Title: Tomography for quantum diagnostics

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Abstract: Quantum information technologies have recorded enormous progress within the recent fifteen years. They have developed from the early stage of thought experiments into nowadays almost ready-to-use technology. In view of many possible applications the question of efficient analysis and diagnostics of quantum systems appears to be crucial. The quantum state is not an observable and as such it cannot be measured in the traditional sense of thisword. Information encoded in a quantum state may be portrayed by various ways yielding the most complete and detailed picture of the quantum object available. Due to the formal similarities between the quantum estimation and medical non-invasive 3D imaging, this method is also called quantum tomography. Many different methods of quantum tomography have been proposed and implemented for various physical systems. Experiments are being permanently improved in order to increase our ability to unravel even the most exquisite and fragile non-classical effects. Progress has been made not only on the detection side of tomography schemes. Mathematical algorithms too have been improved. The original linear methods based on the inverse Radon transformation are prone to producing artifacts and have other serious drawbacks. For example, the positivity of the reconstructed state required by quantum theory is not guaranteed. This may obviously lead to inconsistent statistical predictions about future events. For such reasons, the simple linear methods are gradually being replaced by statistically motivated methods, for example by Bayesian or maximum-likelihood (ML) [1,2] tomography methods.The quantification of all relevant errors is an indispensable but often neglected part of any tomographic scheme used for quantum diagnostic purposes. The result of quantum tomography cannot be reduced merely to finding the most likely state. What also matters is how much the other states, those being less likely ones, would be consistent with the registered data. In this sense, also states lying in the neighborhood of the most likely state should be taken into account for making future statistical predictions. For this purpose we introduce a novel resolution measure, which provides ``error bars\'\' for any inferred quantity of interest. This is illustrated with an example of the diagnostics of non-classical states based on the value of the reconstructed Wigner function at the origin of the phase space. We show that such diagnostics is meaningful only when some prior information on the measured quantum state is available. In this sense quantum tomography based on homodyne detection is more noisy and more uncertain than widely accepted nowadays. Since the error scales with the dimension, the choice of a proper dimension of the reconstruction space is vital for successful diagnostics of non-classical states. There are two concurring tendencies for the choice of this dimension. When the reconstruction space is low-dimensional, the reconstruction noise is kept low, however there may not be enough free parameters left for fitting of a possibly high-dimensional true state. In the case of high-dimensional reconstruction space, the danger of missing important components of the true state is smaller, however the reconstruction errors may easily exceed acceptable levels. These issues will be discussed in the context of penalization and constraints for maximizing the likelihood [3]. The steps described above are the necessary prerequisites for the programme of objective tomography, where all the conclusions should be derived on the basis of registered data without any additional assumptions. New resolution measure based on the Fisher information matrix may be adopted for designing optimized tomography schemes with resolution tuned to a particular purpose. Quantum state tomography may serve as a paradigm for estimating of more complex objects, for example process tomography. [1] Z. Hradil, Phys. Rev. A 55, R1561 (1997). [2] Z. Hradil, D. Mogilevtsev, and J.Rehacek, Phys. Rev. Lett. 96, 230401 (2006). [3] J.Rehacek, D. Mogilevtsev and Z. Hradil, New J. Phys 8. April, 043022 (2008)

Outline

- Motivation: Inversion problems
- Elements of quantum mechanics, estimation and objective MaxLik tomography
- Diagnostics of inferred variables
- Example of homodyne tomography
- Penalized MaxLik and Schwarz info
- Resource analysis: What is feasible?

Linear inverse problems

Tomography = linear inverse problems with constraints

$$I_j = \Sigma_k \, c_{jk} \, \mu_k$$

detected mean values	I_j	$j = 1,2,\ldots M$
reconstructed signal	μ_k	$k = 1,2,\ldots N$

Over-determined problems	**$M > N$**
Well defined problems	**$M = N$**
Under-determined problems	**$M < N$**

Resources for homodyne tomography

- About 10^5 detected events
- Due to the redundancy of non-orthogonal projections about 10^4 events are independent

- **Density matrix might be estimated up to the dimension 100 !?!**
- **If reconstruction is done on 10 dim subspace, only 1% of the potential is used ?!?**

Von Neumann Measurement and its generalization

Signal: density matrix $\rho \geq 0$

Probability in Quantum Mechanics:

$$p_j = \mathrm{Tr}(\rho A_j)$$

Measurement: elements of positive-valued operator measure (POVM)

$$A_j \geq 0$$

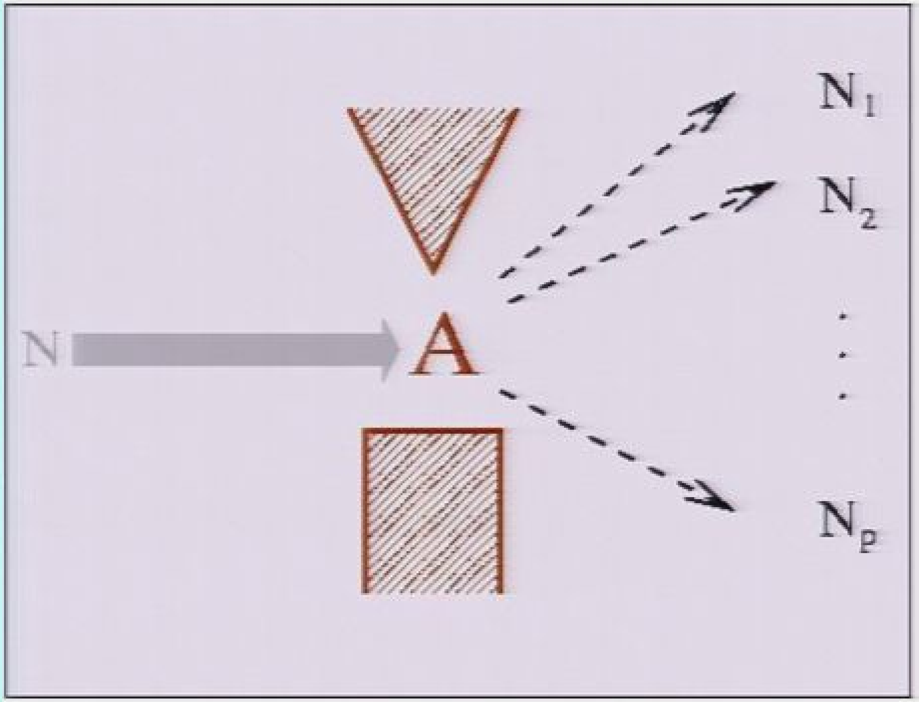
Relation of completeness $\sum_j A_j = 1$

Over/un-completeness $\sum_j A_j = G \geq 0$

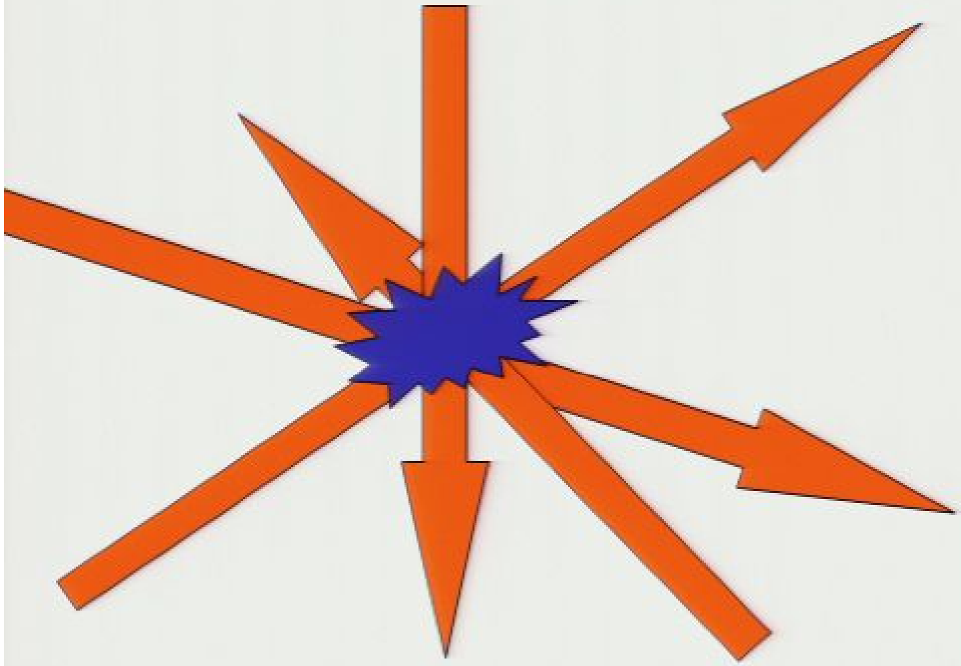
$$G^{-1/2} / \qquad \ldots\ldots\ldots \qquad / G^{-1/2}$$

$$\sum_j G^{-1/2} A_j G^{-1/2} = 1_G$$

Concept of objective tomography



Geometry: overlap of states



Projector

$A_i = |y_i\rangle\langle y_i|$

Overlap of all projectors

$\Sigma_i\ |\langle y_i|\varphi\rangle|^2$

Maximum overlap

$\{\Sigma_i\ |y_i\rangle\langle y_i|\}|\varphi\rangle = \lambda|\varphi\rangle$

Maximum Likelihood Estimation (1922)

Sir Ronald Aylmer Fisher, FRS (17 February 1890 – 29 July 1962)
http://digital.library.adelaide.edu.au/coll/special/fisher/papers.html

- Maximum Likelihood (MaxLik) principle is not a rule that requires justification: Bet Always On the Highest Chance!
- Numerous applications in signal analysis, optics, geophysics, nuclear physics,…
- A. Witten, The application of ML estimator to tunnel detection, Inverse Problems 7(1991), 49.
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Maximum Likelihood Tomography

- Likelihood $\mathcal{L}$ quantifies the degree of belief in certain hypothesis under the condition of the given data.
- MaxLik principle selects the most likely configuration
- Information is updated according to the Bayes rule prior probability → posterior probability

$$P(\rho|D) = P(D|\rho)\, p(\rho)\, [p(D)]^{-1}$$

Generic reconstruction scheme

Log-likelihood for generic measurement

$$\log \mathcal{L} = \sum_i N_j \log p_j / (\sum_k p_k)$$

(probabilities are mutually normalized)

Equivalent formulation: estimation of parameters with Poissonian probabilities and unknown mean λ (constrained MaxLik by Fermi)

$$\log \mathcal{L} = \sum_j N_j \log (\lambda\, p_j) - \lambda \sum_j p_j$$

Easy derivation

Differentiate formally the Log-likelihood with the constraint

$$\log \mathcal{L}(\rho) = \sum_i N_i \log p_j(\rho) - \lambda \, \mathrm{Tr}(\rho) \qquad /\partial\rho_{kl}$$

$$\sum_i N_i/p_j(\rho) \quad (A_i)_{kl} \quad |k\rangle\langle l| - \lambda \, \delta_{kl} \, |k\rangle\langle l| = 0 \qquad /\rho$$

$$\sum_i N_i/p_j(\rho) \, A_i \, \rho = \lambda \, \rho \qquad /\mathrm{Tr}\rho = 1$$

$$R \, \rho = \rho$$

Equivalently $\quad \rho = R \, \rho \, R$

$$R = \sum_i \left(f_i / \mathrm{Tr}(\rho A_i)\right) A_i$$

(Log)-likelihood is convex functional over the convex manifold of density matrices = convex optimization

MaxLik interpretation

Linear inversion

$\sum_k A_k \equiv 1$ $\qquad\qquad$ $\mathrm{Tr}(\rho A_k) = f_k$

MaxLik inversion

$\sum_k A'_k = 1_G$ $\qquad\qquad$ $\mathrm{Tr}(\rho A'_k) \equiv f_k$

where $A'_k = (f_k/p_k)\, A_k$

MaxLik in terms of Quantum Mechanics

Fluctuations in the k-th channel

$$(\Delta\varepsilon_k)^2 = \mathrm{Tr}(\rho A_k)\,[1 - \mathrm{Tr}(\rho A_k)]$$

All the observations cannot be equally trusted!

MaxLik estimation in 3 steps:

1. Re-define POVM elements $A_k \Rightarrow \mu_k A_k$
2. Postulate mean values $\mu_k \mathrm{Tr}(\rho A_k) = f_k$
3. Postulate the closure relation

$$\Sigma_k \mu_k A_k = 1$$

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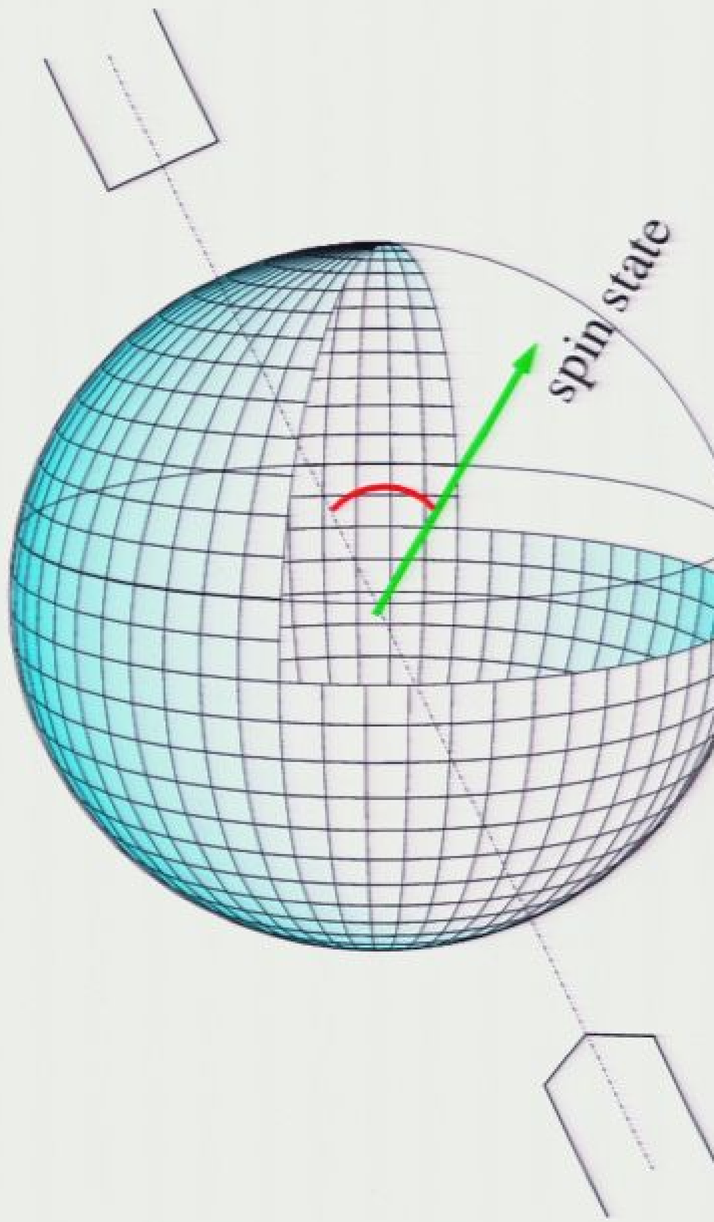
Why the optimal estimation must be nonlinear:

- Various projections are counted with different accuracy.
- Accuracy depends on the unknown quantum state.
- Optimal estimation strategy must re-interpret the registered data and estimate the state simultaneously.
- Optimal estimation should be nonlinear. MaxLik is doing this.



Diagnostics

- MaxLik reconstruction characterizes the estimated state as random variable with Gaussian posterior distribution

- Any prediction based on tomography e.g. fidelity, Wigner function at origin, etc. is uncertain

$$Q = \langle Q \rangle_{ML} \pm \Delta Q$$

- Quantum state = set of $M = d^2-1$ parameters
- Ω_i ... generator basis

$$\rho = \Omega_0/d + \sum_I \rho_i \Omega_i\,, \quad \rho^{ML} = \Omega_0/d + \sum_I \rho_i^{ML} \Omega_i\,,$$

- Relative coordinate $r_i = \rho - \rho^{ML}$, $\mathbf{r} = (r_0, r_1, \dots r_{M-1})$

- Posterior (multi-normal) distribution

$$P^\rho(r) = (2\pi)^{-M/2} (\det F)^{1/2} \exp(-\tfrac{1}{2}\, \mathbf{rFr}\ )$$

Fisher information matrix, $P = \sum_i p_i$

$$F_{jk} = N^2 \sum_i 1/N_i\ \partial r_j\, [p_i/P]\ \ \partial r_k\, [p_j/P]$$

- Performance measure linear in quantum state

$$z = \mathrm{Tr}(\mathbf{Z}\rho)$$

- Wigner function at origin $\quad Z = \sum_n\ (-1)^n\, |n\rangle\langle n|$
- Fidelity $\quad Z = |\psi_{true}\rangle\langle\, \psi_{true}\, |$

- Expansion in fixed operator basis

$$Z = \sum_i z_i \Omega_i ; \ |\mathbf{z}\rangle = (z_0, z_1, \ldots z_{M-1})$$

- Experimental uncertainty

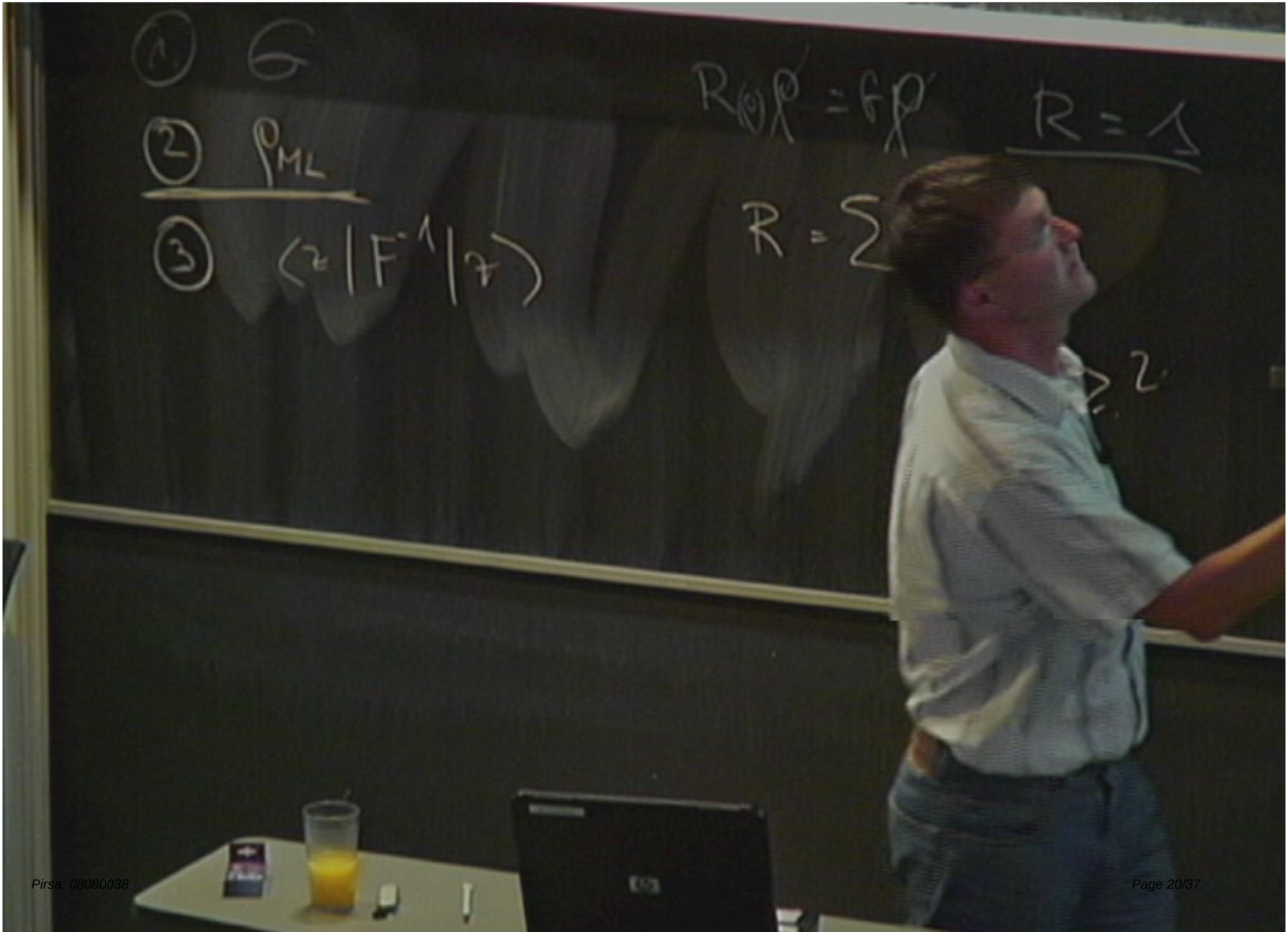
$$(\Delta z)^2 = \langle \mathbf{z}|\mathbf{F}^{-1}|\mathbf{z}\rangle$$

- Statistics of inferred variable

$$z = \mathrm{Tr}(Z\rho_{ML}) \pm \{\langle \mathbf{z}|\mathbf{F}^{-1}|\mathbf{z}\rangle\}^{1/2}$$

- Experimental uncertainty relations

$$(\Delta a)^2 (\Delta b)^2 \geq |\langle a|F^{-1}|b\rangle|^2$$


1 G
2 ρ_{ML}
3 $\langle z|F^{-1}|z\rangle$
$R = 1$
$R = \Sigma$

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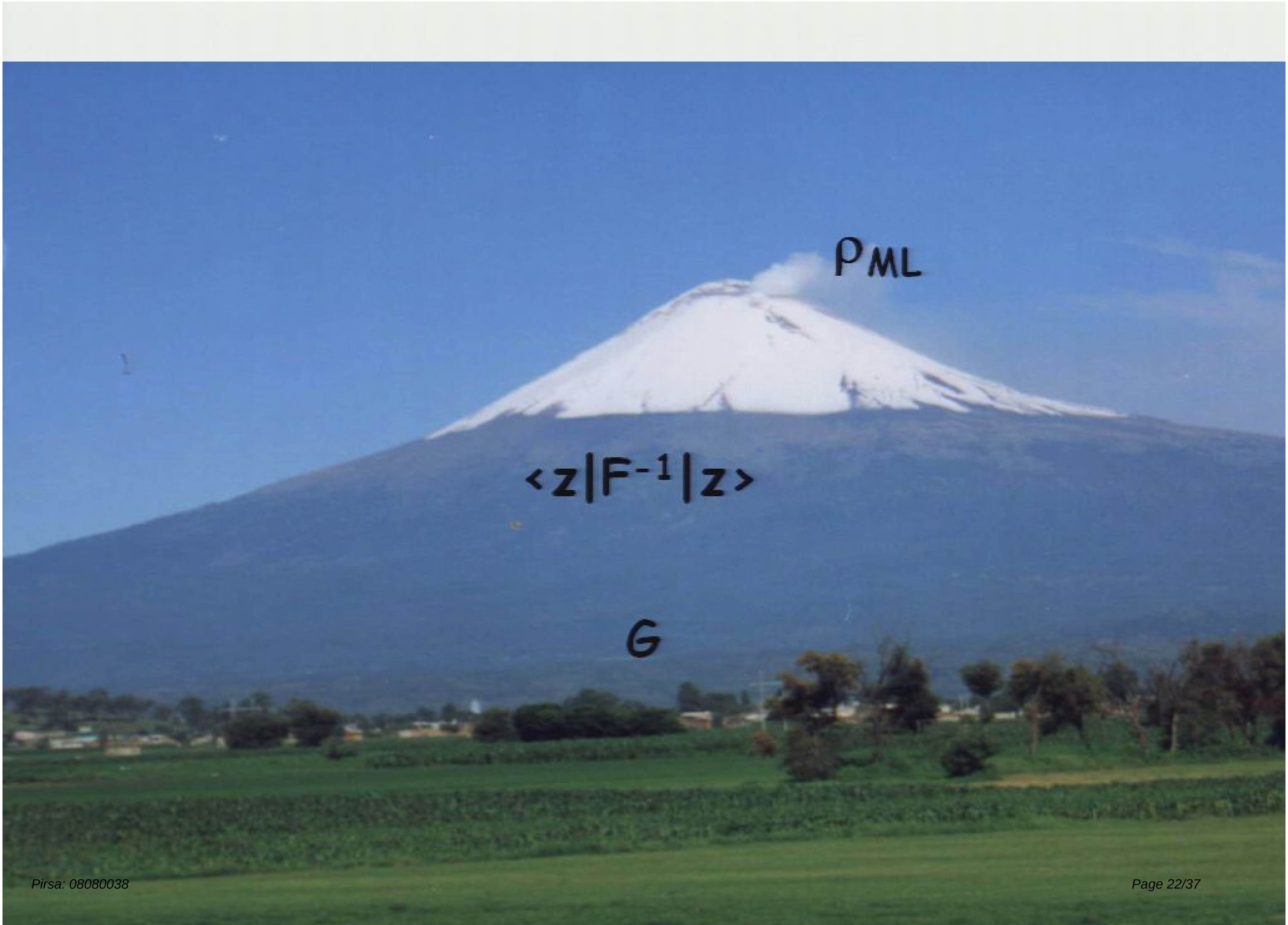

ρ_{ML}
$\langle z|F^{-1}|z\rangle$
G

Several examples

- Phase estimation
- Spin and entangled spin state tomography
- Transmission tomography
- Reconstruction of photocount statistics
- Image reconstruction
- Vortex beam analysis
- Quantification of entanglement
- Operational quantum information
- Reconstruction of neutron wave packet
- Full reconstruction based on on/off detection
- Reconstruction of CP maps
- Reconstruction based on homodyne detection

Simulation of realistic homodyne tomography

- Standard setup used for detection of negative Wigner function
- 6 phase cuts in phase space, efficiency v=0,8
- 1,2 .10^5 detected events
- ML estimation using 1000 iterations
- Simulation repeated 1000 times

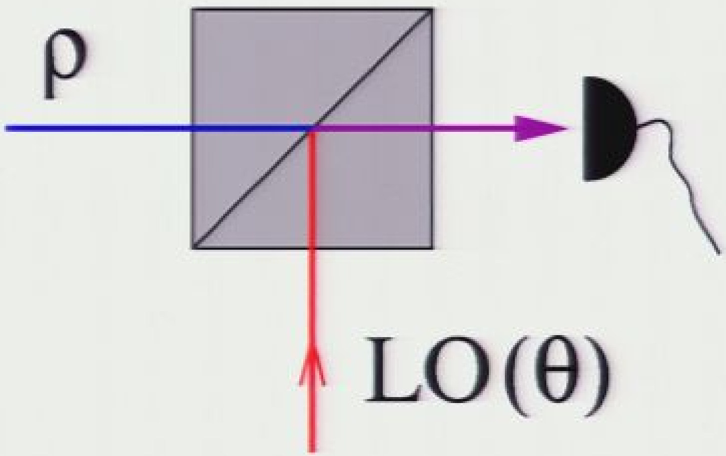
Reconstruction of Wigner function

- rotated quadratures

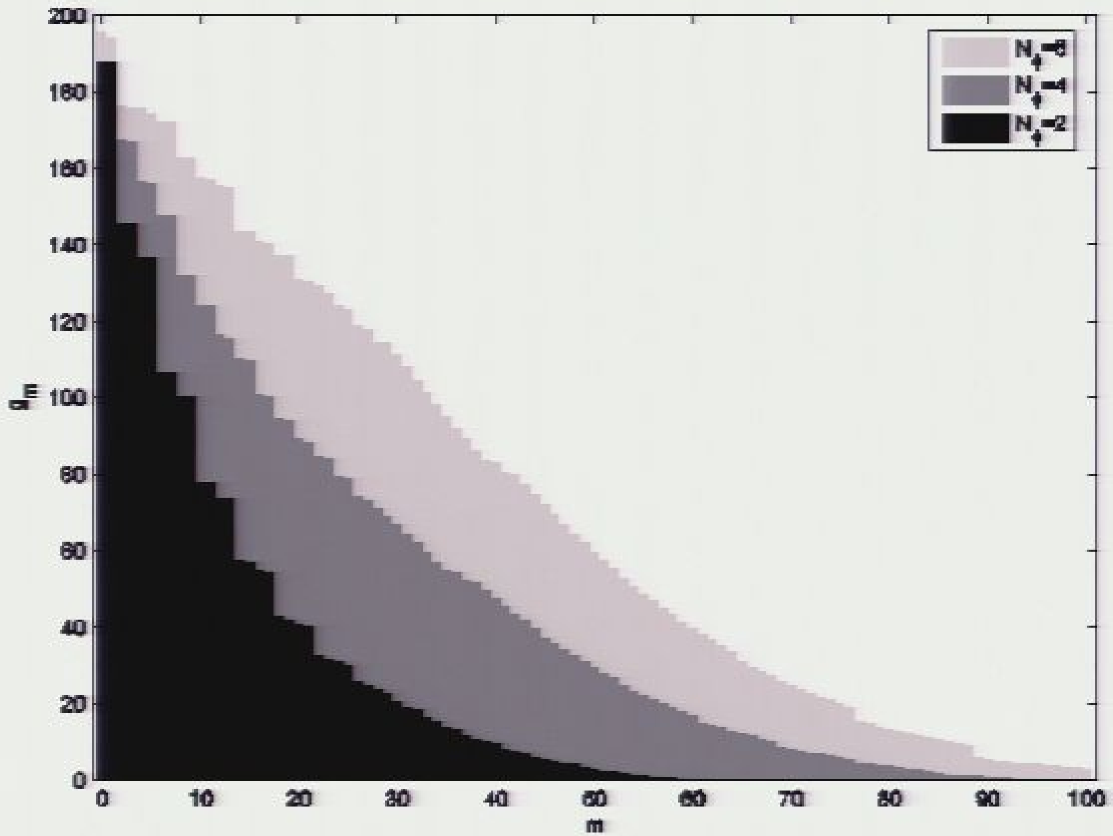
$$\hat{x}_\theta = \sqrt{\frac{\hbar}{2}}\left[\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}\right], \qquad \hat{x}_{\theta+\pi/2} = \frac{\sqrt{\hbar}}{i\sqrt{2}}\left[\hat{a}e^{-i\theta} - \hat{a}^\dagger e^{i\theta}\right]$$

- marginal distributions for $P_\theta(x_\theta)$

Homodyne measurement

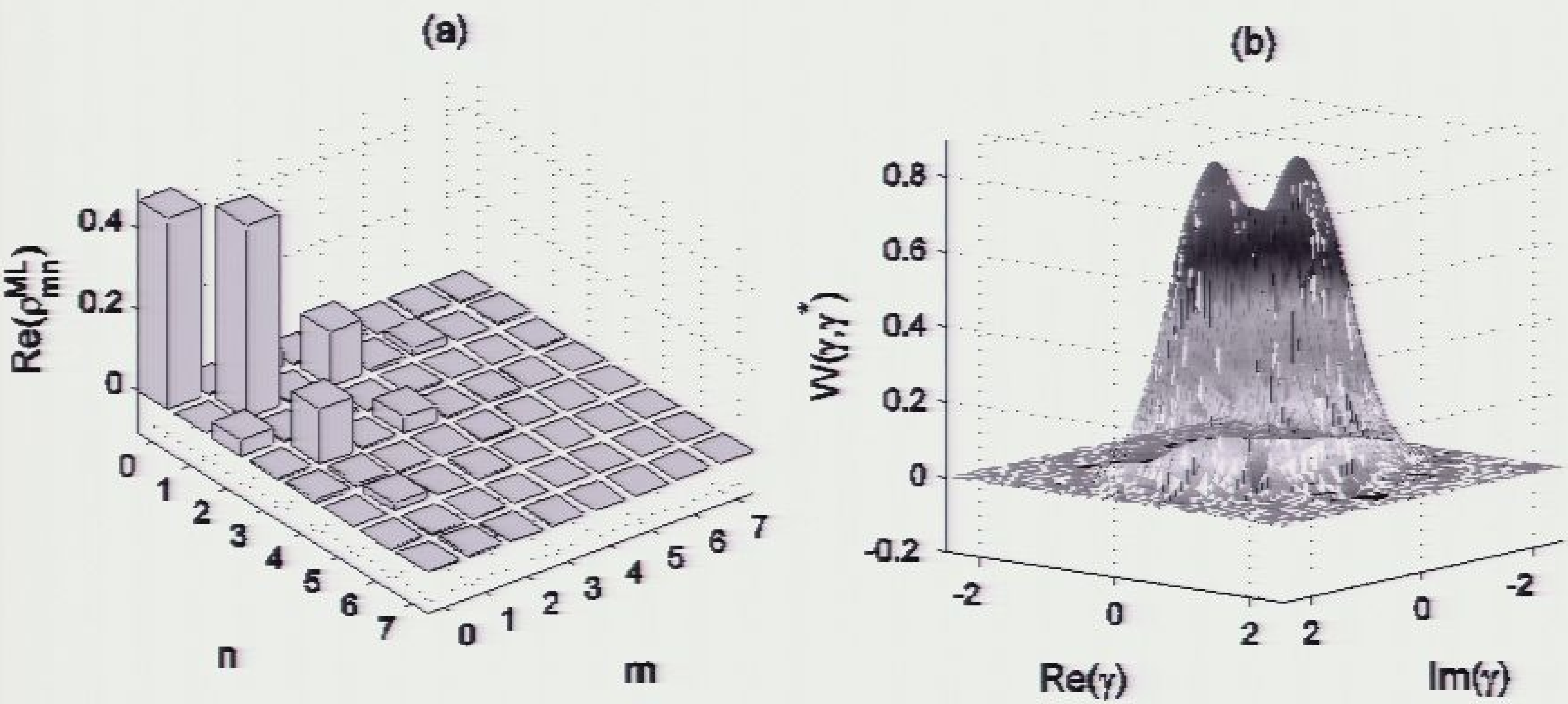


Homodyne tomography: field of view given by G

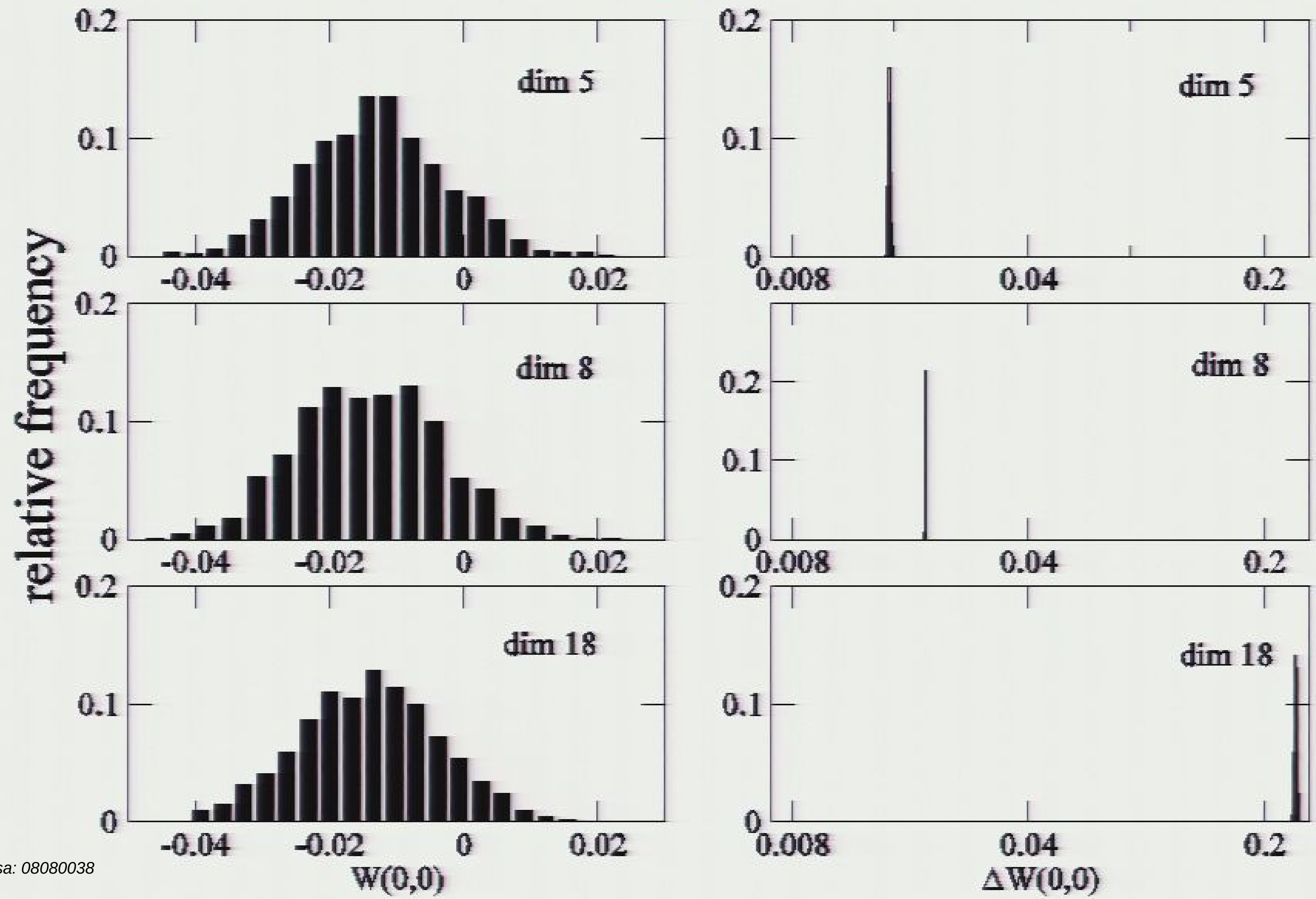


Rationale behind: Projections into rotated "quadrature eigenstates" are not sufficiently resolving. The delimited Hilbert space is always too large for data fitting.

MaxLik reconstruction

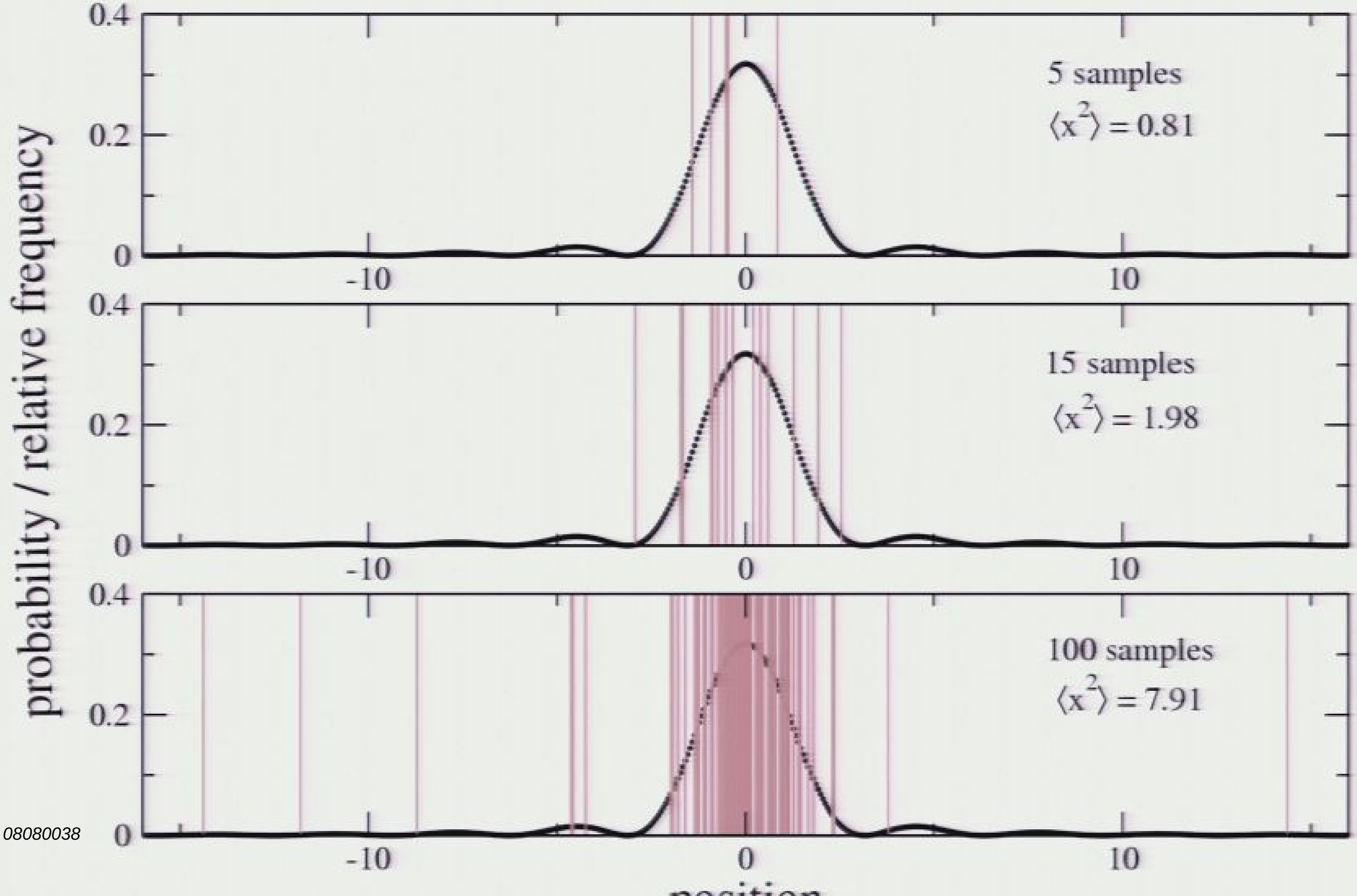


Diagnostics of Wigner at origin



Fingerprint or footprint?

Simple arguments: estimation of the width of $\text{sinc}^2 x$



Why errors cannot be sometimes simulated:

"Classical" example: Assume the statistics of variable s estimated on the basis of N trials

$$\mathbf{s} = (1/N)\ \Sigma_i\ \mathbf{x}_i^2$$

Singular Statistics: $p(x) = 1/\pi\ \mathrm{sinc}^2 x$

$\langle s \rangle_{theory} = \infty$ but

$\langle s \rangle_{exp} \sim N^{1/2}$, $\langle s^2 \rangle_{exp} \sim N^{3/2}$ and

$SNR \sim N^{1/4}$

"Quantum" example:

$$\rho_n = (1-1/n)|0\rangle\langle 0| + 1/n\ |n\rangle\langle n|$$

$\langle n \rangle = 1$ independent of n

but

$\langle k\ \rho\ k \rangle = 0$ for any k and n going to ∞

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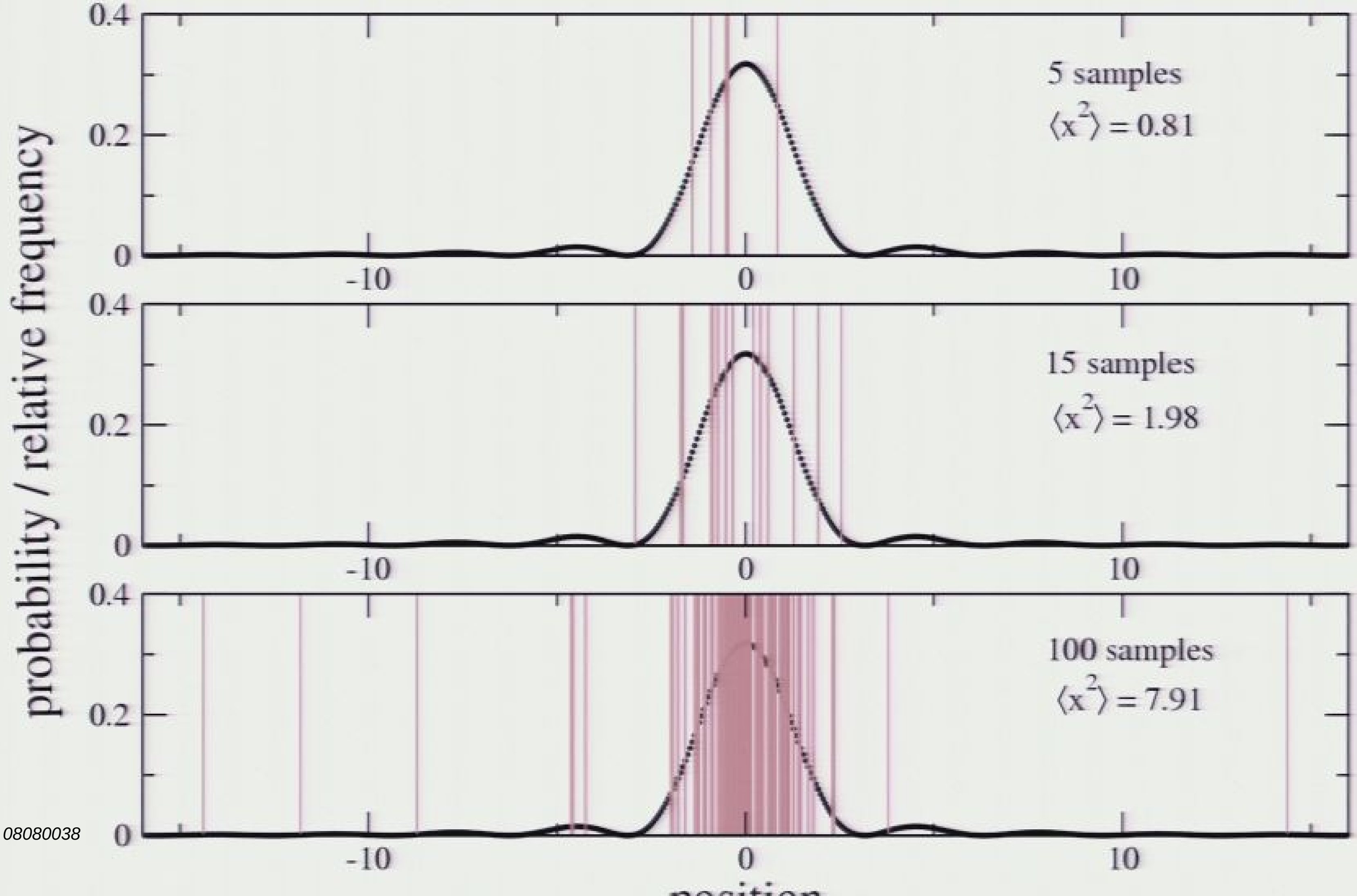

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Penalized MaxLik estimation

Hint: Normalize the Likelihood. The normalization term is state independent but dimension dependent!

Modified Schwarz information

$$I_{MS} = \log \mathcal{L}(\rho) - \tfrac{1}{2} M \log N + \tfrac{1}{2} M \log(2\pi) - \tfrac{1}{2} \log \det F$$

M ... dimension of estimated variable (density matrix)
N ... dimension of data set (# of POVM elements)

Some numerical simulations show that relatively small dimension M is sufficient for successful data fitting.

Resource analysis

To control the quantum system means to control all relevant errors….

•Pure state in dimension d: 2d -1 real parameters
Estimation is not a convex problem…

•Density matrix $d^2 - 1$ real parameters
Fisher info matrix: $\frac{1}{2}(d^2-1)(d^2-2)$ real parameters

•CP maps: $d^2(d^2-1)$ real parameters
Fisher info matrix for CP maps: $\frac{1}{2}d^2(d^2-1)(d^4-d^2-1)$ real parameters

Quantum computation with 5 qbits: $d = 2^5 = 32$
Quantum state: $\sim 10^3$ parameters
Fisher info: $\sim 10^6$ parameters
CP maps: $\sim 10^6$ parameters
Fisher info of CP maps: $\sim 10^{12}$ parameters

MaxLik tomo in few steps

- Specify quantum measurement
- Find field of view given by G operator
- Solve MaxLik equation
- Find Fisher information matrix
- Find spread of desired inferred variable