Title: Multiple observations of quantum systems

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Abstract: Let us assume a following scenario: In a state of a quantum system one qubit is encoded. The first observer has no prior knowledge about the state of the qubit. He performs an optimal measurement on the system and based on the measured data he estimates the state on the qubit. After performing the measurement the first observer leaves the measured quantum system in a lab. I will study the question whether the second observer who has no knowledge about the measurement setup and the measurement outcome of the first observation can learn anything about the original preparation of the qubit.

QUANTUM OBSERVATIONS:

1) “Recycling” of QI
2) Compression of QI via position coding
3) Approximation of non-physical maps

28.08.2008

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Knowledge about physical situation

existing quantum theory must be supplemented with some principle that
lls us how to translate, or encode, the results of measurements into a
efinite state description $\hat{\rho}$. Note that the problem is not to find $\hat{\rho}$ which
rrectly describes "true physical situation". That is unknown, and always
mains so, because of incomplete information. In order to have a usable
eory we must ask the much more modest question: **What $\hat{\rho}$ best
escribes our state of knowledge about the physical situation?**

E.T.Jaynes

T. Jaynes: "Information theory and statistical mechanics" in *1962 Brandeis Lectures*, p.181

Qubit – encoding orientation

Pure state of a spin -1/2 particle

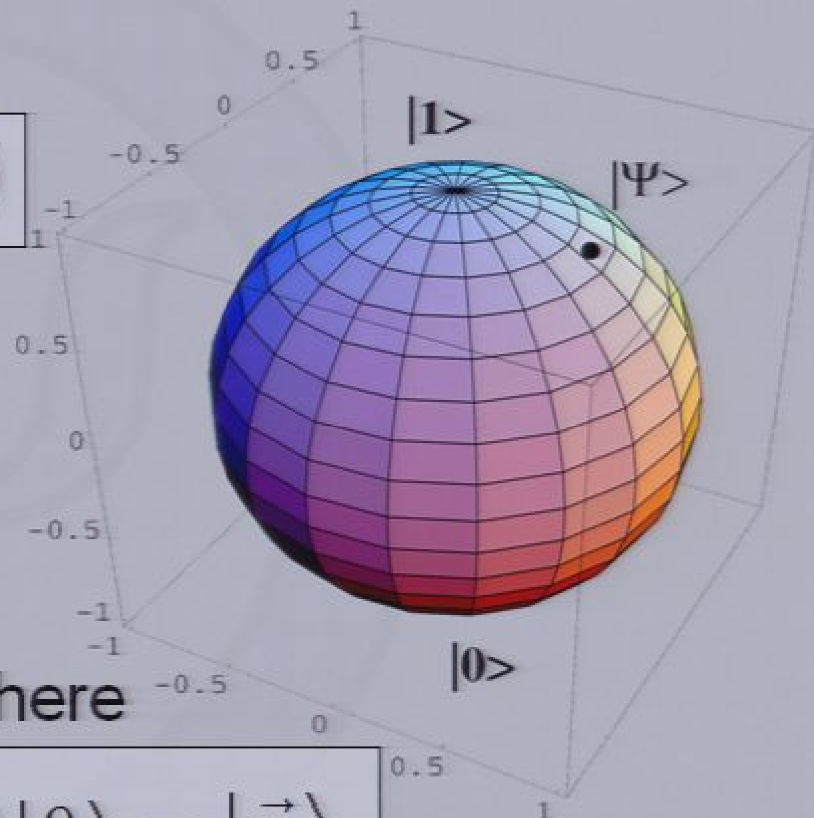
$$|\psi\rangle = \cos\vartheta/2\,|1\rangle + e^{i\varphi}\sin\vartheta/2\,|0\rangle$$

2-d Hilbert space

density operator

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{n}.\vec{\sigma}\right) = \frac{1}{2}\left(\hat{I} + n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z\right)$$

$$\rho = \frac{1}{2}(1+\vec{n}.\vec{\sigma}) = \frac{1}{2}\begin{pmatrix}1\\ n_x\\ n_y\\ n_z\end{pmatrix} \leftrightarrow \vec{n} = \left(n_x, n_y, n_z\right)$$



State space – Bloch (Poincare) sphere

$$|\psi\rangle = \cos\vartheta/2\,|1\rangle + e^{i\varphi}\sin\vartheta/2\,|0\rangle = |\vec{n}\rangle$$

$$\vec{n}.\vec{\sigma}\,|\vec{n}\rangle = \vec{n}\,|\vec{n}\rangle$$

Quantum clickology

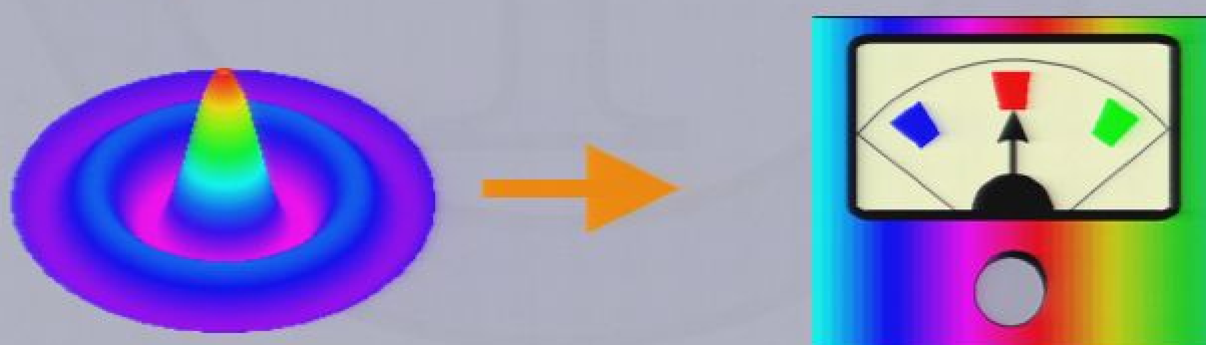
- Measurement: conditional distribution on a discrete state space of the aparatus *A*: $\hat{O}$ observables with eigenvalues λ_j
- a priori distribution $p_0\left(\hat{\rho}\right)$ on the state space of the system
- joint probability distribution

$$p\left(\hat{O},\lambda_i \middle| \hat{\rho}\right) = \mathrm{Tr}\left(\hat{P}_{\lambda_i,\hat{O}}\hat{\rho}\right)$$

$$p\left(\hat{O},\lambda_i;\hat{\rho}\right) = p\left(\hat{O},\lambda_i \middle| \hat{\rho}\right) p_0\left(\hat{\rho}\right)$$

System

Apparatus



Measurement

Quantum Bayesian inference

- Bayesian inversion from distribution on *A* to distribution on Ω

$$p\left(\hat{\rho}|\hat{O},\lambda_i\right) = \frac{p\left(\hat{O},\lambda_i\middle|\hat{\rho}\right)p_0\left(\hat{\rho}\right)}{\int_\Omega p\left(\hat{O},\lambda_i;\hat{\rho}\right)\mathrm{d}\Omega}$$

- Reconstructed density operator given the result λ_i

$$\hat{\rho}_{est} = \int_\Omega \hat{\rho}\left(\vartheta,\varphi\right)p\left(\hat{\rho}\middle|\hat{O},\lambda_i\right)\mathrm{d}_\Omega$$

- d_Ω – invariant integration measure

K.R.W. Jones, *Ann. Phys. (N.Y.)* 207, 140 (1991)

V.Bužek, R.Derka, G.Adam, and P.L.Knight, *Annals of Physics (N.Y.)* 266, 454 (1998)

Single-qubit measurement

- von Neumann measurement: projectors on eigenvectors of the apparatus

$$\hat{O}_{+} = |+\vec{m}\rangle\langle+\vec{m}| \quad ;\lambda = +1$$

$$\hat{O}_{-} = |-\vec{m}\rangle\langle-\vec{m}| \quad ;\lambda = -1$$

$|\vec{n}\rangle$

$$|+\vec{m}\rangle \quad ; P_{+} = |\langle+\vec{m}|\vec{n}\rangle|^2 = (1+\vec{n}.\vec{m})/2$$

$$|-\vec{m}\rangle \quad ; P_{-} = |\langle-\vec{m}|\vec{n}\rangle|^2 = (1-\vec{n}.\vec{m})/2$$

- one bit of information
- once the measurement is performed no more info can be gained

Single-qubit measurement

- Prior knowledge: state is pure

$$p_0(\hat{\rho}) = const.$$

- Density operator

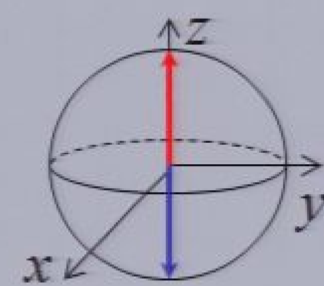
$$= \frac{1}{2}\left(\hat{I} + \sin\vartheta\cos\varphi\hat{\sigma}_x + \sin\vartheta\sin\varphi\hat{\sigma}_y + \cos\vartheta\hat{\sigma}_z\right)$$

- Invariant measure

$$d_\Omega = d_{\vec{n}} = \frac{1}{4\pi}\sin\vartheta\, \mathrm{d}\,\vartheta\, \mathrm{d}\,\varphi$$

- Projectors on eigenvectors of the apparatus

$$\hat{O}_z = \frac{1}{2}\left(\hat{I} + \lambda\hat{\sigma}_z\right); \quad \lambda = \pm 1$$



- Distribution on Ω

$$p\left(\hat{\rho}\middle|\hat{O},\lambda_i = +1\right) = \tfrac{1}{2}(1 + \cos\vartheta)$$

$$\hat{\rho}_{est} = \int_\Omega \hat{\rho}(\vartheta,\varphi)\, p\left(\hat{\rho}\middle|\hat{O},\lambda\right) d_\Omega = \frac{1}{2}\left(\hat{I} + \frac{1}{3}\hat{\sigma}_z\right)$$

Figure of merit: Mean fidelity

- Fidelity of the guess when $|\pm\vec{m}\rangle$ is measured

$$|+\vec{m}\rangle \quad ; F_+ = |\langle +\vec{m}|\vec{n}\rangle|^2 = (1+\vec{n}.\vec{m})/2$$

$$|-\vec{m}\rangle \quad ; F_- = |\langle -\vec{m}|\vec{n}\rangle|^2 = (1-\vec{n}.\vec{m})/2$$

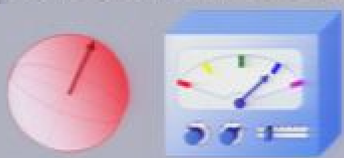
- Mean fidelity (averaged over all input states)

$$\overline{F} = \int d_{\vec{n}} \left[\frac{1+\vec{n}.\vec{m}}{2} \left|\langle \vec{n}|\vec{m}\rangle^2\right| + \frac{1-\vec{n}.\vec{m}}{2} \left|\langle \vec{n}|-\vec{m}\rangle^2\right| \right] = \frac{2}{3}$$

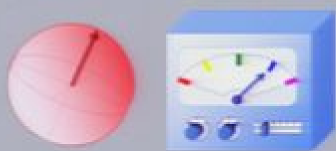
Massar & S.Popescu: *Phys. Rev. Lett.* 74, 1259 (1995)
W.Helstrom: *Quantum Detection and Estimation Theory* (Academic, NY, 1976)

Two-qubits

independent measurements of two qubits:



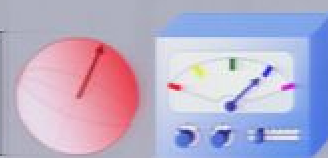
$$\hat{O}_{\pm} = \left|\pm\vec{m}\right\rangle\left\langle\pm\vec{m}\right| \quad ;\lambda = \pm 1$$



$$\hat{O}_{\pm} = \left|\pm\vec{k}\right\rangle\left\langle\pm\vec{k}\right| \quad ;\lambda = \pm 1$$

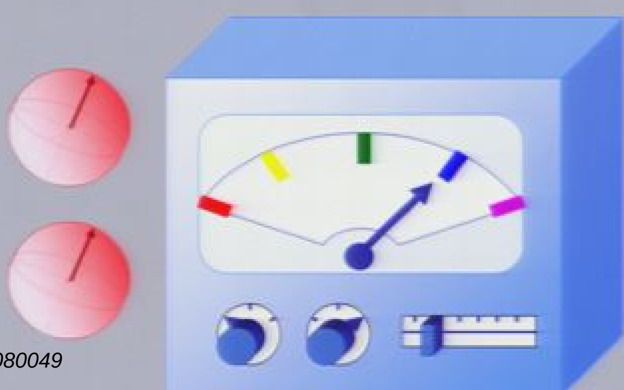
adaptive measurements of two qubits:

$$\hat{O}_{\pm} = \left|\pm\vec{m}\right\rangle\left\langle\pm\vec{m}\right|$$



$$\hat{O}_{\pm} = \left|\pm\vec{k}_{\pm\vec{m}}\right\rangle\left\langle\pm\vec{k}_{\pm\vec{m}}\right|$$

simultaneous measurements of two qubits:



$$\hat{P}_j = \left|\Phi_j\right\rangle_{12}\left\langle\Phi_j\right| \quad ;\sum_j \hat{P}_j = \hat{I}$$

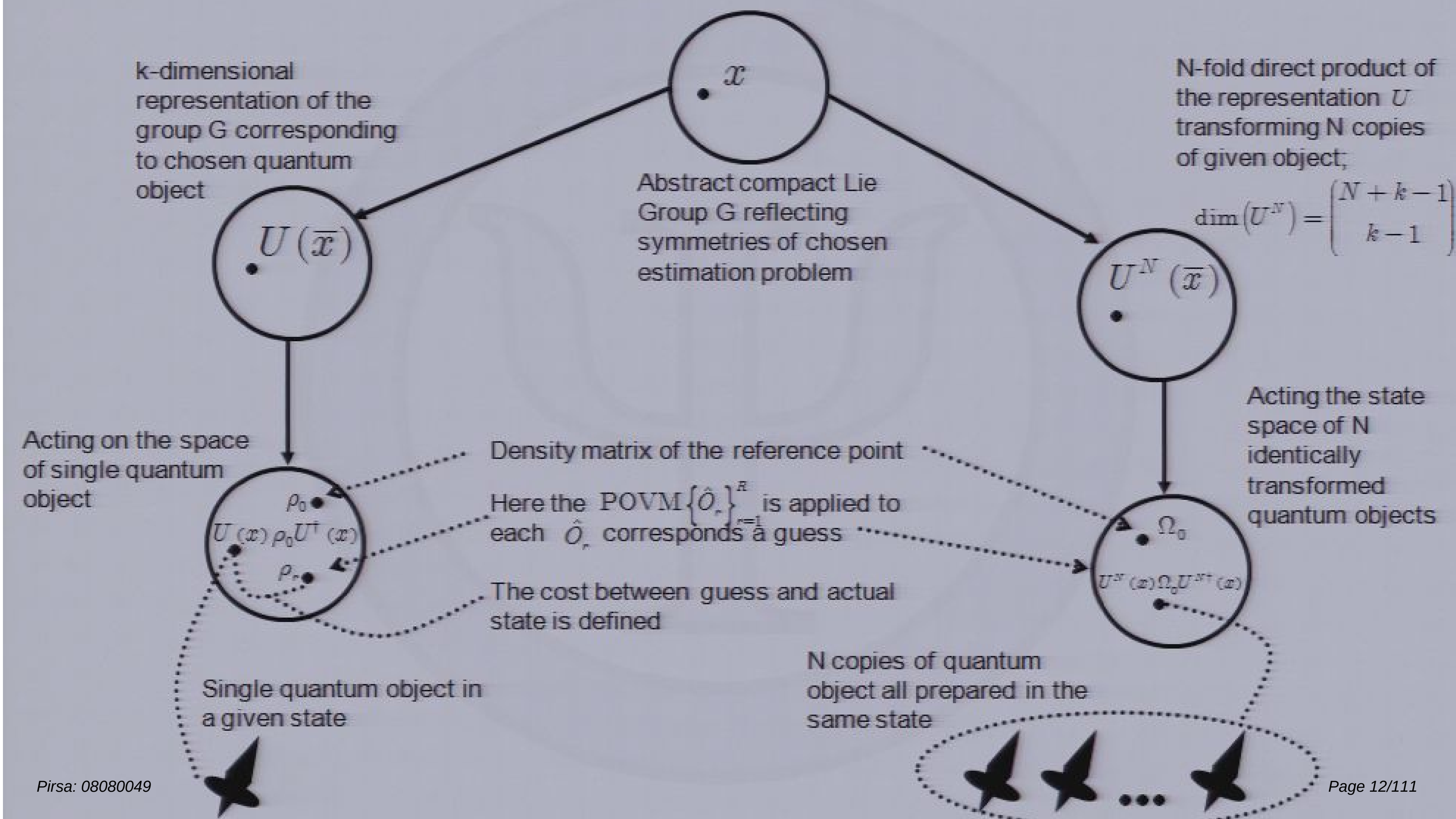
Generalized quantum measurements

- **Positive operators $\hat{O}_r$ – not projectors; $\sum_r \hat{O}_r = 1$**
- **Mean fidelity *F* via the cost function**

$$F = \sum_r \int_\Omega \mathrm{dx}\,\mathrm{Tr}\left[\hat{O}_r \overbrace{U(\mathrm{x})\,\hat{\rho}_0 U^\dagger(\mathrm{x}) \otimes \ldots \otimes U(\mathrm{x})\,\hat{\rho}_0 U^\dagger(\mathrm{x})}^{\text{N times}}\right] \mathrm{Tr}\left[U(\mathrm{x})\,\hat{\rho}_0 U^\dagger(\mathrm{x}) U_r \hat{\rho}_0 U_r^\dagger\right]$$

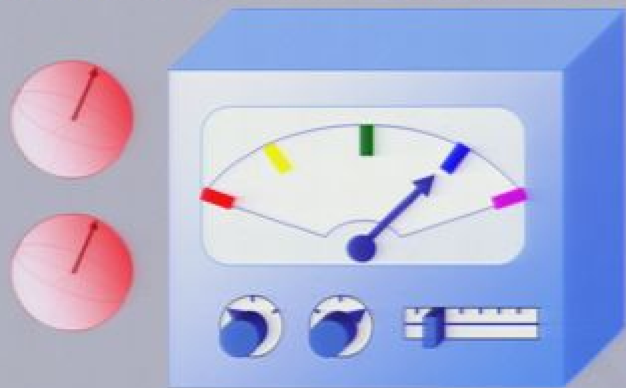
W.Holstrom, *Quantum detection and estimation theory* (Academic Press, New York, 1976)
S.Holevo, *Probabilistic and statistical aspects of quantum theory* (North Holland, Amsterdam, 1982)
G.A.Paris and J.Rehacek, *Quantum estimations* (Springer, Berlin, 2004)

Optimal quantum measurements



Optimal two-qubit measurement

simultaneous measurements of two qubits:



$$\hat{P}_j = \left|\Phi_j\right\rangle_{12}\left\langle\Phi_j\right| \quad ; \sum_j \hat{P}_j = \hat{I}$$

$$\left|\Phi_j\right\rangle_{12} = \frac{\sqrt{3}}{2}\left|\vec{n}_j;\vec{n}_j\right\rangle_{12} + \frac{1}{2}\left|\psi^s\right\rangle_{12}$$

$$= (0,0,1)$$

$$= \left(\frac{\sqrt{8}}{3}, 0, -\frac{1}{3}\right)$$

$$= \left(\frac{-\sqrt{2}}{3}, \frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$$

$$= \left(\frac{-\sqrt{2}}{3}, \frac{-\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$$

- Four vectors point to the four vertices of the tetrahedron
- Mean fidelity of estimation

$$F = \frac{3}{4}$$

Massar & S Popescu: Phys. Rev. Lett. 74, 1259 (1995)

Optimal reconstructions of qubits

- average fidelity of estimation

$$F=\frac{N+1}{N+2}=\frac{1}{2}\left[1+\frac{N}{N+2}\right]$$

$$F=\frac{2}{3}=\frac{1}{2}\left(1+\frac{1}{3}\right)$$

Estimated density operator on average

$$\hat{\rho}_{est}=s\hat{\rho}+\frac{1-s}{2}\hat{I};\ s=2F-1=\frac{N}{N+2}$$

- Construction of optimal (& finite-dimensional) POVM's – maximize the fidelity *F*
- POVM via von Neumann projectors – Naimark theorem
- Optimal decoding of information
- Optimal preparation of quantum systems
- Recycling of q-information

Massar and S.Popescu, *Phys. Rev. Lett.* 74, 1259 (1995)
Latorre, P.Pascual, and R.Tarrach, *Phys. Rev. Lett.* 81, 1351 (1998)
Gill and S.Massar, *Phys. Rev. A* 61, 042312 (2000)
Hayashi, *Asymptotic Theory of Quantum Statistical Inference* (Academic Press, NY, 2005)

“Recycling” of quantum information

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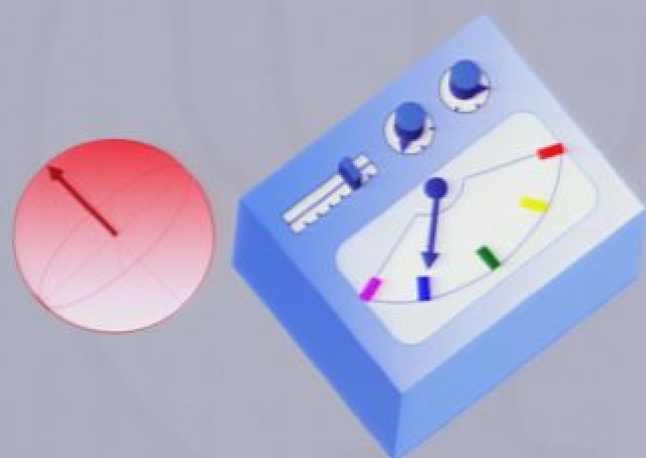
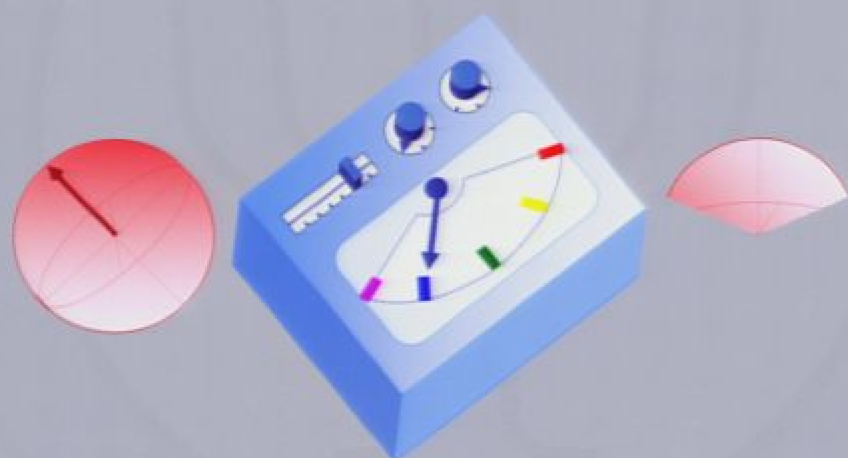
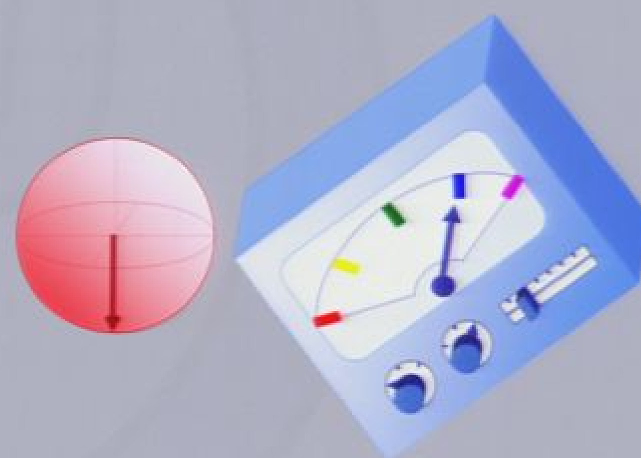


“Recycling” of quantum information



"Recycling" of quantum information



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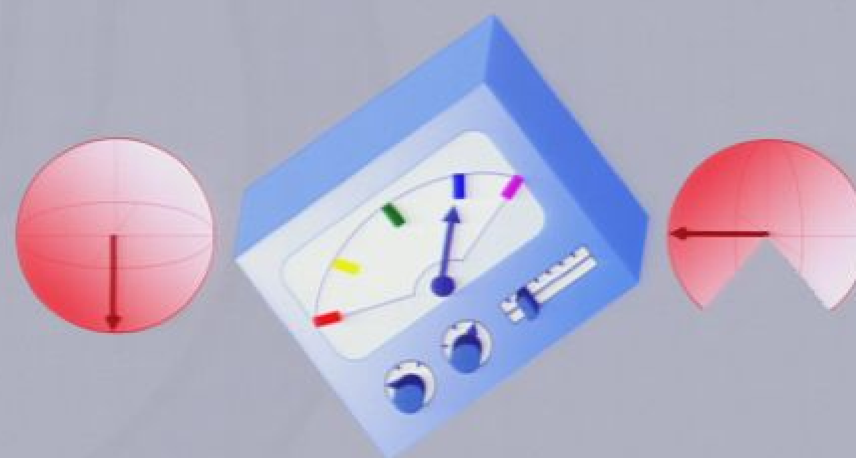
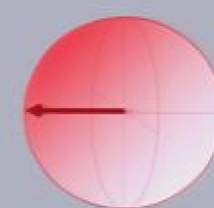


“Recycling” of quantum information



Recycling of q-information

Trade-off:
information gain *vs* disturbance

How much information is left in the system after it has been measured?

Recycling of q-information

Trade-off:
information gain *vs* disturbance

How much information is left in the system after it has been measured?

- Sequence of observers
- No classical communication
- Observers use the "same" measurement device (up to its "orientation") though with no knowledge of previous results
- Optimal observation of single qubit
- Non-trivial estimation of everybody?

$$F = \frac{1}{2}\left(1 + \frac{1}{3^k}\right)$$

Robustness of quantum information

Sequence of observers: *k*

Size of ensemble: *N*

$$F = \frac{1}{2}\left(1 + \left(\frac{N}{N+2}\right)^k\right) \sim 1 - \frac{k}{N}$$

- Robustness against observations
- Trade-off gain vs. disturbance
- Non-trivial estimation by everybody

Sending q-information via q-channel

Alice sends info to Bob about direction $\vec{n}$ encoded in qubit $|\vec{n}\rangle$

Given the preparator (information source) and the quantum channel there must exist an optimal encoding-decoding procedure that maximizes the knowledge of Bob about $\vec{n}$

Coding of direction: one qubit

Two preparators: original one

$$\left|\vec{n}\right\rangle$$

$$\hat{O}_{\pm} = \left|\pm\vec{m}\right\rangle\left\langle\pm\vec{m}\right| \quad ;\lambda = \pm 1$$

or the flipped preparator

$$\left|-\vec{n}\right\rangle$$

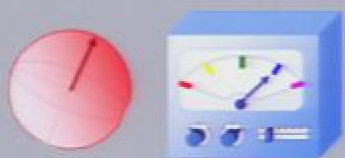
$$\hat{O}_{\mp} = \left|\mp\vec{m}\right\rangle\left\langle\mp\vec{m}\right| \quad ;\lambda = \mp 1$$

Average fidelity is the same – optimal coding given the resources

$$F = \frac{2}{3}$$

Coding of direction: two qubits

One preparator generates two copies; independent measurements



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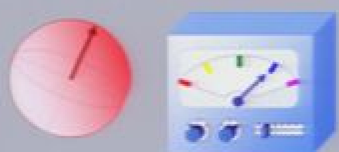
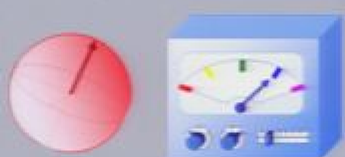


$$\hat{O}_{\pm} = \left|\pm\vec{k}\right\rangle\left\langle\pm\vec{k}\right| \quad ;\lambda = \pm 1$$

Two preparators – each generate one qubit; independent measurements



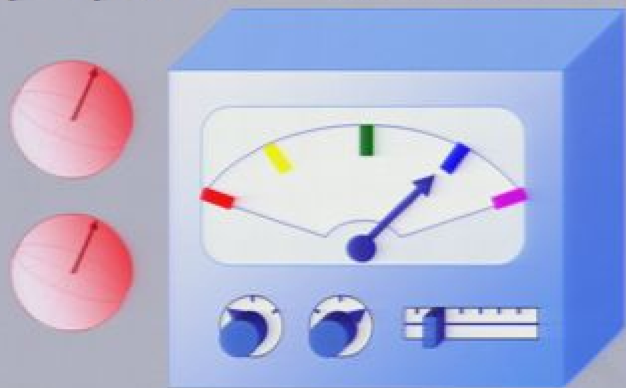
$$\hat{O}_{\pm} = \left|\pm\vec{m}\right\rangle\left\langle\pm\vec{m}\right| \quad ;\lambda = \pm 1$$

$$\hat{O}_{\mp} = \left|\mp\vec{k}\right\rangle\left\langle\mp\vec{k}\right| \quad ;\lambda = \mp 1$$

Average fidelity is the same, though not optimal.

Coding of direction: Two qubits

One preparator: simultaneous measurements of two qubits:



$$= (0,0,1)$$

$$= \left(\frac{\sqrt{8}}{3}, 0, -\frac{1}{3}\right)$$

$$= \left(\frac{-\sqrt{2}}{3}, \frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$$

$$= \left(\frac{-\sqrt{2}}{3}, \frac{-\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$$

$$\hat{P}_j = \left|\Phi_j\right\rangle_{12} \left\langle \Phi_j \right| \quad ; \sum_j \hat{P}_j = \hat{I}$$

$$\left|\Phi_j\right\rangle_{12} = \frac{\sqrt{3}}{2}\left|\vec{n}_j;\vec{n}_j\right\rangle_{12} + \frac{1}{2}\left|\psi^s\right\rangle_{12}$$

- Four vectors point to the four vertices of the tetrahedron
- Mean fidelity of estimation

$$F = \frac{3}{4} = 0.75$$

Massar & S Popescu: Phys. Rev. Lett 74 1259 (1995)

Gisin & Popescu: anti-parallel qubits

Two preparators & simultaneous measurements of two qubits:

$$\hat{P}_j = \left|\Theta_j\right\rangle_{12}\left\langle\Theta_j\right| \quad ; \sum_j \hat{P}_j = \hat{I}$$

$$\left|\Theta_j\right\rangle_{12} = \alpha\left|\vec{n}_j;-\vec{n}_j\right\rangle_{12} - \beta\sum_{k\neq j}\left|\vec{n}_k;-\vec{n}_k\right\rangle_{12}$$

$$_1 = (0,0,1)$$

$$_2 = \left(\frac{\sqrt{8}}{3},0,-\frac{1}{3}\right)$$

$$_3 = \left(\frac{-\sqrt{2}}{3},\frac{\sqrt{2}}{\sqrt{3}},-\frac{1}{3}\right)$$

$$_4 = \left(\frac{-\sqrt{2}}{3},\frac{-\sqrt{2}}{\sqrt{3}},-\frac{1}{3}\right)$$

$$= 1.095; \beta = 0.129$$

- Mean fidelity of estimation

$$F = \frac{5\sqrt{3}+33}{3\left(3\sqrt{3}-1\right)^2} \approx 0.789$$

Gisin & S.Popescu: ***Phys. Rev. Lett.*** **83, 432 (1999)**

Massar: ***Phys. Rev. A*** **62, 040101R (2000)**

Generalization of GP by Bagan et al.

BBBM have shown that if one is allowed to perform only space rotations on 2 qubits then the best fidelity one can achieve is the one obtained in the case of two anti-parallel spins.

N	1	2	3	4	5	6
$\vert\vec{n}\rangle^{\otimes N}$	0.666	0.750	0.800	0.833	0.855	0.875
GP-BBBM	0.666	0.789	0.845	0.911	0.931	0.943

$$F_{GP-BBBM} \sim 1 - \frac{\xi^2}{N^2}$$

$$F_{\vert\vec{n}\rangle^{\otimes N}} \sim 1 - \frac{1}{N}$$

$$\xi \sim 2.4$$

Bagan, M Baig, A Brey, & R Munoz-Tapia: Phys. Rev. Lett 85, 5230 (2000)

Can we do better?

Assume just one single-qubit preparator $|\vec{n}\rangle$

Nevertheless can we encode information in two qubits better than GP?

Compression of quantum information

just one single-qubit preparator $\left|\vec{n}\right\rangle$

Nevertheless can we encode information in two qubits better than GP?

$$\left|\Phi\left(\vec{n}\right)\right\rangle_{23}$$

Quantum compression: Three qubits

$$\left|\vec{n}\right\rangle^{\otimes 3} = \left(\alpha\left|0\right\rangle + \beta\left|1\right\rangle\right)^{\otimes 3}$$
$$= \left[\alpha^3\left|3;0\right\rangle_{123} + \sqrt{3}\alpha^2\beta\left|3;1\right\rangle_{123} + \sqrt{3}\alpha\beta^2\left|3;2\right\rangle_{123} + \beta^3\left|3;3\right\rangle_{123}\right]$$

- **dimension of the Hilbert space of 3 qubits = 8**
- **dimension of symmetric subspace = 4**
- **basis vectors of symmetric subspace** $\left|N;k\right\rangle$
- **can we map symmetric subspace of 3 qubits on 2-qubit HS?**

"Position" coding

$$|3;0\rangle_{123} \rightarrow |0\rangle_1 |00\rangle_{23}$$
$$|3;1\rangle_{123} \rightarrow |0\rangle_1 |01\rangle_{23}$$
$$|3;2\rangle_{123} \rightarrow |0\rangle_1 |10\rangle_{23}$$
$$|3;3\rangle_{123} \rightarrow |0\rangle_1 |11\rangle_{23}$$

$$|\vec{n}\rangle_{123}^{\otimes 3} \rightarrow |0\rangle_1 \left(\alpha^3 |00\rangle_{23} + \sqrt{3}\alpha^2\beta|01\rangle_{23} + \sqrt{3}\alpha\beta^2|10\rangle_{23} + \beta^3|11\rangle_{23} \right)$$

- mapping of symmetric subspace of 3 qubits on 2-qubit HS?
- does there exist a *state-independent* unitary transformation?

Required unitary transformation

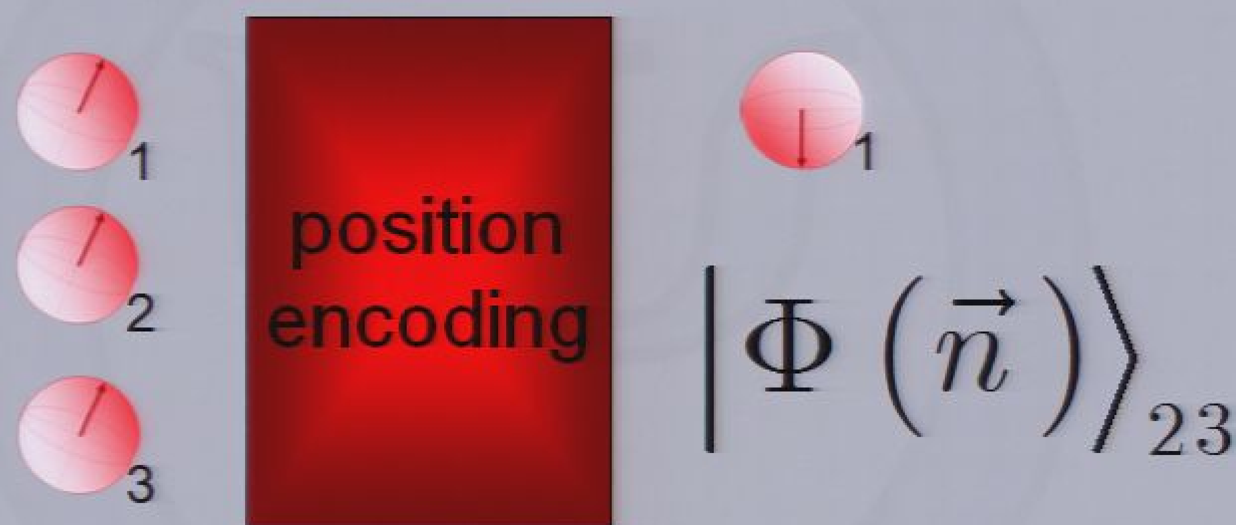
$$\begin{pmatrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & \sqrt{3} & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{pmatrix}$$

Required unitary transformation

$$
\begin{pmatrix} \alpha^3 \\ \sqrt{3}\alpha^2\beta \\ \sqrt{3}\alpha\beta^2 \\ \beta^3 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
= \frac{1}{\sqrt{6}}
\begin{pmatrix}
\sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{2} & \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{2} & 0 & \sqrt{2} & \sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\
0 & \sqrt{3} & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & -\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & -2 & 0
\end{pmatrix}
\begin{pmatrix} \alpha^3 \\ \alpha^2\beta \\ \alpha^2\beta \\ \alpha\beta^2 \\ \alpha^2\beta \\ \alpha\beta^2 \\ \alpha\beta^2 \\ \beta^3 \end{pmatrix}
$$

Efficient two-qubit encoding

$$\left|\Phi(\vec{n})\right\rangle_{23} = \left(\alpha^3\left|00\right\rangle_{23} + \sqrt{3}\alpha^2\beta\left|01\right\rangle_{23} + \sqrt{3}\alpha\beta^2\left|10\right\rangle_{23} + \beta^3\left|11\right\rangle_{23}\right)$$



Mean fidelity: two-qubit decoding

$$F = \frac{4}{5} = 0.8 > F_{GP} = 0.789 > F = 0.75$$

Generalization to N qubits

$$|\vec{n}\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\vec{n}\rangle^{\otimes N} = \sum_{k=0}^{N} \alpha^{N-k}\beta^{k} \sum_{P} P\left[|0\rangle^{\otimes(N-k)}|1\rangle^{\otimes k}\right]$$

$$= \sum_{k=0}^{N} \alpha^{N-k}\beta^{k} \sqrt{\binom{N}{k}}|N;k\rangle$$

- transformation on symmetric subspace

$$|0\rangle^{\otimes N} \rightarrow |0\rangle^{\otimes N}$$

$$|N;k\rangle \rightarrow |0\rangle^{\otimes(N-\log(N+1))} \otimes |(k)_2\rangle$$

- $(k)_2$ is a binary representation of number of “ones” k

Position encoding of symmetric states

- first encode the symmetric state into the position of 1
- Second encode the position of 1 into the binary number
- Defined for every symmetric states
- Then specify transformation for non-symmetric states

$$|N;k\rangle$$

$$\downarrow$$

$$|\underbrace{000\ldots01}_{N-k}\underbrace{0\ldots000}_{k-1}\rangle$$

$$\downarrow$$

$$|0\rangle^{\otimes(N-\log N)}\otimes|(k)_2\rangle$$

- transformation on symmetric subspace is efficient – quadratic in *N*

$$F=\frac{2^N}{2^N+1}$$

Comparison of fidelities

N	1	2	3	4	5	6
$\|\vec{n}\rangle^{\otimes N}$	0.666	0.750	0.800	0.833	0.855	0.875
GP-BBBM	0.666	0.789	0.845	0.911	0.931	0.943
compression	0.666	0.800	0.889	0.941	0.970	0.992

Gisin & S.Popescu: *Phys. Rev. Lett.* 83, 432 (1999)
Massar: *Phys. Rev. A* 62, 040101R (2000)

Asymptotics

Position encoding of information about direction would allow for exponential asymptotics of the fidelity

$$F_{|\vec{n}\rangle^{\otimes N}} \sim 1 - \frac{1}{N}$$

$$F_{GP-BBBM} \sim 1 - \frac{\xi^2}{N^2} \qquad \xi \sim 2.4$$

$$F = \frac{2^N}{2^N + 1} = 1 - \frac{1}{2^N}$$

Black-box problem

- Having a black box (with no memory) processing one qubit in a time, how can we determine this channel?

W.Holstrom, *Quantum detection and estimation theory* (Academic Press, New York, 1976)
S.Holevo, *Probabilistic and statistical aspects of quantum theory* (North Holland, Amsterdam, 1982)
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General operations (maps, channels)

- The density operator $$\rho = \frac{1}{2}\left(1 + \vec{r}.\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} \leftrightarrow \vec{r} = (x, y, z)$$

- The general operation is an affine transformation of the Bloch sphere

$$\vec{r} \rightarrow \vec{r}' = T\vec{r} + \vec{t}$$

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ \vec{t} & \vec{T} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & \alpha_1 & \alpha_2 & \alpha_3 \\ y & \beta_1 & \beta_2 & \beta_3 \\ z & \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

$$\rho' = \varepsilon[\rho] = \sum_l A_l \rho \, A_l^\dagger$$

- With $x^2 + y^2 + z^2 \leq 1$ and $\forall j \quad (x - \alpha_j) + (y - \beta_j) + (z - \gamma_j) \leq 1$

Clickology: Maximum likelihood

- ML works with finite sets of data, not with infinite ensembles
- In case of quantum operations, the relevant data are
  - Input state specification ρ_i
  - Measurement direction $|\psi\rangle_i$
  - Measurement outcome (binary) p_i
- We build a functional

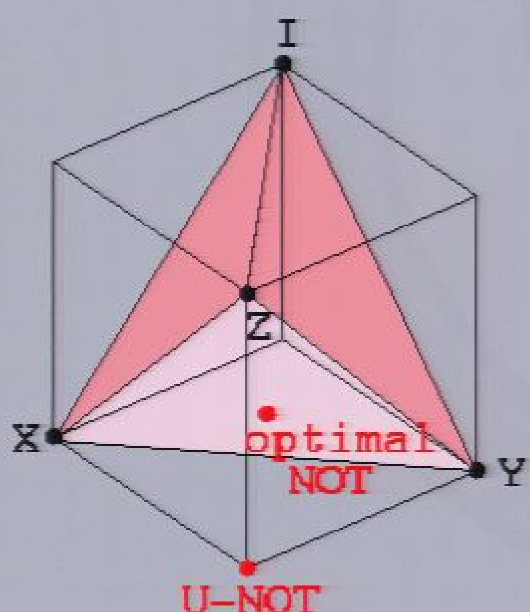
$$F = \prod_i \left[\langle\psi|_i \, \varepsilon(\rho_i) |\psi\rangle_i \, p_i + \left(1 - \langle\psi|_i \, \varepsilon(\rho_i) |\psi\rangle_i\right)(1 - p_i) \right]$$

- The numerical task is to find the ε, for which this functional reaches the maximum (using the logarithm of functional)
- Trace-preservation is obtained automatically from the parameterization, CP has to be checked in the algorithm

Unital operations

- Displacement $\vec{t}=0$
- Affine transformation specified as
- Positivity $\forall j \quad |\lambda_j| \le 1$
- Complete positivity $|\lambda_1 \pm \lambda_2| \le |1 \pm \lambda_3|$

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \vec{T} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}$$



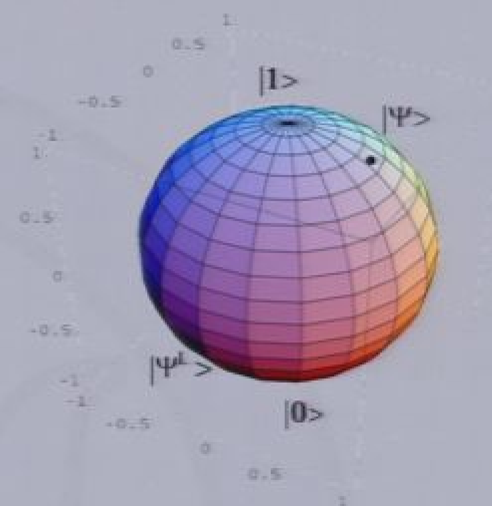
Unital CP maps are embeded in the set of all positive unital maps (cube). The CP maps form a tetrahedron with four unitary transformations in its corners (extremal points) I,x,y,z corresponding to the Pauli sigma-matrices.

The unphysical U-NOT operation $\lambda_1=\lambda_2=\lambda_3=-1$ and its optimal completely positive approximation quantum universal NOT gate $\lambda_1=\lambda_2=\lambda_3=-1/3$ are shown.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi^{\perp}\rangle = \beta^{*}|0\rangle - \alpha^{*}|1\rangle$$

Universal NOT Gate: Problem

$|\psi^{\perp}\rangle$ **is antipode of** $|\psi\rangle$



- Spin flipping is an inversion of the Poincare sphere
- This inversion preserves angels
- The Wigner theorem - spin flip is either unitary or anti-unitary operation
- Unitary operations are equal to proper rotations of the Poincare sphere
- Anti-unitary operations are orthogonal transformations with det=-1
- Spin flip operation is anti-unitary and is not CP
- In the unitary world the ideal universal NOT gate which would flip a qubit in an arbitrary (unknown) state does not exist

Measurement-based vs *Quantum Scenario*

Measurement-based scenario: optimally measure and estimate the state then on a level of classical information perform flip and prepare the flipped state of the estimate

Quantum scenario: try to find a unitary operation on the qubit and ancillas that at the output generates the best possible approximation of the spin-flipped state. The fidelity of the operation should be state independent (universality of the U-NOT)

Unital operations

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- Positivity $\forall j \quad |\lambda_j| \le 1$
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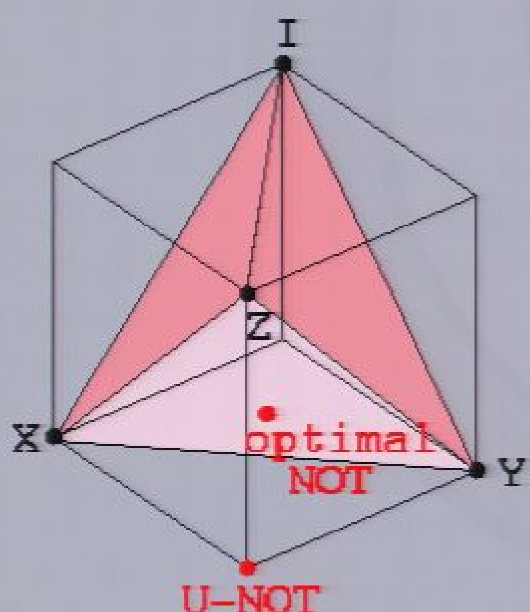

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Ziman & V.Buzek, PRA 72, 022110 (2005)

Theorem: Optimal Universal NOT Gate

Among all completely positive trace preserving maps

$$T : S\left(H_{+}^{\otimes N}\right) \to S\left(H\right)$$

The measurement-based U-NOT scenario attains the highest possible fidelity, namely

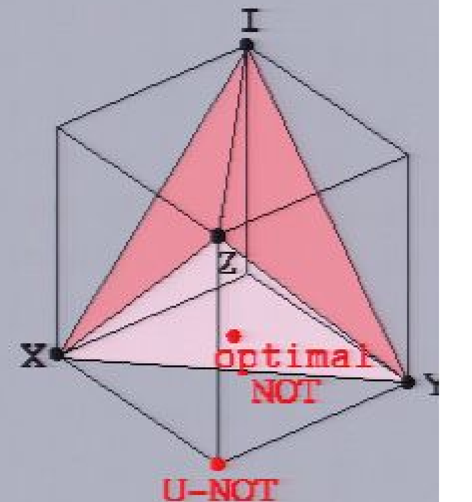
$$F = (N+1)/(N+2).$$

Bechmann-Pasquinucci and N.Gisin, Phys. Rev, A 59, 4238 (1999)
Bužek, M.Hillery, and R.F.Werner *Phys. Rev. A* 60, R2626 (1999)

Approximation of non-physical maps I

Universal NOT gate $\varepsilon = \mathrm{diag}\,(1,-1,-1,-1)$

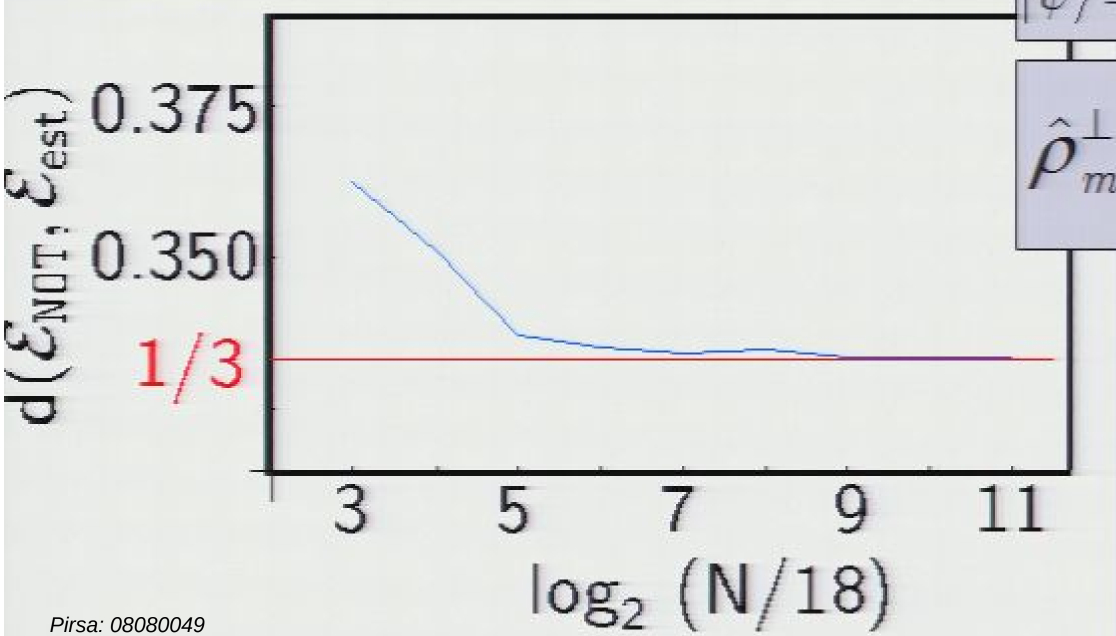
Best approximation $\varepsilon = \mathrm{diag}\,(1,-1/3,-1/3,-1/3)$

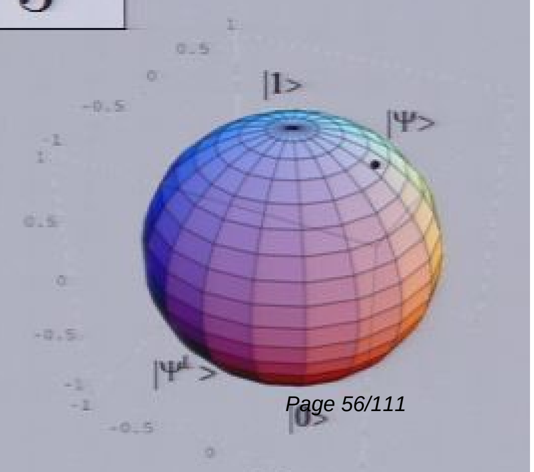


6 input states – eigenstates of σ_j
3 measurements σ_j
N=100 x 18 clicks

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi^{\perp}\rangle = \beta^{*}|0\rangle - \alpha^{*}|1\rangle$$

$$\hat{\rho}^{\perp}_{meas} = \frac{1}{3}\hat{\rho}^{\perp} + \frac{1}{3}\hat{I}$$



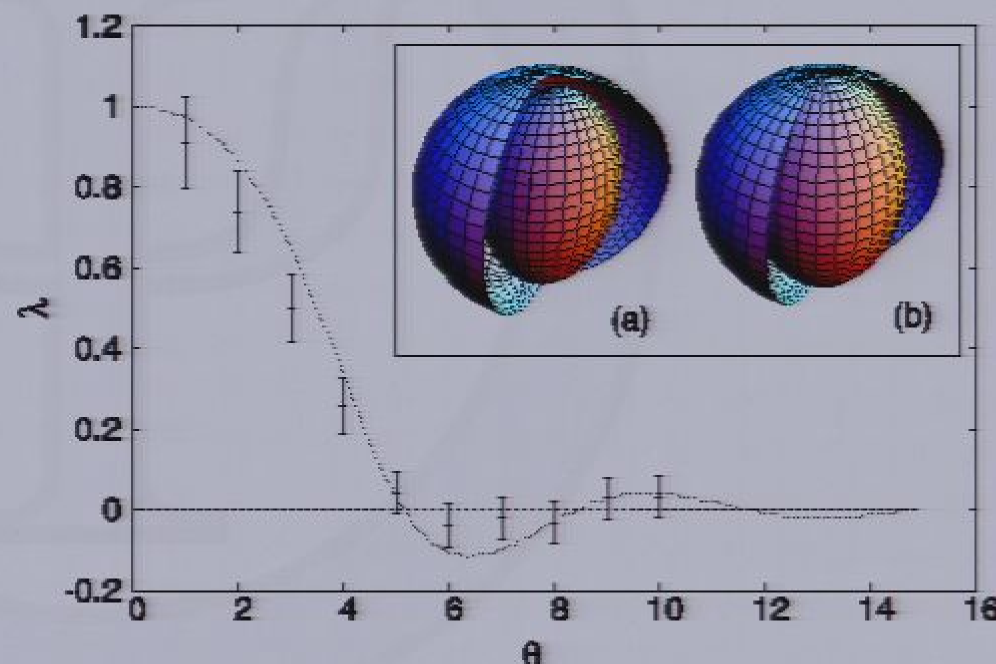


Approximation of non-physical maps II

- **Nonlinear polarization rotation**
- 1800 input states
- 3 measurements

$$\varepsilon[\rho] = \exp\left(i\frac{\Theta}{2}\langle\sigma_z\rangle_\rho \sigma_z\right)\rho \exp\left(-i\frac{\Theta}{2}\langle\sigma_z\rangle_\rho \sigma_z\right)$$

$$\boldsymbol{\varepsilon = \mathrm{diag}\,(1,\lambda,\lambda,1)}$$



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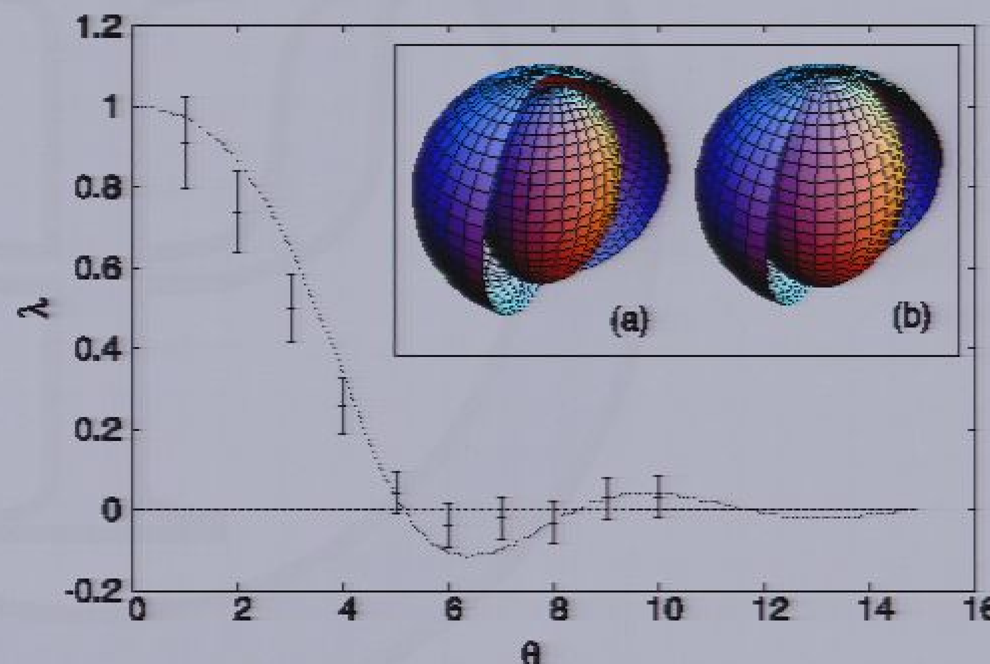

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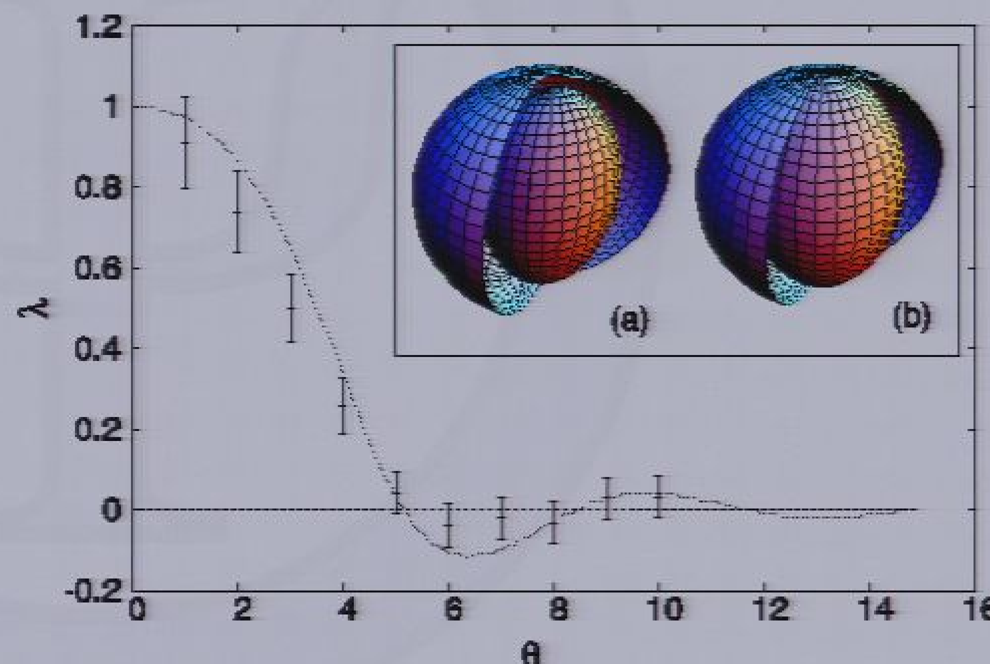

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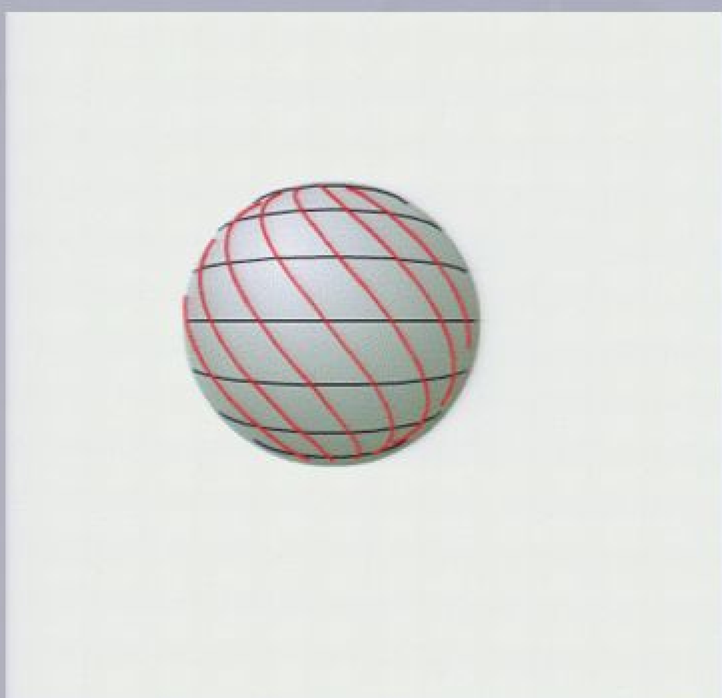

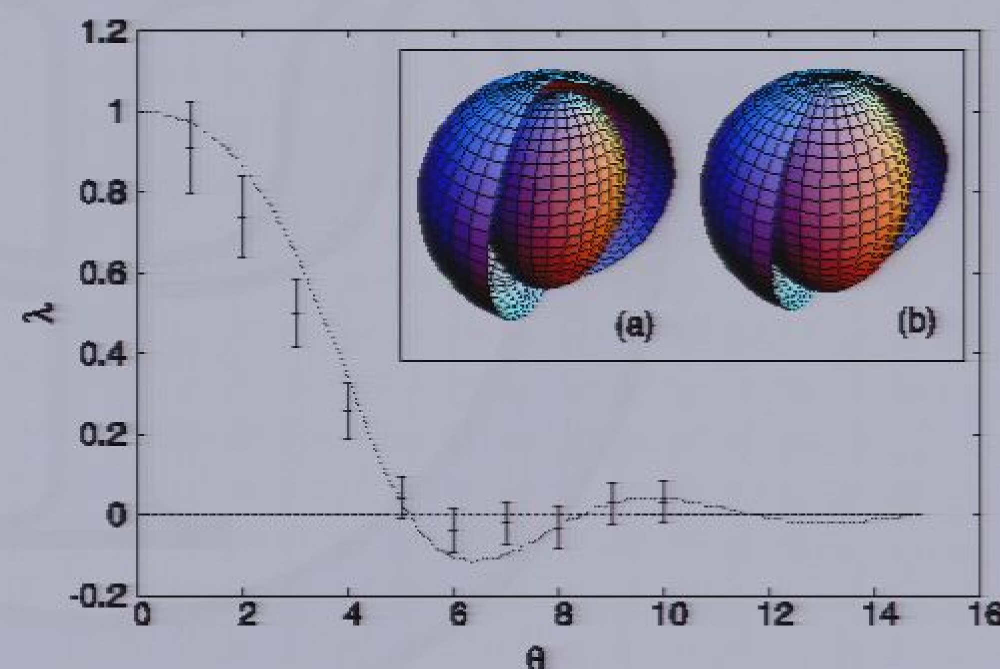

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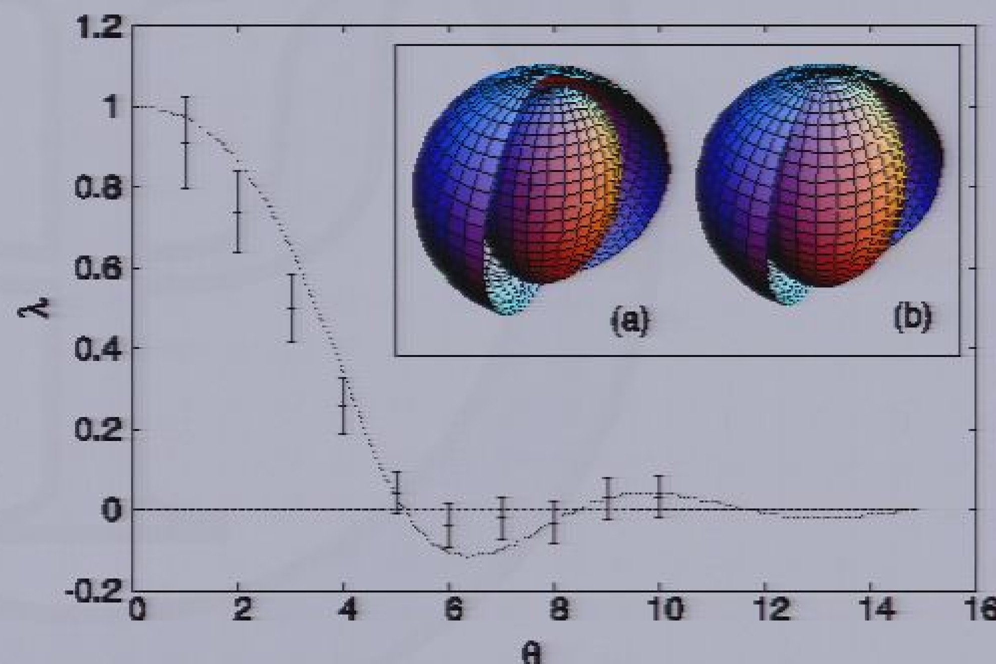

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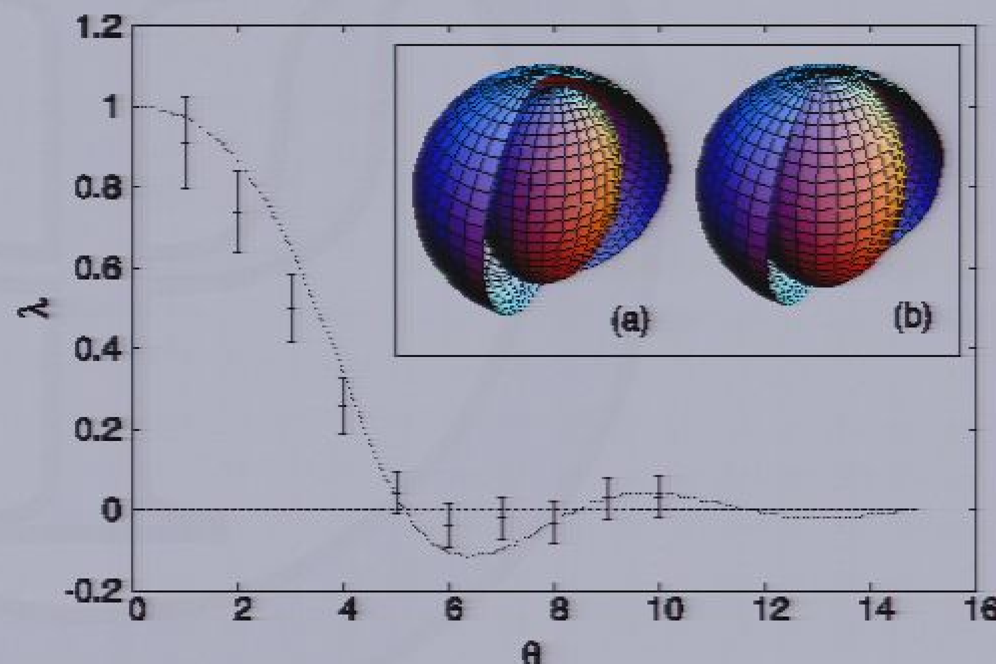

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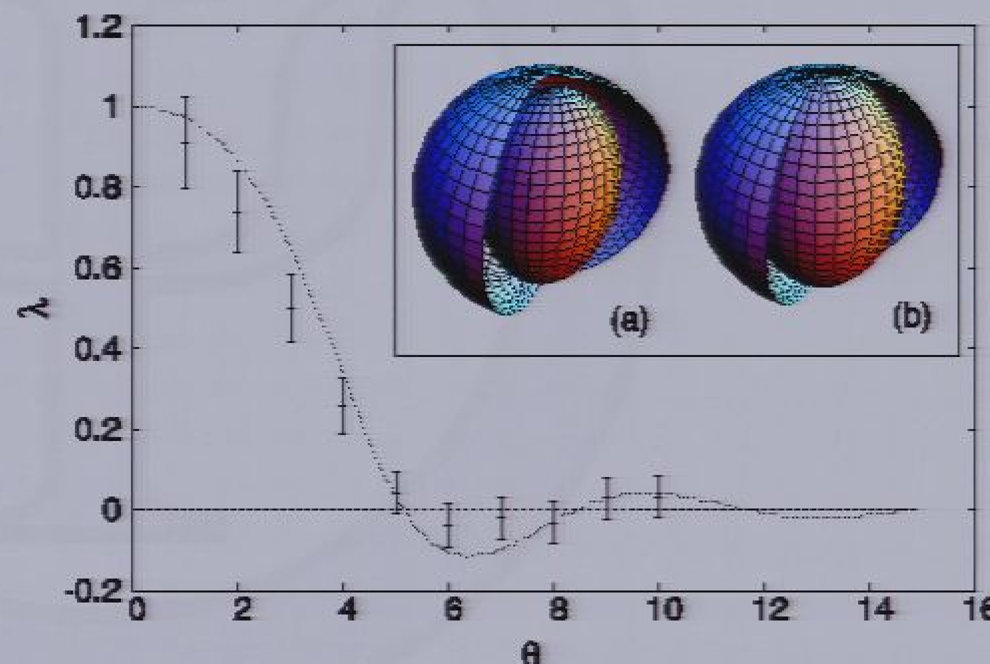

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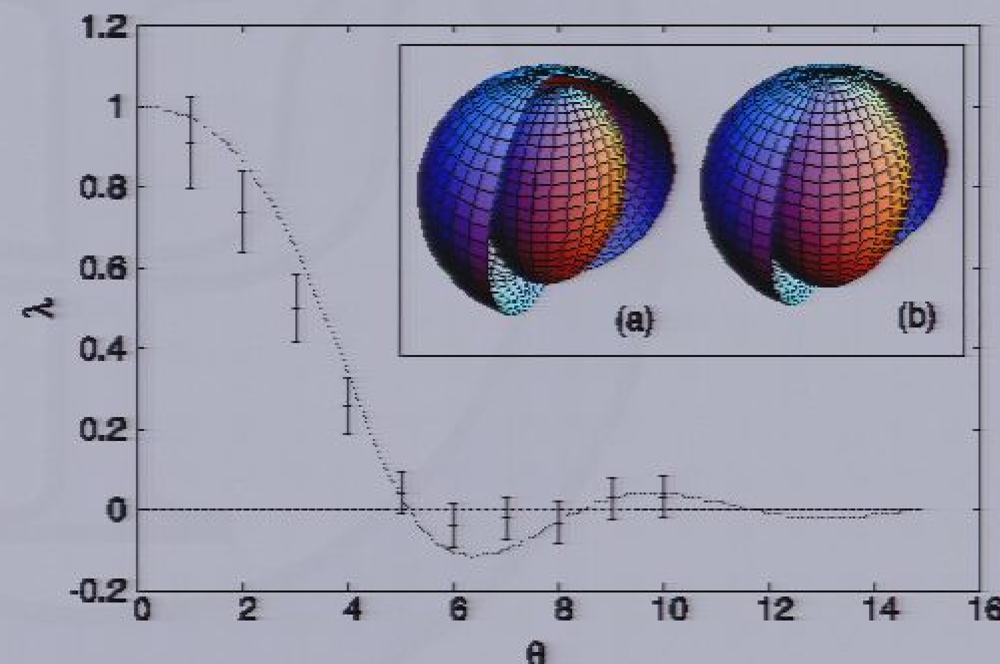

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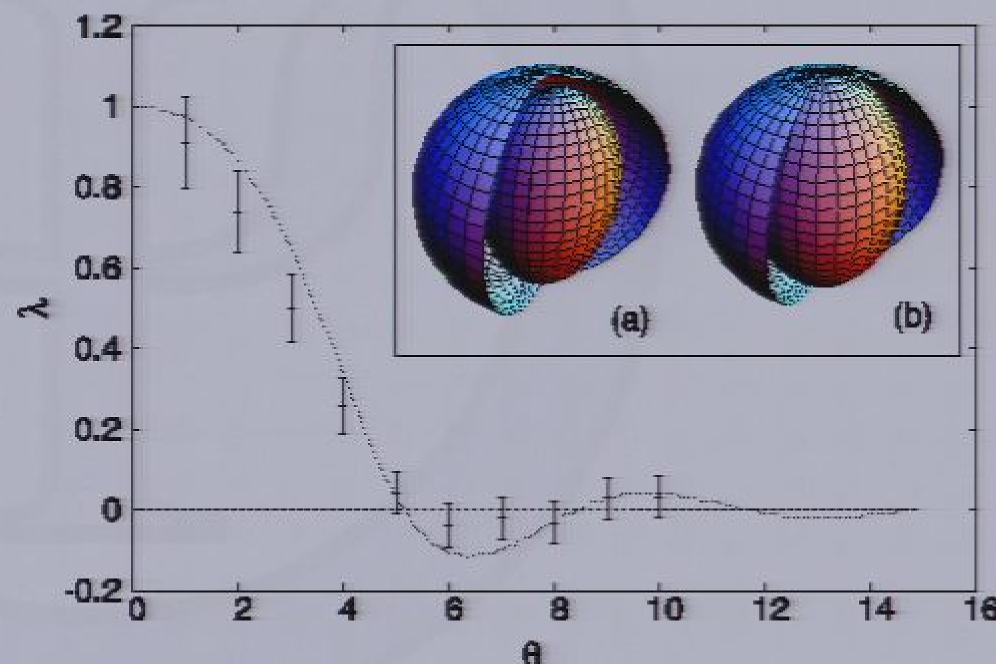

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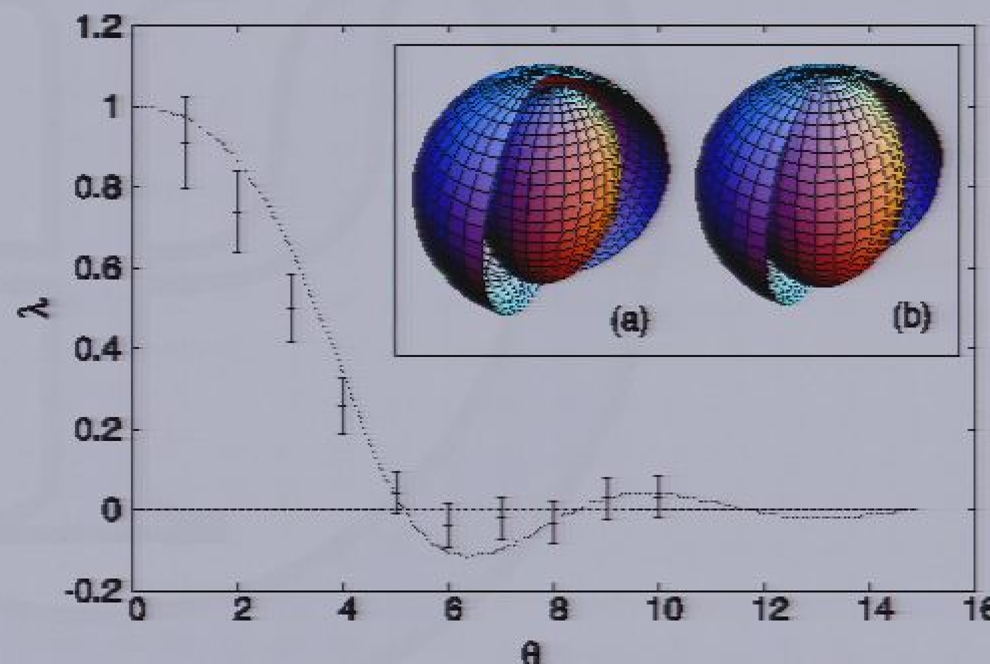

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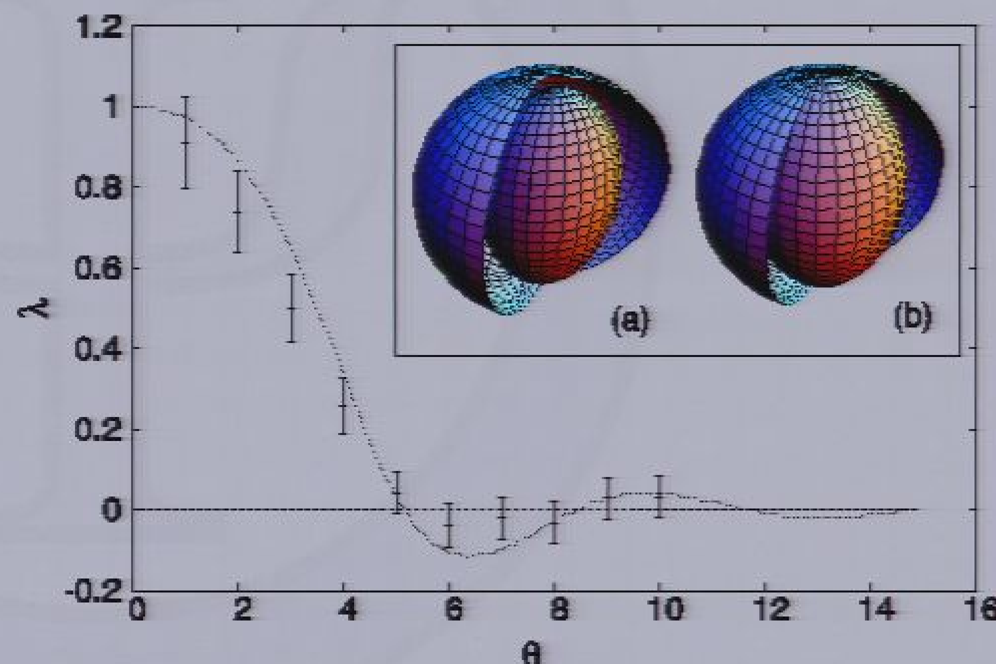

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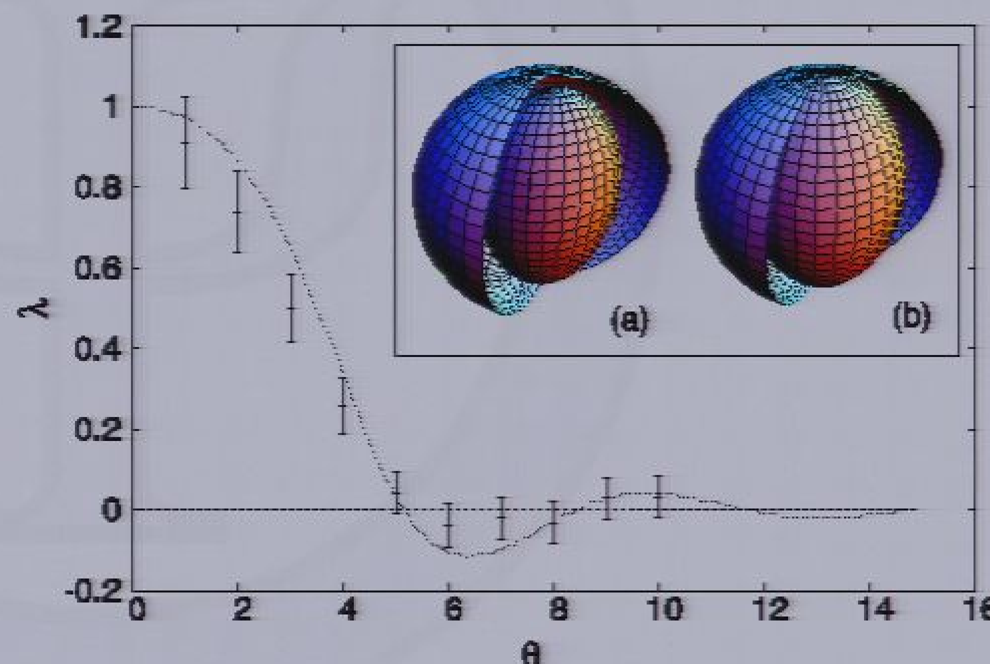

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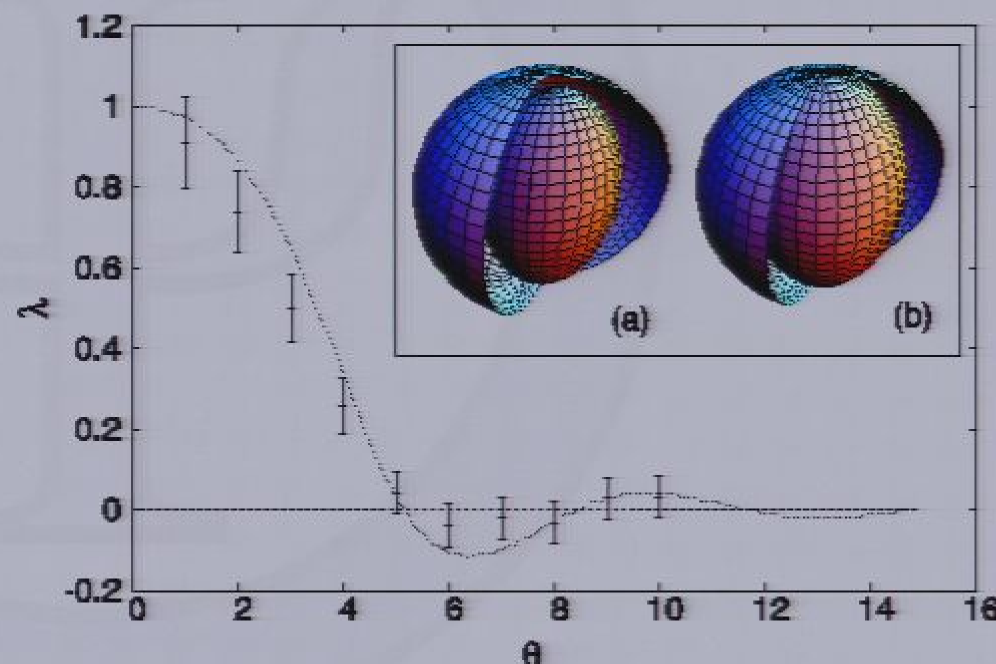

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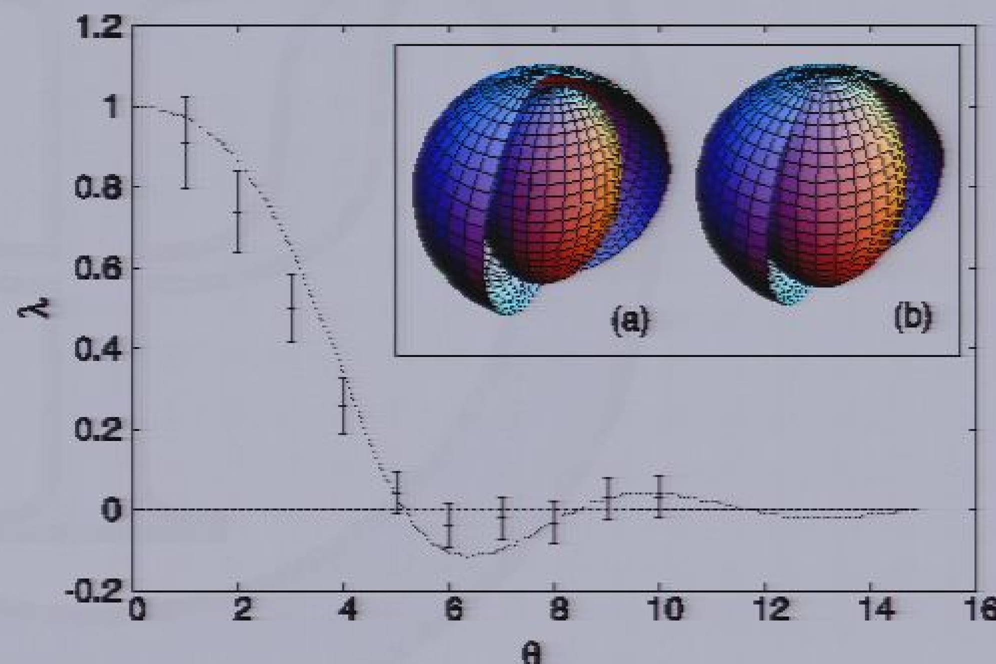

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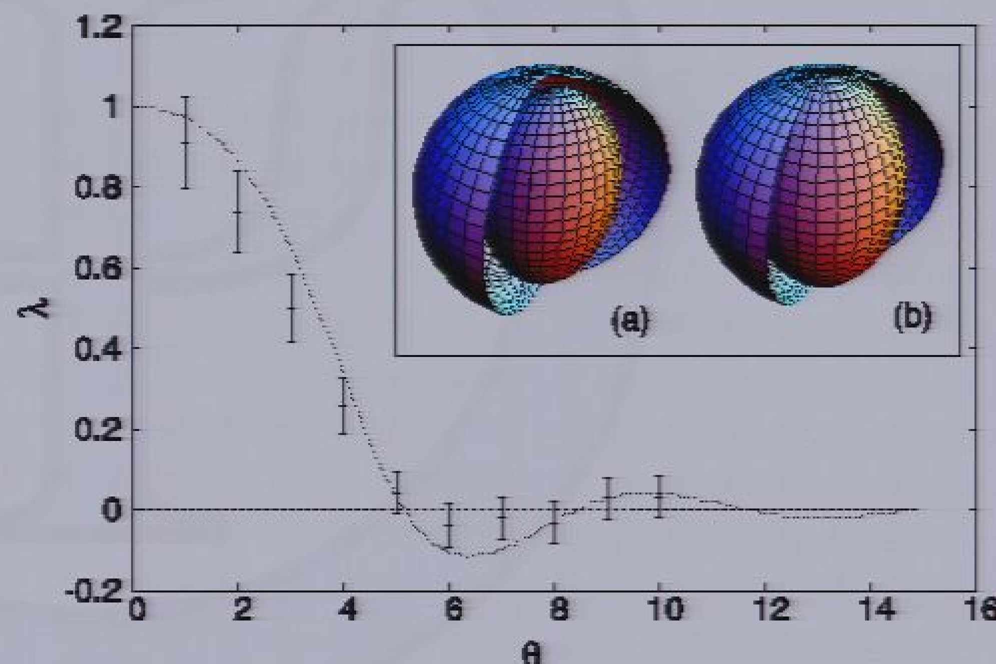

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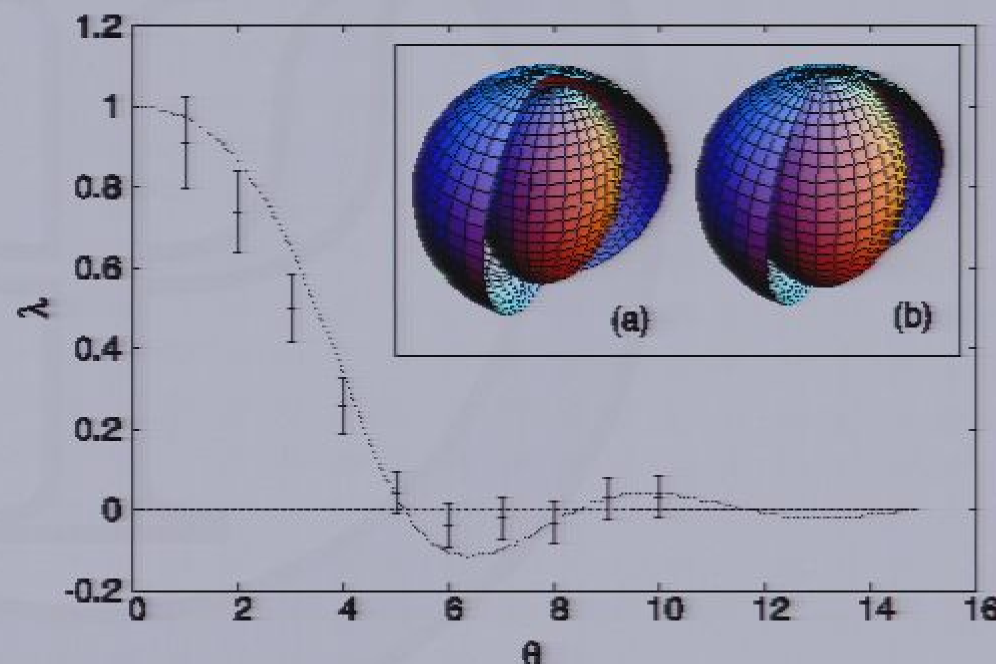

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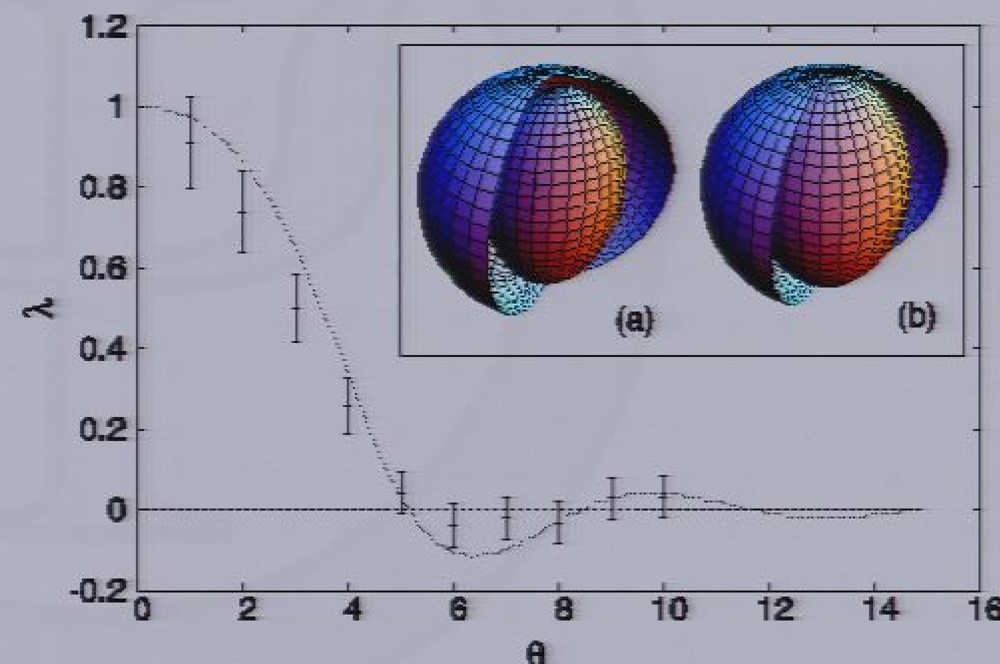

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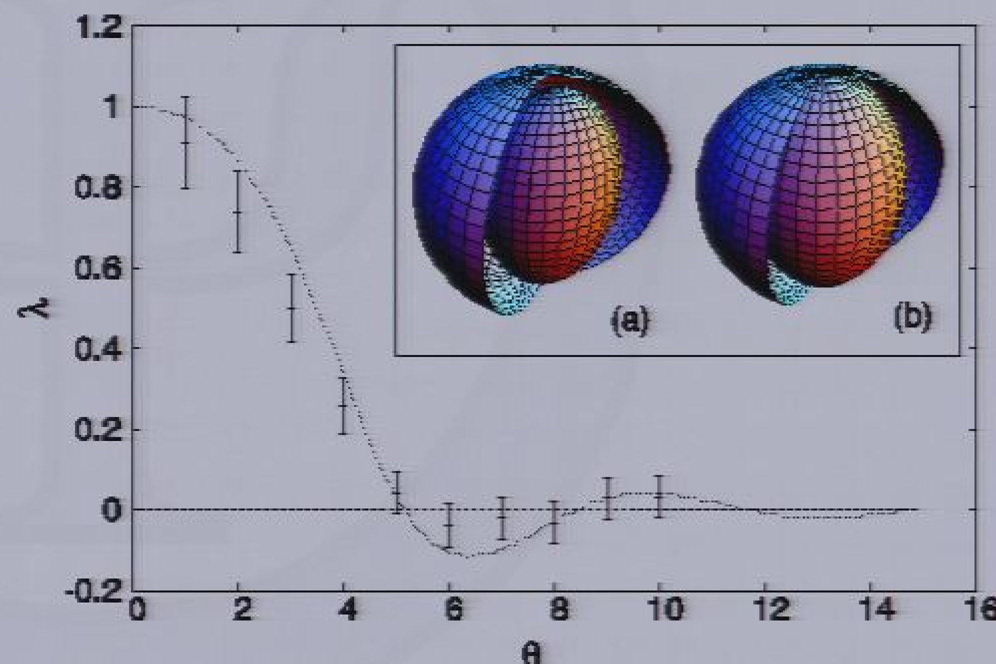

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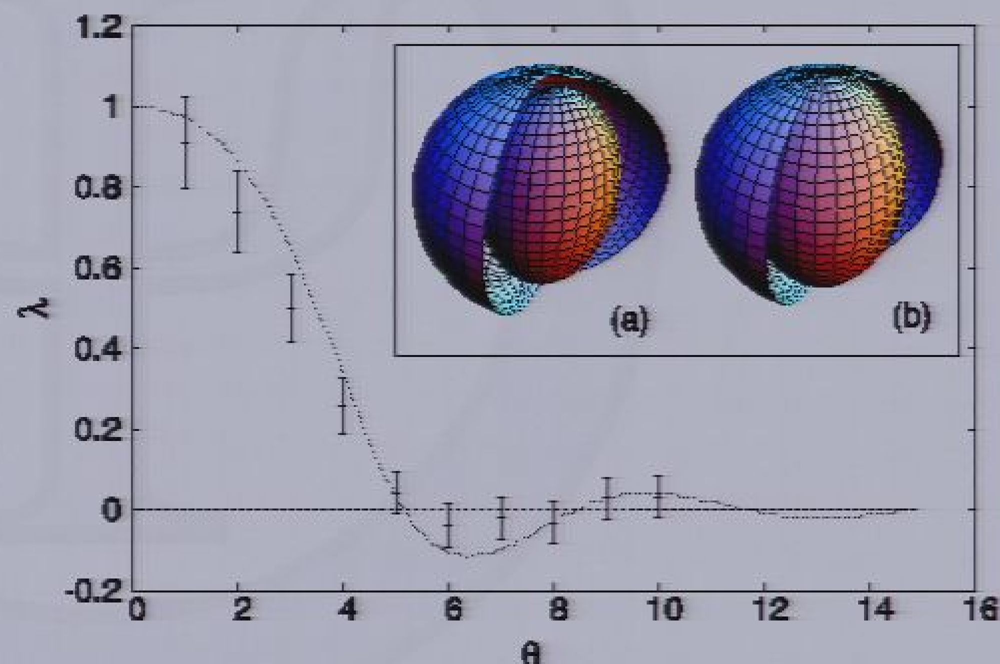

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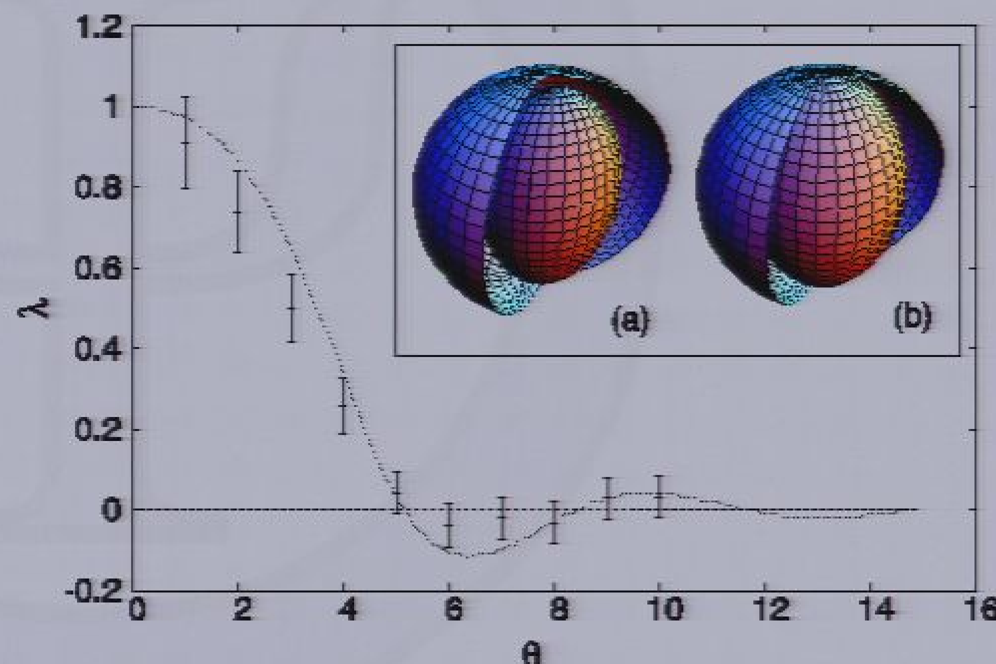

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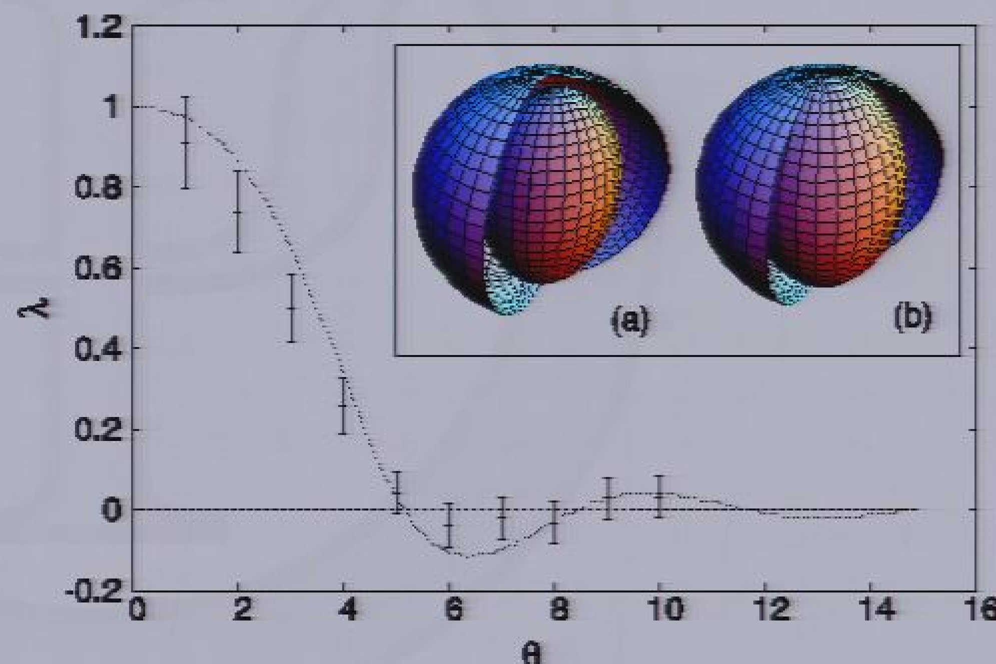

Approximation of non-physical maps II

- **Nonlinear polarization rotation**
- 1800 input states
- 3 measurements

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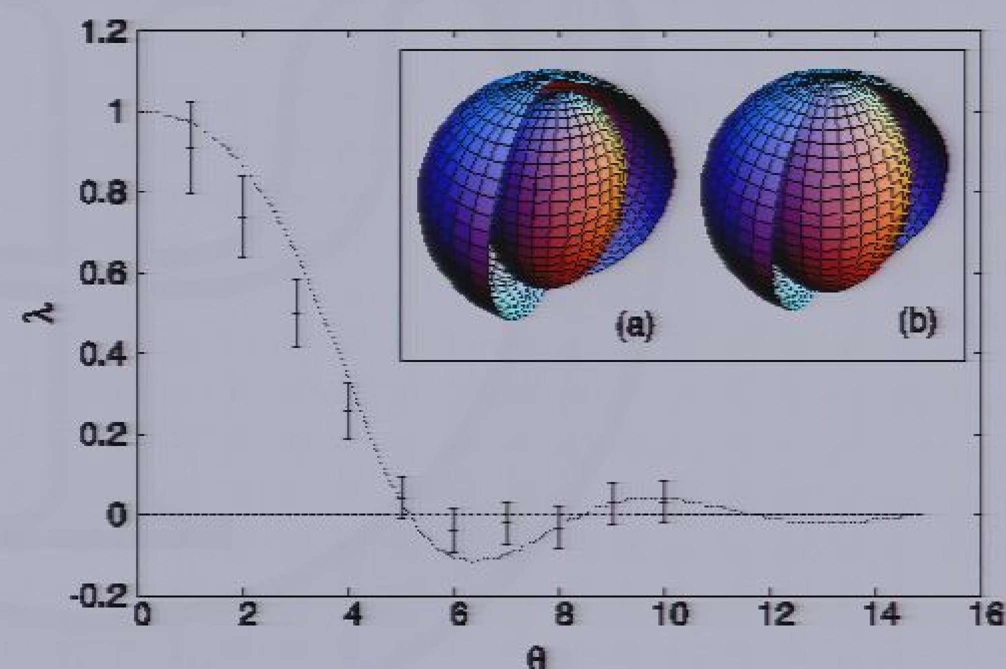

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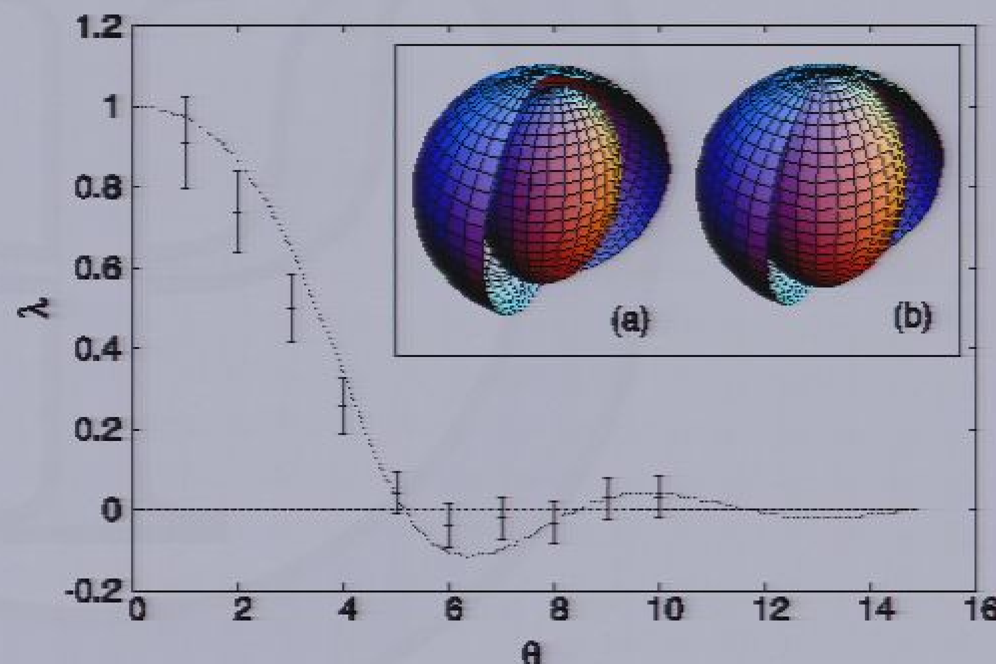

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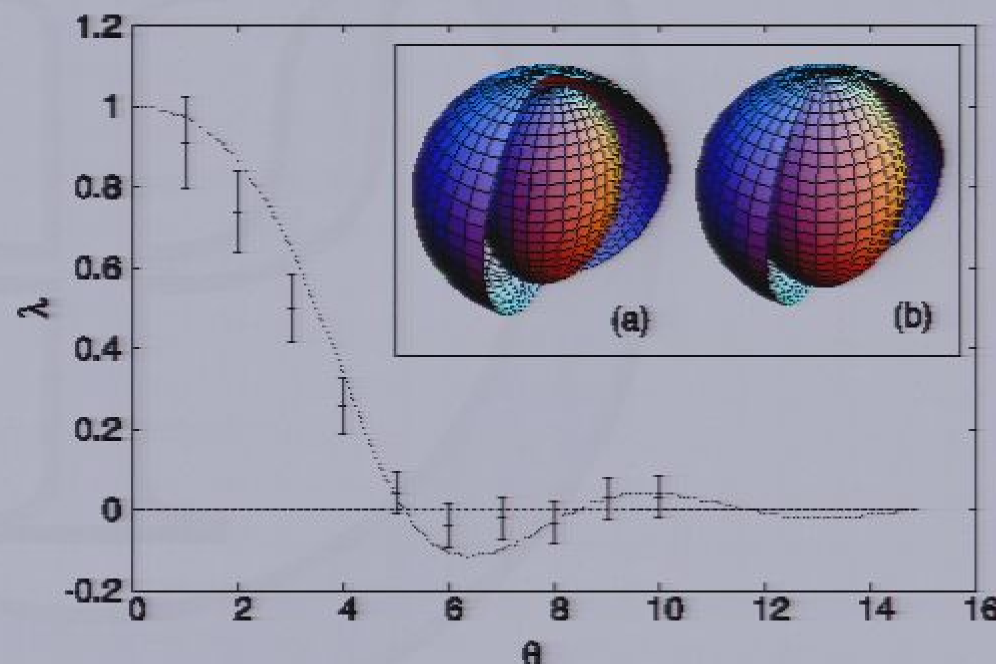

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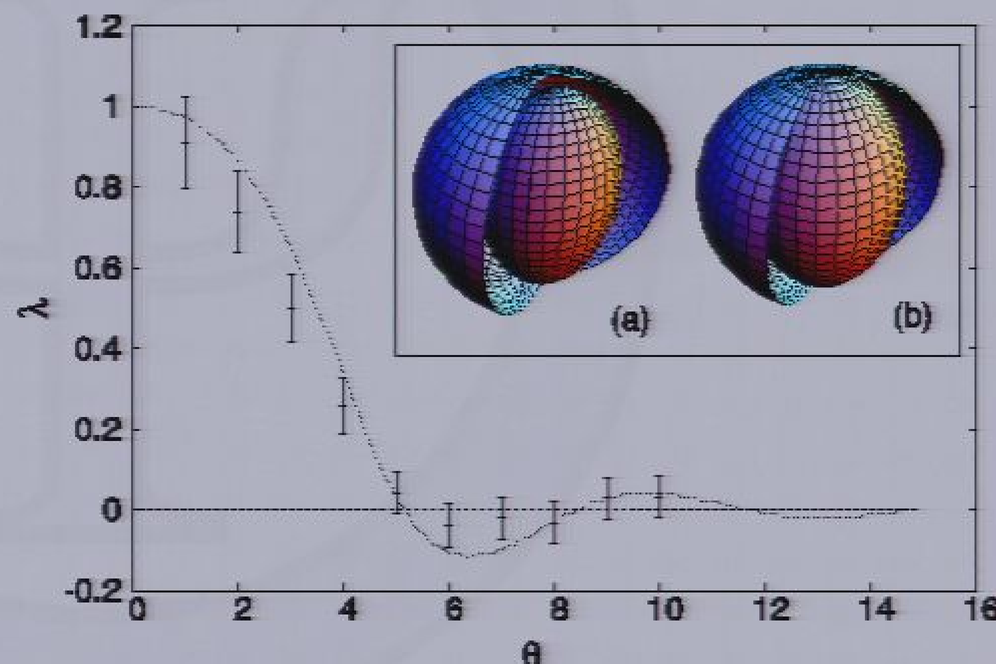

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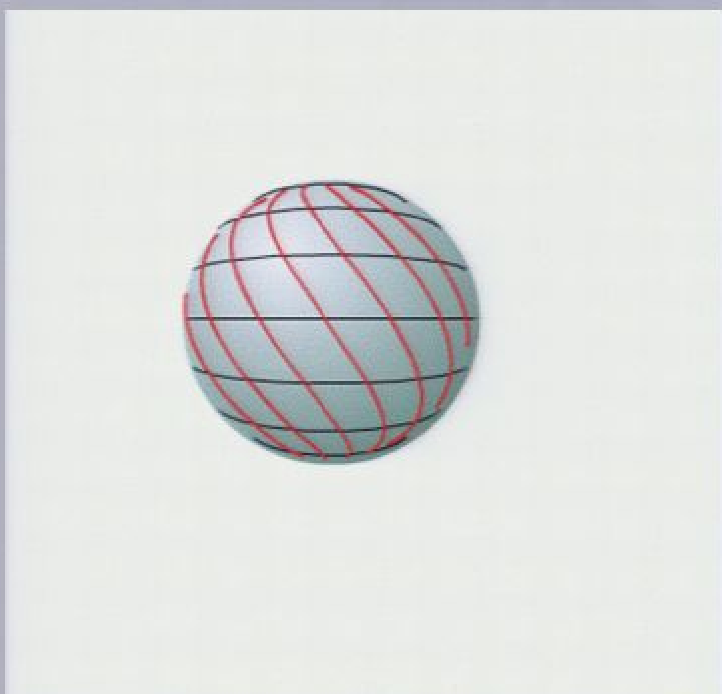

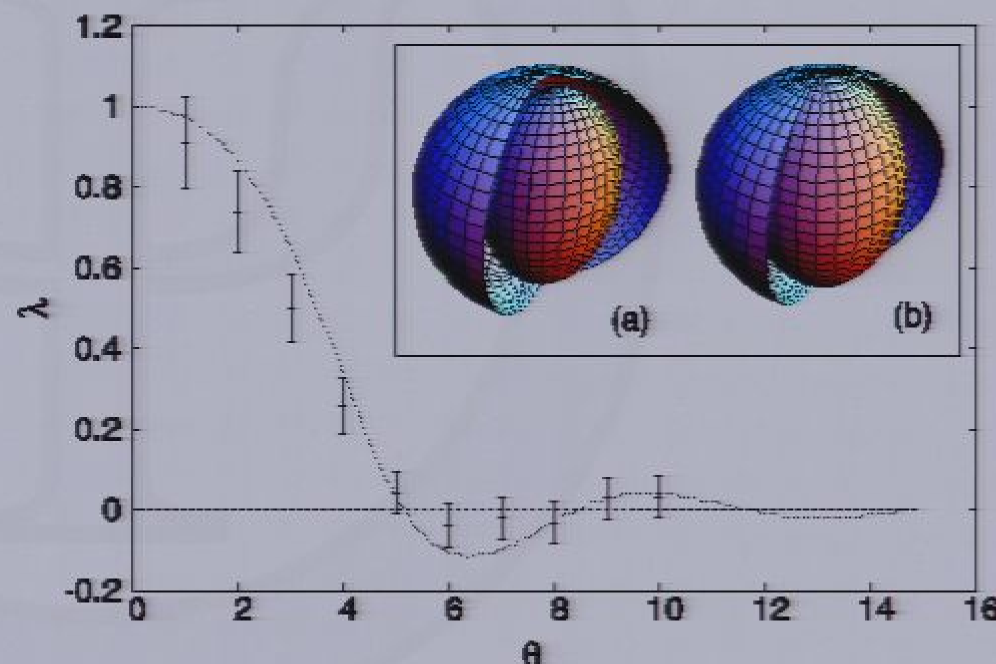

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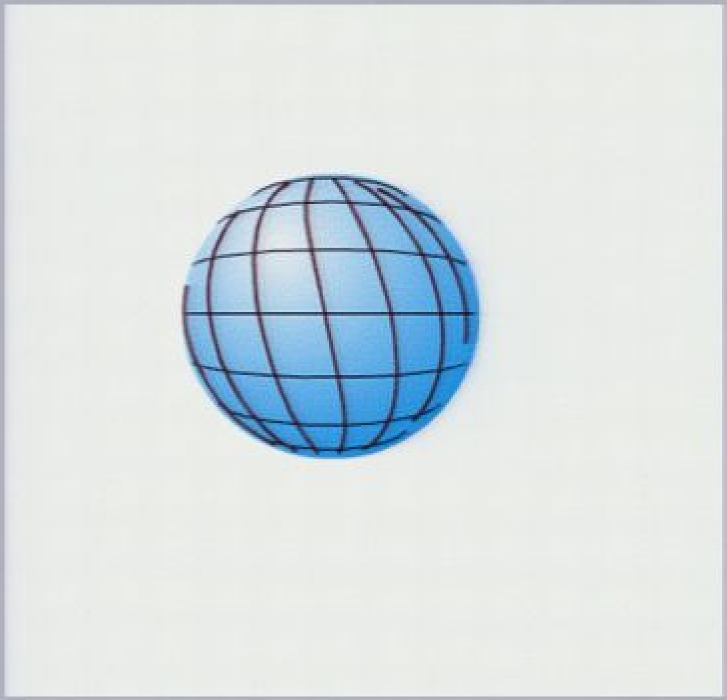

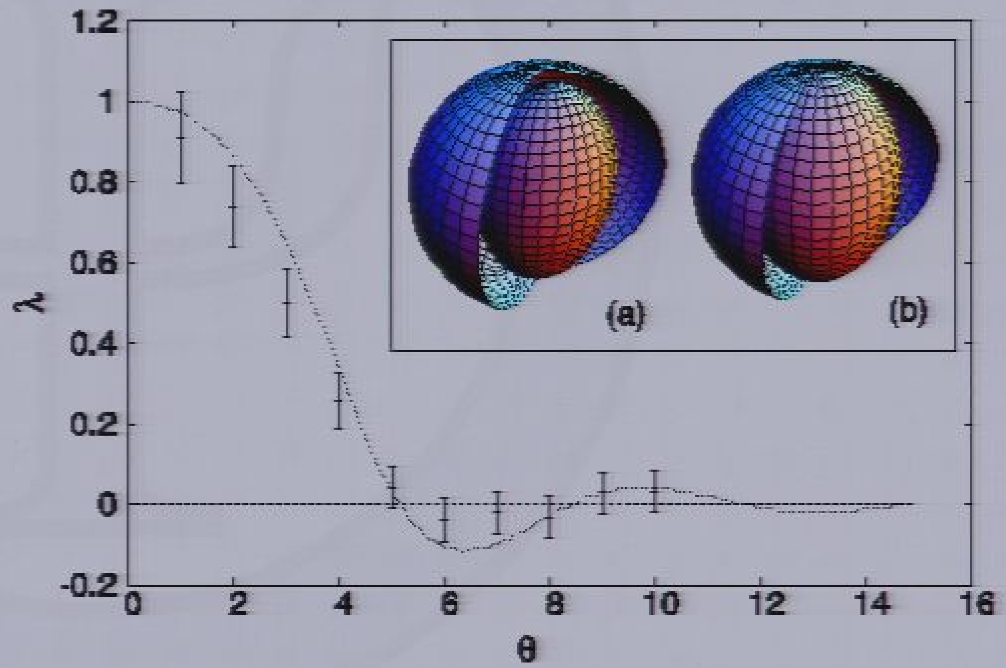

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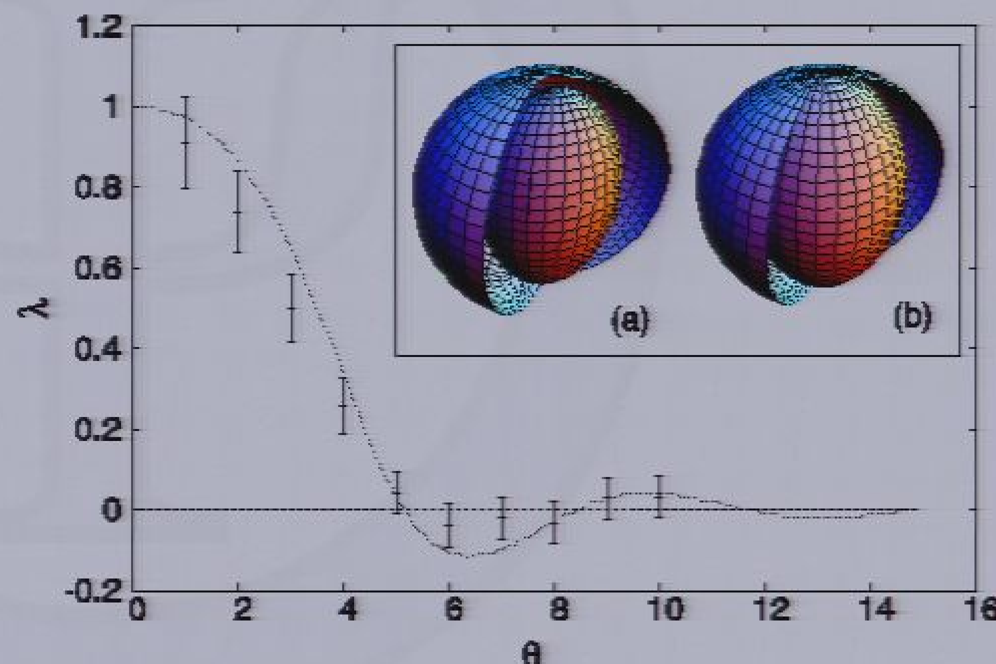

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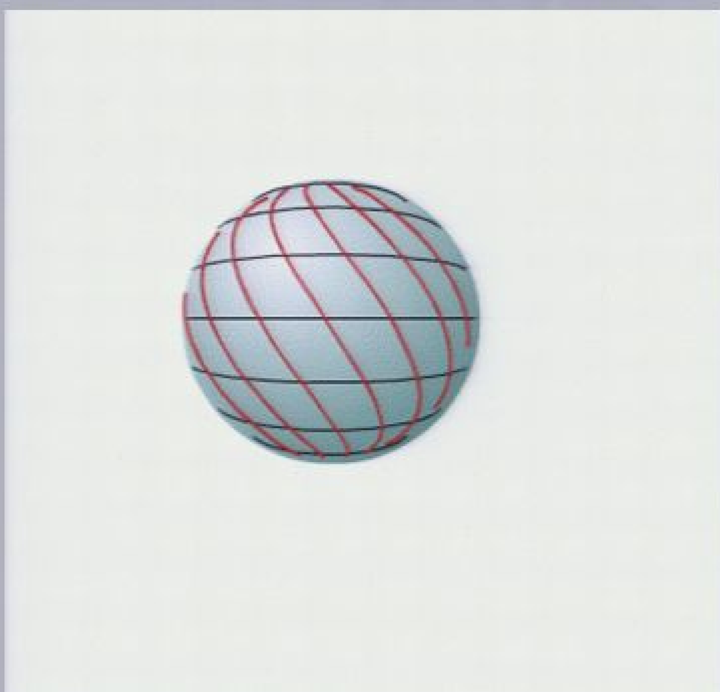

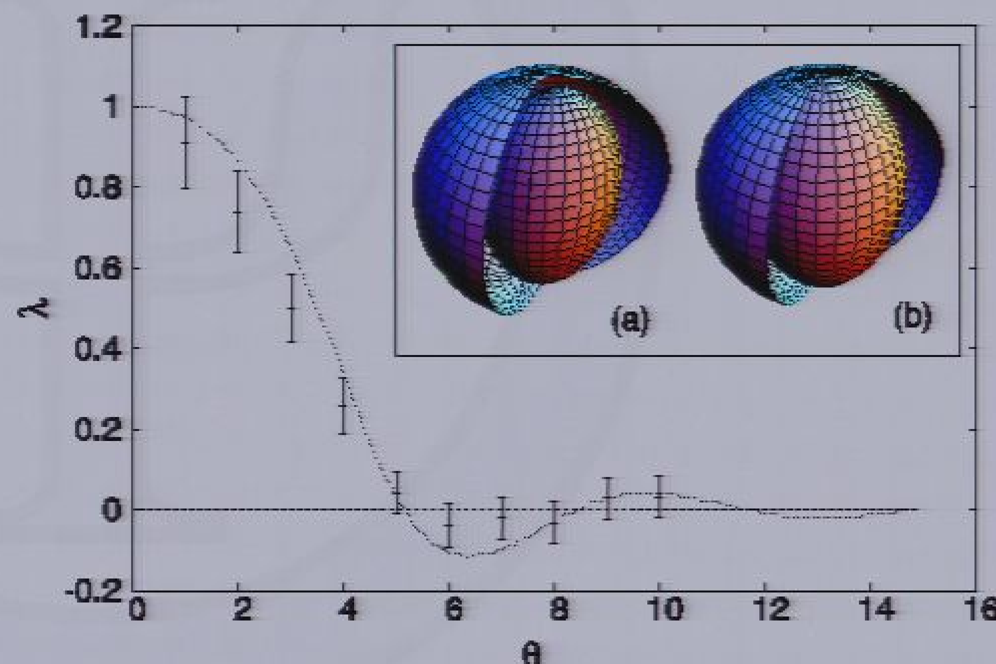

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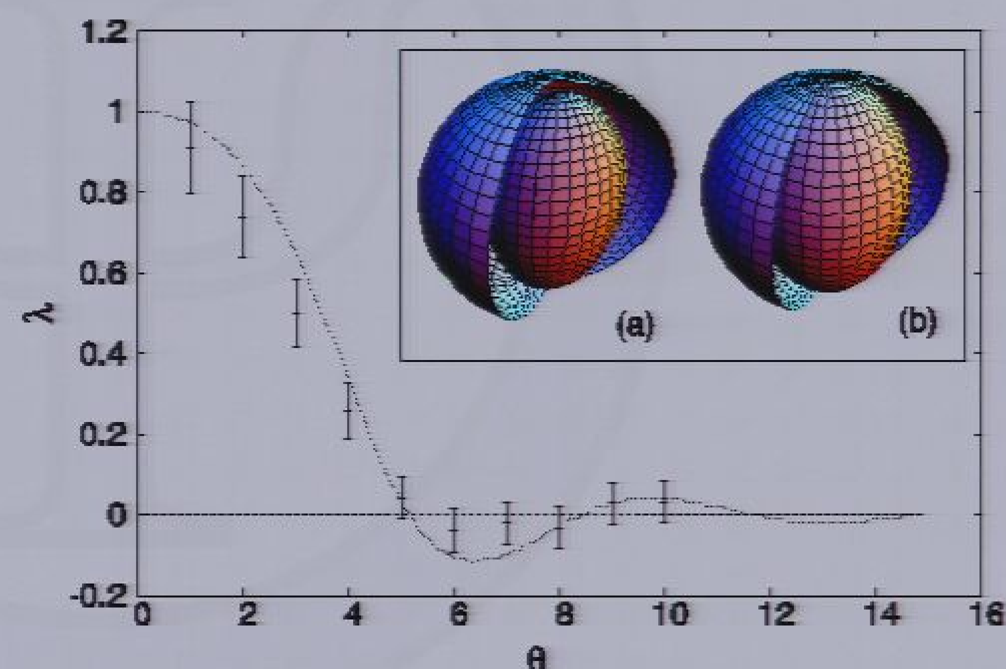

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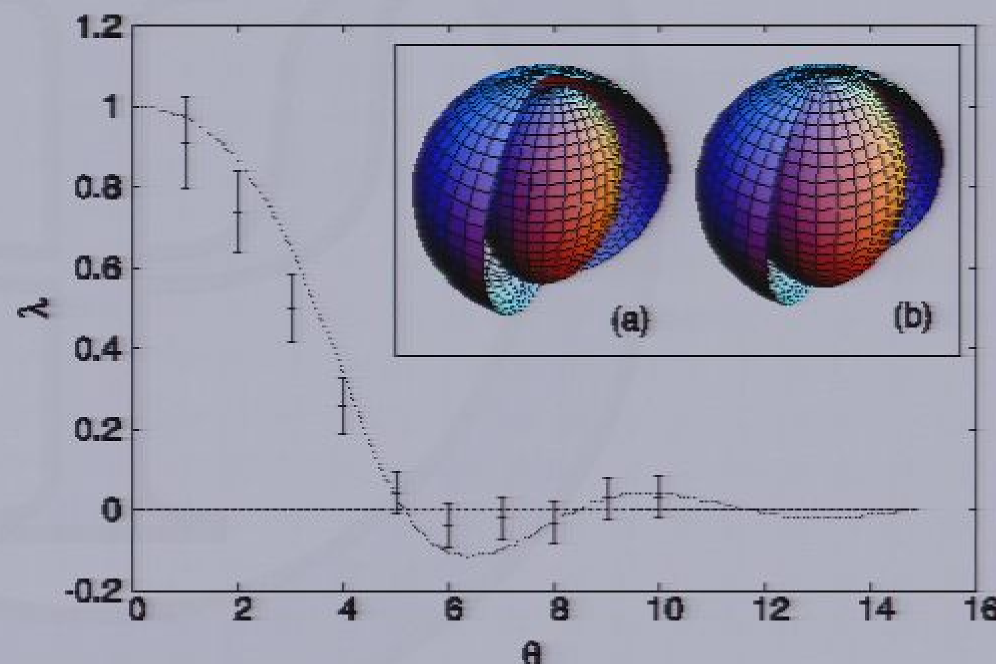

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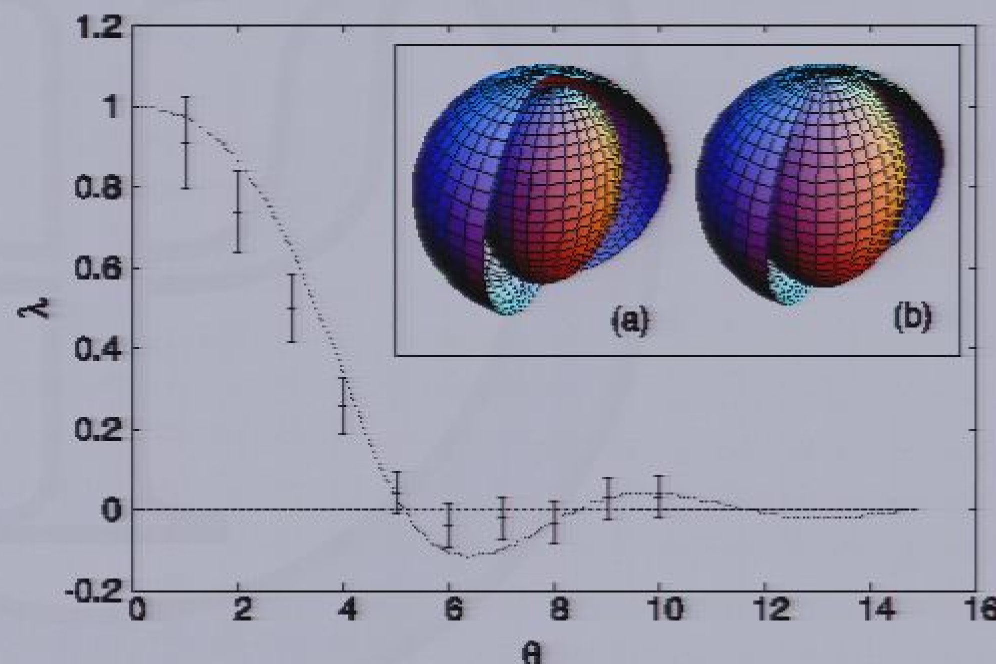

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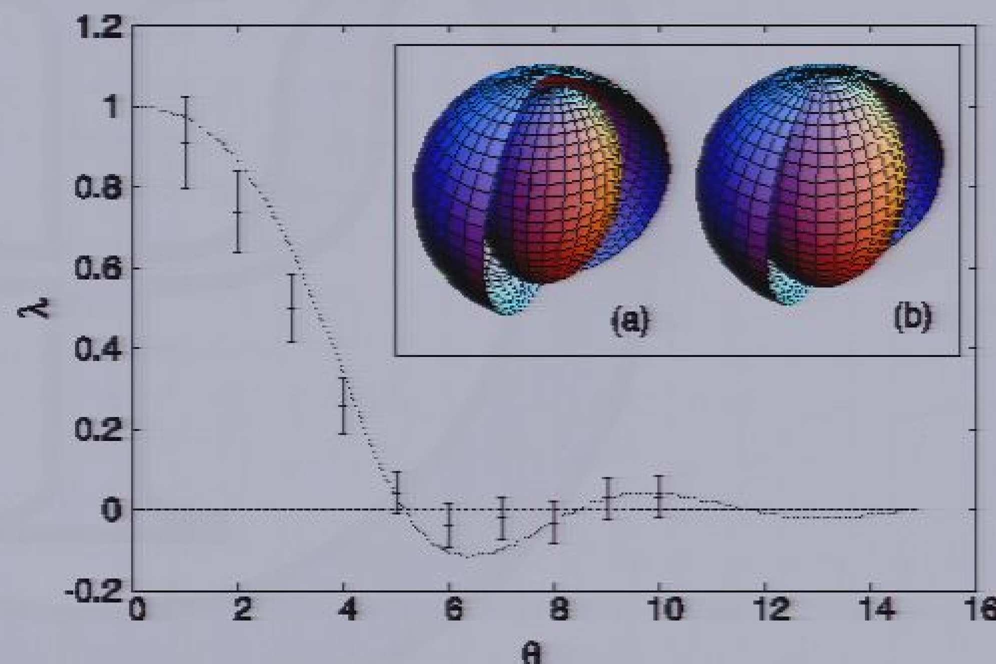

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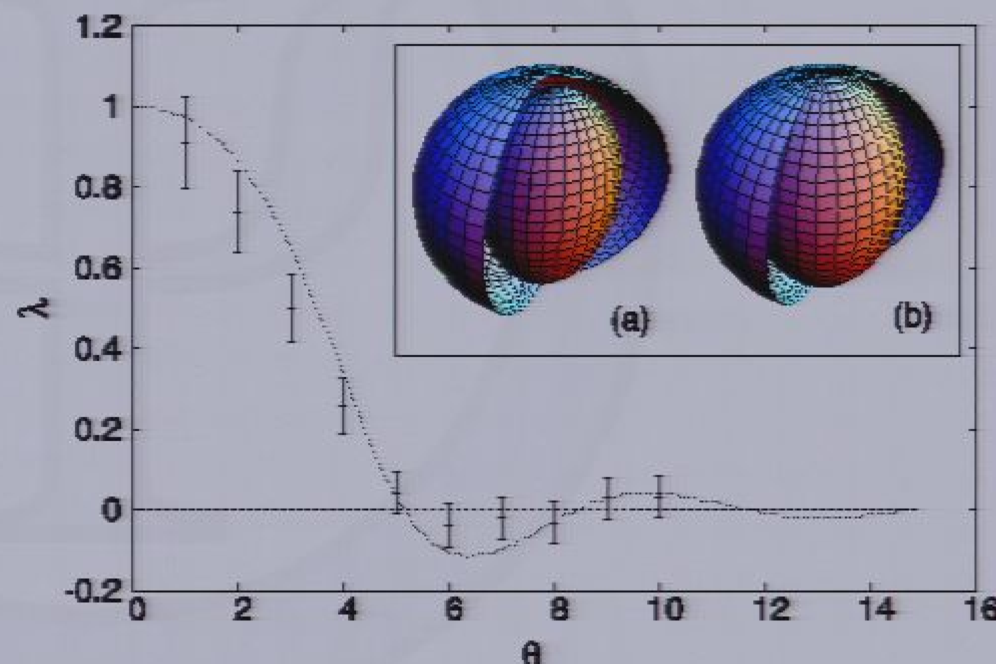

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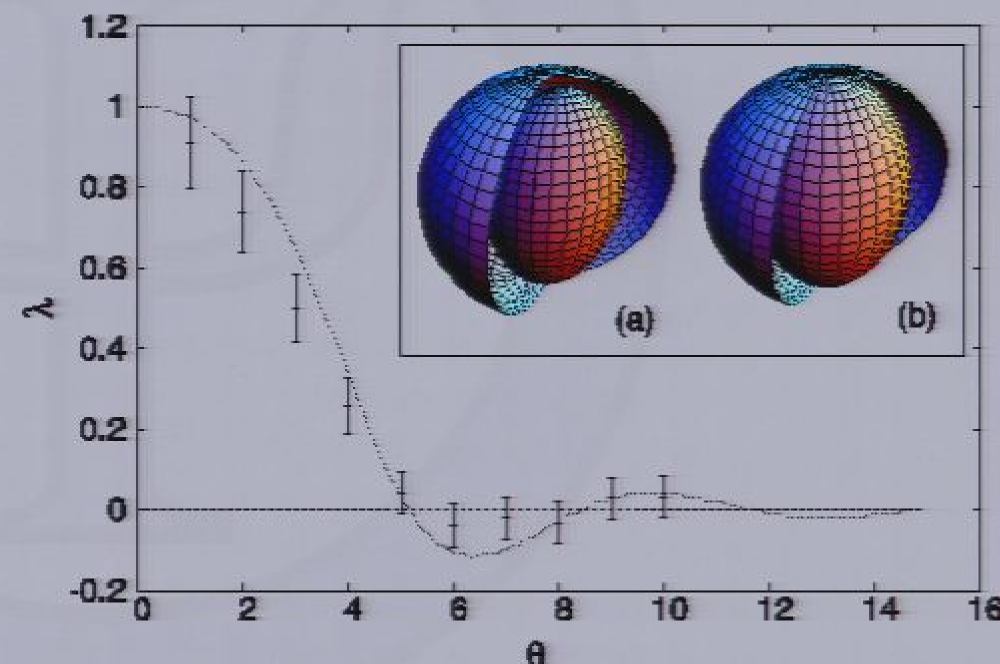

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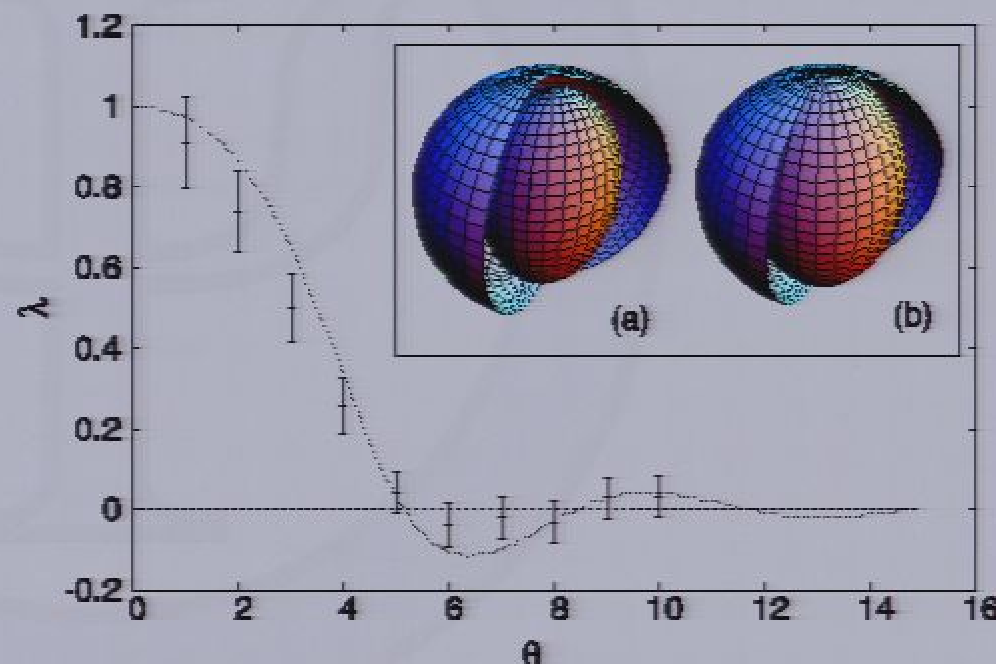

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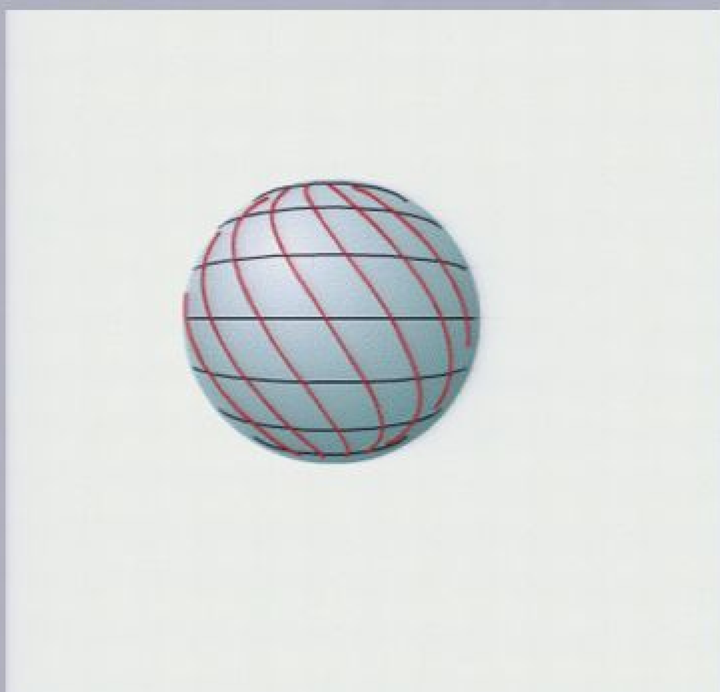

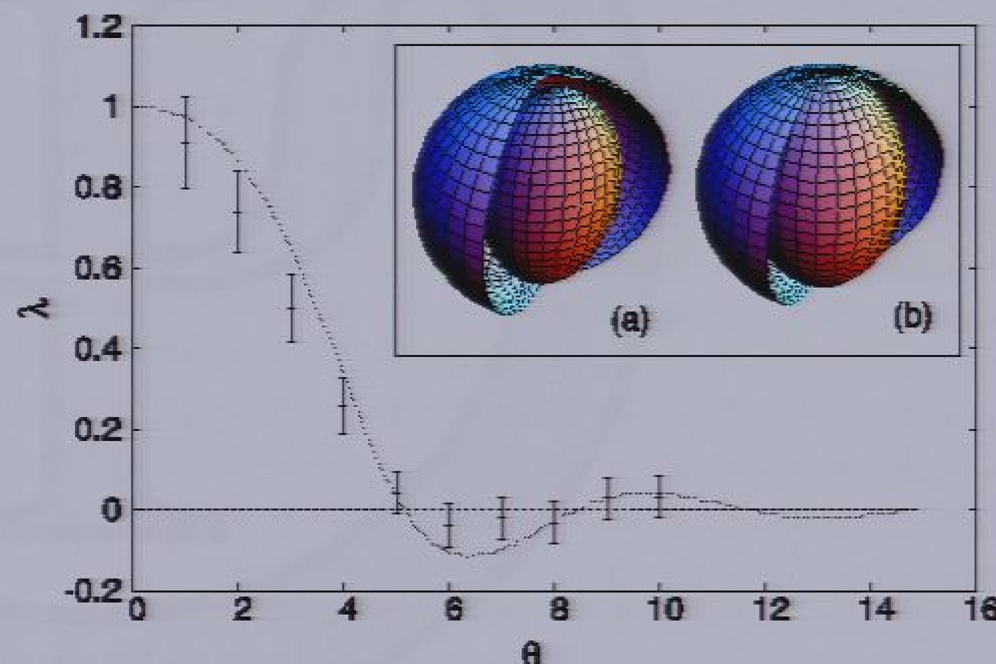

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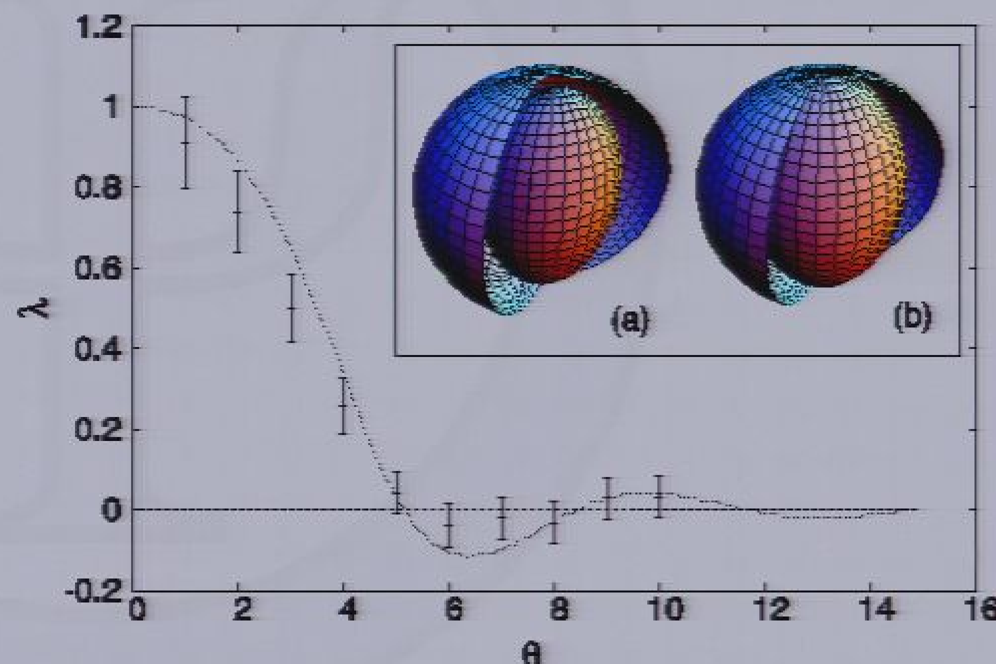

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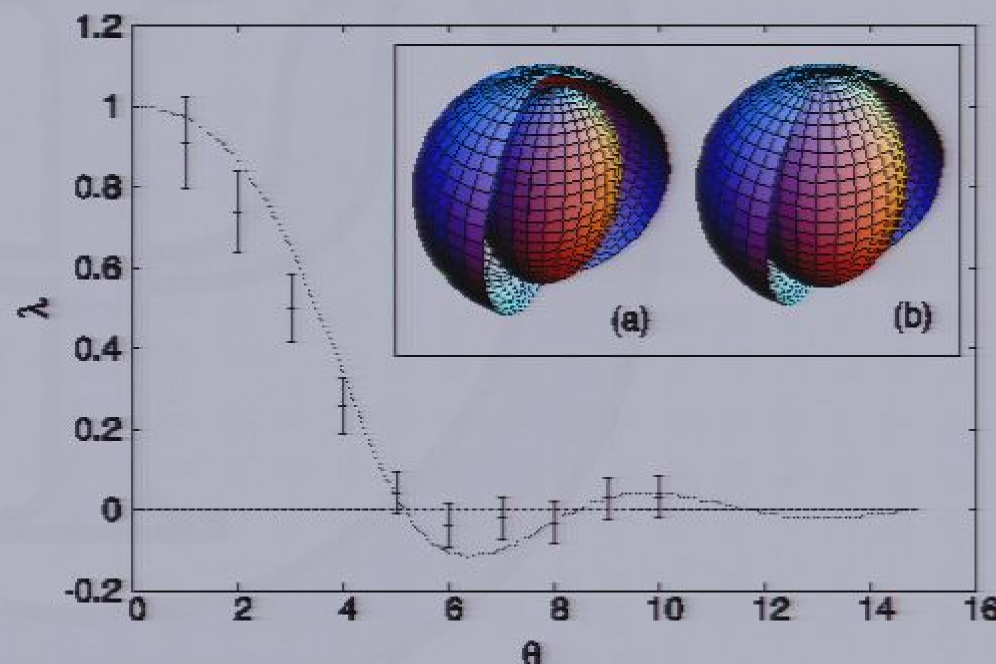

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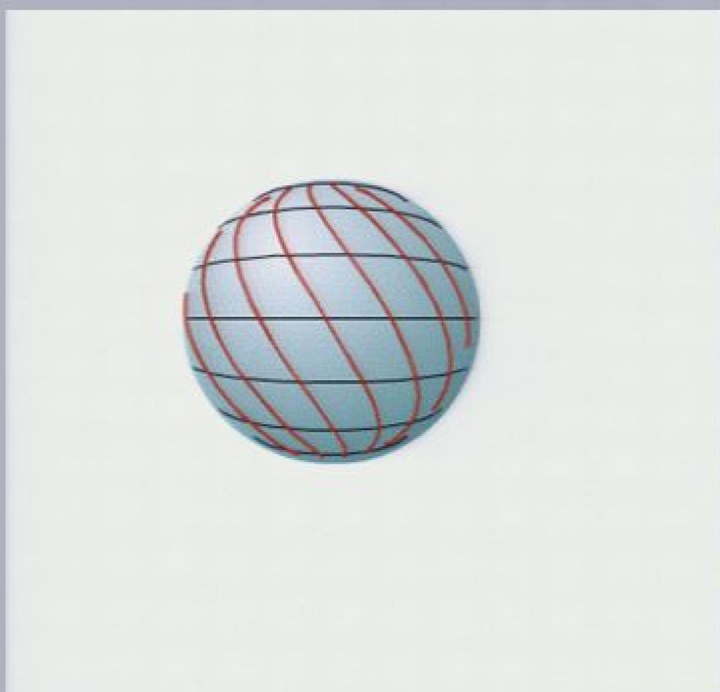

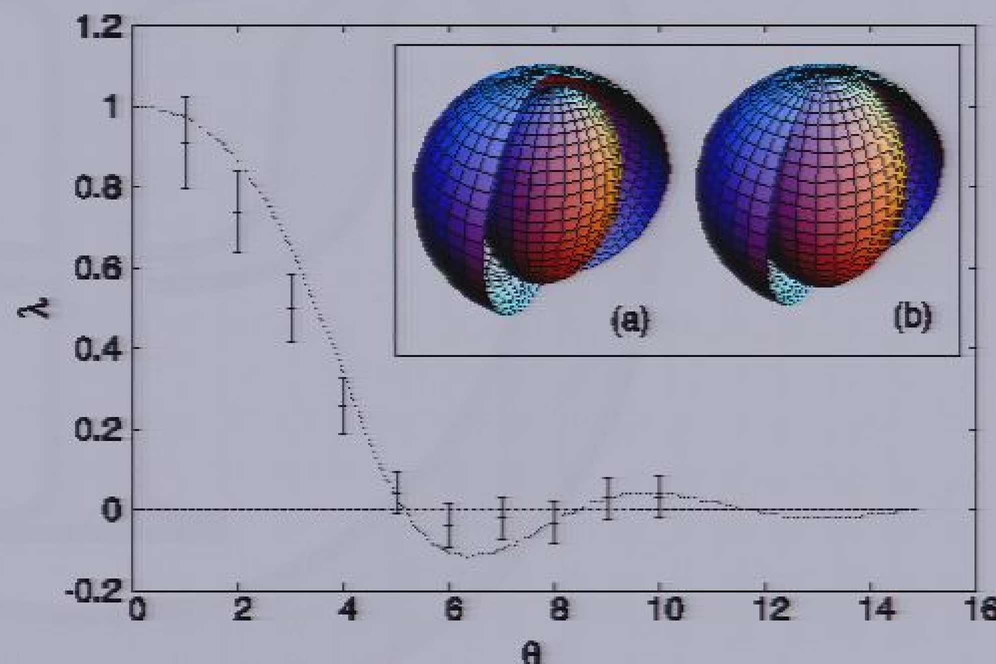

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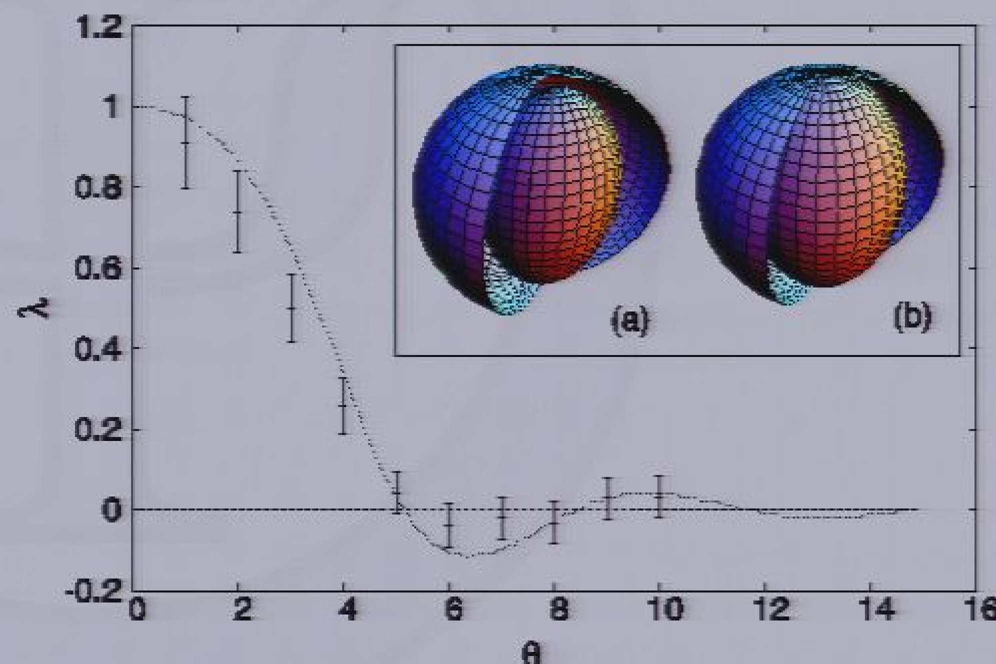

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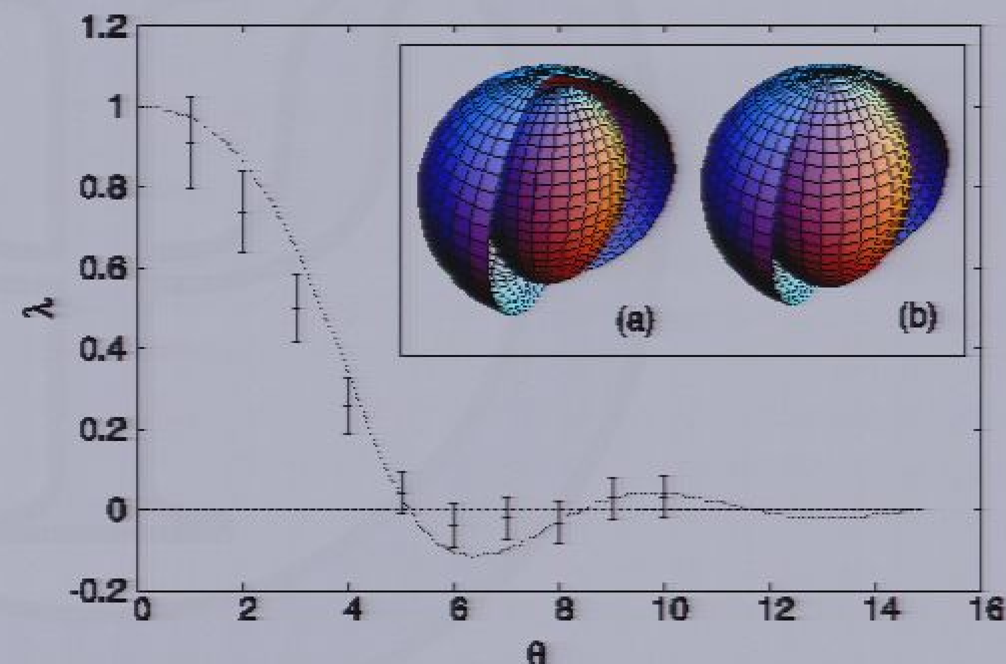

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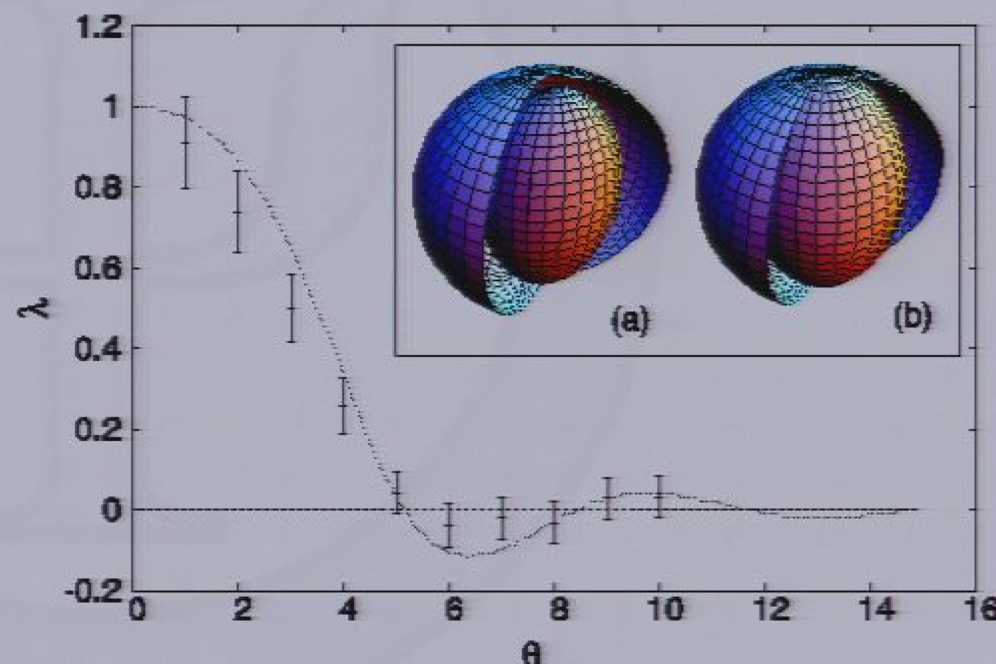

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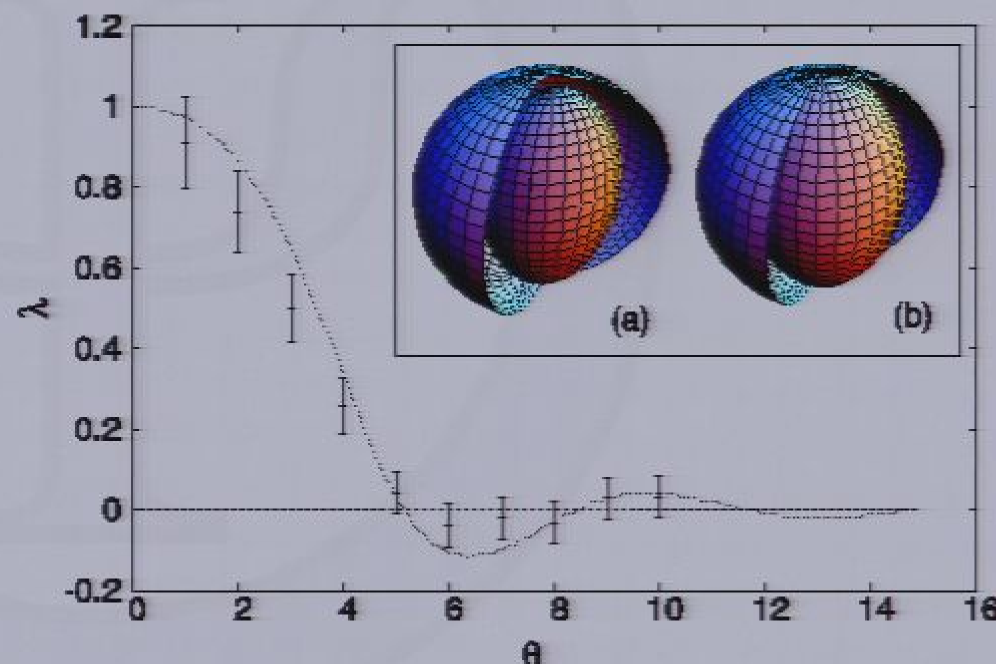

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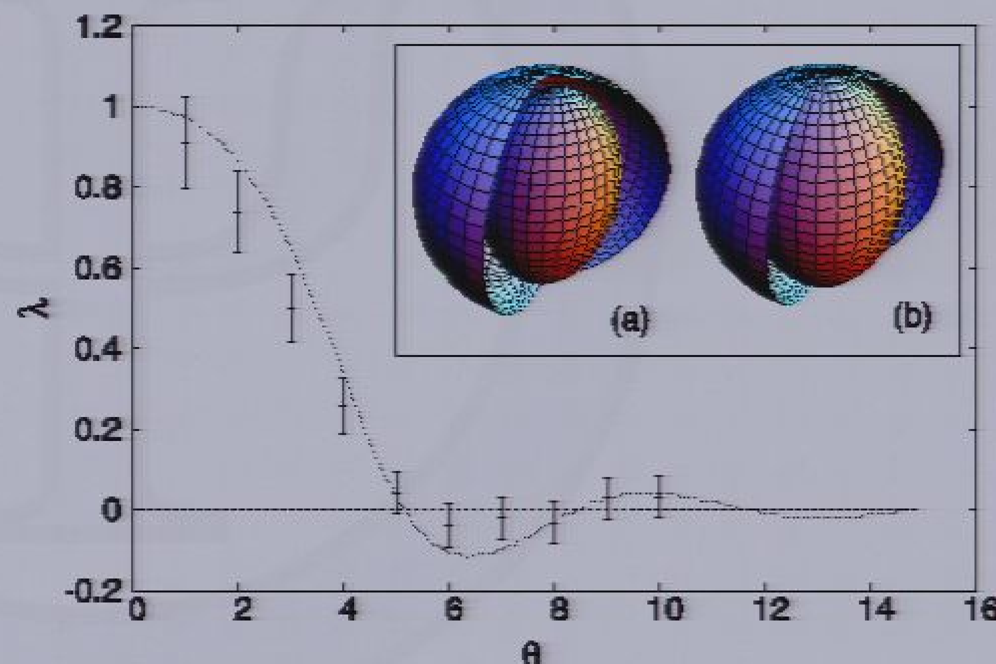

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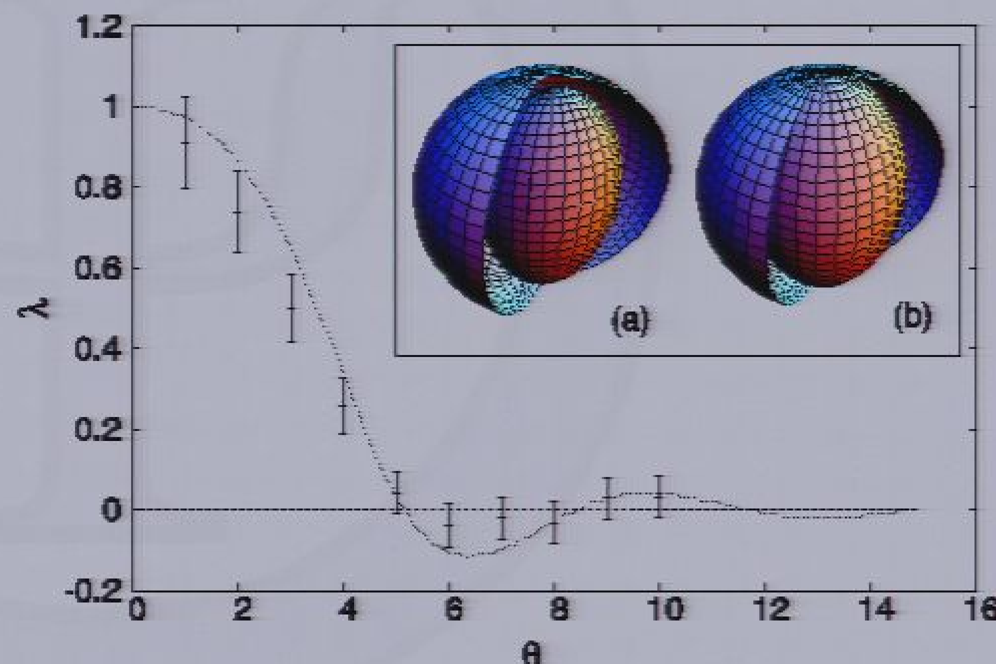

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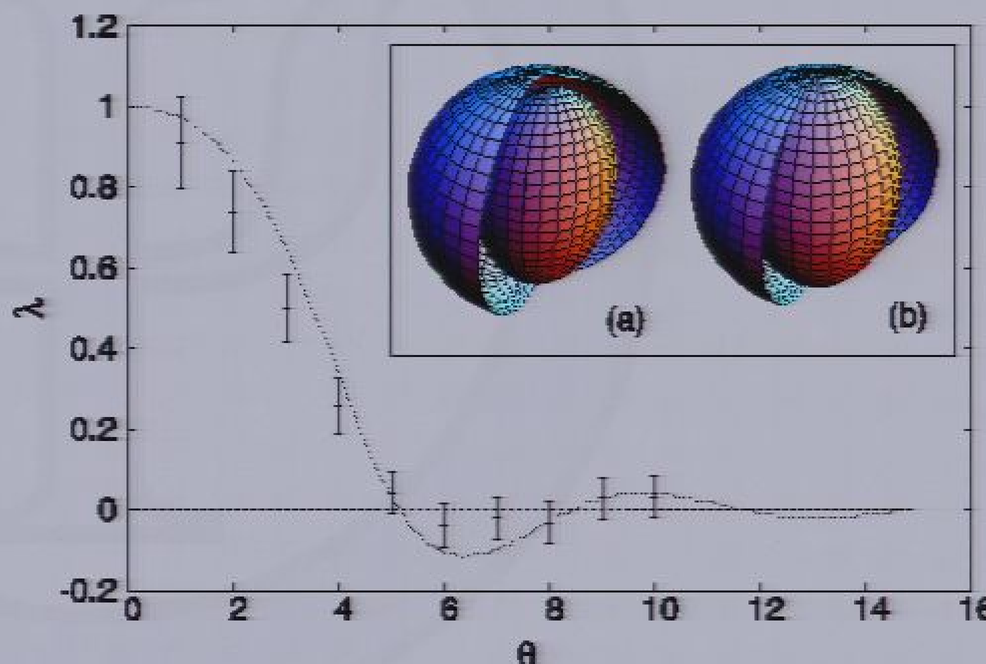

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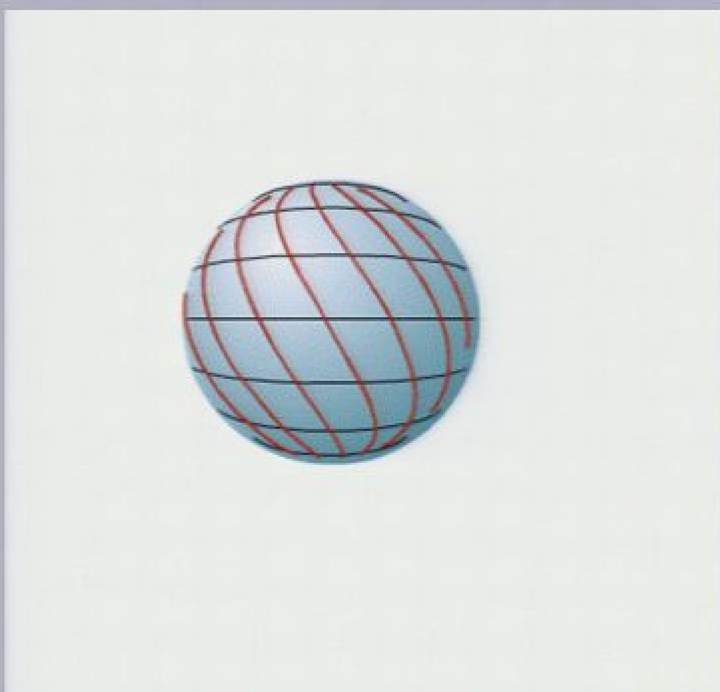

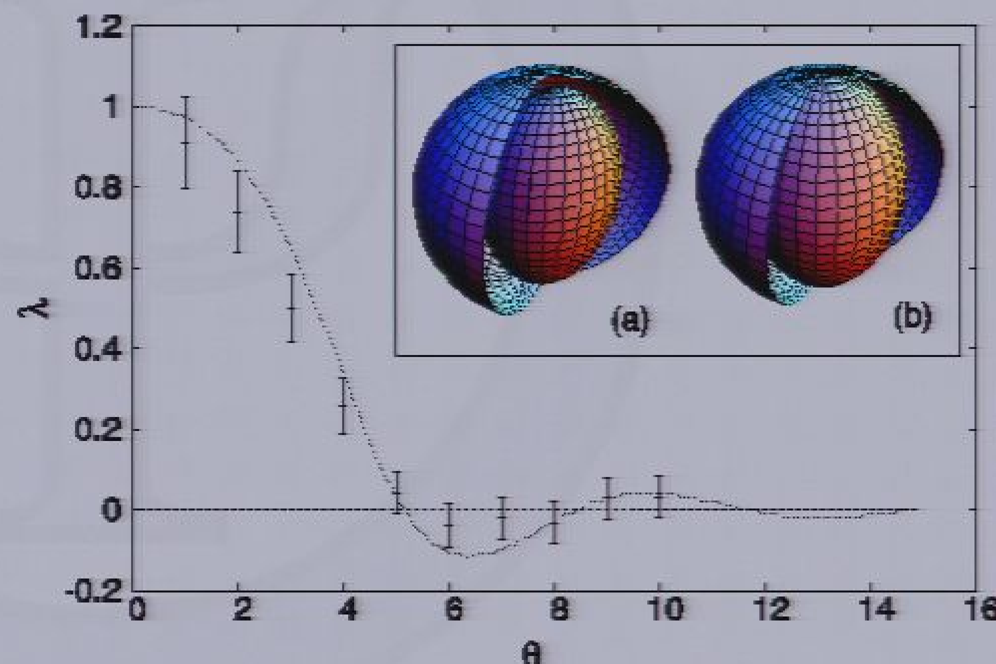

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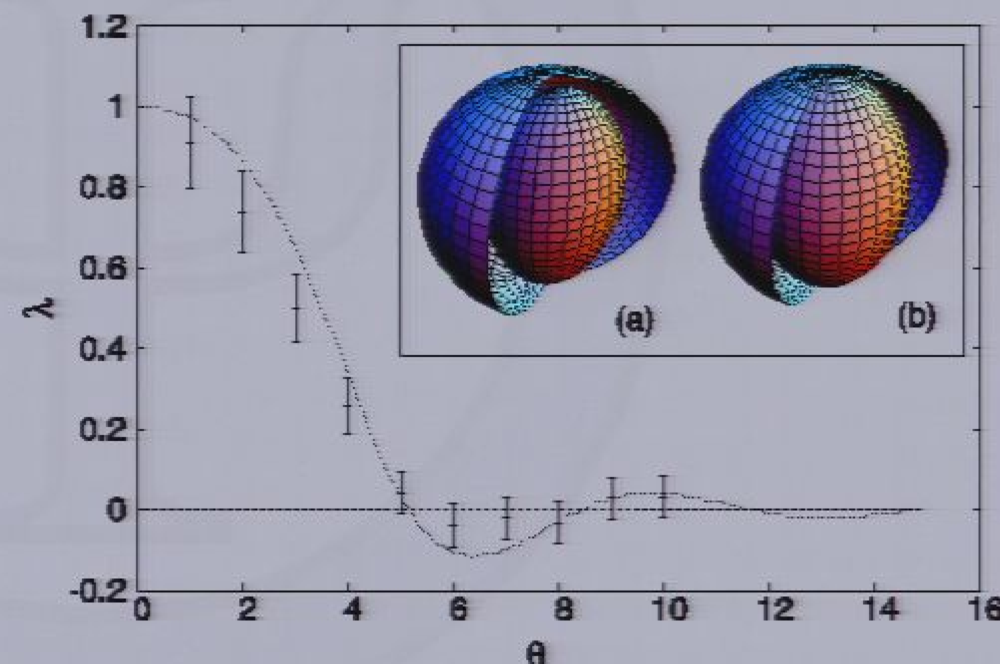

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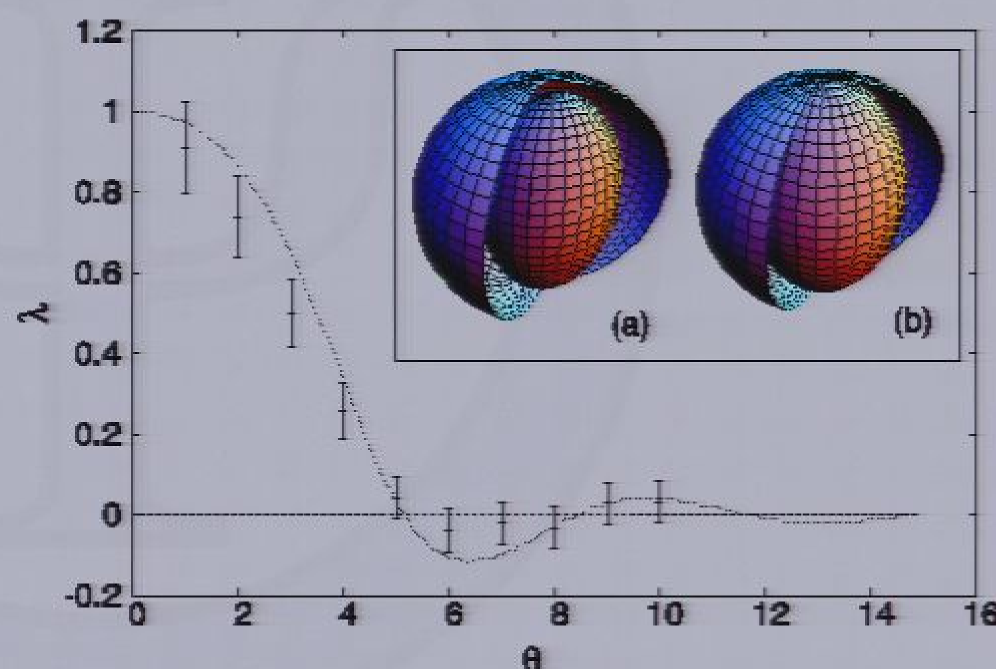

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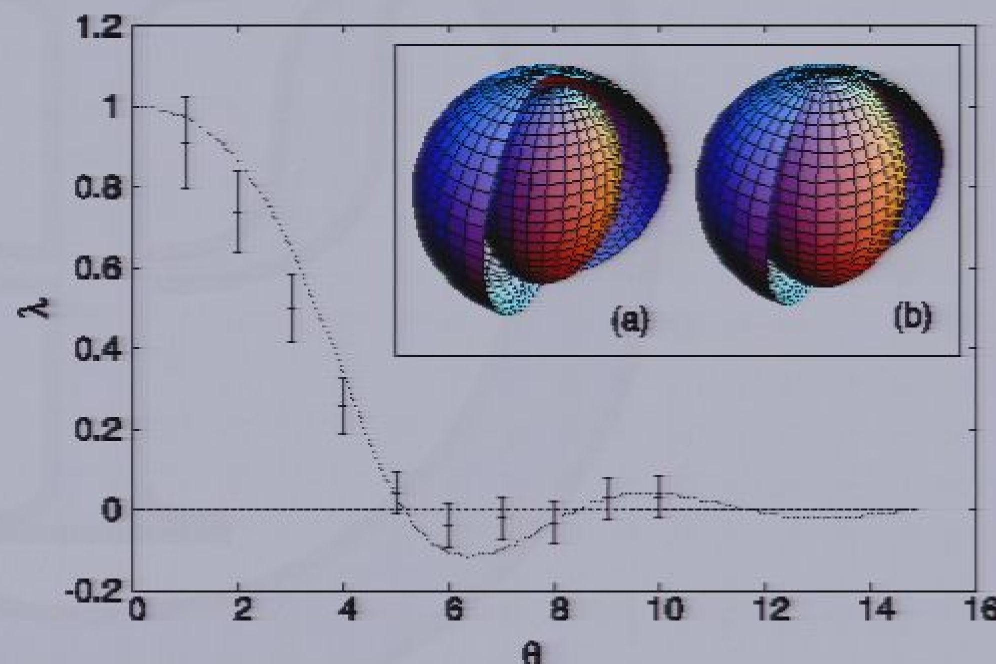

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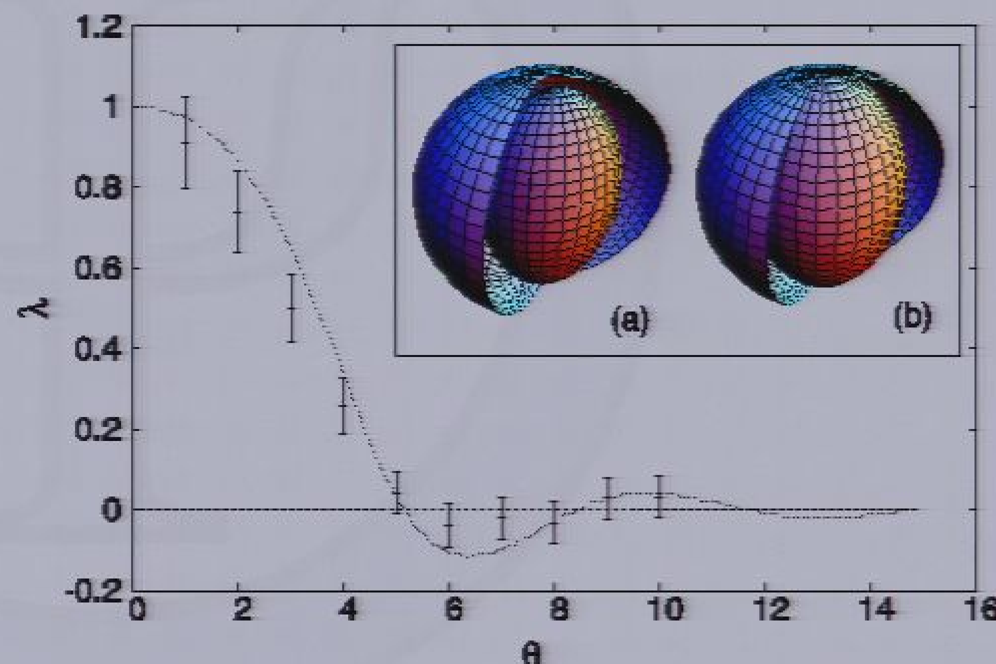

Summary

- quantum Bayesian inference from finite ensembles
- optimal measurements of finite ensembles
- recycling of quantum information
- coding and decoding of quantum information
- simulation of non-physical maps

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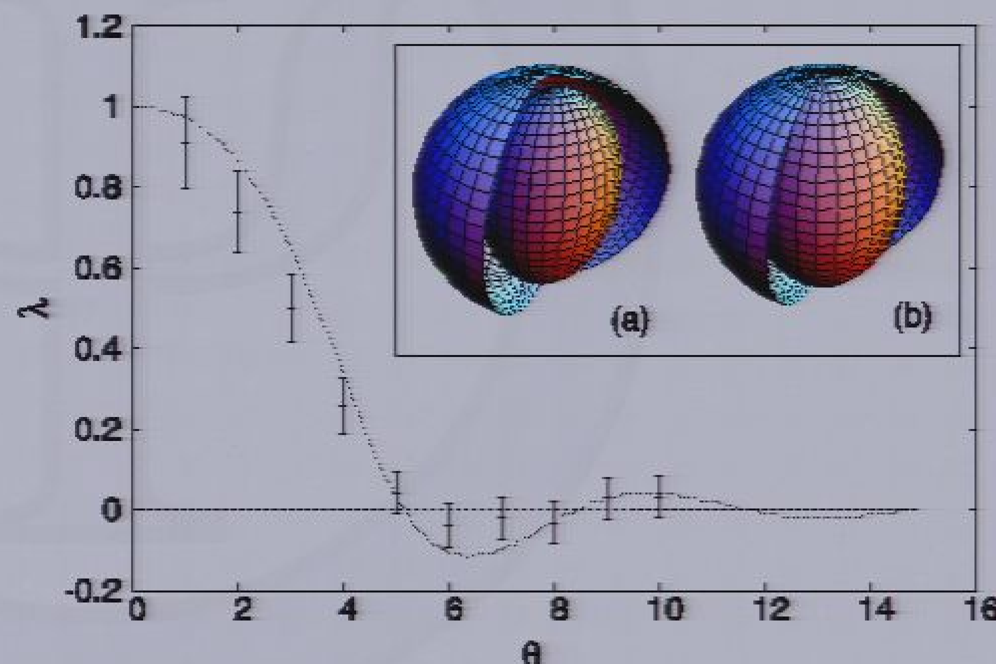
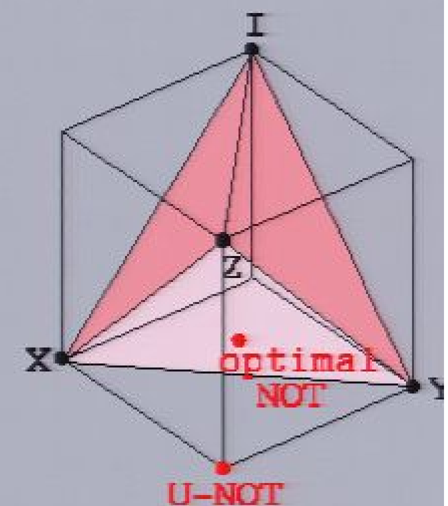


Approximation of non-physical maps I

Universal NOT gate $\varepsilon = \mathrm{diag}\ (1,-1,-1,-1)$

Best approximation $\varepsilon = \mathrm{diag}\ (1,-1/3,-1/3,-1/3)$



6 input states – eigenstates of σ_j

3 measurements σ_j

N=100 x 18 clicks

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi^{\perp}\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$$

$$\hat{\rho}^{\perp}_{meas} = \frac{1}{3}\hat{\rho}^{\perp} + \frac{1}{3}\hat{I}$$