Title: How to measure fidelity between two mixed quantum states?

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Abstract: Assume one laboratory designed a technique to produce quantum states in a given state $ ho$. The other lab wants to generate exactly the same state and they produce states $sigma$. If we want to know how well the second lab is doing we need to characterize the distance between $sigma$ and $ ho$ by some means,e.g. by trying to measure their fidelity, which allows us to find the Bures distance between them. The task is simple if the given state is pure, $ ho=|psi angle langle psi|$, since then fidelity reduces to the expectation value, $F=langlepsi| sigma| psi angle$. If $ ho$ is mixed the explicit formula for fidelity contains the trace of an absolute value of an operator which is not simple to compute nor to measure. Therefore we provide lower and upper bounds for fidelity and propose schemes to measure them. These experimental schemes require much less effort than the full quantum tomography of both states in question. The bounds for fidelity are called {sl sub-} and {sl super-fidelity}, respectively, due to their properties: as fidelity is multiplicative with respect to the tensor product, the sub-fidelity is sub-multiplicative, while super-fidelity is shown to be super-multiplicative. In the case of any two states of a one qubit system the bounds are strict and all three quantities coincide. The super-fidelity allowes us to define a modified Bures distance which for larger systems induces an alternative geometry of the space of quantum states.

How to measure the fidelity between two mixed quantum states ?

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Motivation

A short visit in a quantum shop

Assume you need a quantum state ρ,

you go to a quantum shop, pay for it and

...

you get a state σ instead !

How good the quantum shop is doing ?

Is the state σ we bought at least ϵ–close

to the state ρ we have ordered??

Close with respect to which metric?

If the desired state is pure, $\rho = |\psi\rangle\langle\psi|$

the situation is simple:

You need to maximize the overlap (expectation value:)
$F = \langle\psi|\sigma|\psi\rangle$,

What should one do, if the ordered state ρ is mixed?

How to measure the distance between ρ and σ?

The set $\mathcal{M}_N$ of mixed states of size N

definition

$$\mathcal{M}_N := \{\rho : \mathcal{H}_N \to \mathcal{H}_N; \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}\rho = 1\}$$

Distances in the set of quantum states

a) Hilbert–Schmidt distance, $D_{\mathrm{HS}}(\rho,\sigma) := [\mathrm{Tr}(\rho-\sigma)^2]^{1/2}$

b) trace distance, $D_{\mathrm{tr}}(\rho,\sigma) := \frac{1}{2}\mathrm{Tr}|\rho-\sigma|$

c) Bures distance, $D_{\mathrm{B}}(\rho,\sigma) := \left(2[1-\sqrt{F(\rho,\sigma)}]\right)^{1/2}$,

where fidelity between two states reads (Uhlmann '76, Jozsa '94),

$$F(\rho,\sigma) := [\mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|]^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)^2 .$$

Warning! An alternative definition of fidelity (without the square!) is sometimes used, $F' = \sqrt{F} = \mathrm{Tr}|\sqrt{\rho}\sqrt{\sigma}|$

Metrics in the space $\mathcal{M}_N$ of quantum states

properties

a) **Riemannian metric** - related to a geodesic distance
b) **monotone metric** - the corresponding monotone distance $D_{\rm mon}$ does not grow under the action of any quantum operation Φ,

$$D_{\rm mon}(\rho, \sigma) \geq D_{\rm mon}(\Phi(\rho), \Phi(\sigma)) \quad (1)$$

Metric	Hilbert–Schmidt	Trace	Bures
Is it Riemannian ?	Yes	No	Yes
Is it monotone ?	No	Yes	Yes

Theorem of **Morozova and Chentsov** '90

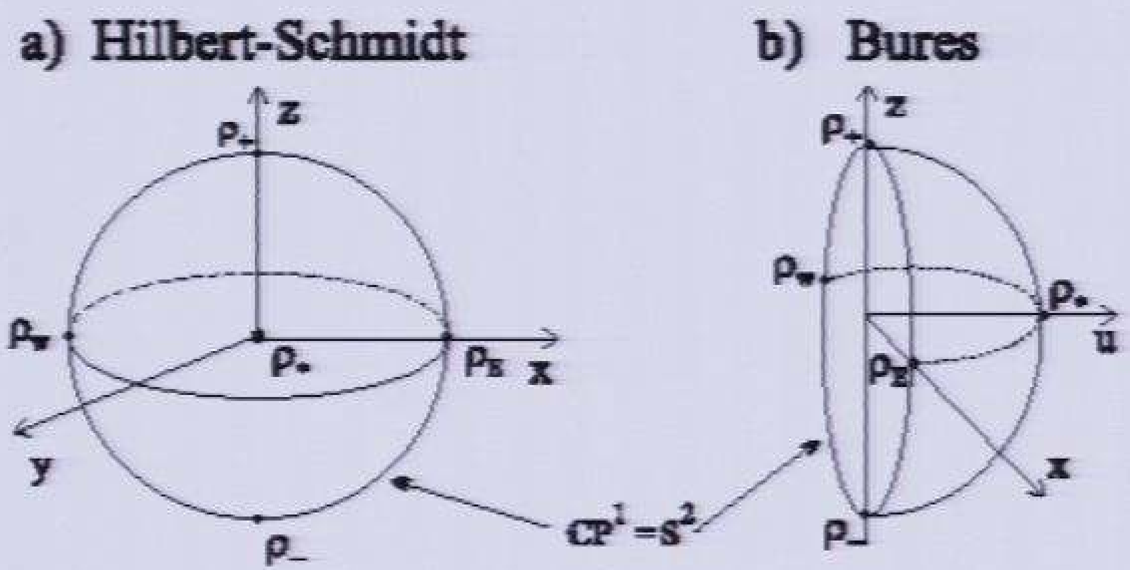
There exist infinitely many monotone Riemannian metrics on $\mathcal{M}_N$ ...

Geometry of the set Quantum States depends on the metric used:

Example: $N = 2$ – quantum states for a one–qubit system

$\mathcal{M}_2 \equiv B_3$ – Bloch ball for Hilbert–Schmidt (Euclidean) metric

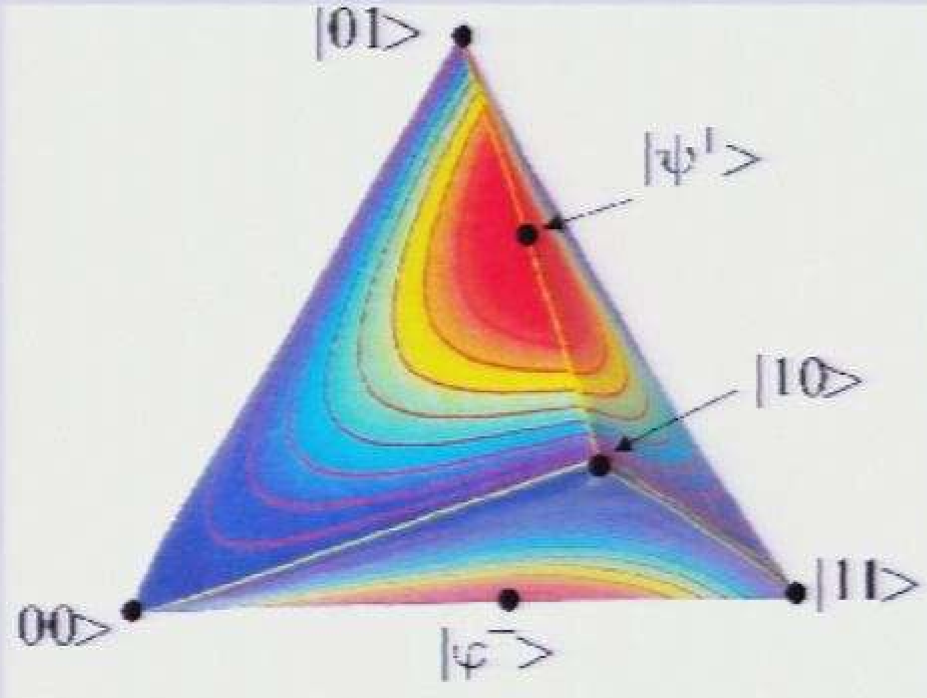
$\mathcal{M}_2 \equiv \frac{1}{2} S^3$ – Uhlmann hemisphere for Bures metric



(one dimension supressed)

more about these issues in:

Geometry of Quantum States
an Introduction to Quantum Entanglement



I. Bengtsson and K. Życzkowski

Cambridge University Press, 2006 (hardcover)
Cambridge 2007 (improved version in paperback)

Bounds between distances

a) Hilbert-Schmidt and trace distances

$$\frac{2}{N}D_{\rm tr}(\rho,\sigma) \;\leq\; D_{\rm HS}(\rho,\sigma) \;\leq\; 2D_{\rm tr}(\rho,\sigma) \qquad (2)$$

b) Bures and trace distances:

$$\sqrt{2-2\sqrt{1-[D_{\rm tr}(\rho,\sigma)]^2}} \;\leq\; D_B(\rho,\sigma) \;\leq\; \sqrt{2D_{\rm tr}(\rho,\sigma)} \qquad (3)$$

implied by the inequality of **Fuchs** and **van de Graaf** (1999)

$1-\sqrt{F(\rho,\sigma)} \;\leq\; D_{\rm tr}(\rho,\sigma) \leq \sqrt{1-F(\rho,\sigma)} \;=:\; C(\rho,\sigma)$ 'root infidelity'

($C(\rho,\sigma)$ also forms a distance; Gilchrist, Langford, Nielsen 2005)

conclusion: they generate the same **topology**

Since one distance can be estimated by the other one, if σ is close to ρ with respect to one distance, the other distance between them will also be small...

Bures distance – a function of fidelity, $D_{\rm B}^2 = 2(1-\sqrt{F})$

Fidelity has several nice properties (**Jozsa** 1994)

1. Normalisation, $0 \le F(\rho_1,\rho_2) \le 1$
2. Symmetry, $F(\rho,\sigma) = F(\rho,\sigma)$
3. Concavity, $F(\sigma, a\rho_1 + (1-a)\rho_2) \ge aF(\sigma,\rho_1) + (1-a)F(\sigma,\rho_2)$
4. Multiplicativity, $F(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) = F(\rho_1,\rho_3)F(\rho_2,\rho_4)$
5. Unitary invariance, $F(\rho,\sigma) = F(U\rho U^\dagger, U\sigma U^\dagger)$
6. Monotonicity, $F(\Phi(\rho),\Phi(\sigma)) \ge F(\rho,\sigma)$ where Φ is a quantum operation.

Purification property: $F(\rho,\sigma)$ equals the maximal transition probability between a pair of purifications of ρ and σ, (**Uhlmann** 1974).

but due to its formula, $F = [\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}]^2$,

fidelity is not easy to compute and to measure

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Useful bounds for fidelity

a) trace bounds

$$\mathrm{Tr}\rho\sigma \leq F(\rho,\sigma) \leq \mathrm{Tr}|\rho\sigma| = \mathrm{Tr}\sqrt{\rho^2\sigma^2} \tag{4}$$

b) bounds by classical fidelities (**Markham** et al. 2008)

$$F(p^{\uparrow}, q^{\downarrow}) \leq F(\rho,\sigma) \leq F(p^{\uparrow}, q^{\uparrow}), \tag{5}$$

where symbols $p^{\uparrow}$ and $q^{\downarrow}$ denote the spectra of both states with all eigenvalues in the increasing (decreasing) order.

c) upper bound by determinants (**Miszczak** et al 2008)

$$F(\rho,\sigma) \geq \mathrm{Tr}\rho\sigma + N(N-1)\sqrt[N]{\det\rho\det\sigma} =: E'(\rho,\sigma). \tag{6}$$

How to measure the **fidelity** between two mixed states?

One could

- perform full quantum tomography on both states ρ and σ,
- get all elements of both density matrices ρ_{ij} and σ_{ij},
- and use the explicit formula $F = [\mathrm{Tr}\sqrt{\rho}\,\sigma\sqrt{\rho}]^2$.

This procedure is rather *expensive*:
it requires measuring $2 \times (N^2 - 1)$ quantities.....

Is this measurement scheme optimal ??

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Is this measurement scheme optimal ??

we do not know...

Key idea: find good bounds for fidelity, which can be measured

main result of this work

Proof of the upper bound for fidelity:

$$F(\rho,\sigma) \;\leq\; \mathrm{Tr}\rho\sigma + \sqrt{(1-\mathrm{Tr}\rho^2)(1-\mathrm{Tr}\sigma^2)} \;=:\; G(\rho,\sigma)$$

Remark: a function of the upper bound $G(\rho,\sigma)$ was earlier analyzed by **Chen, Fu, Ungar**, and **Zhao** (Phys. Rev. A. 2002),

– some properties of G where independently studied in a recent preprint of **Mendonça, Napolitano, Marchiolli, Foster**, and **Liang** (June 2008).

A similar lower bound for fidelity of **Uhlmann** (2000)

$$F(\rho,\sigma) \;\geq\; \mathrm{Tr}\rho\sigma + \sqrt{2[(\mathrm{Tr}\rho\sigma)^2 - \mathrm{Tr}\rho\sigma\rho\sigma]} =: \; E(\rho,\sigma)$$

Sketch of the proof of the inequality $F \leq G$

For any matrix X oof size N with spectrum $\{\lambda_1, \ldots, \lambda_N\}$ one defines elementary symmetric polinomials $s_m(X)$. For instance

$$s_2(X) := \sum_{i<j} \lambda_i \lambda_j, \tag{7}$$

$$s_3(X) := \sum_{i<j<k} \lambda_i \lambda_j \lambda_k \ , \tag{8}$$

The proof is based on two **algebraic lemmas**

Lemma 1:

$$s_2\left(\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right) \leq s_2\left(\sqrt{\hat{\mathbf{diag}}(p)\,\hat{\mathbf{diag}}(q)}\right) ,$$

Lemma 2:

$$s_2\left(\sqrt{\hat{\mathbf{diag}}(p)\,\hat{\mathbf{diag}}(q)}\right) \leq \sqrt{s_2(\hat{\mathbf{diag}}(p))s_2(\hat{\mathbf{diag}}(q))} = \sqrt{s_2(\rho)s_2(\sigma)}.$$

Properties with respect to the tensor product

Both bounds for fidelity,

$$E(\rho,\sigma) \le F(\rho,\sigma) \le G(\rho,\sigma)$$

can be called **sub–** and **super–fidelity**, respectively.

- a) sub–fidelity is **sub–multiplicative**,

$$E(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \le F(\rho_1, \rho_3) F(\rho_2, \rho_4)$$

- b) **fidelity** is **multiplicative**,

$$E(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) = F(\rho_1, \rho_3) F(\rho_2, \rho_4)$$

- c) **super–fidelity** is **super–multiplicative**,

$$G(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \ge G(\rho_1, \rho_3) G(\rho_2, \rho_4)$$

Properties of sub–fidelity E and super–fidelity G

- **Bounds:** $0 \leq E(\rho,\sigma) \leq 1$ and $0 \leq G(\rho,\sigma) \leq 1$.
- **Symmetry:** $E(\rho,\sigma) = E(\sigma,\rho)$ and $G(\rho,\sigma) = G(\rho,\sigma)$.
- **Unitary invariance:** $E(\rho,\sigma) = E(U\rho U^\dagger, U\sigma U^\dagger)$ and $G(\rho,\sigma) = G(U\rho U^\dagger, U\sigma U^\dagger)$, for any unitary operator U.
- **Concavity:** for any states ρ_1, ρ_2 and σ and any $a \in [0,1]$ the following is true

$$E(\sigma, a\rho_1 + (1-a)\rho_2 \geq aE(\sigma,\rho_1) + (1-a)E(\sigma,\rho_2),$$

$$G(\sigma, a\rho_1 + (1-a)\rho_2 \geq aG(\sigma,\rho_1) + (1-a)G(\sigma,\rho_2),$$

- **Geometry:** Super–fidelity G induces
  a) modified root infidelity, $D_G(\sigma,\rho) := \sqrt{1 - G(\sigma,\rho)}$
  a) modified Bures length, $D_M(\sigma,\rho) := \arccos G(\sigma,\rho)$
- **Equivalence for one qubit system:**
  For $N = 2$ the equality holds, $E(\rho,\sigma) = F(\rho,\sigma) = G(\rho,\sigma)$.

Example

Consider a family of mixed quantum states

$$\rho_a = a|\psi\rangle\langle\psi| + (1-a)I/N.$$

and the maximally mixed state $\rho_* := I/N$. Simple calculation gives

$$F(\rho_a, \rho_*) = \frac{1}{N^2}\left(\sqrt{(N-1)a+1} + (N-1)\sqrt{1-a}\right)^2,$$

$$E(\rho_a, \rho_*) = \frac{1}{N} + \sqrt{2}\frac{1}{N}\sqrt{1-\frac{1}{N}\sqrt{1-a^2}}$$

$$E'(\rho_a, \rho_*) = \frac{1}{N} + \left(1-\frac{1}{N}\right)\sqrt[N]{((N-1)a+1)(1-a)^{N-1}}$$

$$G(\rho_a, \rho_*) = \frac{1}{N} + \left(1-\frac{1}{N}\right)\sqrt{1-a^2}.$$

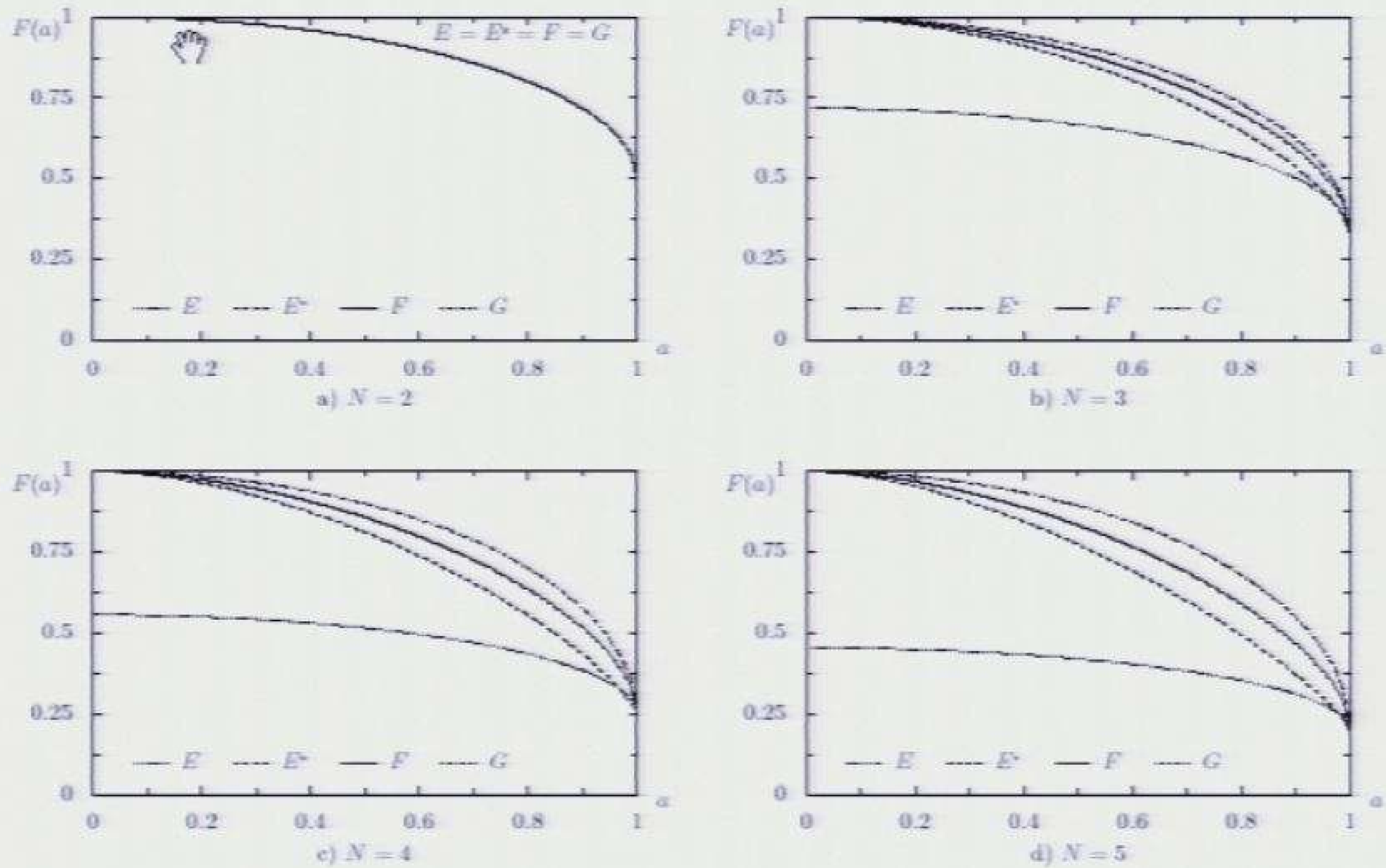

Figure: Comparison of sub–fidelity E, alternative lower bound E' based on determinants, fidelity F and super–fidelity G for $N = 2, 3, 4$ and 5.

Measurement of sub– and superfidelity

Trick with trace of the SWAP operator

Identity of **Werner** holds for any operators A and B

$$\mathrm{Tr} AB = \mathrm{Tr} V(A \otimes B)$$

where the SWAP operator V satisfies $V|x\rangle \otimes |y\rangle = |y\rangle \otimes |x\rangle$.

Hence to compute $\mathrm{Tr}\rho^2$ it is enough to take two copies of this state and to measure $\mathrm{Tr} V(\rho \otimes \rho)$, **Ekert, Alves, Oi, Horodecki^2, Kwek** (2002). Analogously to measure $\mathrm{Tr}\rho^k$ we need k copies of ρ.

Advantages of sub– and superfidelity

Super–fidelity G is a function of traces:

$\mathrm{Tr}\rho\sigma$ and purities $\mathrm{Tr}\rho^2$ and $\mathrm{Tr}\sigma^2$.

Subfidelity E is a function of traces: $\mathrm{Tr}\rho\sigma$ and $\mathrm{Tr}\sigma\rho\sigma\rho$.

For each quantity one needs not more than pairs (for G) or quadruples of states (for E)!

Measurement of super–fidelity G

Since the subfidelity reads

$$G(\rho,\sigma) \;=\; \operatorname{Tr} V\rho\otimes\sigma + \sqrt{1-\operatorname{Tr} V\rho\otimes\rho}\,\sqrt{1-\operatorname{Tr} V\sigma\otimes\sigma}$$

it is enough to measure expectation value of the SWAP operator V for three pairs of states: $\{\rho,\sigma\},\{\rho,\rho\}$ and $\{\sigma,\sigma\}$.

Measurement of sub–fidelity E

Quadruples of states are needed. One may use a programmable network. Depending on the *'program state'* $|\Psi_{12}\rangle$ different quantities necessary to obtain E are produced by a single measurement of the operator J_z:

(i) $\operatorname{Tr}\rho\sigma$ if $|\Psi_{12}\rangle = |0\rangle|0\rangle$,

(ii) $\operatorname{Tr}\rho\sigma\rho\sigma$ if $|\Psi_{12}\rangle = |1\rangle|0\rangle$,

(iii) $\frac{1}{2}\left(\operatorname{Tr}\rho\sigma\rho\sigma - (\operatorname{Tr}\rho\sigma)^2\right)$ if $|\Psi_{12}\rangle = (|0\rangle|1\rangle + |1\rangle|0\rangle)/\sqrt{2}$ (Bell state).

Programable network to masureme sub–fidelity E

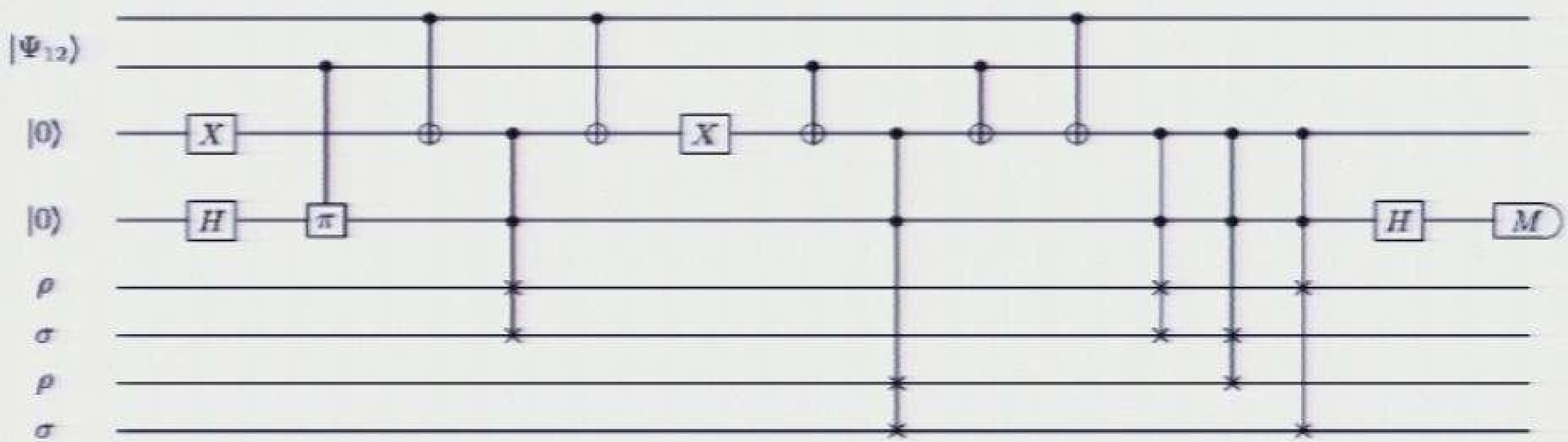

Figure: Programmable network acting on two pairs of the states investigated ρ and σ, two control qubits (initially in the ground state $|0\rangle$) and a two–qubit program state $|\Psi_{12}\rangle$. Symbol M represents the measurement of the J_z component of the lower **control qubit**, which provides all quantities necessary to get the sub–fidelity E.

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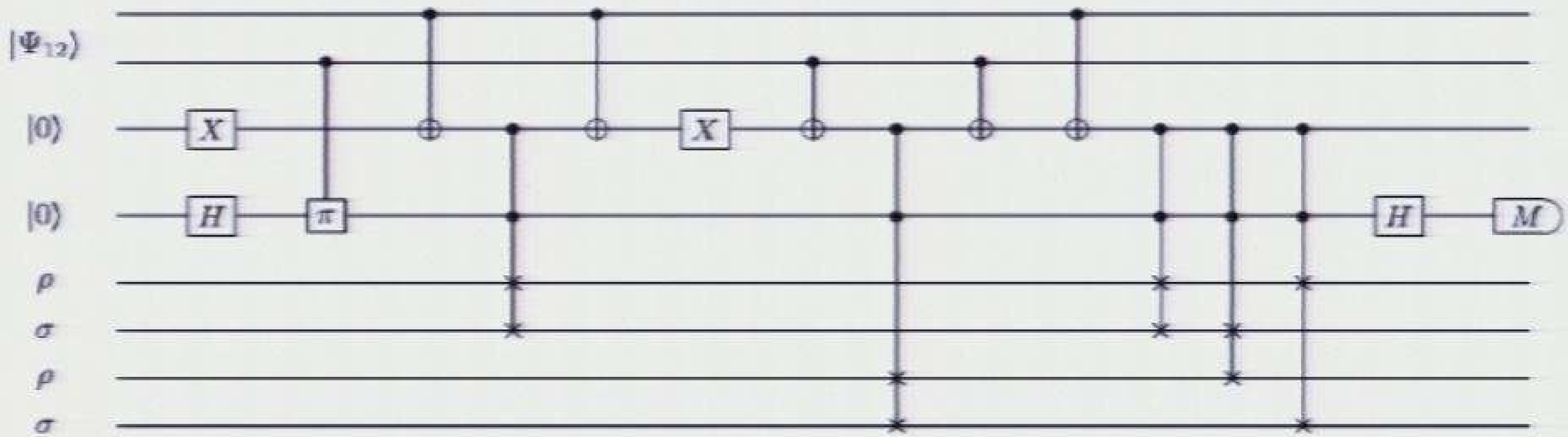

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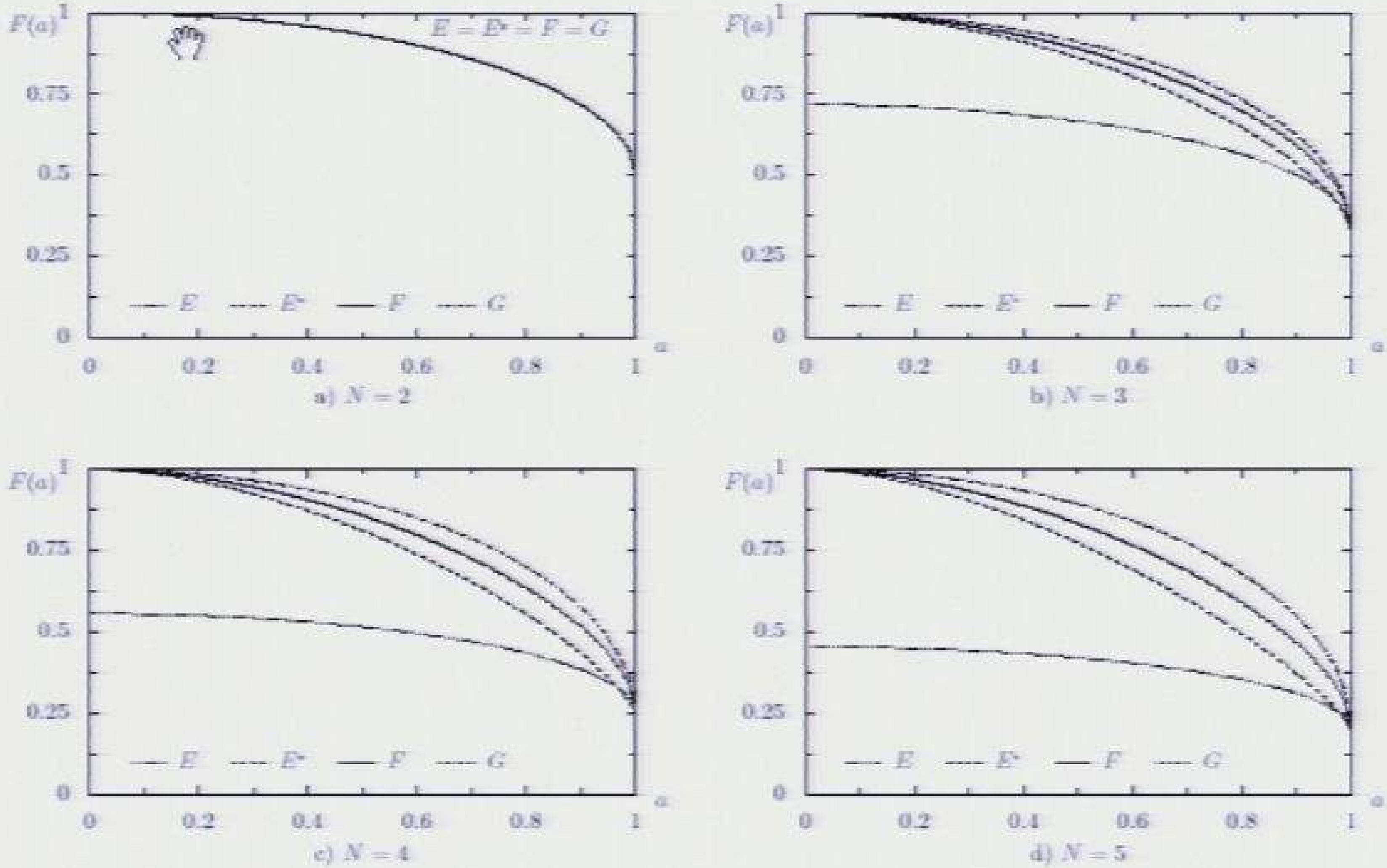
Concluding Remarks

- Geometry of the set of quantum states:
  depends on the metric used.
- The standard distances (Hilbert-Schmidt, trace, Bures)
  generate the same **topology**.
- **Bures distance** is induced by the minimal Riemannian monotone metric and it is related to distinguishability between states.
  It is a function of fidelity, $F = [\mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}]^2$,
  which in general is not easy to compute and to measure.
- We find both bounds for fidelity, $E \leq F \leq G$, where
  a) **sub–fidelity** $E(\rho,\sigma) = \mathrm{Tr}\rho\sigma + \sqrt{2[(\mathrm{Tr}\rho\sigma)^2 - \mathrm{Tr}\rho\sigma\rho\sigma]}$
  is sub–multiplicative, while
  b) **super–fidelity** $G(\rho,\sigma) = \mathrm{Tr}\rho\sigma + \sqrt{(1-\mathrm{Tr}\rho^2)(1-\mathrm{Tr}\sigma^2)}$
  is super–multiplicative

Both quantities E and G can be experimentally measured in an set–up involving a few copies of both states investigated for an arbitrary size of the system.



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