Title: Efficient tomography of generalized coherent states

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Abstract: Quantum tomography and fidelity estimation of multi-partite systems is generally a time-consuming task. Nevertheless, this complexity can be reduced if the desired state can be characterized by certain symmetries measurable with the corresponding experimental setup. In this talk I could explain an efficient way (i.e., in polylog(d) time, with d the dimension of the Hilbert space) to perform tomography and estimate the fidelity of generalized coherent state (GCS) preparation. GCSs differ from the well known coherent states in that the associated Hilbert space is finite dimensional. In particular, the class of GCSs is very important in condensed matter applications. These results are useful to experimentalists seeking the simulations of some quantum systems, such as the Ising model in a transverse field. I\'d prefer to give a 30\' + talk late in the week, maybe on Thursday afternoon. Part of this work has been done in collaboration with ion-trap experimentalists J. Chiaverini and D. Berkeland, at Los Alamos National Laboratory. Rolando Somma.

EFFICIENT TOMOGRAPHY OF GENERALIZED COHERENT STATES

①

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Motivation: • full state tomography takes, in general, exp. time

Can we push it to poly time for special cases?

→ I will show that tomography of GCSs is an example.

GCSs: Definition

$h = \{O_1, \ldots, O_M\}$ $\quad O_j = O_j^\dagger \; ; \; O_j : \mathcal{H} \to \mathcal{H}$

↑ operator set

h : Semisimple Lie Algebra

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$\dim(\mathcal{H}) = d < \infty$

$$[O_j, O_k] = \sum_l \lambda_{jk}^{\ l} O_l$$

define the adjoint representation

$$O_j \longrightarrow \bar{O}_j = \begin{bmatrix} \lambda_{j1}^{\ 1} & \cdots & \lambda_{j1}^{\ M} \\ \vdots & & \\ \lambda_{jM}^{\ 1} & \cdots & \lambda_{jM}^{\ M} \end{bmatrix}$$

$$[\bar{O}_j, \bar{O}_k] = \sum_l \lambda_{jk}^{\ l} \bar{O}_l$$

matrix commutation

Semisimple + orthogonality $\Rightarrow$ $\underline{\mathrm{Tr}[\bar{O}_j \bar{O}_k] = \delta_{jk}}$

CARTAN – WEYL BASIS OF h

③

$$h = \{\underbrace{K_1, \dots, K_n}_{\text{CARTAN SUBALGEBRA}}, \underbrace{E_1^+, \dots, E_m^+}_{\text{"RAISING" OPERATORS}}, E_1^-, \dots, E_m^-\}$$

$[K_i, K_j] = 0$; $(E_i^+)^\dagger = E_i^- \longrightarrow$ "LOWERING" ops.

$[K_i, E_j^+] = \alpha_{ij} E_j^+$

so, an observable basis would be

$O_1 = K_1$

$\vdots$

$O_n = K_n$

$O_{n+1} = E_1^+ + E_1^-$

$O_{n+2} = i(E_1^+ - E_1^-)$

$\vdots$

GCSs:

$$|GCSs\rangle = \underbrace{e^{i\sum_{j=1}^{M}\gamma_j O_j}}_{\text{unitary group operation induced by } h} |HW\rangle$$

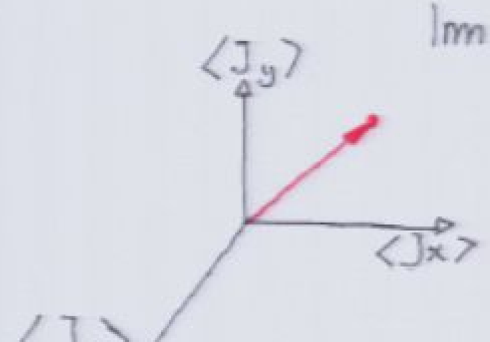
EXAMPLE: $SU(2) = \{J_z, J_x, J_y\}$

spin 1/2 rep. $\longrightarrow$ all single qubit (pure) states are GCSs

spin 1 rep $\longrightarrow$ $|GCSs\rangle = e^{i[\gamma_x J_x + \gamma_y J_y + \gamma_z J_z]} \underbrace{|HW\rangle}_{|m=+1\rangle}$

$|m=\pm 1\rangle$ is a $|GCS\rangle$

$|m=0\rangle$ is not a $SU(2)$-GCS



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EXAMPLE:

$$SU(2) = \{ J_z, J_x, J_y \}$$

$$SU(2) \to \{ J_z, J^+, J^- \} \qquad J^{+(-)} = \frac{J_x \pm i J_y}{\sqrt{2}}$$

WEIGHT STATES $|W_s\rangle$

$K_i |W_s\rangle = \beta_{is} |W_s\rangle$ → eigenstates of CARTAN ops

HIGHEST WEIGHT STATE $|HW\rangle$

$|HW\rangle \in \{ |W_s\rangle \}$

and $E_j^+ |HW\rangle = 0 \quad \forall j$

SU(2) EXAMPLE FOR SPIN J

$|HW\rangle = |m = J\rangle$

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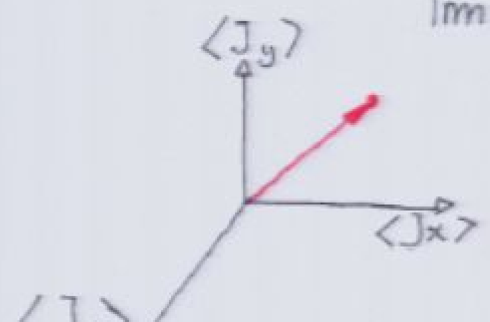

SOME FUN PROPERTIES OF GCSs

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(fixed irrep.)

- MINIMUM UNCERTAINTY :

$$\sum_j \left[\langle GCS | O_j^2 | GCS \rangle - \left(\langle GCS | O_j | GCS \rangle \right)^2 \right] \text{ is minimum}$$

because $\sum O_j^2$ is a casimir operator 'C'

$$\rightarrow [C, O_j] = 0 \quad \forall j$$

GCSs maximize the "relative" purity (h-purity)

$$\mathcal{P}_h = \sum_j \langle O_j \rangle^2$$

Example: in SU(2) $\quad C = J_x^2 + J_y^2 + J_z^2 = J^2$

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⑦

$\mathcal{P}_h$ is maximum for any GCS

$\Rightarrow$ given (many copies) state $|\psi\rangle$, obtain $\langle\psi|O_j|\psi\rangle \ \forall j$,

Compute $\mathcal{P}_h \longrightarrow$ decide wether $|\psi\rangle$ is a GCS or not

- Each GCS of h is uniquely determined (up to irrelevant phases) by $\langle O_j \rangle \ \forall j$

$\rightarrow$ GCSs are extremal points in the set ($\in \mathbb{R}^M$, given by $[\langle O_1\rangle, \dots, \langle O_M\rangle]$

Efficient tomography

Assume $|\psi\rangle$ is a GCS of $h = \{O_1, \dots, O_M\}$,

and $\langle\psi| O_j |\psi\rangle = x_j$ is known (arb. prec.) for all j

$\rightarrow$ determine $|\psi\rangle$ in time poly (M)

$$\rightarrow \quad |\psi\rangle = \underbrace{e^{i \sum_{j=1}^{M} \gamma_j O_j}} |HW\rangle = U |HW\rangle$$

determine U, the quantum circuit that prepares $|\psi\rangle$

solution :

1) Build the Hamiltonian

$$H = -\sum_{j=1}^{M} x_j O_j$$

Note that $\langle\psi|H|\psi\rangle = -\sum x_j^2 = -P_h$

$\Rightarrow |\psi\rangle$ is the unique ground state of H.

because if $|\psi'\rangle$ is GCS, $\langle\psi'|O_j|\psi'\rangle = y_j$

$$|\langle\psi'|H|\psi'\rangle| = |\sum_j x_j \cdot y_j| = P_h \cdot \cos\theta_{(x,y)}$$
$$< P_h$$

(10)

2) Diagonalize $H = -\sum \chi_j O_j$ to find

The ground state $|\psi\rangle$

$$\rightsquigarrow \quad U^{+} H U = H_D = -\sum_{j=1}^{n} \overbrace{\varepsilon_j}^{<0} \underbrace{K_j}_{\text{CARTAN OPERATORS}}$$

in general, we diagonalize by writing

H as a $d \times d$ matrix. For semisimple Lie Algebras, we can carry the diagonalization in any faithful representation like the adjoint representatio of $M \times M$ matrices :

$$\bar{U}^{+} \bar{H} \bar{U} = \bar{H}_D = -\sum_{j=1}^{n} \varepsilon_j \bar{K}_j$$

in the adjoint representation, diagonalization outputs $\bar{U} = e^{i \sum_j \gamma_j \bar{O}_j}$ and ξ_j

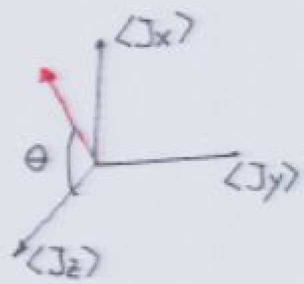
in $M \cdot \log M/\underbrace{\epsilon}_{\text{final precision}}$ operations

$$\longrightarrow \boxed{U = e^{i \sum \gamma_j O_j}}$$

Example: $SU(2) = \{ J_z, J_x, J_y \}$

spin $|J| = 100$

$|\psi\rangle$ a GCS of $SU(2)$



$\langle \psi | J_z | \psi \rangle = 100 \cdot \cos\theta$

$\langle \psi | J_y | \psi \rangle = 0$

$\langle \psi | J_x | \psi \rangle = 100 \cdot \cos\theta$

what other predictions can we make ? (13)

$|\psi\rangle$ is a GCS of $h = \{O_1, \ldots, O_M\}$

we know $\langle\psi| O_j |\psi\rangle = x_j$

⇒ we build $U = e^{i \sum \gamma_j O_j}$ such that

$U|HW\rangle = |\psi\rangle$ in time poly$(M \cdot \log M)$

→ Can we compute $\langle\psi| O_j O_k |\psi\rangle$?

yes:

$$\langle\psi| O_j O_k |\psi\rangle = \langle HW| U^+ O_j \underbrace{U U^+}_{1} O_k U |HW\rangle$$

in the adjoint representation compute

$\bar{U}^+ \bar{O}_j \bar{U}$ and $\bar{U}^+ \bar{O}_k \bar{U}$

That takes poly(M) time.

Note that $\bar{U}^{+}\bar{O}_j\bar{U} = \sum a_{jj'}\bar{O}_{j'}$

$$= \sum_{r=1}^{n} b_r \bar{K}_r + \sum_{j=1}^{m} \left(c_j \bar{E}_j^{+} + c_j^{*} \bar{E}_j^{-}\right)$$

The same for $\bar{U}^{+}\bar{O}_K\bar{U}$

→ we go back to the original representation and compute

$$\langle\psi|O_j O_k|\psi\rangle$$

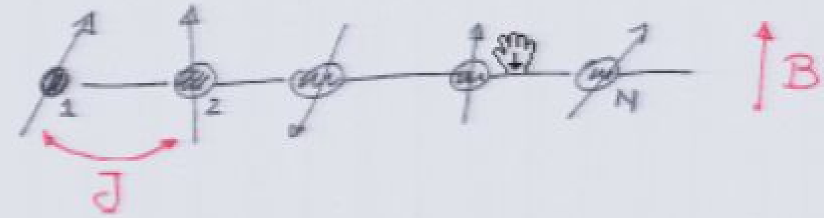
by using the fact that

$$\langle HW| E_j^{\pm} |HW\rangle = 0$$

and the commutation relations of the algebra

For the Ising Model:

$$H_{\text{ising}} = J \sum_{i=1}^{N} \sigma_x^i \sigma_x^{i+1} + B \sum_{i=1}^{N} \sigma_z^i$$



$$|HW\rangle = |\uparrow \ldots \uparrow\rangle$$

$$e^{-iH_{\text{ising}} \cdot t} |HW\rangle \longrightarrow$$ Generalized Coherent state of $SO(2N)$

→ We can do tomography by measuring poly(N) observable expectations.