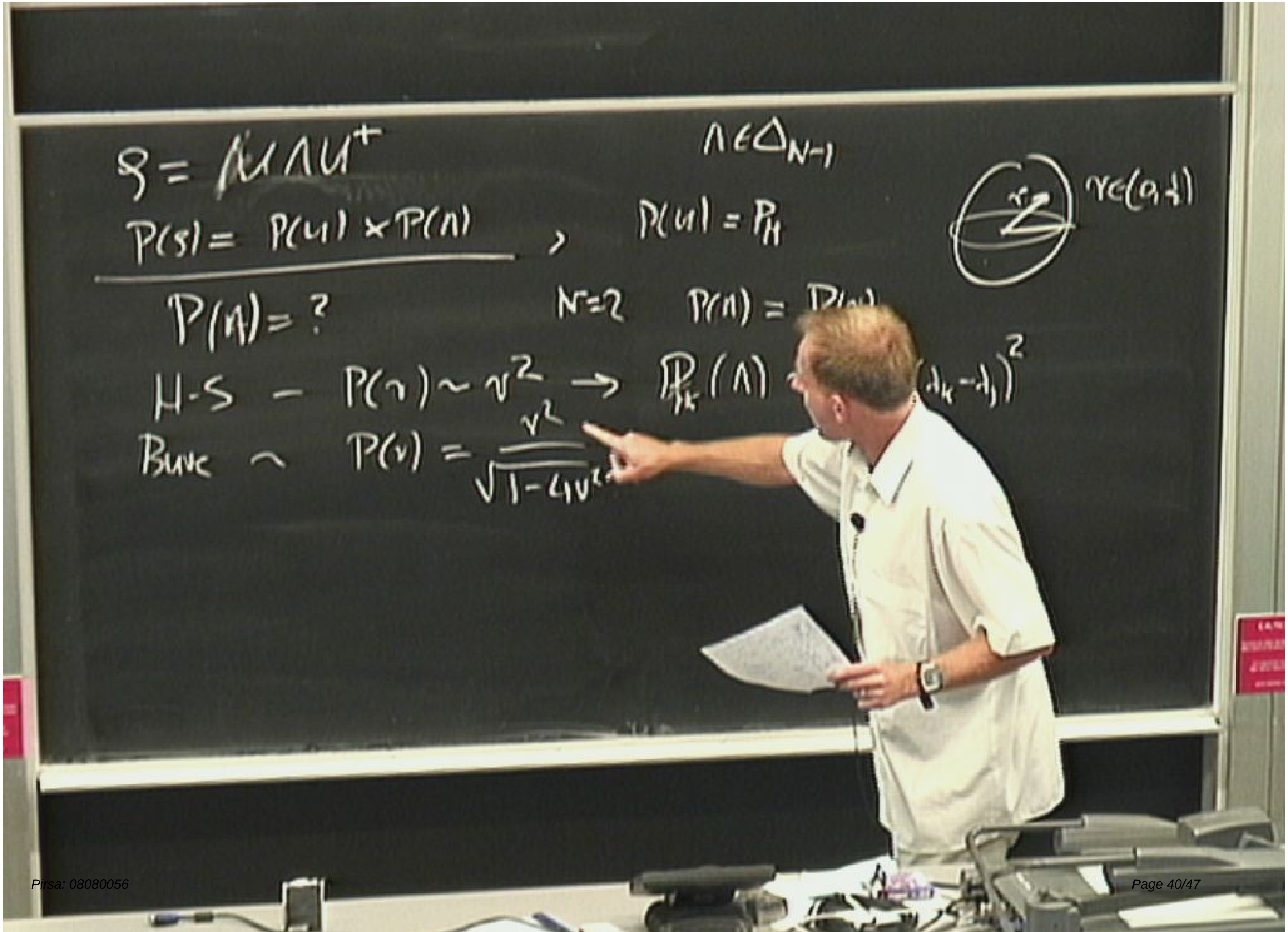
$\rho = U \Lambda U^{\dagger}$ $\qquad \Lambda \in \Delta_{N-1}$

$r \in (0,1)$

$P(\rho) = P(U) \times P(\Lambda)$ , $\qquad P(U) = P_H$

$P(\Lambda) = ?$ $\qquad N=2 \quad P(\Lambda) = P(r)$

$$H\text{-}S - P(r) \sim r^2 \rightarrow P_{HS}(\Lambda) \sim \prod_{j<k} (\lambda_k - \lambda_j)^2$$


ρ = UΛU⁺
Λ∈Δ_{N-1}
r∈(0,½)
P(ρ) = P(U) × P(Λ) ,
P(U) = P_H
P(Λ) = ?
N=2 P(Λ) = P(r)
H-S — P(r) ~ r² → P_HS(Λ)
Bure ~ P(r) = r²/√(1-4r²)

$\rho = U \Lambda U^{+}$ $\qquad \Lambda \in \Delta_{N-1}$

$r \in (0, \tfrac{1}{2})$

$P(\rho) = P(U) \times P(\Lambda)$ , $\quad P(U) = P_H$

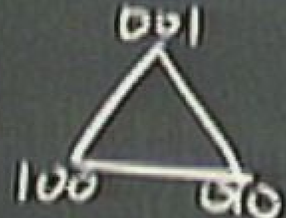
$P(\Lambda) = ?$ $\qquad N=2 \quad P(\Lambda) = P(r)$

H-S — $P(r) \sim r^2 \rightarrow P_{HS}(\Lambda) \sim \prod_{j<k} (\lambda_k - \lambda_j)^2$

Bures $\sim$ $P(r) = \dfrac{r^2}{\sqrt{1-4r^2}}$

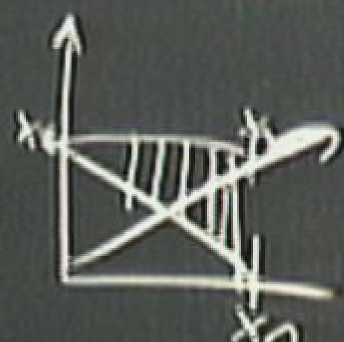
$M,\quad \vec{x} = \{x_1 \cdots x_n\} \in \Delta_{M-1}$

$M=3$




Dirichlet $\quad D_\alpha(\vec{x}) = C_\alpha\, \delta\left(\sum_{i=1}^{M} x_i - 1\right) \prod_{i=1}^{M} x_i^{\alpha-}$

$$\begin{cases} \alpha = 1 \rightarrow \text{flat.} \\ \alpha = 1/2 \rightarrow \text{Jeff. stat.} \end{cases}$$




$\alpha = 1$

$\alpha = 1/2$

$\rho = U \Lambda U^{\dagger}$ $\Lambda \in \Delta_{N-1}$

$r \in (0, \tfrac{1}{2})$

$P(\rho) = P(U) \times P(\Lambda)$ , $P(U) = P_H$

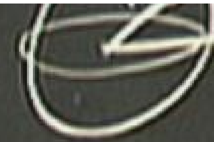
$P(\Lambda) = ?$ $N=2$ $P(\Lambda) = P(r)$

fld

H-S $-$ $P(r) \sim r^2 \longrightarrow P_{HS}(\Lambda) \sim \prod_{j<k} (\lambda_k - \lambda_j)^2 \longrightarrow \alpha = 1$

Bures $\sim$ $P(r) = \dfrac{r^2}{\sqrt{1-4r^2}}$ $\longrightarrow$ $\longrightarrow \alpha = 1/2$

Stat



$P(\rho) = P(U) \times P(\Lambda)$ , $P(U) = P_H$

$P(\Lambda) = ?$ $\quad N=2 \quad P(\Lambda) = P(r)$

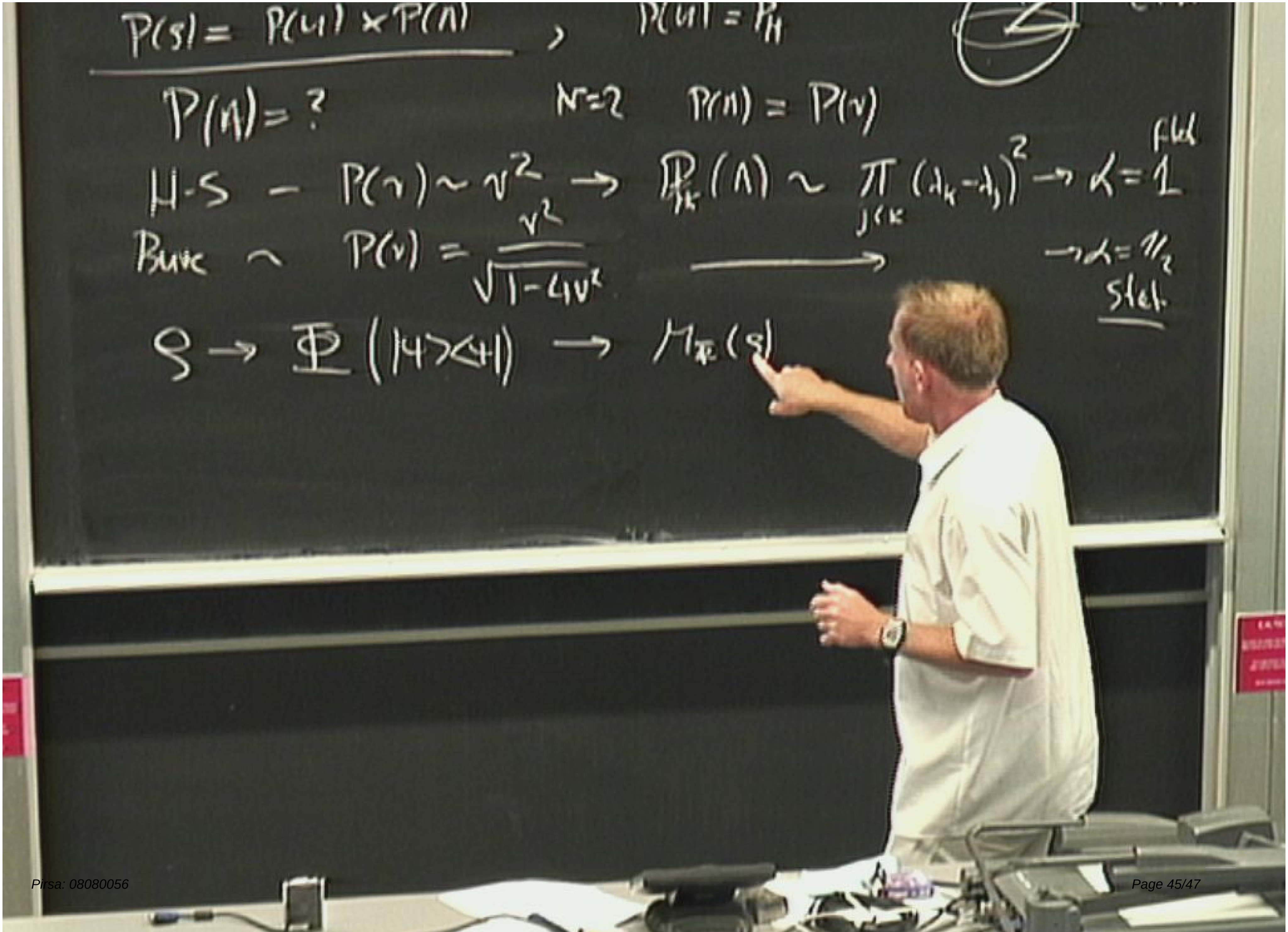
flat

H-S $-$ $P(r) \sim r^2 \rightarrow P_N(\Lambda) \sim \prod_{j<k} (\lambda_k - \lambda_j)^2 \rightarrow \alpha = 1$

Bure $\sim$ $P(r) = \dfrac{r^2}{\sqrt{1-4r^2}} \longrightarrow \quad \rightarrow \alpha = 1/2$

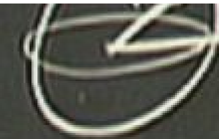
Stat.

$\rho \rightarrow \Phi(|\psi\rangle\langle\psi|)$



P(ρ) = P(U) × P(Λ) , P(U) = P_H
P(Λ) = ?
N=2 P(Λ) = P(ν)
H-S – P(ν) ~ ν² → P_HS(Λ) ~ Π_{j<k} (λ_k − λ_j)² → α = 1
Bures ~ P(ν) = ν²/√(1−4ν²) → α = 1/2
Stat.
ρ → Φ(|ψ⟩⟨ψ|) → μ_Φ(ρ)

$$P(\rho) = P(U) \times P(\Lambda) \quad , \quad P(U) = P_H$$



$$P(\Lambda) = ? \qquad N=2 \quad P(\Lambda) = P(r)$$

Fild

$$H\text{-}S \; - \; P(r) \sim r^2 \;\to\; P_{1K}(\Lambda) \sim \prod_{j<k} (\lambda_k - \lambda_j)^2 \to \alpha = 1$$

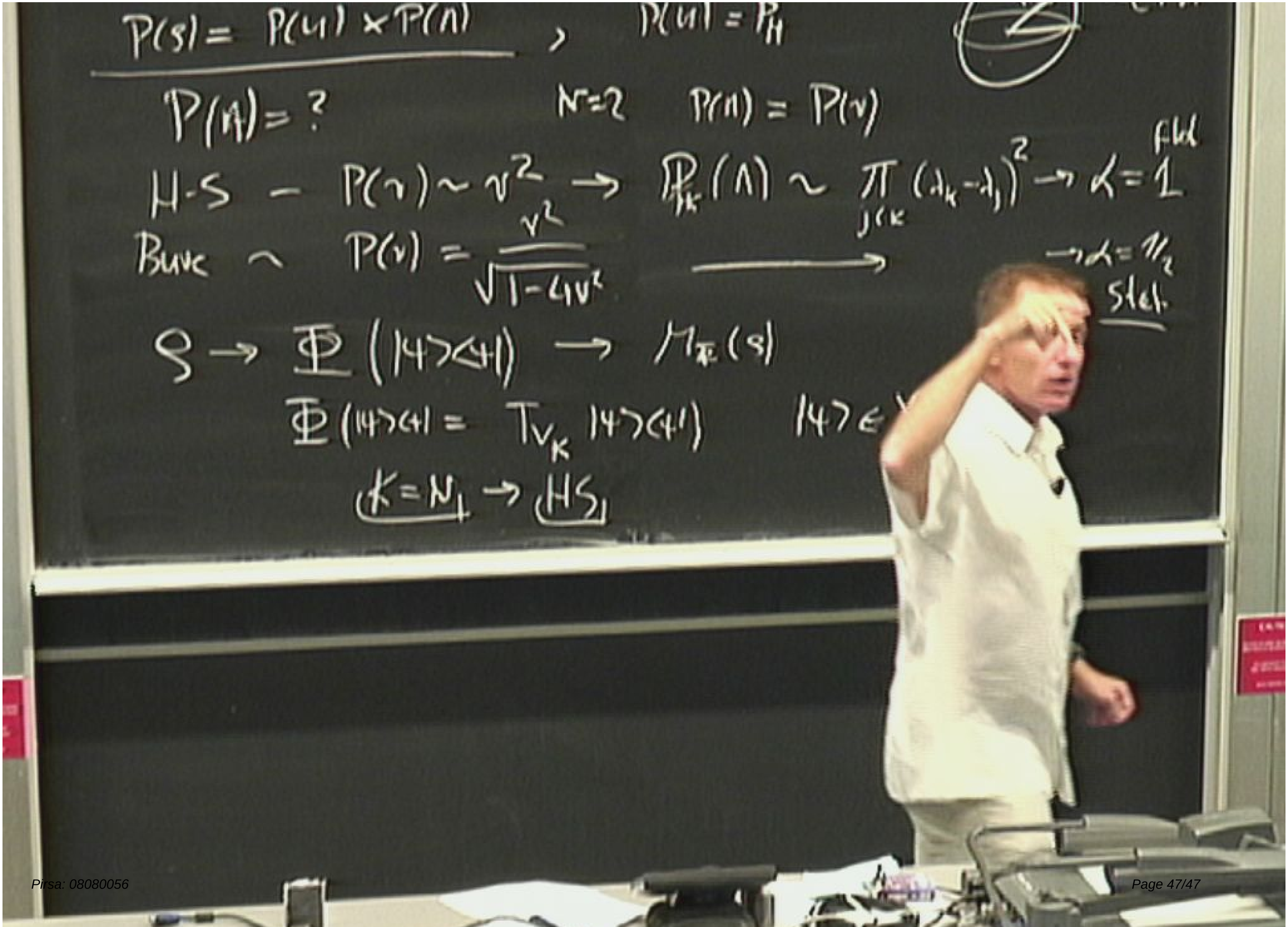
$$\text{Bure} \sim \; P(r) = \frac{r^2}{\sqrt{1-4r^2}} \longrightarrow \quad \to \alpha = 1/2$$

Stat.

$$\rho \to \Phi\left(|\psi\rangle\langle\psi|\right) \longrightarrow \mu_{\Phi}(\rho)$$

$$\Phi\left(|\psi\rangle\langle\psi|\right) = \mathrm{Tr}_{K} \, |\psi\rangle\langle\psi| \qquad |\psi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K$$

$$K = N \to HS$$


P(ρ) = P(ψ) × P(Λ) , P(ψ) = P_H
P(Λ) = ?
N=2 P(Λ) = P(r)
H-S – P(r) ~ r^2 → P(Λ) ~ Π (λ_k − λ_j)^2 → α = 1
Bure ~ P(r) = r^2 / √(1−4r^2) → α = 1/2
ρ → Φ(|ψ⟩⟨ψ|) → μ_Φ(ρ)
Φ(|ψ⟩⟨ψ| = Tr_{V_K} |ψ⟩⟨ψ|)
K = N → HS