Title: Optimal linear tomography of quantum states and processes with tight POVMs

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Abstract: We introduce the concept of tight POVMs. In analogy with tight frames, these are POVMs that are as close as possible to orthonormal bases for the space they span. We show that tight rank-one POVMs define the exact class of optimal measurements for linear tomography of quantum states. In this setting they are equivalent to complex projective 2-designs. We also show that tight POVMs define the optimal class of measurements on the probe state for ancilla-assisted process tomography of unital channels. In this setting they are equivalent to unitary 2-designs.

Optimal linear tomography of quantum states and processes with tight POVMs

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Outline

1. States and POVMs in real Hilbert space. ie. "Bloch sphere" for qudits.

2. Informational completeness and state reconstruction.

3. Tight POVMs. "Almost" orthonormal bases for quantum states.

   – complex projective designs

4. Optimal linear tomography. Tight POVMs (when they exist) describe the exact optimal class of measurements.

   – unitary designs

States and POVMs in real Hilbert space

- States ρ are positive Hermitian operators on $\mathbb{C}^d$.
- POVMs $\{F(x)\}_{x\in\mathscr{X}}$ are sets of positive Hermitian operators on $\mathbb{C}^d$.

  (lets assume a finite number of outcomes $|\mathscr{X}| < \infty$ and a finite dimension $d < \infty$)

- Hermitian operators live in a real Hilbert space:

$$\mathrm{H}(\mathbb{C}^d) := \{A \in \mathrm{End}(\mathbb{C}^d) : A^\dagger = A\} \cong \mathbb{R}^{d^2}$$

with Hilbert-Schmidt inner product: $(A|B) := \mathrm{tr}(A^\dagger B)$

inducing a norm: $\|A\| := \sqrt{(A|B)}$

and distance: $\|A - B\|$

- But states have unit trace. They in fact live in an affine subspace of H:

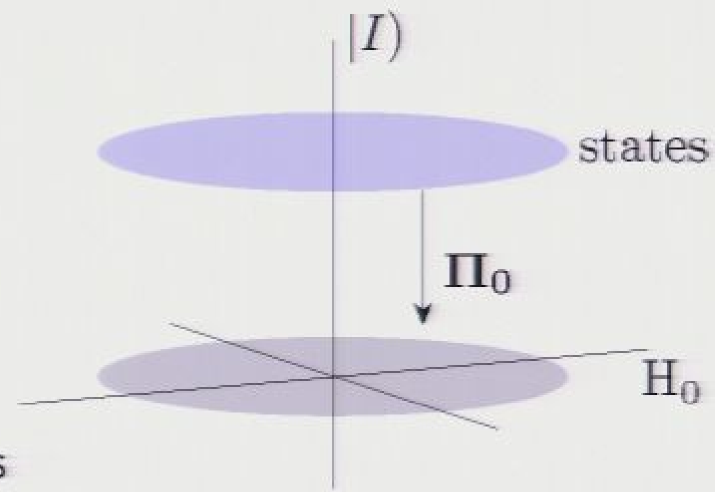
$$(I|\rho) = \mathrm{tr}(\rho) = 1$$

States and POVMs in real Hilbert space

- Define the vector subspace of traceless Hermitian operators:

$$\mathrm{H}_0(\mathbb{C}^d) := \{A \in \mathrm{H}(\mathbb{C}^d) : \mathrm{tr}(A) = 0\} \cong \mathbb{R}^{d^2-1}$$

- We prune a dimension by projecting into H_0:



$$|\rho) \hookrightarrow |\rho_0) := \mathbf{\Pi}_0|\rho) = |\rho - I/d)$$

where $\mathbf{\Pi}_0 := \mathbf{I} - \frac{1}{d}|I)(I|$

$|A)$ and $(A|$ are operator kets and bras

$\mathbf{I}$ is the identity superoperator: $\mathbf{I}|A) = |A)$

- Distances are preserved: $\|\rho_0 - \sigma_0\| = \|\rho - \sigma\|$

States and POVMs in real Hilbert space

- States are now embedded into a ball centered at the origin of $\mathrm{H}_0 \cong \mathbb{R}^{d^2-1}$

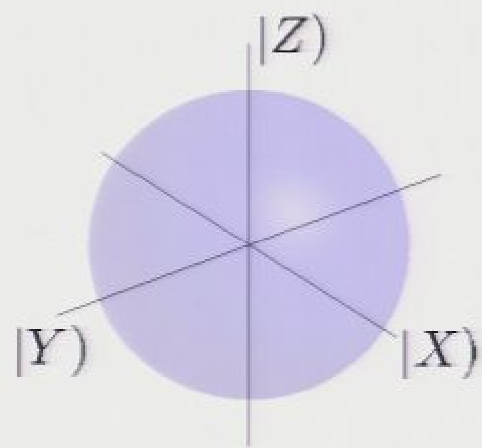
$$\|\rho_0\| \leq \sqrt{\frac{d-1}{d}}$$

- Pure states lie on the surface of the ball and mixed states within.

- This is of course just a fancy description of the Bloch-sphere representation of a qubit:

$$|\rho_0) = \frac{x}{2}|X) + \frac{y}{2}|Y) + \frac{z}{2}|Z)$$

$$(x, y, z) \in \mathbb{R}^3, \qquad x^2 + y^2 + z^2 \leq 1$$



- When $d = 2$ the embedding is bijective, but otherwise only injective, i.e., for $d > 2$ not every point in the ball represents a state.

States and POVMs in real Hilbert space

- POVMs can be embedded into H_0 by first rewriting them in terms of

a scalar-valued trace measure: $\tau := \mathrm{tr}(F)$

and positive operator-valued density (POVD): $P(x) := F(x)/\tau(x)$

$$\Rightarrow \qquad \mathrm{tr}(P) = 1, \qquad \sum\nolimits_x \tau(x) = d, \qquad \sum\nolimits_x \tau(x) P(x) = I$$

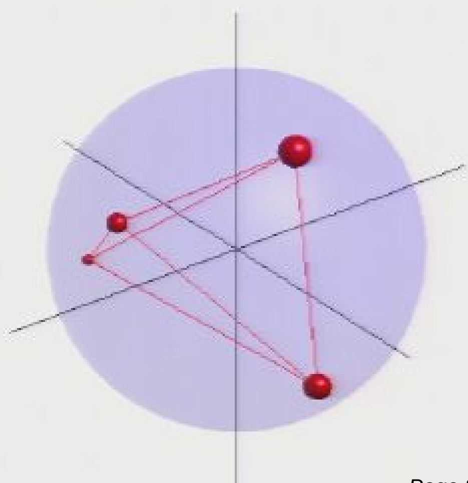
(more generally, define P through the Radon-Nikodym derivative: $F(\mathscr{E}) = \int_{\mathscr{E}} \mathrm{d}\tau(x) P(x)$)

- Embed the POVD elements as if they were states:

$$|P) \hookrightarrow |P_0) := \mathbf{\Pi}_0 |P) = |P - I/d)$$

- In this case the points $P_0(x)$ are weighted by $\tau(x)$ and satisfy

$$\sum\nolimits_x \tau(x) P_0(x) = 0$$



Informational completeness and state reconstruction

- Distinct $\rho \in \mathrm{F} := \mathrm{span}\{F(x)\}_{x\in\mathscr{X}}$ have distinct statistics $p(x) := (F(x)|\rho)$, ie. Every POVM is informationally complete with respect to the subspace it spans.

- State reconstruction then follows a standard procedure:

Define $\mathcal{F} := \sum_x \tau(x)|P(x))(P(x)|$ and $\tilde{\mathcal{F}} := (\mathcal{F} + \mathbf{\Pi}_{\mathrm{F}\perp})^{-1}\mathbf{\Pi}_{\mathrm{F}}$

(so that $\tilde{\mathcal{F}}\mathcal{F} = \mathbf{\Pi}_{\mathrm{F}} =$ projector onto F)

Next define the reconstruction density $|R) := \tilde{\mathcal{F}}|P)$, which satisfies

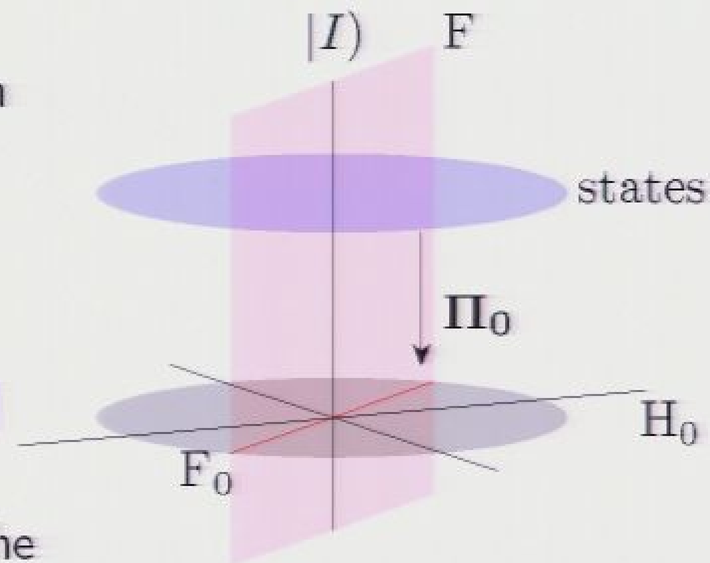
$$\sum_x \tau(x)|R(x))(P(x)| = \tilde{\mathcal{F}}\sum_x \tau(x)|P(x))(P(x)| = \tilde{\mathcal{F}}\mathcal{F} = \mathbf{\Pi}_{\mathrm{F}}$$

($\{R(x)\}_x$ is the canonical dual frame in F, with respect to τ, to the operator frame $\{P(x)\}_x$)

Multiplying on the right by $|\rho)$ then gives a state-reconstruction formula:

$$\rho = \sum_x \tau(x)(P(x)|\rho)R(x) = \sum_x (F(x)|\rho)R(x) = \sum_x p(x)R(x)$$

Tight POVMs



- Embedding into H_0 gives a better picture: We can rewrite

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{d}|I)(I|$$

where $\mathcal{F}_0 := \mathbf{\Pi}_0 \mathcal{F} \mathbf{\Pi}_0 = \sum_x \tau(x)|P_0(x))(P_0(x)|$

- Notice that $\mathcal{F}$ decomposes into a piece fixed by the normalization of F (expressed as $\mathcal{F}|I) = |I)$) and a piece $\mathcal{F}_0$ acting within H_0, an invariant subspace. It is $\mathcal{F}_0$ which can be adjusted.

- What might be a good choice? Let $\mathrm{F}_0 := \mathbf{\Pi}_0 \mathrm{F} = \mathrm{span}\{P_0(x)\}_{x\in\mathscr{X}}$.

A POVM F is called tight if $\{P_0(x)\}_{x\in\mathscr{X}}$ is a tight operator frame in F_0:

$$\mathcal{F}_0 = \sum_x \tau(x)|P_0(x))(P_0(x)| = a\mathbf{\Pi}_{\mathrm{F}_0} \qquad \text{(some } a > 0\text{)}$$

Tight POVMs

Frames are a generalization of bases that includes "overcomplete" sets. For every frame $\{A_k\}_k$ there is a dual frame $\{B_k\}_k$ such that $\sum_k |B_k)(A_k| \propto \mathbf{I}$.

Tight frames are those with $\sum_k |A_k)(A_k| \propto \mathbf{I}$ and are thus a generalization of orthonormal bases. A basis is a tight frame iff it is an orthonormal basis.

- Just like for orthonormal bases, state reconstruction for tight POVMs is trivial:

$$\rho_0 = \frac{1}{a}\sum_x p_0(x) P_0(x) \quad \text{where} \quad p_0(x) = \tau(x)(P_0(x)|\rho_0) = p(x) - \tau(x)/d$$

Or lifting back into H:

$$\rho = \frac{1}{a}\sum_x p(x) P(x) + \left(1 - \frac{1}{a}\right) I/d$$

- But we cannot choose a basis for H_0 since POVM normalization requires linear dependence:

$$\sum_x \tau(x) P_0(x) = 0$$

Tight rank-one POVMs

- The constant a is a measure of the purity of the POVM:

$$a = \frac{1}{\delta - 1}\left(\sum\nolimits_x \tau(x)(P(x)|P(x)) - 1\right) \leq \frac{d-1}{\delta - 1} \qquad \text{where } \delta := \dim \mathrm{F}$$

$$a = \frac{d-1}{\delta - 1} \quad \text{iff the tight POVM is also a rank-one POVM}$$

- Tomographic optimality will require a to take this maximum value.

- Tight POVMs are easy to construct for a chosen small enough.

  But as a is increased we encounter regions inside the Bloch sphere that correspond to nonpositive operators (if $d > 2$).

  This makes tight rank-one POVMs very difficult to construct.

- Do tight rank-one POVMs even exist?

Tight rank-one POVMs spanning $\mathrm{H}(\mathbb{C}^d)$

- For $\mathrm{F} = \mathrm{H}(\mathbb{C}^d)$ we have a second characterization of tight rank-one POVMs:

Let $\mathscr{X} \subset \mathbb{C}P^{d-1}$ and then set $P(x) = |x\rangle\langle x|$.

A rank-one POVM $F(x) = \tau(x)P(x)$ with $\mathrm{F} = \mathrm{H}(\mathbb{C}^d)$ is tight if and only if

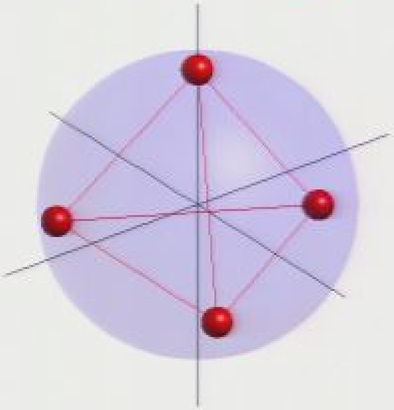
$$\frac{1}{d}\sum_{x\in\mathscr{X}} \tau(x)\,|x\rangle\langle x| \otimes |x\rangle\langle x| = \int_{\mathbb{C}P^{d-1}} \mathrm{d}x\,|x\rangle\langle x| \otimes |x\rangle\langle x|$$

ie. the outcome set $\mathscr{X}$ specifies a weighted complex projective 2-design.

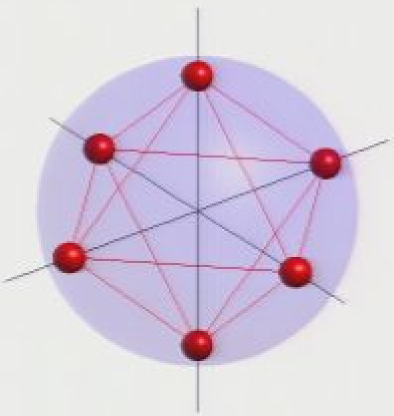
From the theory of t-designs:

- There exist 2-designs of size $|\mathscr{X}| \leq (d^4 + 2d^3 + d^2)/4$ for all d.
- But it is necessary that $|\mathscr{X}| \geq d^2$. $\quad (= \dim \mathrm{H}(\mathbb{C}^d))$
- All 2-designs with $|\mathscr{X}| = d^2$ are equiangular: $|\langle x|y\rangle|^2 = \frac{1}{d+1}$ $(x \neq y)$, $\tau(x) = \frac{1}{d}$.

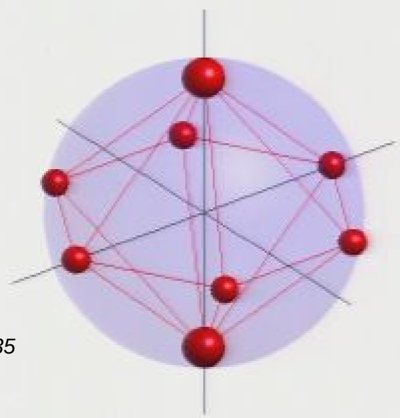
Tight rank-one POVMs spanning $\mathrm{H}(\mathbb{C}^d)$



- SIC-POVM: $|\mathscr{X}| = d^2$. Constructions known only for $d \leq 10$ and $d = 12, 13, 19$, but conjectured to exist in all dimensions. The unique minimal tight rank-one POVMs spanning $\mathrm{H}(\mathbb{C}^d)$ (when they exist).



- $d+1$ MUBs: $|\mathscr{X}| = d^2 + d$. Constructions known for all prime-power dimensions $d = p^m$, but conjectured not to exist in other dimensions. The unique minimal tight rank-one POVMs spanning $\mathrm{H}(\mathbb{C}^d)$ that can be implemented by orthogonal measurements (when they exist).



- A new family of weighted 2-designs: $|\mathscr{X}| = d^2 + 2d$. Construction generalizes to all $d = p^m - 1$, covering many dimensions missed by MUBs ($d = 6, 10, 12, 15, ..$), and can be implemented by $d+2$ orthogonal measurements. [Roy and Scott, J Math Phys 48 (2007) 072110]

Optimal linear tomography

- Suppose we want to perform tomography on a subset of states that span some subspace $\mathrm{H}' \leq \mathrm{H}(\mathbb{C}^d)$. Assume that $I \in \mathrm{H}'$ and let $\delta' = \dim \mathrm{H}'$.

Examples:

- $\mathrm{H}^{\mathrm{qu}} := \mathrm{H}(\mathbb{C}^d)$: Contains all quantum states. $\delta' = d^2$.
- $\mathrm{H}^{\mathrm{cl}} := \{A \in \mathrm{H}(\mathbb{C}^d) : A \text{ is diagonal}\}$: Contains all classical states. $\delta' = d$.

Set $|\psi_{\mathrm{e}}\rangle = \sqrt{1/n} \sum_k |k\rangle \otimes |k\rangle$, the maximally entangled state in $\mathbb{C}^d \cong \mathbb{C}^n \otimes \mathbb{C}^n$, ie. $d = n^2$, and consider entanglement-assisted process tomography in terms of state tomography on the output state $(\mathcal{E} \otimes \mathcal{I})(|\psi_{\mathrm{e}}\rangle\langle\psi_{\mathrm{e}}|)$:

- $\mathrm{H}^{\mathrm{uc}} := \{A \in \mathrm{H}(\mathbb{C}^d) : \mathrm{tr}_1(A) = \mathrm{tr}_2(A) = \mathrm{tr}(A)I/d\}$: Contains all output states for the process tomography of unital channels. $\delta' = (n^2-1)^2+1 = (d-1)^2+1$.
- $\mathrm{H}^{\mathrm{gc}} := \{A \in \mathrm{H}(\mathbb{C}^d) : \mathrm{tr}_1(A) = \mathrm{tr}(A)I/d\}$: Contains all output states for the process tomography of general channels. $\delta' = n^2(n^2-1)+1 = d(d-1)+1$

Optimal linear tomography

- We need the POVM F to span a space that includes H' as a subspace: $\mathrm{F} \geq \mathrm{H}'$.
- Assume a linear state-reconstruction formula valid for all $\rho \in \mathrm{H}'$:

$$\rho = \sum_x p(x) Q'(x) \; = \sum_x (F(x)|\rho) Q'(x) = \sum_x \tau(x)(P(x)|\rho) Q'(x)$$

where we can assume $Q'(x) \in \mathrm{H}'$ without loss of generality.

- This means Q' is a dual frame in H' to the projected frame $|P'(x)) := \mathbf{\Pi}_{\mathrm{H}'}|P(x))$:

$$\sum_x \tau(x) |Q'(x))(P'(x)| = \mathbf{\Pi}_{\mathrm{H}'}$$

- There are generally many different choices for the dual frame. Our first task will be to show that the canonical dual is optimal.
- Note that although $P \geq 0$ we generally have $P' \not\geq 0$. Assuming $\mathrm{F} = \mathrm{H}'$ above would thus neglect this possibility.

Optimal linear tomography

- Suppose we are given N copies of an unknown state ρ, and we perform measurements on each copy, all described by the same POVM F, and with outcomes $y_1, \ldots, y_N$.

- Take the linear tomographic estimate of ρ:

$$\hat{\rho}(y_1, \ldots, y_N) := \sum_x \hat{p}(x; y_1, \ldots, y_N) Q'(x)$$

where $$\hat{p}(x; y_1, \ldots, y_N) := \frac{\text{number of } y_1, \ldots, y_N \text{ equal to } x}{N}$$

- Use the mean squared Hilbert-Schmidt distance to estimate the error in $\hat{\rho}$:

$$\mathrm{err}(\rho) := \mathop{\mathrm{E}}_{y_1, \ldots, y_N} \|\rho - \hat{\rho}\|^2$$

$$\vdots$$

$$= \frac{1}{N}\left(\sum_x p(x)(Q'(x)|Q'(x)) - \mathrm{tr}(\rho^2)\right)$$

Optimal linear tomography

- Now define an average error, where the average is taken over some set $\mathscr{O} \subseteq \mathrm{U}(d)$, of possible orientations O between the state and measuring instrument, with the property that $\mathrm{E}_O\, OAO^\dagger = \mathrm{tr}(A)I/d$:

$$\mathrm{err}_{\mathrm{av}}(\rho) := \mathop{\mathrm{E}}_{O\in\mathscr{O}} \mathrm{err}(O\rho O^\dagger)$$
$$= \frac{1}{Nd}\Big(\sum\nolimits_x \tau(x)(Q'(x)|Q'(x)) - \mathrm{tr}(\rho^2)\Big)$$

Examples:

- $\mathrm{H}^{\mathrm{qu}} = \mathrm{H}(\mathbb{C}^d)$: Take $\mathscr{O} = \mathrm{U}(d)$. Then $\mathrm{E}_U\, UAU^\dagger = \mathrm{tr}(A)I/d$ by Schur's lemma.

- H^{cl}: Take $\mathscr{O} =$ set of $d!$ permutation matrices P. Then $\mathrm{E}_P\, PAP^\dagger = \mathrm{tr}(A)I/d$ for any diagonal A.

- $\mathrm{H}^{\mathrm{uc}}, \mathrm{H}^{\mathrm{gc}}$: Take $\mathscr{O} = \mathrm{U}(n) \otimes \mathrm{U}(n)$, the set of local orientations U between the system and measuring instrument and local orientations V between the system and ancilla. Then $\mathrm{E}_{U,V}(U\otimes V)A(U\otimes V)^\dagger = \mathrm{tr}(A)I/d$ by Schur's lemma.

Optimal linear tomography

- We now want to prove our main result:

Theorem. For all POVMs F spanning a space $\mathrm{F} \geq \mathrm{H}'$ (where $I \in \mathrm{H}'$), and for all duals Q',

$$\mathrm{err}_{\mathrm{av}}(\rho) \geq \frac{1}{Nd}\left(\frac{(\delta'-1)^2}{d-1} + 1 - d\,\mathrm{tr}(\rho^2)\right)$$

with equality if and only if F is a tight rank-one POVM spanning $\mathrm{F} = \mathrm{H}'$ and $Q' = R$ (the canonical dual).

Optimal linear tomography

Proof step 1:

- We first minimize the quantity $\sum_x \tau(x)(Q'(x)|Q'(x))$ over all duals Q' satisfying $\sum_x \tau(x)|Q'(x))(P'(x)| = \mathbf{\Pi}_{\mathrm{H}'}$ while keeping the POVM fixed:

> Lemma 1. For all duals Q'
>
> $$\sum_x \tau(x)(Q'(x)|Q'(x)) \geq \sum_x \tau(x)(R'(x)|R'(x))$$
>
> with equality if and only if $Q' = R'$ where R' is the canonical dual.

- The canonical dual in H' is $|R') := \tilde{\mathcal{F}}'|P')$ where $\tilde{\mathcal{F}}' := (\mathcal{F}' + \mathbf{\Pi}_{\mathrm{H}'\perp})^{-1}\mathbf{\Pi}_{\mathrm{H}'}$ with

$$\mathcal{F}' := \sum_x \tau(x)|P'(x))(P'(x)| = \mathbf{\Pi}_{\mathrm{H}'}\mathcal{F}\,\mathbf{\Pi}_{\mathrm{H}'}$$

Note $\sum_x \tau(x)|R'(x))(R'(x)| = \tilde{\mathcal{F}}' \sum_x \tau(x)|P'(x))(P'(x)|\,\tilde{\mathcal{F}}' = \tilde{\mathcal{F}}'\mathcal{F}'\tilde{\mathcal{F}}' = \tilde{\mathcal{F}}'$

Optimal linear tomography

Proof of Lemma 1. Let $D' = Q' - R'$ which satisfies

$$\begin{aligned}\sum_x \tau(x)(D'(x)|R'(x)) &= \sum_x \tau(x)(Q'(x)|R'(x)) - \sum_x \tau(x)(R'(x)|R'(x))\\ &= \sum_x \tau(x)(Q'(x)|\tilde{\mathcal{F}}'|P'(x)) - \mathrm{Tr}[\tilde{\mathcal{F}}']\\ &= \mathrm{Tr}[\tilde{\mathcal{F}}'\mathbf{\Pi}_{\mathrm{H}'}] - \mathrm{Tr}[\tilde{\mathcal{F}}']\\ &= 0\end{aligned}$$

Then

$$\begin{aligned}\sum_x \tau(x)(Q'(x)|Q'(x)) &= \sum_x \tau(x)(R'(x)|R'(x)) + \sum_x \tau(x)(R'(x)|D'(x))\\ &\quad + \sum_x \tau(x)(D'(x)|R'(x)) + \sum_x \tau(x)(D'(x)|D'(x))\\ &= \sum_x \tau(x)(R'(x)|R'(x)) + \sum_x \tau(x)(D'(x)|D'(x))\\ &\geq \sum_x \tau(x)(R'(x)|R'(x))\end{aligned}$$

with equality if and only if $D' = 0$.

Optimal linear tomography

Proof step 2:

- Now minimize the quantity $\sum_x \tau(x)(R'(x)|R'(x)) = \mathrm{Tr}[\tilde{\mathcal{F}}']$ over all POVMs:

Lemma 2. For all POVMs F spanning a space $\mathrm{F} \geq \mathrm{H}'$

$$\mathrm{Tr}[\tilde{\mathcal{F}}'] \geq 1 + \frac{(\delta' - 1)^2}{d - 1}$$

with equality if and only if

$$\mathcal{F} = \mathcal{F}' = \frac{d-1}{\delta' - 1} \mathbf{\Pi}_{\mathrm{H}'_0} + \frac{1}{d} |I)(I|$$

ie. F is a tight rank-one POVM spanning $\mathrm{F} = \mathrm{H}'$.

- Here $\mathrm{H}'_0 := \mathbf{\Pi}_0 \mathrm{H}'$. Recall that we assumed $I \in \mathrm{H}'$ and defined $\delta' = \dim \mathrm{H}'$.

- Note that equality requires $\mathrm{F} = \mathrm{H}'$, meaning $P' = P$, $R' = R$, etc.

Optimal linear tomography

Proof of Lemma 2. Since $\mathrm{F} \geq \mathrm{H}'$ the positive superoperator $\mathcal{F}' = \mathbf{\Pi}_{\mathrm{H}'} \mathcal{F} \mathbf{\Pi}_{\mathrm{H}'}$ has exactly $\delta' = \dim \mathrm{H}'$ nonzero eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_{\delta'} > 0$.

But one eigenvalue is fixed at unity, $\lambda_1 = 1$ say, since we assumed $I \in \mathrm{H}'$ and POVM normalization requires $\mathcal{F}|I) = |I)$: $\mathcal{F}'|I) = \mathbf{\Pi}_{\mathrm{H}'} \mathcal{F} \mathbf{\Pi}_{\mathrm{H}'}|I) = |I)$.

The remaining satisfy $$\sum_{k=2}^{\delta'} \lambda_k = \mathrm{Tr}[\mathcal{F}'] - 1 \leq \mathrm{Tr}[\mathcal{F}] - 1 \leq d - 1 \qquad (1)$$

given that $\mathrm{Tr}[\mathcal{F}'] = \mathrm{Tr}[\mathbf{\Pi}_{\mathrm{H}'} \mathcal{F} \mathbf{\Pi}_{\mathrm{H}'}] \leq \mathrm{Tr}[\mathcal{F}]$ with equality iff $\mathrm{F} = \mathrm{H}'$, and

$\mathrm{Tr}[\mathcal{F}] = \sum_x \tau(x)(P(x)|P(x)) \leq \sum_x \tau(x) = d$ with equality iff P is rank-1.

Under (1), $\mathrm{Tr}[\tilde{\mathcal{F}}'] = 1 + \sum_{k=2}^{\delta'} \frac{1}{\lambda_k}$ takes its minimum of $1 + (\delta' - 1)^2/(d-1)$

iff $\lambda_2 = \cdots = \lambda_{\delta'} = (d-1)/(\delta'-1)$, requiring (1) to take its maximum and, moreover, requiring a tight rank-1 POVM spanning H': $\mathcal{F}_0 = \mathcal{F}_0' = \frac{d-1}{\delta'-1} \mathbf{\Pi}_{\mathrm{H}'}$.

Optimal linear tomography

- Lemma 1 and 2 thus give our main result:

Theorem. For all POVMs F spanning a space $\mathrm{F} \geq \mathrm{H}'$ (where $I \in \mathrm{H}'$)

$$\mathrm{err}_{\mathrm{av}}(\rho) := \mathop{\mathrm{E}}_{O \in \mathcal{O}} \mathrm{err}(O \rho O^\dagger) \geq \frac{1}{Nd} \left(\frac{(\delta' - 1)^2}{d-1} + 1 - d\, \mathrm{tr}(\rho^2) \right)$$

for all duals Q', with equality if and only if F is a tight rank-one POVM spanning $\mathrm{F} = \mathrm{H}'$ and $Q' = R$ (the canonical dual).

- This theorem is unchanged if we replace $\mathrm{err}_{\mathrm{av}}(\rho)$ with the worst-case error:

Corollary. For all POVMs F spanning a space $\mathrm{F} \geq \mathrm{H}'$ (where $I \in \mathrm{H}'$)

$$\mathrm{err}_{\mathrm{wc}}(\rho) := \sup_{O \in \mathcal{O}} \mathrm{err}(O \rho O^\dagger) \geq \frac{1}{Nd} \left(\frac{(\delta' - 1)^2}{d-1} + 1 - d\, \mathrm{tr}(\rho^2) \right)$$

for all duals Q', with equality if and only if F is a tight rank-one POVM spanning $\mathrm{F} = \mathrm{H}'$ and $Q' = R$ (the canonical dual).

Optimal linear tomography

- Tight rank-one POVMs in fact achieve the optimal error rate independent of O:

$$\mathrm{err}(O\rho O^\dagger) = \frac{1}{Nd}\left(\frac{(\delta'-1)^2}{d-1} + 1 - d\,\mathrm{tr}(\rho^2)\right)$$

Examples:

- $\mathrm{H}^{\mathrm{qu}} = \mathrm{H}(\mathbb{C}^d)$: $\delta' = d^2 \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(d^2 + d - 1 - \mathrm{tr}(\rho^2)\right)$
- H^{cl}: $\delta' = d \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(1 - \mathrm{tr}(\rho^2)\right)$ (which vanishes for pure states)
- H^{uc}: $\delta' = (d-1)^2 + 1 \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(d^2 - 3d + 3 - \mathrm{tr}(\rho^2)\right)$
- H^{gc}: $\delta' = d(d-1) + 1 \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(d^2 - d + 1/d - \mathrm{tr}(\rho^2)\right)$

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Corollary. For all POVMs F spanning a space $\mathrm{F} \geq \mathrm{H}'$ (where $I \in \mathrm{H}'$)

$$\mathrm{err}_{\mathrm{wc}}(\rho) := \sup_{O \in \mathcal{O}} \mathrm{err}(O \rho O^\dagger) \geq \frac{1}{Nd} \left(\frac{(\delta' - 1)^2}{d-1} + 1 - d\,\mathrm{tr}(\rho^2) \right)$$

for all duals Q', with equality if and only if F is a tight rank-one POVM spanning $\mathrm{F} = \mathrm{H}'$ and $Q' = R$ (the canonical dual).

Optimal linear tomography

- Tight rank-one POVMs in fact achieve the optimal error rate independent of O:

$$\mathrm{err}(O\rho O^\dagger) = \frac{1}{Nd}\left(\frac{(\delta'-1)^2}{d-1} + 1 - d\,\mathrm{tr}(\rho^2)\right)$$

Examples:

- $\mathrm{H}^{\mathrm{qu}} = \mathrm{H}(\mathbb{C}^d)$: $\delta' = d^2 \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(d^2 + d - 1 - \mathrm{tr}(\rho^2)\right)$
- H^{cl}: $\delta' = d \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(1 - \mathrm{tr}(\rho^2)\right)$ (which vanishes for pure states)
- H^{uc}: $\delta' = (d-1)^2 + 1 \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(d^2 - 3d + 3 - \mathrm{tr}(\rho^2)\right)$
- H^{gc}: $\delta' = d(d-1) + 1 \quad \Rightarrow \quad \mathrm{err} = \frac{1}{N}\left(d^2 - d + 1/d - \mathrm{tr}(\rho^2)\right)$

When do tight rank-one POVMs exist?

- We already know that tight rank-one POVMs spanning $\mathrm{H}^{\mathrm{qu}} = \mathrm{H}(\mathbb{C}^d)$ are equivalent to weighted complex projective 2-designs. These exist in every dimension.

- Tight rank-one POVMs spanning H^{cl} are trivial. In this case $\delta' = d$ and our definition becomes

$$\begin{aligned}\mathcal{F} = \sum\nolimits_x \tau(x)|P(x))(P(x)| &= \frac{d-1}{\delta'-1}\mathbf{\Pi}_{\mathrm{H}_0^{\mathrm{cl}}} + \frac{1}{d}|I)(I| \\ &= \mathbf{\Pi}_{\mathrm{H}_0^{\mathrm{cl}}} + \frac{1}{d}|I)(I| \\ &= \mathbf{\Pi}_{\mathrm{H}^{\mathrm{cl}}}\end{aligned}$$

  which means this type of tight rank-one POVM also corresponds to a tight frame for H^{cl} (along with $\mathrm{H}_0^{\mathrm{cl}}$ by definition). An orthonormal basis will do the job:

$$P(e_k) = |e_k\rangle\langle e_k|, \quad \tau(e_k) = 1, \quad \mathscr{X} = \{e_1, \ldots, e_d\}$$

- What about tight rank-one POVMs spanning H^{uc} and H^{gc}?

Tight rank-one POVMs spanning H^{uc}

- For $\mathrm{F} = \mathrm{H}^{\mathrm{uc}}$ we again have a second characterization:

Let $\mathscr{X} \subset \mathrm{U}(n)$ and then set $P(U) = (U \otimes I)|\psi_{\mathrm{e}}\rangle\langle\psi_{\mathrm{e}}|(U \otimes I)^\dagger$ ($d = n^2$), which parametrizes the (normalized) rank-one members of H^{uc} in terms of unitary matrices.

A rank-one POVM $F(U) = \tau(U)P(U)$ with $\mathrm{F} = \mathrm{H}^{\mathrm{uc}}$ is tight if and only if

$$\frac{1}{d}\sum_{U \in \mathscr{X}} \tau(U)\, U \otimes U \otimes U^\dagger \otimes U^\dagger = \int_{\mathrm{U}(n)} \mathrm{d}U\, U \otimes U \otimes U^\dagger \otimes U^\dagger$$

ie. the outcome set $\mathscr{X}$ specifies a weighted unitary 2-design.

From the theory of $\mathrm{U}(n)$ t-designs:

- There exist 2-designs with $|\mathscr{X}| \leq (d^4 - 6d^3 + 25d^2 - 28d + 16)/4$ for all $d = n^2$.
- But it is necessary that $|\mathscr{X}| \geq d^2 - 2d + 2$. $\quad (= \dim \mathrm{H}^{\mathrm{uc}})$
- The only 2-designs with $|\mathscr{X}| = d^2 - 2d + 2$ are equiangular.

Tight rank-one POVMs spanning H^{uc}

- $\mathrm{H}^{\mathrm{uc}} / \mathrm{U}(n)$ versions of SIC-POVMs?: $|\mathscr{X}| = d^2 - 2d + 2 = n^4 - 2n^2 + 2$ will require equiangularity:
$$|\operatorname{tr}(U^\dagger V)|^2 = 1 - \tfrac{1}{n^2-1} \quad \text{for all } U \neq V \in \mathscr{X}$$
Such designs (when they exist) specify the unique minimal tight rank-one POVMs spanning H^{uc}. They do not exist in dimension $n = 2$.

- $\mathrm{H}^{\mathrm{uc}} / \mathrm{U}(n)$ versions of MUBs?: $|\mathscr{X}| = d^2 - d = n^4 - n^2$. These are (maximal) sets of $n^2 - 1$ mutually unbiased unitary-operator bases (MUUBs):
$$|\operatorname{tr}(U^\dagger V)|^2 = 1 \quad \text{for } U \text{ and } V \text{ taken from different bases}$$
Constructions are known for $n = 2, 3, 5, 7, 11$ [Chau, 2005] and specify (when they exist) the unique minimal tight rank-one POVMs spanning H^{uc} that can be implemented by orthogonal measurements.

- Clifford group designs: $|\mathscr{X}| = n(d^2 - d) = n^5 - n^3$. The projective Clifford group is a 2-design that exists for all prime-powers $n = p^m$.

- Open problem: Find a family of $\mathrm{U}(n)$ 2-designs with the optimal size $O(n^4)$.

Tight rank-one POVMs spanning H^{gc}

- We know that $\mathrm{H}^{\mathrm{gc}} := \{A \in \mathrm{H}(\mathbb{C}^d) : \mathrm{tr}_1(A) = \mathrm{tr}(A)I/d\}$ contains $\mathrm{H}^{\mathrm{uc}} := \{A \in \mathrm{H}(\mathbb{C}^d) : \mathrm{tr}_1(A) = \mathrm{tr}_2(A) = \mathrm{tr}(A)I/d\}$ as a proper vector subspace: $\mathrm{H}^{\mathrm{uc}} < \mathrm{H}^{\mathrm{gc}}$.

  But all rank-one members of H^{gc} are also members of H^{uc}. They are the maximally entangled states: $A \propto (U \otimes I)|\psi_{\mathrm{e}}\rangle\langle\psi_{\mathrm{e}}|(U \otimes I)^\dagger$.

  Thus there is no way to span H^{gc} with the rank-one members of H^{gc}. We can, at most, span only the subspace H^{uc}.

- Tight rank-one POVMs spanning H^{gc} do not exist:

$$\mathrm{err}_{\mathrm{av}}(\rho) > \frac{1}{N}\left(d^2 - d + 1/d - \mathrm{tr}(\rho^2)\right)$$

- Bisio, Chiribella, D'Ariano, Facchini, and Perinotti [arXiv:0806.1172] have recently derived the tight bound:

$$\mathrm{err}_{\mathrm{av}}(\rho) \geq \frac{1}{N}\left(d^2 + (2\sqrt{2}-3)d + (5-4\sqrt{2}) + 2(\sqrt{2}-1)/d - \mathrm{tr}(\rho^2)\right)$$

Summary and open problems

- Tight POVMS are a natural choice for "orthonormal bases" for quantum states.
- When they exist, tight rank-one POVMs describe the most robust measurements against statistical error for linear tomography.
- What is the exact class of optimal measurements for general channels? What are the simplest members of this class?
- Tight POVMs are likely to remain good choices for tomography when the linear tomographic estimate of the state is replaced by something more sophisticated, or when the Hilbert-Schmidt distance is replaced by something more appropriate, at least in the limit of large numbers of measurements. But how good?
- Within the class of tight POVMs, there will still be better choices. Eg. t-designs. What can be proven?

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