Title: Pretty-Good Tomography

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Abstract: I\'ll survey recent results from quantum computing theory showing that,if one just wishes to learn enough about a quantum state to predictthe outcomes of most measurements that will actually be made, then itoften suffices to perform exponentially fewer measurements than wouldbe needed in quantum state tomography. I\'ll then describe the resultsof a numerical simulation of the new quantum state learning approach.The latter is joint work with Eyal Dechter.

Pretty-Good Tomography

Scott Aaronson

MIT

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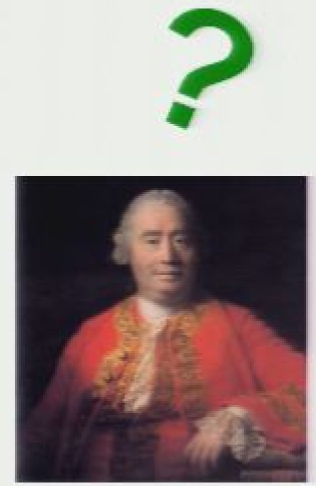
Seems to bolster the arguments of quantum computing skeptics who think quantum mechanics will break down in the "large N limit"

Really we're talking about Hume's Problem of Induction…

You see 500 ravens. Every one is black. Why does that give you any grounds whatsoever for expecting the next raven to be black?



The answer, according to computational learning theory: In practice, we always restrict attention to some class of hypotheses vastly smaller than the class of all logically conceivable hypotheses

Probably Approximately Correct (PAC) Learning

Set S called the **sample space**

Probability distribution D over S

Class C of **hypotheses**: functions from S to {0,1}

Unknown function $f \in C$

Goal: Given $x_1, \ldots, x_m$ drawn independently from D, together with $f(x_1), \ldots, f(x_m)$, output a hypothesis $h \in C$ such that

$$\Pr_{x \in D}\left[h(x) = f(x)\right] \geq 1 - \varepsilon,$$

with probability at least $1-\delta$ over $x_1, \ldots, x_m$

Occam's Razor Theorem

Valiant 1984: If the hypothesis class C is finite, then any hypothesis consistent with

$$m = O\left(\frac{1}{\varepsilon}\log\frac{|C|}{\delta}\right)$$

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But the number of quantum states is infinite!

And even if we discretize, it's still **doubly** exponential in the number of qubits!

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A Hint of What's Possible…

Theorem [A. 2004]: Any n-qubit quantum state can be "simulated" using O(n log n log m) classical bits, where m is the number of (binary) measurements whose outcomes we care about.

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Let $E=(E_1,\ldots,E_m)$ be two-outcome POVMs on an n-qubit state ρ. Then given (classical descriptions of) E and ρ, we can produce a classical string of

$$\tilde{O}\left(\frac{n\log n}{\varepsilon^2}\cdot\log m\right)$$

bits, from which $\mathrm{Tr}(E_i\rho)$ can be estimated to within additive error ε given any E_i (without knowing ρ).

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Quantum Occam's Razor Theorem
[A. 2006]

Let ρ be an n-qubit state, and let D be a distribution over two-outcome measurements.

Suppose we draw measurements $E_1,\ldots,E_m$ independently from D, and then find a hypothesis state σ that minimizes

$$\sum_{i=1}^{m}\left(\mathrm{Tr}\left(E_i\sigma\right)-b_i\right)^2 \qquad (b_i = \text{outcome of } E_i)$$

Then $\Pr_{E\in D}\left[\left|\mathrm{Tr}\left(E\sigma\right)-\mathrm{Tr}\left(E\rho\right)\right|\leq\gamma\right]\geq 1-\varepsilon$

with probability at least 1-δ over $E_1,\ldots,E_m$, provided

$$m\geq\frac{C}{\gamma^4\varepsilon^2}\left(\frac{n}{\gamma^4\varepsilon^2}\log^2\frac{1}{\gamma\varepsilon}+\log\frac{1}{\delta}\right) \qquad (C \text{ a constant})$$

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No!

Suppose $U|x\rangle=(-1)^{f(x)}|x\rangle$, for some random Boolean function $f:\{0,1\}^n \rightarrow \{0,1\}$

Then the values of f(x) constitute 2^n **independently accessible** bits to be learned about

Yet each measurement provides at most n of the bits, by Holevo's Theorem

Hence, no analogue of my learning theorem is going to be true

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Custom Convex Programming Method

[E. Hazan, 2008]

Let $f(\sigma) = \sum_{i=1}^{m} \left(\mathrm{Tr}(E_i \sigma) - b_i\right)^2$

Set $S_0 := I/N$

For $t:=0$ to ∞

Compute smallest eigenvector v_t of $\nabla f(S_t)$

Compute step size α_t that minimizes $f(S_t+\alpha_t(v_t v_t^* - S_t))$

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Theorem (Hazan): This algorithm returns an ε-optimal solution after only $\log(m)/\varepsilon^2$ iterations.

Implementation
[A. & Dechter 2008]

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Code available on request

Using MIT's computing cluster, we then did numerical simulations to check experimentally that the learning theorem is true

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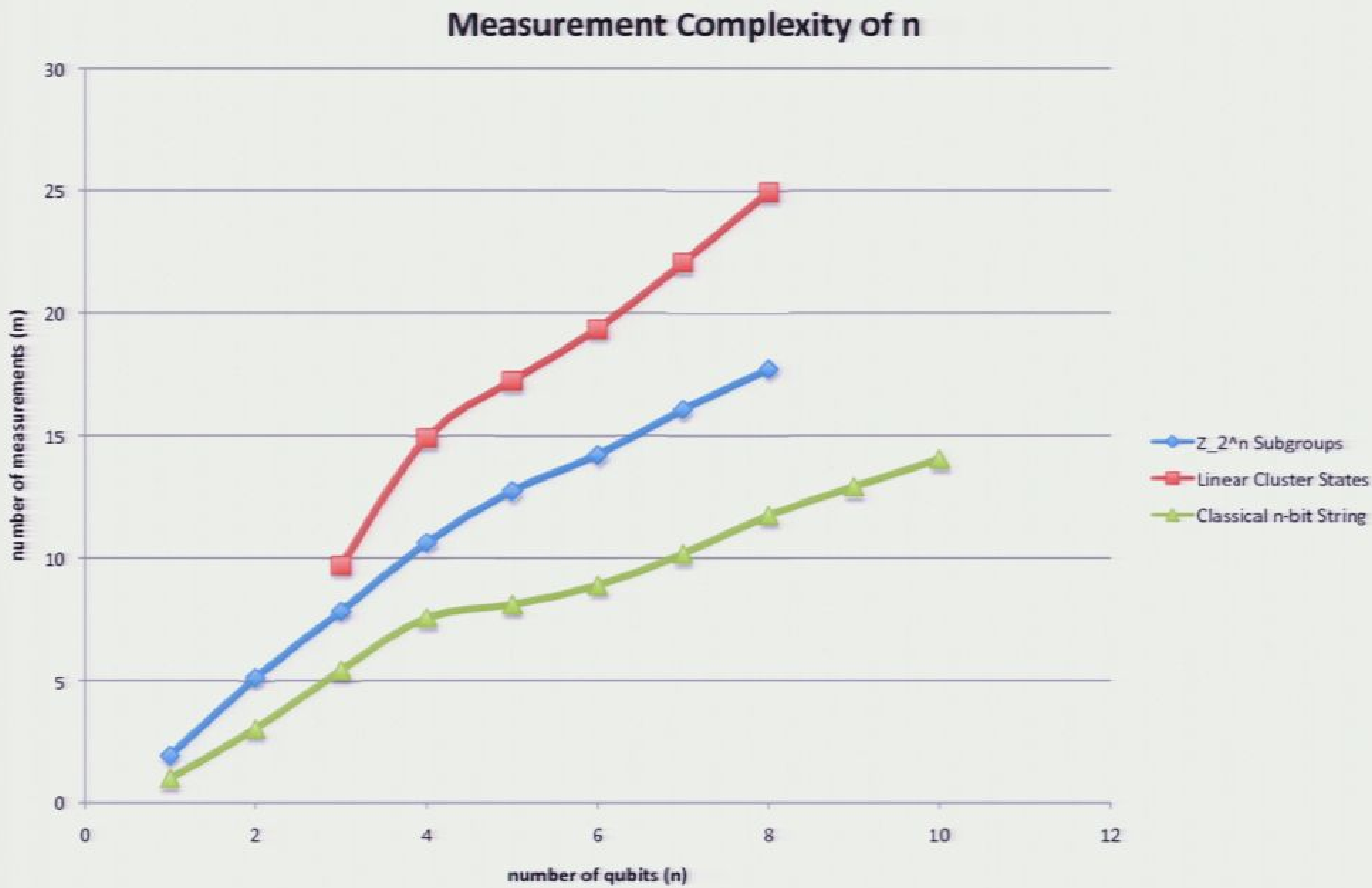
2. **Linear Cluster States.** States are n qubits, prepared by starting with $|+\rangle^{\otimes n}$ and then applying conditional phase ($P|xy\rangle=(-1)^{xy}|xy\rangle$) to each neighboring pair. Measurements check three randomly-chosen neighboring qubits, in a basis like $\{|0\rangle|+\rangle|0\rangle,|1\rangle|+\rangle|1\rangle,|0\rangle|-\rangle|1\rangle\}$. Acceptance probability is always ¾.

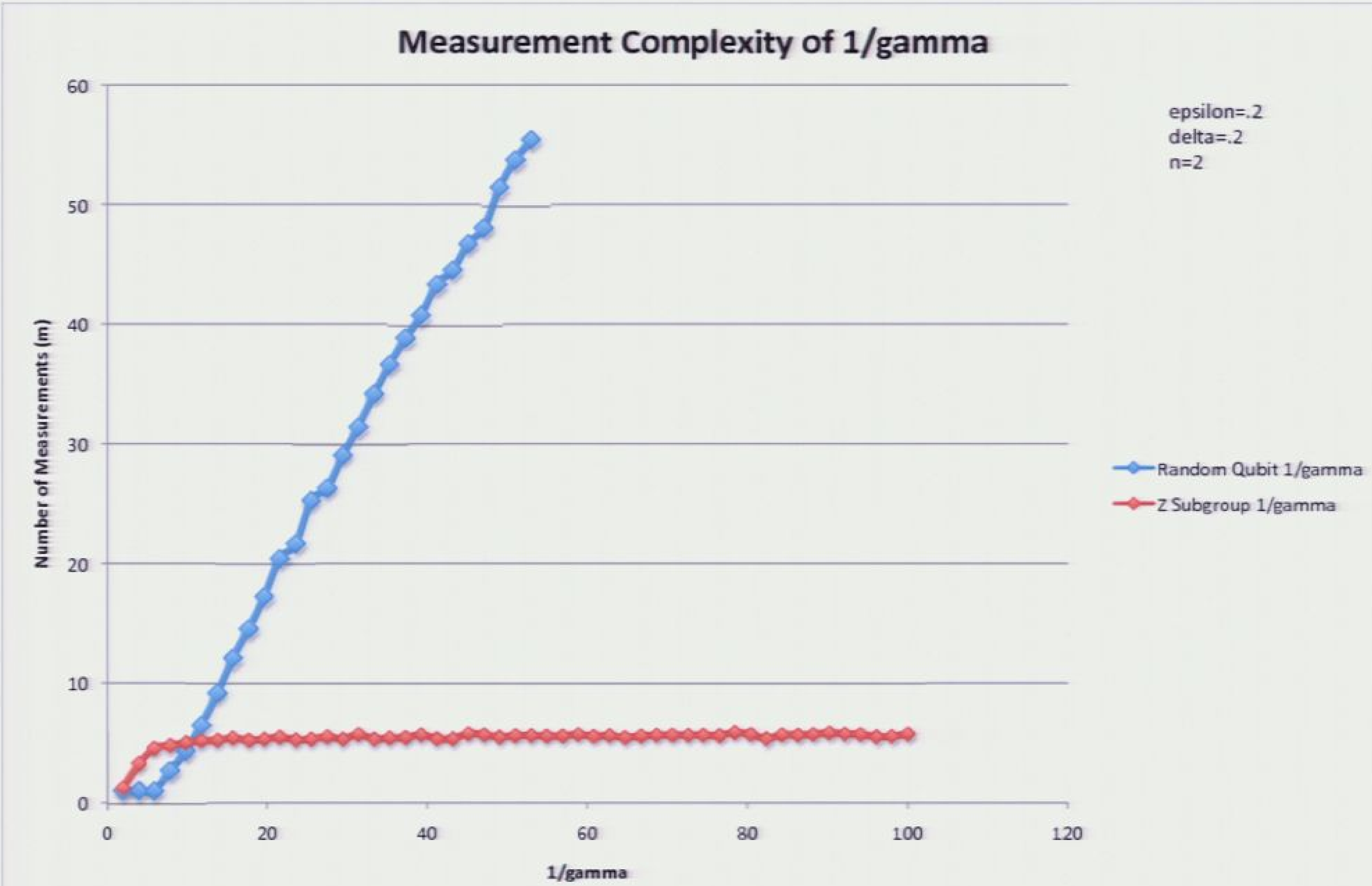
3. Z_2^n Subgroup States. Let H be a subgroup of $G=Z_2^n$ of order 2^{n-1}. States $\rho=|H\rangle\langle H|$ are equal superpositions over H. There's a measurement E_g for each element $g\in G$, which checks whether $g\in H$:

$$E_g = \frac{1}{2} I_n + \frac{1}{4} U_g + \frac{1}{4} U_g^*$$

where $U_g|h\rangle=|gh\rangle$ for all $h\in G$. E_g accepts with probability 1 if $g\in H$, or ½ if $g\notin H$.

Inspired by [Watrous 2000]; meant to showcase pretty-good tomography with non-commuting measurements.


Measurement Complexity of n
number of measurements (m)
number of qubits (n)
0
5
10
15
20
25
30
0
2
4
6
8
10
12
Z_2^n Subgroups
Linear Cluster States
Classical n-bit String


Measurement Complexity of 1/gamma
epsilon=.2
delta=.2
n=2
Number of Measurements (m)
1/gamma
Random Qubit 1/gamma
Z Subgroup 1/gamma
0
10
20
30
40
50
60
0
20
40
60
80
100
120

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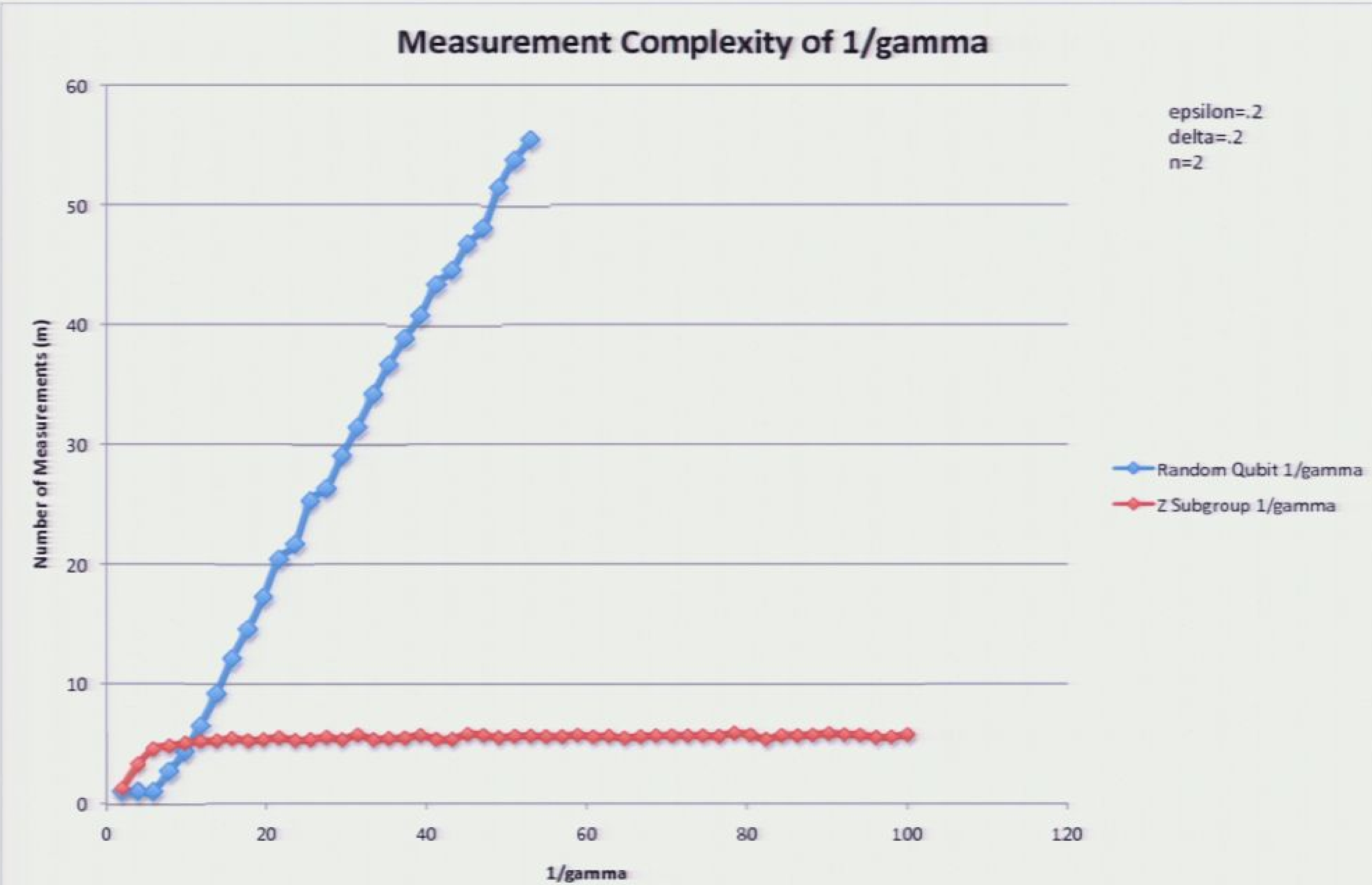
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