Title: Phase estimation and Quantum Benchmarks for phase-covariant states

Date: Aug 27, 2008 04:15 PM

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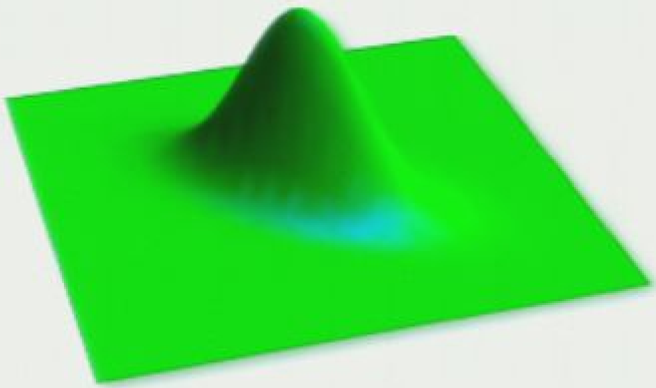
Abstract: We study two related estimation problems involving phase- covariant quantum states. We first address the problem of phase estimation. We give optimal bounds for pure and mixed Gaussian states and find that for a fixed squeezing parameter a larger temperature can enhance the estimation fidelity. In addition we use state estimation concepts to give a benchmark that asses whether experimental implementations of quantum storage and teleportation protocols could be reproduced by classical means, i.e., by a measure and prepare strategy.

Gaussian States



- A family of continuos variable quantum states defined by a Gaussian characteristic function.

- Canonical form:

$$\rho = D(\alpha)S(r,\phi)\rho_\beta S(r,\phi)^\dagger D(\alpha)^\dagger$$

$$\rho_\beta = \frac{\mathrm{e}^{-\beta\hat{n}}}{\mathrm{tr}\left(\mathrm{e}^{-\beta\hat{n}}\right)} \quad D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a} \quad S(r,\phi) = \exp\left[r/2(a^2\mathrm{e}^{-i2\phi} - a^{\dagger 2}\mathrm{e}^{i2\phi})\right]$$

- Very good description of states of light produced in labs (laser produces coherent state + passive/active optical operations)

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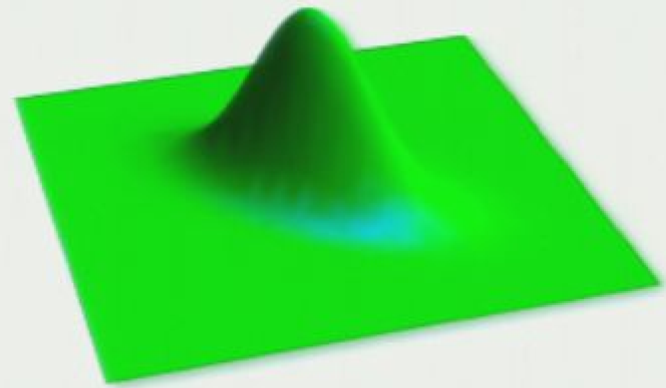

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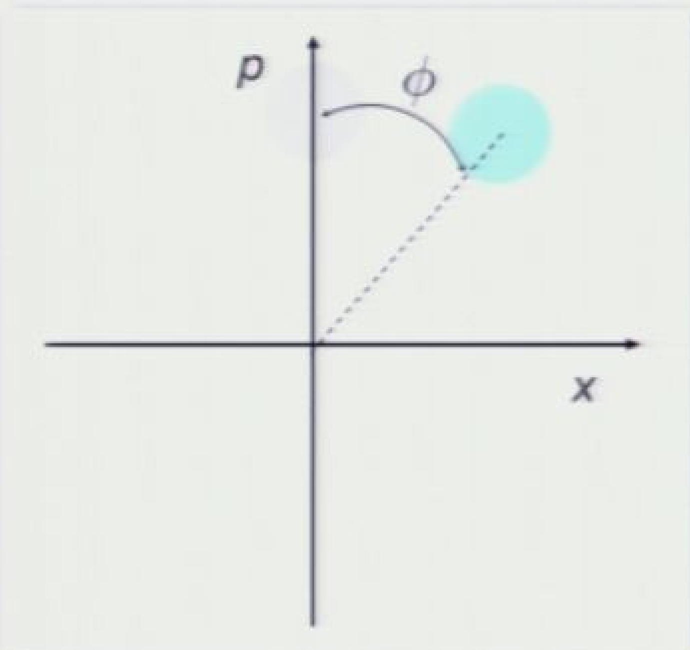
Phase estimation

$$\rho \longrightarrow \boxed{U(\phi)} \longrightarrow \rho(\phi) = U(\phi)\rho U(\phi)^\dagger$$

$$U(\phi) = \mathrm{e}^{i\phi a^\dagger a}$$



$$\{O_\chi\}$$

$$\rho \longrightarrow \boxed{U(\phi)} \longrightarrow$$ 

$$\rightarrow \phi_\chi$$

Figure of merit: $f(\phi, \phi_\chi) = \cos(\phi - \phi_\chi)$

$$\mathcal{F} = \sum_\chi \int \frac{d\phi}{2\pi} f(\phi, \phi_\chi) \mathrm{tr}[\rho(\phi) O_\chi]$$

General solution for phase estimation:

$$f(\phi, \phi_\chi) = \sum_k a_k \cos[k(\phi - \phi_\chi)] \qquad a_k \geq 0$$

$$\mathcal{F} = \sum_k a_k \sum_n |\rho_{n,n+k}| = a_1 \sum_n |\rho_{n,n+1}| + a_2 \sum_n |\rho_{n,n+2}| + \dots$$

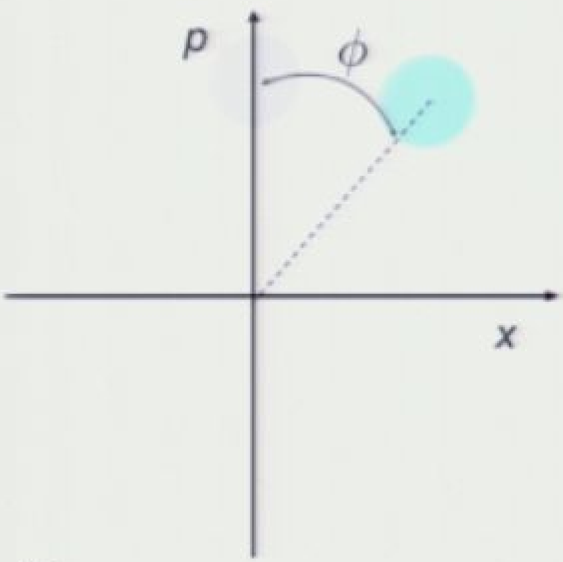
This optimal fidelity can be obtained with a covariant state-independent* POVM:

$$O_\theta = \frac{1}{2\pi} U(\theta) |\chi_0\rangle\langle\chi_0| U(\theta)^\dagger \qquad |\chi_0\rangle = \sum_{n=0}^{\infty} |n\rangle$$

Completeness relation $\int d\theta O_\theta = \mathbb{1}$

*for states with positive matrix elements

Coherent Thermal States

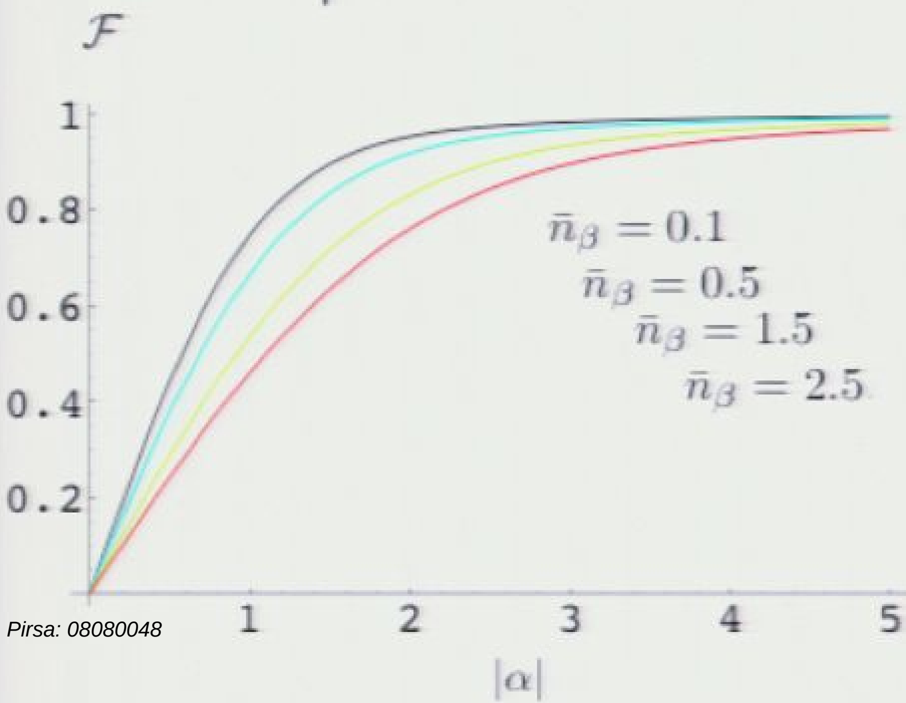


$$\rho = D(\alpha)\rho_\beta D(\alpha)^\dagger$$

$$\rho_\beta = \frac{1}{\pi \bar{n}_\beta} \int d^2\tilde{\alpha}\, e^{-\frac{|\tilde{\alpha}|^2}{\bar{n}_\beta}} |\tilde{\alpha}\rangle\langle\tilde{\alpha}|$$

$$\mathcal{F} = \sum_k \int \frac{d^2\tilde{\alpha}}{\pi \bar{n}_\beta} e^{-\frac{|\tilde{\alpha}|^2}{\bar{n}_\beta}} e^{-|\tilde{\alpha}+\alpha|^2} \frac{|\tilde{\alpha}+\alpha|^{2k}|\tilde{\alpha}+\alpha|}{k!\sqrt{k+1}}$$

$$\frac{1}{\sqrt{k+1}} = \frac{1}{\sqrt{\pi}} \int \frac{dt}{\sqrt{t}} e^{-t(k+1)}$$

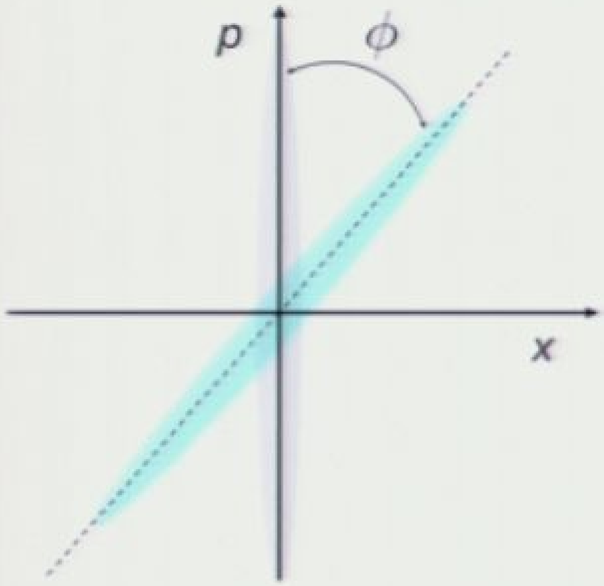


$$\mathcal{F} = \frac{|\alpha|}{\sqrt{\pi}} \int_0^1 dx \frac{\exp(-\frac{x}{1+\bar{n}_\beta x}|\alpha|^2)}{(1+\bar{n}_\beta x)^2\sqrt{-\log(1-x)}}$$

$$\mathcal{F} \simeq 1 - \frac{1+8\bar{n}_\beta}{8\bar{n}}$$

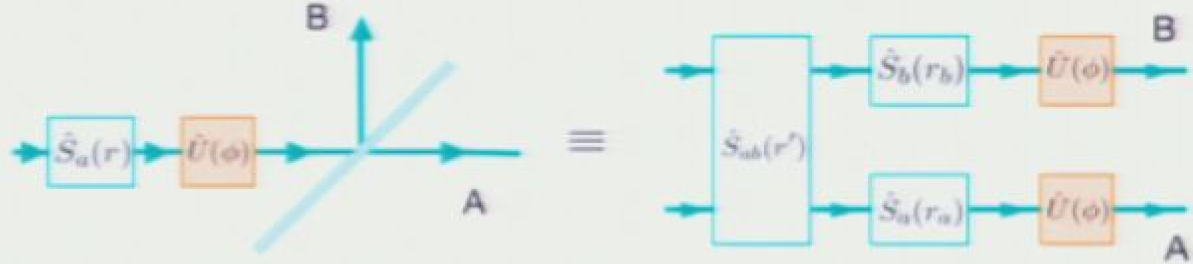
$$\bar{n} = |\alpha|^2 + \bar{n}_\beta$$

Squeezed Thermal States



$$\rho = S(r)\rho_\beta S(r)^\dagger$$

No fully general solution, but for important subclass



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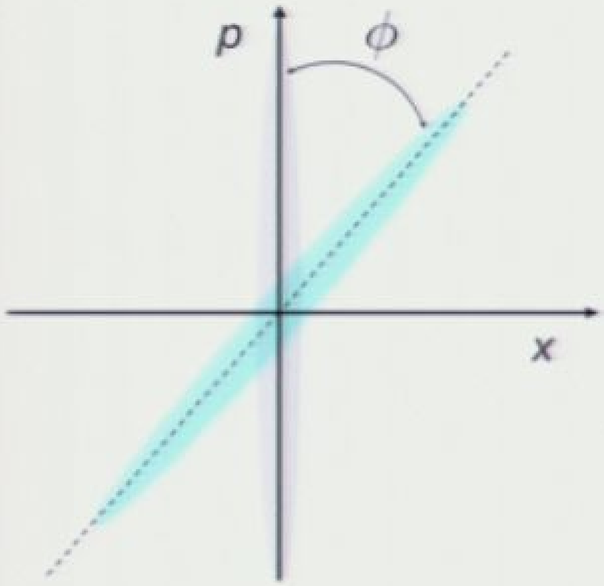

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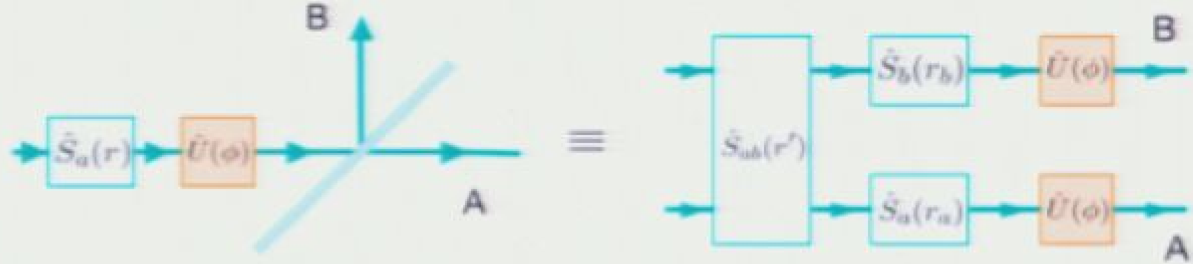

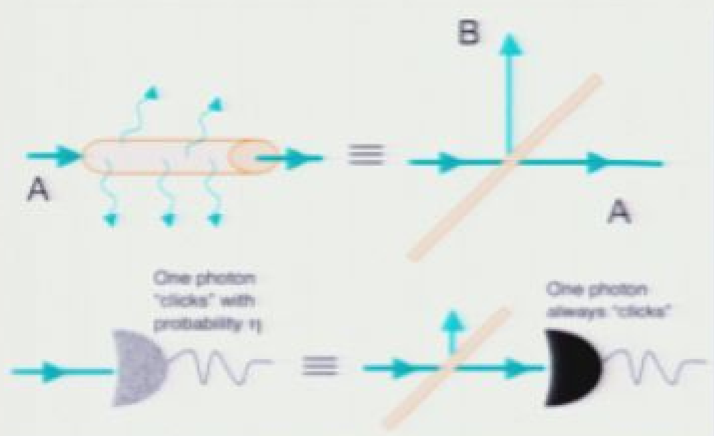

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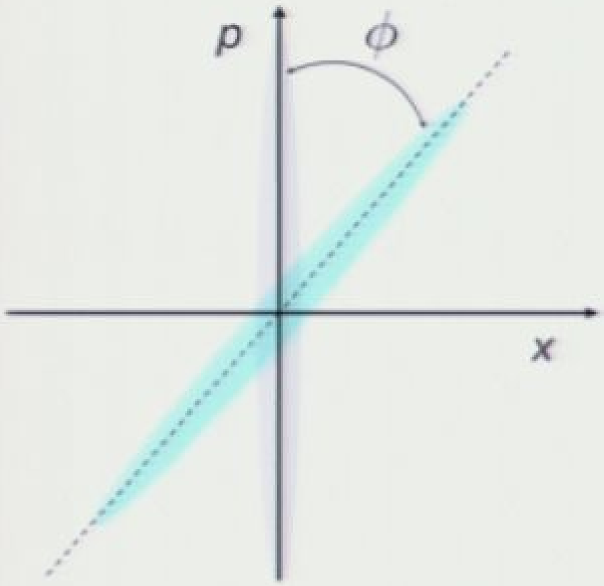

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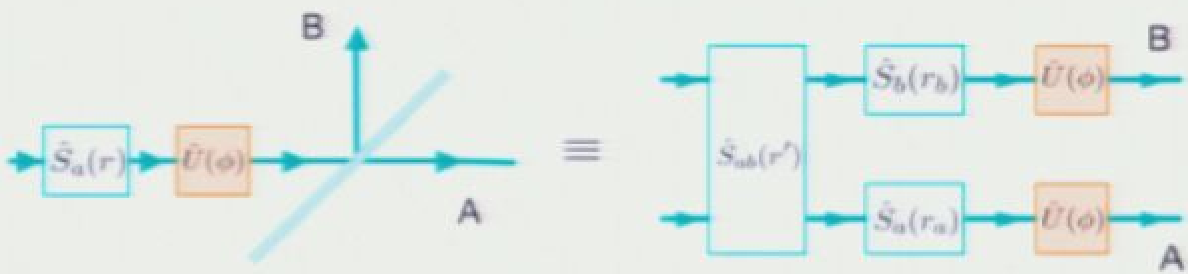

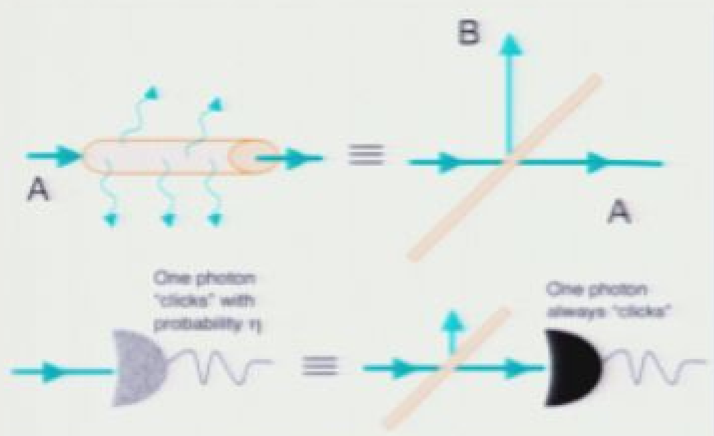

$$\mathcal{F} = \lambda\sqrt{1-\lambda^2}T\int_0^\infty dt \frac{e^{-\frac{3}{2}t}I_0(t/2)}{\{1-e^{-2t}[T+(1-T)e^t]^2\lambda^2\}^{3/2}}$$

$$\mathcal{F}_{T=1} \approx 1 - \frac{0.55}{\sqrt{n_r}} \quad \mathcal{F}_{T=1/2} \approx 1 - \frac{1.20}{\sqrt{n_r}} \quad \mathcal{F}_{T=1/3} \approx 1 - \frac{1.62}{\sqrt{n_r}}$$

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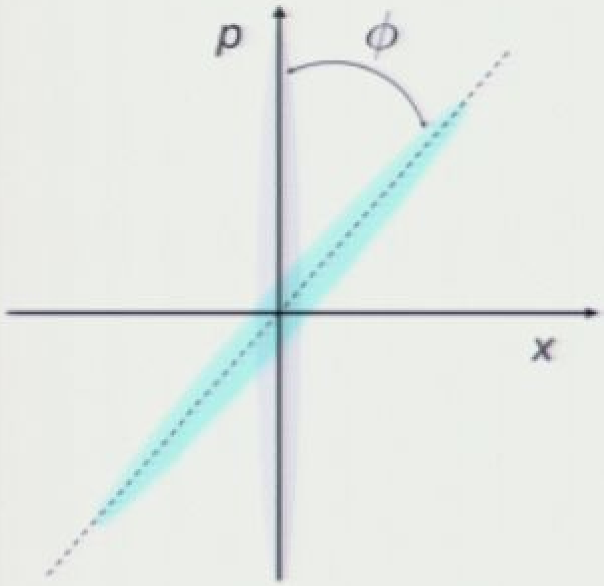

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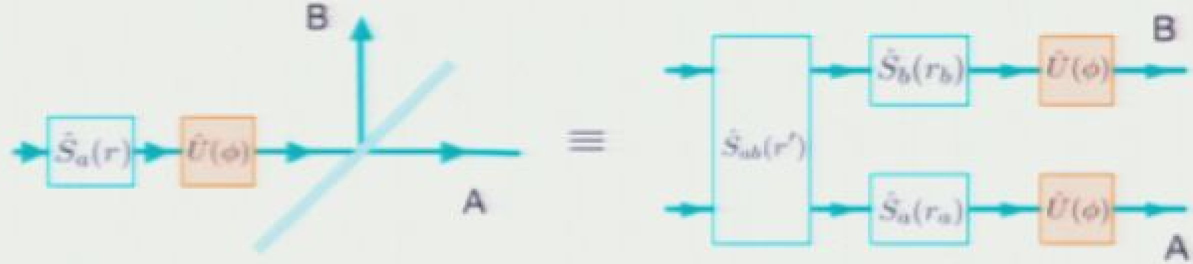

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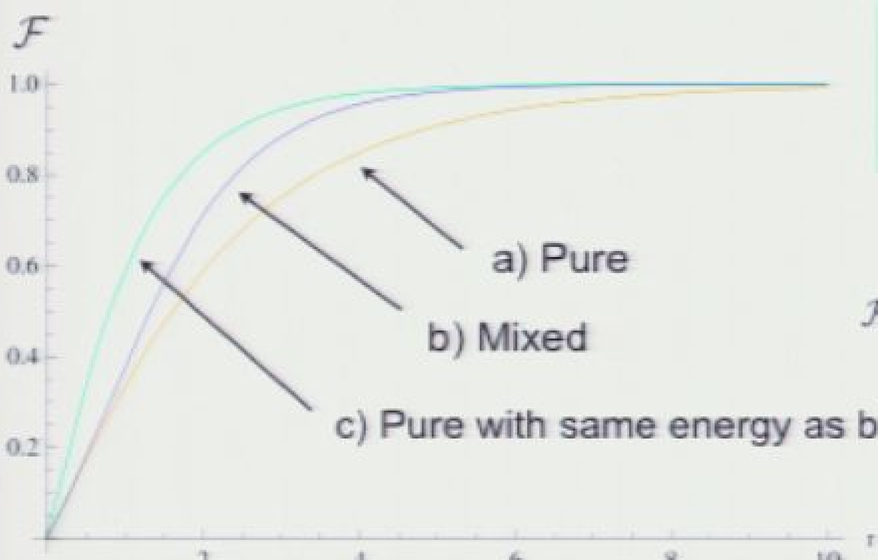

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$$\langle n \rangle = |\alpha|^2 + n_\beta + (2n_\beta + 1)\overbrace{\sinh^2 r}^{n_r}$$

Asymptotically many copies

$$\mathcal{F}_{\text{coh}} = 1 - \text{Var}[\phi]/2 \quad \text{and} \quad \mathcal{F}_{\text{sq}} = 1 - 2\text{Var}[\phi]$$

From the Cramér-Rao bound

Fisher Information

$$I(\phi, \{O_\chi\}) \equiv \int d\chi \, \text{tr}[\rho(\phi) O_\chi] \left(\frac{\partial \ln \text{tr}[\rho(\phi) O_\chi]}{\partial \phi} \right)^2$$

$$\text{Var}[\phi] \leq [N I(\phi)]^{-1} = [4 \text{d}s_{BU}^2 / \text{d}\theta^2]^{-1}$$

Braunstein & Caves
PRL 1994

Coherent Thermal

$$\text{Var}[\phi] = \frac{1}{4N} \frac{1}{|\alpha|^2 \tanh(\beta/2)} = \frac{1}{4N} \frac{2\bar{n}_\beta + 1}{n_\alpha}$$

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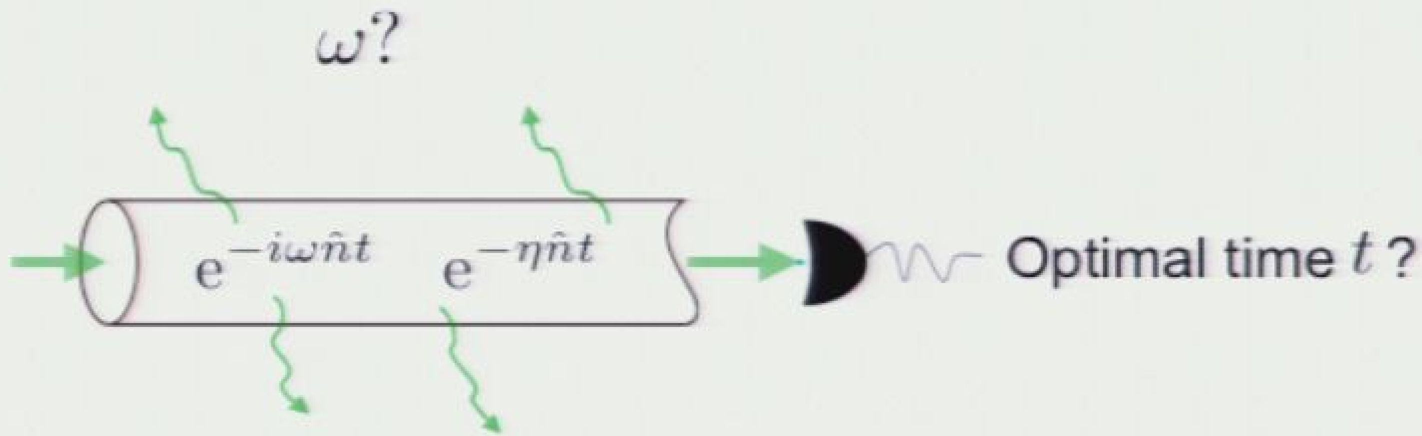
$$\text{Var}[\phi] = \frac{1}{2N} \frac{1}{(1 + \cosh^{-1} \beta) \sinh^2 2r} \simeq \frac{1}{16 N \bar{n}_r^2} \left(1 + \frac{1}{2\bar{n}_\beta + 1} \right)$$

- Estimation in the presence of decoherence



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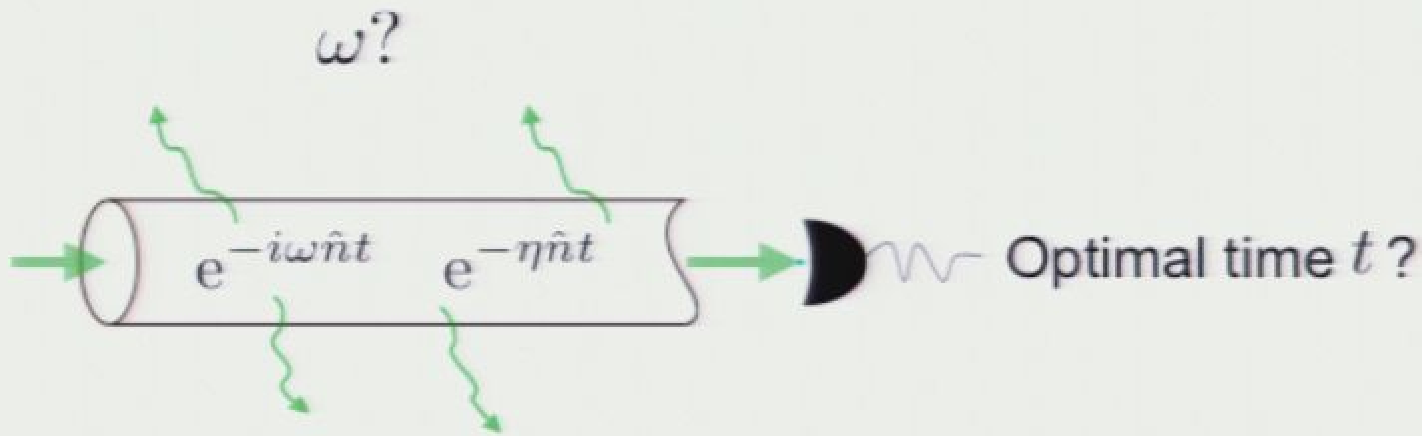

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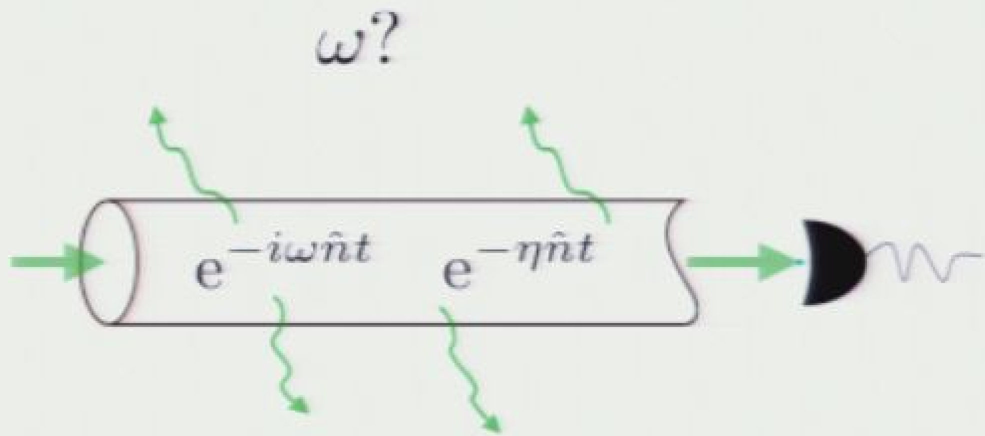

Fixed number of copies *N* optimize *t* $\mathrm{Var}[\omega]|_N = [Nt^2 I]^{-1}$

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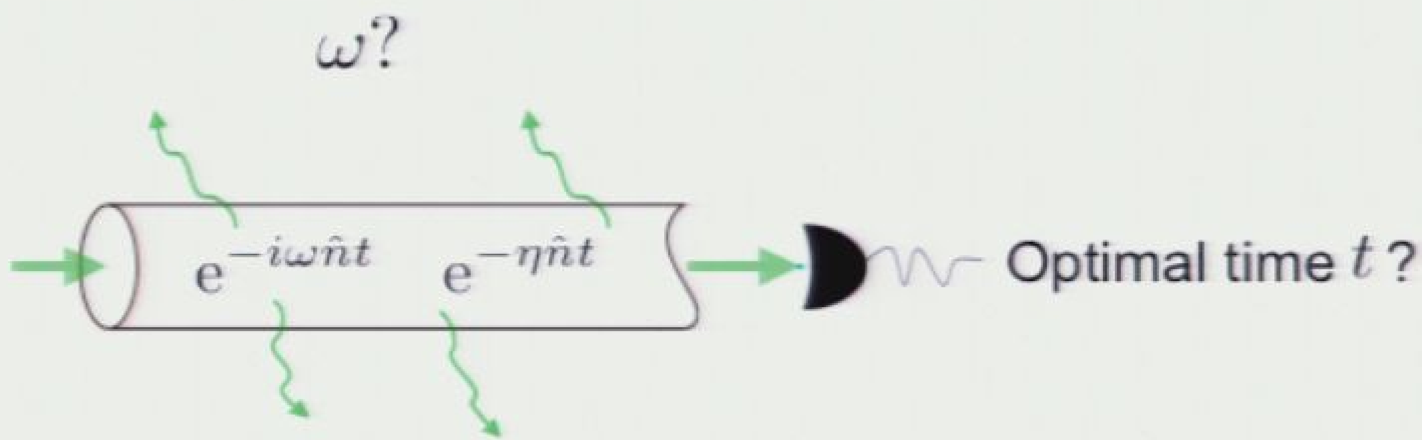

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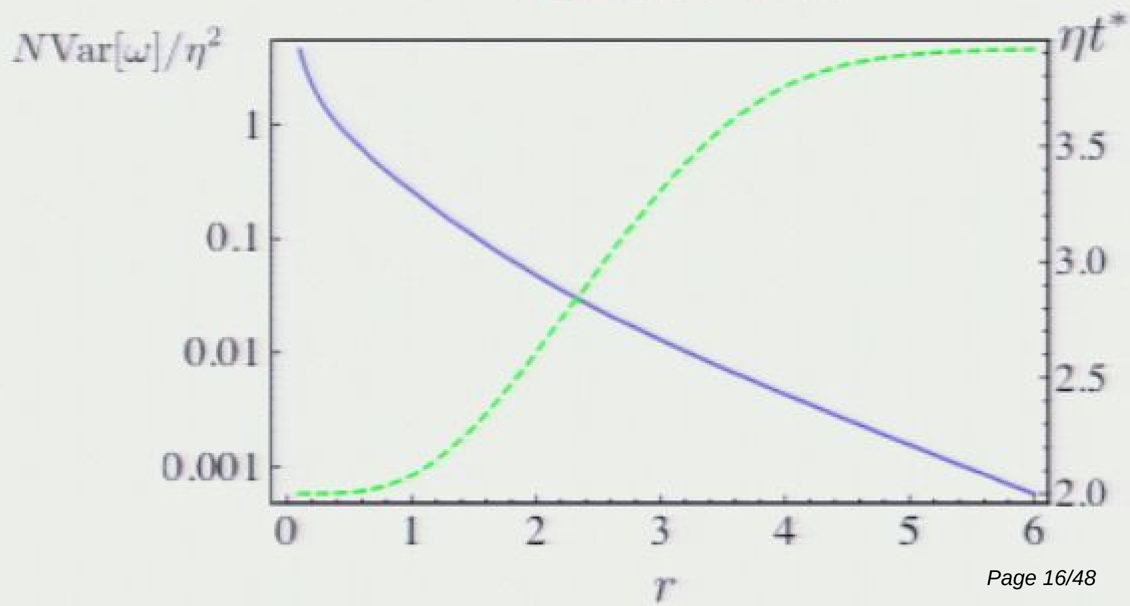

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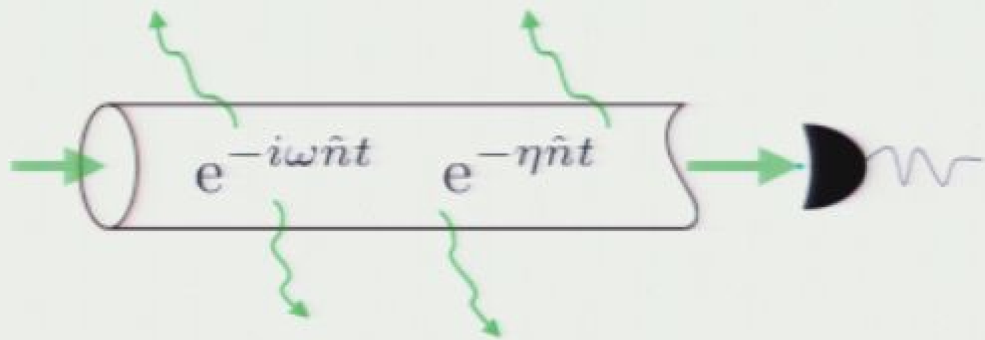
$\omega?$



Optimal time t ?

Fixed number of copies *N* optimize *t* $\quad \mathrm{Var}[\omega]|_N = [N t^2 I]^{-1}$

Coherent I ... zed Input

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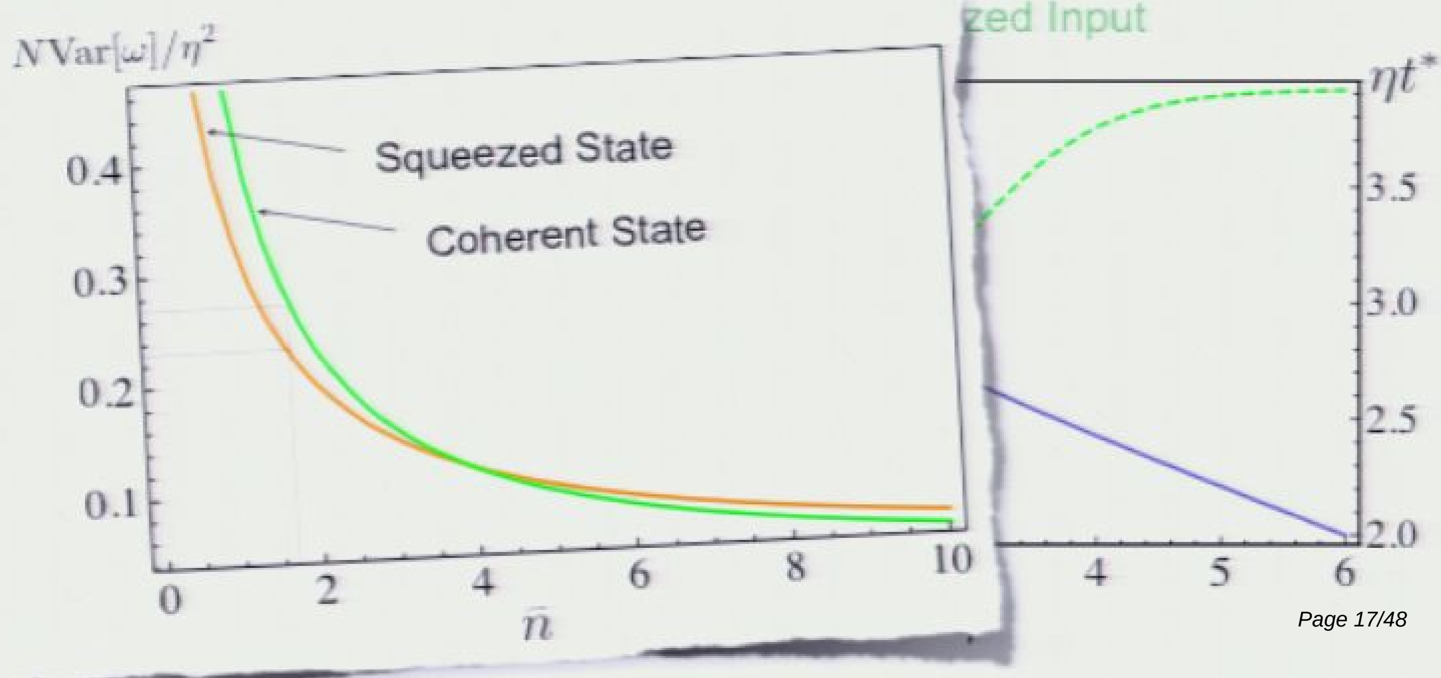

Quantum Benchmarks

$$\rho \longrightarrow \boxed{\mathcal{E}(\rho)} \longrightarrow \tilde{\rho}$$

Noisy identity-channel

e.g. quantum teleportation
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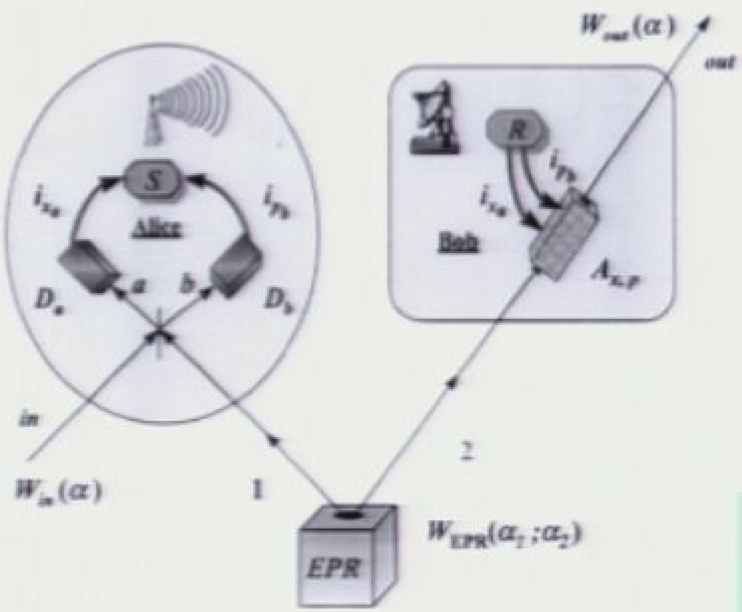


First threshold for quantum teleportation of coherent states: Fidelity of output state when no quantum correlations are used.

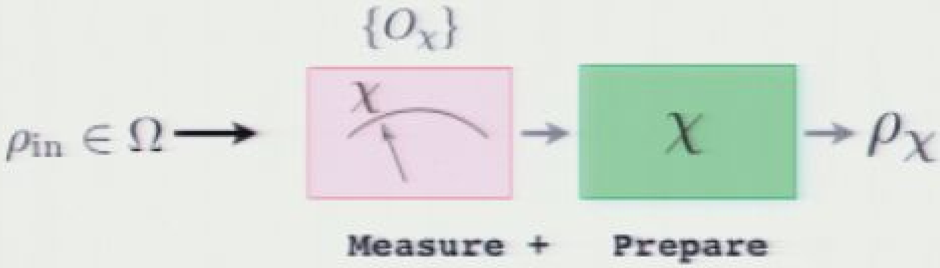
$$\mathcal{F}_{tel} = \langle\alpha|\rho_{out}|\alpha\rangle = \langle\alpha|\left(\int\frac{1}{\pi}\mathrm{tr}\left(E_\beta|\alpha\rangle\langle\alpha|\right)|\beta\rangle\langle\beta|\right)|\alpha\rangle =$$

$$= \frac{1}{\pi}\int d^2\beta|\langle\alpha|\beta\rangle|^4 = \frac{1}{\pi}\int d^2\beta e^{2|\alpha-\beta|^2} = \frac{1}{2}$$

$$\mathcal{F} = \langle\alpha|\rho_{\text{out}}|\alpha\rangle > 1/2$$ → Quantum resources are being used.

A. Furusawa et. al. Science 1998

More rigorous quantum benchmark (Braunstein, Fuchs & Kimble JMO 2000):



$$\longrightarrow \rho_{\text{out}} = \sum_{\chi} \text{tr}(\rho_{\text{in}} O_{\chi}) \rho_{\chi}$$

$$\mathcal{F} = \int \langle \psi_{\text{in}} | \rho_{\text{out}} | \psi_{\text{in}} \rangle P(|\psi_{\text{in}}\rangle) d|\psi_{\text{in}}\rangle$$

Different choices of input-state families:

- 2 non-orthogonal states: $|\psi_0\rangle, |\psi_1\rangle$

$x = \langle \psi_0 | \psi_1 \rangle$

$$\mathcal{F} = \frac{1}{2}\left(1 + \sqrt{1 - x^2 + x^4}\right) \geq 0.933$$

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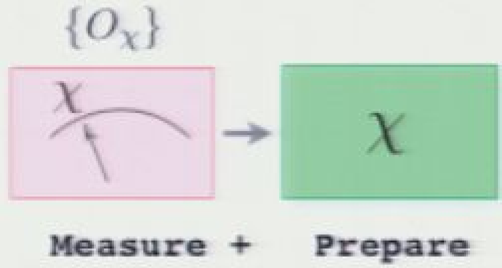
$\rho_{\text{in}} \in \Omega \longrightarrow$



$\rightarrow \rho_\chi \longrightarrow \rho_{\text{out}} = \sum_\chi \text{tr}(\rho_{\text{in}} O_\chi)\rho_\chi$

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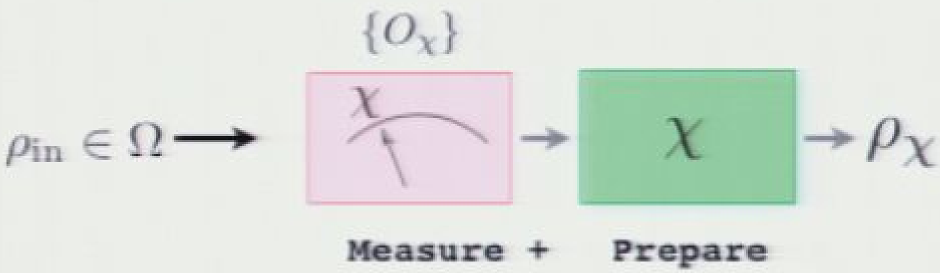

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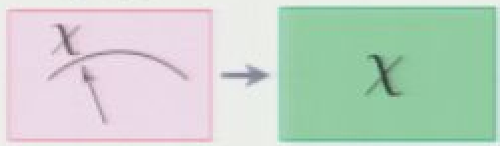
$\rho_{\text{in}} \in \Omega \longrightarrow$  $\rightarrow \rho_\chi \longrightarrow \rho_{\text{out}} = \sum_\chi \text{tr}(\rho_{\text{in}} O_\chi)\rho_\chi$

Measure + Prepare

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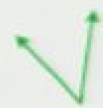
- 2 non-orthogonal states: $|\psi_0\rangle, |\psi_1\rangle$  $x = \langle \psi_0 | \psi_1 \rangle$ $\mathcal{F} = \frac{1}{2}\left(1 + \sqrt{1 - x^2 + x^4}\right) \geq 0.933$
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Hammerer et. al. PRL 2005

- Micro-canonical ensemble of pure Gaussian states (Serafini et al PRL 2007)

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- Recently Adesso & Chiribella [PRL 2008] have also studied benchmarks with mixed states taking as input family: $\rho_r = \hat{S}(r)\rho_{\beta}\hat{S}(r)^{\dagger}$

$$\mathcal{F}_{AC} = \sum_{\chi} \int dr\, P(r) p(\chi|r) F[\rho(r), \rho(r_{\chi})] \leq \mathcal{F}^{Cl}$$

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- Phase-covariant family is large enough to give reasonable benchmarks, and is easy to produce experimentally.
- Recently Adesso & Chiribella [PRL 2008] have also studied benchmarks with mixed states taking as input family: $\rho_r = \hat{S}(r)\rho_\beta\hat{S}(r)^{\dagger}$

$$\mathcal{F}_{AC} = \sum_{\chi} \int dr\, P(r)p(\chi|r)F[\rho(r), \rho(r_\chi)] \leq \mathcal{F}^{Cl}$$

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* Very recently: phase covariant and displaced squeezed states (Owari et al.)

- Covariant strategies

Given a strategy $\{O_\chi = |\xi_\chi\rangle\langle\xi_\chi|\,,\rho_\chi\}$ one can define a *covariant strategy* by $\{O_{\chi,\theta} = 1/(2\pi)\,U_\theta O_\chi U_\theta^\dagger, \rho_{\chi,\theta} = 1/(2\pi)\,U_\theta \rho_\chi U_\theta^\dagger\}$ with at least the same fidelity*:

$$\mathcal{F} = (\mathrm{tr}|\sqrt{\rho_0}\sqrt{\rho_{\mathrm{av}}}|)^2 \quad \text{with} \quad \rho_{\mathrm{av}} = \int d\theta \sum_\chi p(\chi,\theta|\rho_0)\rho_{\chi,\theta}$$

*fidelity is concave and for any two unitaries $\mathrm{tr}|UBV| = \mathrm{tr}|B$

The optimal classical fidelity (or quantum benchmark) can be conveniently written as,

$$\mathcal{F} = \max_K \left(\mathrm{tr}_B \sqrt{\mathrm{tr}_A \sqrt{\rho_0}\otimes\sqrt{\rho_0} K \sqrt{\rho_0}\otimes\sqrt{\rho_0}}\right)^2$$

with $K = \int d\theta \sum_\chi O_{\chi,\theta}\otimes\rho_{\chi,\theta}$ (i.e. $K \geq 0,\ \mathrm{tr}_B K = \mathbb{I}_A, U_\theta \otimes U_\theta$ invariant & separable)

For pure states: $\rho_0 = |\psi_0\rangle\langle\psi_0|$ $\quad \mathcal{F} = \langle\psi_0|\langle\psi_0|\,K\,|\psi_0\rangle|\psi_0\rangle$ Navascues PRL 2008

Qubits

Can be solved with full generality.

$$\mathcal{F} = \frac{1}{2}\left(1 + r\frac{\cos\theta}{\cos\zeta}\right)$$

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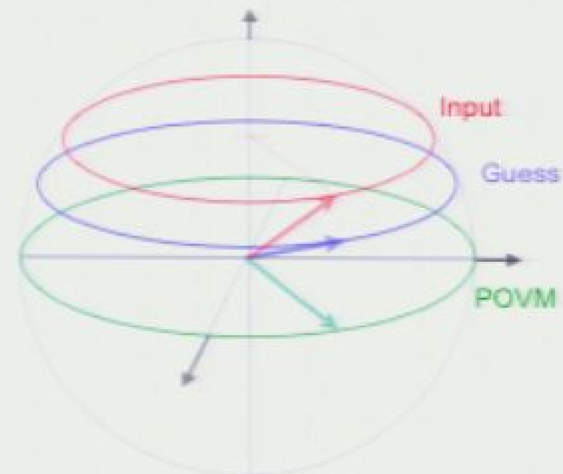

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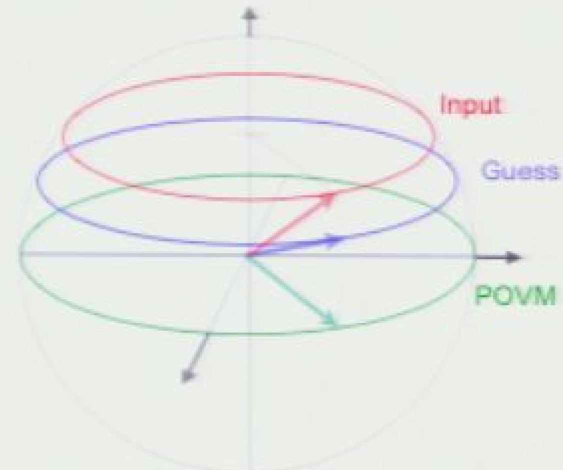

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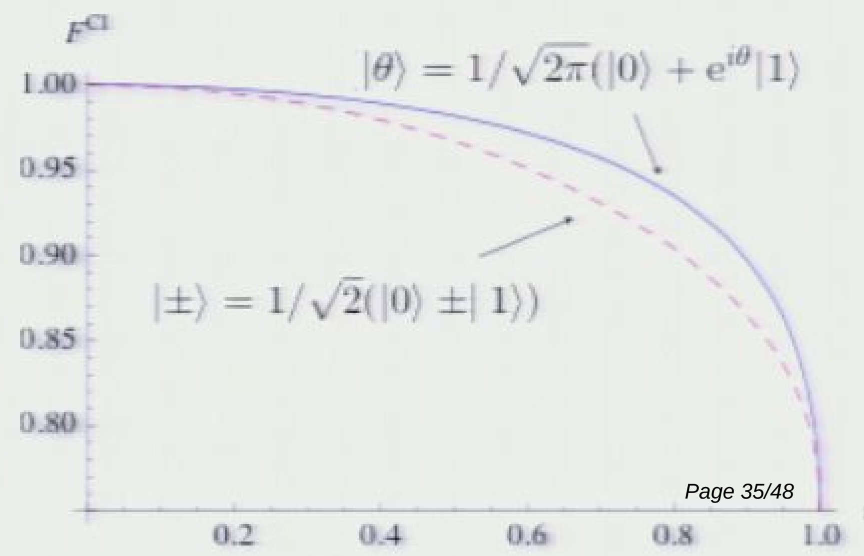
- Continuos POVM overcomes discrete S-G measurement.
- Mixedness improves classical Fidelity



- For a given POVM $\{|\xi_\chi\rangle\langle\xi_\chi|\}$ the optimal fidelity can be written as,

$$\mathcal{F} = \sum_\chi \sup_{\psi_\chi} \langle\psi_\chi| A_\chi |\psi_\chi\rangle = \sum_\chi \|A_\chi\|_\infty \quad \text{with} \quad A_\chi = \int d\phi/(2\pi) |\langle\xi_\chi|\psi_\phi\rangle|^2 |\psi_\phi\rangle\langle\psi_\phi|$$

$$A_\chi = \langle\xi_\chi|\Lambda|\xi_\chi\rangle \quad \text{with} \quad \Lambda = \int \frac{d\phi}{2\pi} |\psi_\phi\rangle|\psi_\phi\rangle\langle\psi_\phi|\langle\psi_\phi|$$

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- In general, no assumptions about the POVM nor the guess can be made, and we have to resort on numerical methods.

Semidefinite programming (SDP)

Minimize a linear *objective function* subject to *semidefiniteness* constraints involving symmetric matrices that are affine in the variables.

Primal problem:

$$p^* = \min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \;\text{ subject to }\; F(\mathbf{x}) = F_0 + \sum_i x_i F_i \geq 0$$

Dual problem:

$$d^* = \max_Z -\text{tr}(ZF_0) \;\text{ subject to }\; Z \geq 0 \;\text{ and }\; c_i = \text{tr}(ZF_i)$$

- $d^* \leq p^*$ (equality if feasible point exist such that $F(\vec{\mathbf{x}}) > 0$).

- Pure states: $\mathcal{F} = \max_K \operatorname{tr}(K \rho_0 \otimes \rho_0)$ $\rho_0 = |\psi_0\rangle\langle\psi_0|$

$$\begin{cases} K \geq 0 \\ \operatorname{tr}_B K = \mathbb{1}_A \\ K \text{ invariant under bilateral } U \otimes U \\ K \text{ separable}^{*} \end{cases}$$

* Hierarchy of constrains based on PPT symmetric extensions.

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Here we stay at first level of hierarchy, i.e., PPT $K^{\Gamma} \geq 0$. Hence, $\mathcal{F}^{\Gamma} \geq \mathcal{F}$

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- **Mixed states:** The objective function becomes non-linear, but we can linearize it by making use of *Uhlmann's Theorem*:

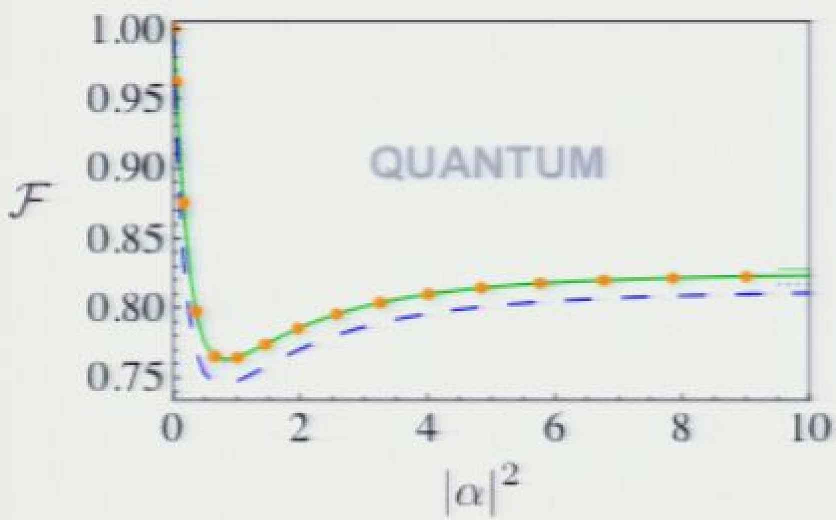
$$\mathcal{F} = \max_{\Psi_{\rm av}} |\langle\Psi_0|\Psi_{\rm av}\rangle|^2 = -\min_{\sigma_{\rm av}}(-\langle\Psi_0|\,\sigma_{\rm av}\,|\Psi_0\rangle)$$

where $|\Psi_0\rangle$ and $|\Psi_{\rm av}\rangle$ are *purifications** of ρ_0 and $\rho_{\rm av}$. * $\rho_A = \operatorname{tr}_B |\Psi\rangle_{AB}\langle\Psi|$

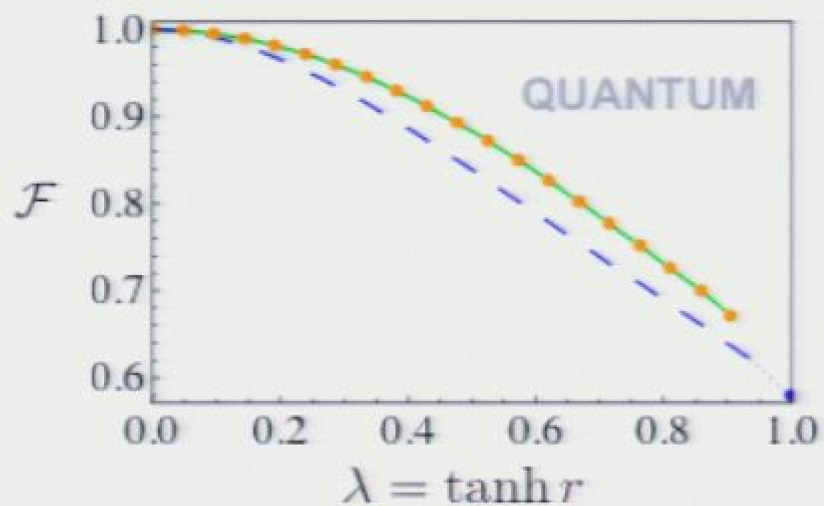
$$\begin{cases} \text{(i) } \operatorname{tr}_B \sigma_{\rm av} = \rho_{\rm av} = \operatorname{tr}_A(\rho_0 \otimes K) \\ \text{(ii) } \sigma_{\rm av} \geq 0 \text{ and } \operatorname{tr}\sigma_{\rm av} = 1 \\ \text{(iii) the same conditions on } K \text{ as above.} \end{cases}$$

Results for CV gaussian states

Coherent States



Squeezed States



- SDP results (PPT constrain) and truncation: $|\alpha\rangle \approx e^{-\alpha^2/2} \sum_{n=0}^{N} \alpha^n/\sqrt{n!}\,|n\rangle$
- Phase-measurement+optimal guess (max. eigenvalue of A)
- Guess from input ensemble.

$$\mathcal{F} = \int \frac{d\phi}{2\pi} |\langle \xi | \alpha e^{i\phi} \rangle|^2 |\langle \alpha | \alpha e^{i\phi} \rangle|^2.$$

- Analytic results for asymptotic limits

Restricted guess: $$\mathcal{F} = \int \frac{d\phi}{2\pi} |\langle \xi | \alpha e^{i\phi} \rangle|^2 |\langle \alpha | \alpha e^{i\phi} \rangle|^2 \xrightarrow{\alpha \to \infty} \sqrt{\frac{2}{3}}$$

$$|\langle \xi | \alpha e^{i\phi} \rangle| = e^{-\alpha^2/2} \sum_n e^{in\phi} \alpha^n / \sqrt{n!} \simeq (2\alpha^2/\pi)^{\frac{1}{4}} \exp[-\alpha^2 \phi^2]$$

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Optimal guess for phase-measurement: $$\mathcal{F} = ||A||_\infty = \lim_{p \to \infty} (\mathrm{tr} A^p)^{1/p}$$

$$(||A||_p)^p = \mathrm{tr} A^p = \int \prod_{j=1}^{p} d\phi_j \, p(\chi|\phi_j) \langle \alpha_j | \alpha_{j+1} \rangle. \qquad \alpha_{p+1} \equiv \alpha_1$$

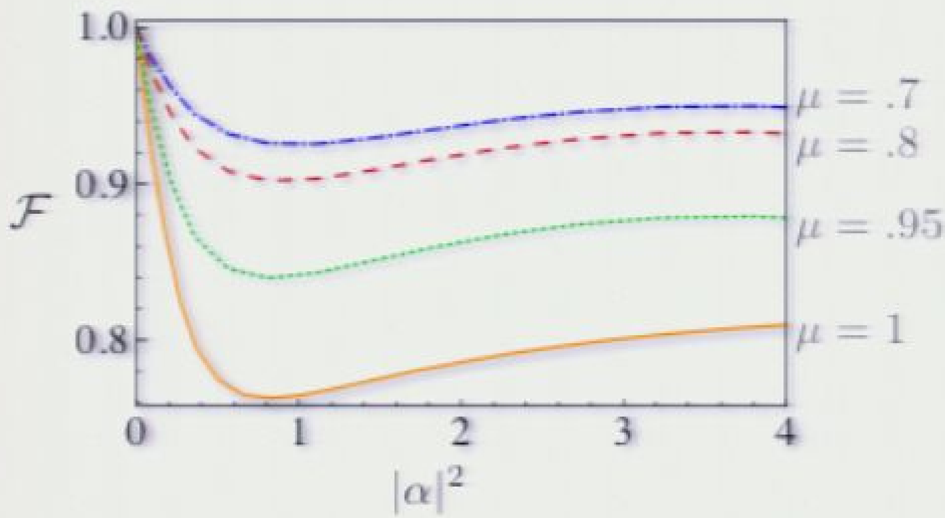
$$\langle \alpha_i | \alpha_j \rangle \approx \exp\{-\alpha^2 [i(\phi_i - \phi_j) + 1/2(\phi_i - \phi_j)^2]\}$$

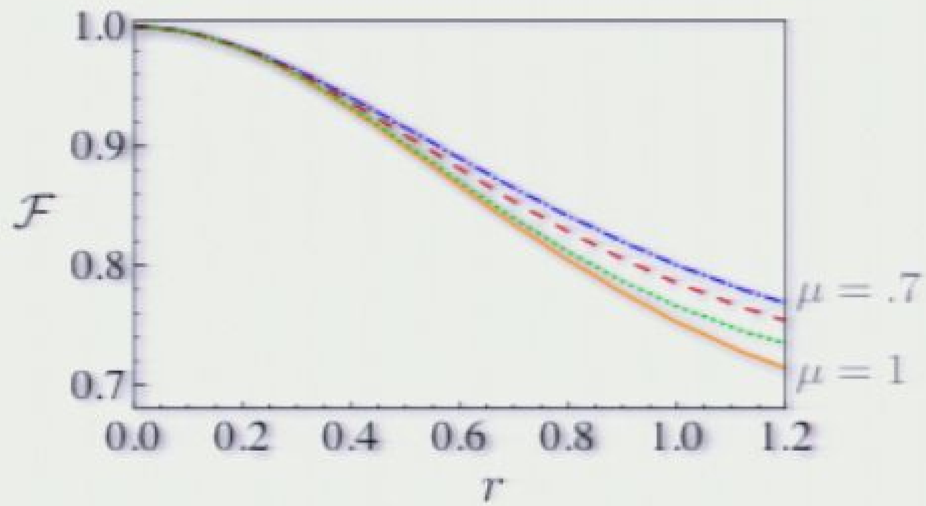
$$\mathrm{tr} A^p \simeq \left(\frac{2\alpha^2}{\pi}\right)^{p/2} \int d^p\phi \, e^{-\frac{\alpha^2}{2} \phi^t \cdot C_p \cdot \phi} = \frac{2^p}{\sqrt{\det C_p}}.$$

$$[C_p]_{ij} = 6\delta_{ij} - \delta_{i+1,j} - \delta_{i,1}\delta_{j,p}$$

$$\mathcal{F} = \lim_{p \to \infty} 2(\det C_p)^{1/2p} = 2(\sqrt{2} - 1) \approx 0.8284$$

• Mixed gaussian states





- Benchmark becomes higher with mixedness (for same displacement and squeezing parameters)

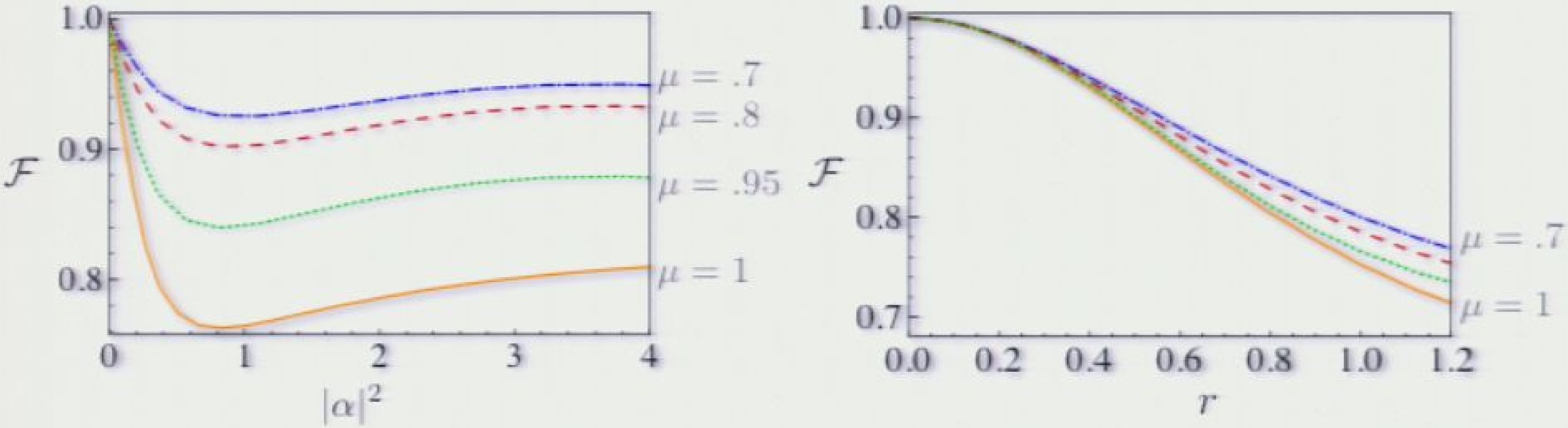
- For phase-measurement & guess in Ω, the effect is the opposite ($\mathcal{F}$ decreases with μ).

Conclusions

- Phase estimation of gaussian states (pure and mixed).
- Benchmarks for CV quantum storage & teleportation experiments
  - Phase covariant family of test states. Easy to implement.
  - Valid for mixed test states.
  - quantum state estimation revised.

Thanks for your attention

- **Mixed gaussian states**



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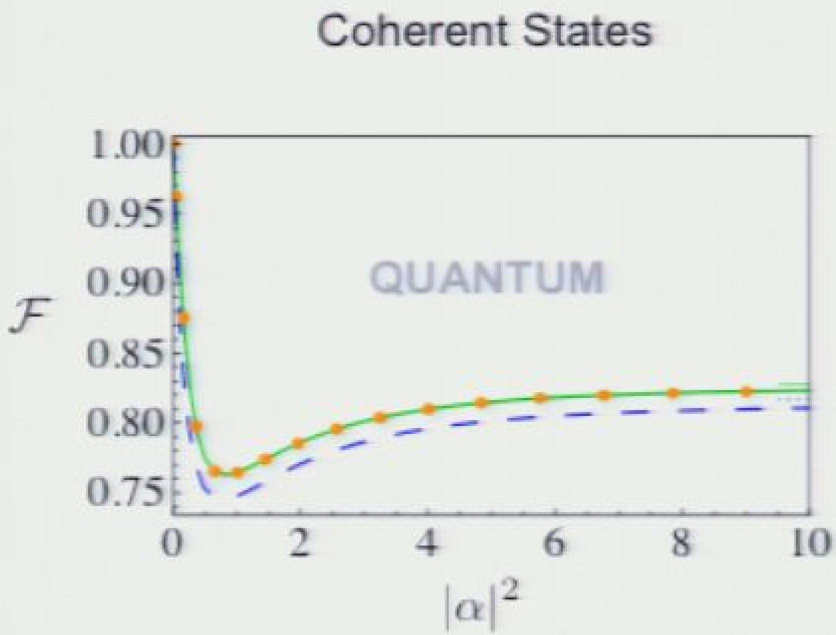

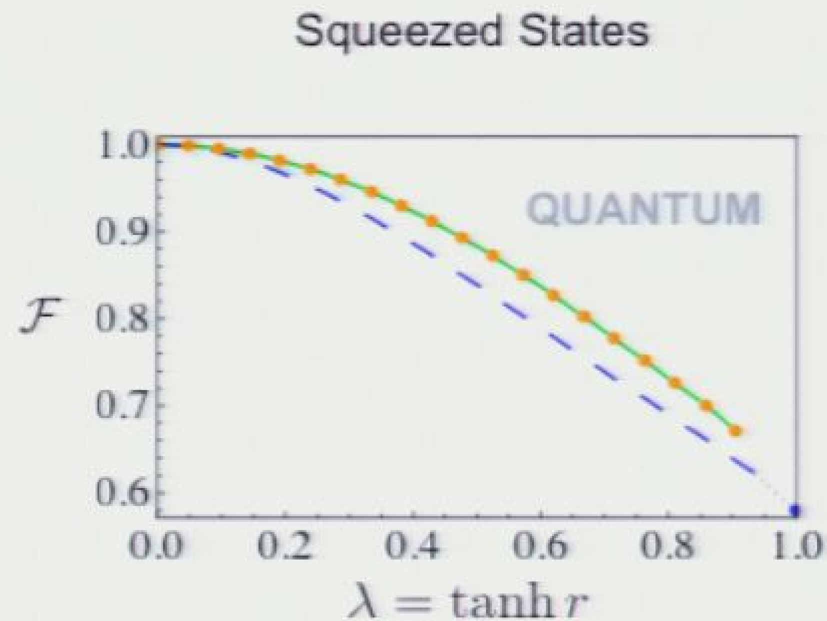

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