Title: Quantum Tomographic Reconstruction with Error Bars: a Kalman Filter Approach

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Abstract: We present a novel quantum tomographic reconstruction method based on Bayesian inference via the Kalman filter update equations. The method not only yields the maximum likelihood/optimal Bayesian reconstruction, but also a covariance matrix expressing the measurement uncertainties in a complete way. From this covariance matrix the error bars on any derived quantity can be easily calculated. This is a first step towards the broader goal of devising an omnibus reconstruction method that could be adapted to any tomographic setup with little effort and that treats measurement uncertainties in a statistically well-founded way. We restrict ourselves to the important subclass of tomography based on measurements with discrete outcomes (as opposed to continuous ones), and we also ignore any measurement imperfections (dark counts, less than unit detector efficiency, etc.), which will be treated in further work. We illustrate the general theory on two real tomography experiments of quantum optical information processing elements.

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I like them "the sunny side up"

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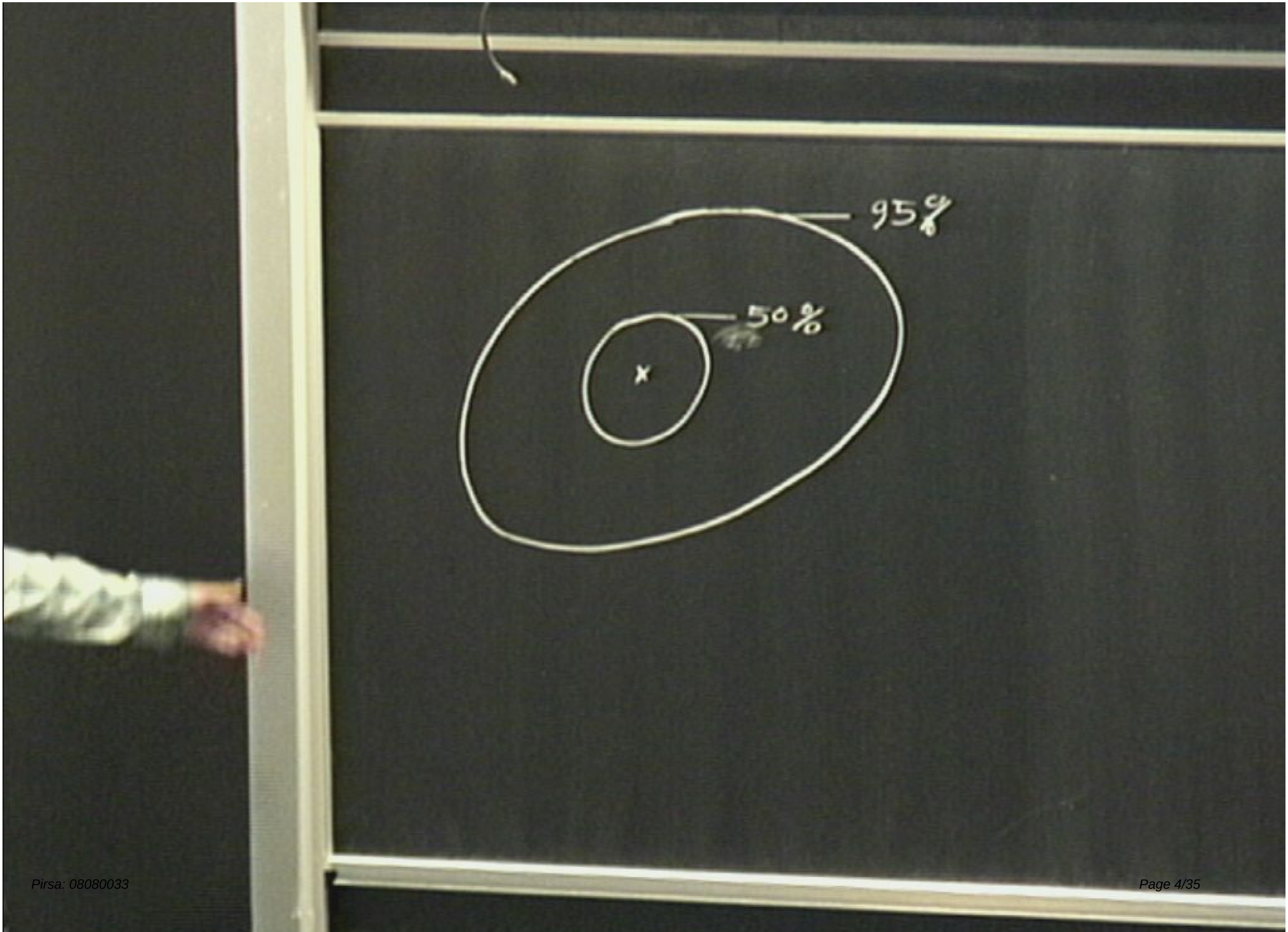
Goals

- Tomographic reconstruction, with error bars
- As always: reconstruction must be physical
- Fast method, even for large systems
- Statistically sound; avoid "zero-eigenvalue problem"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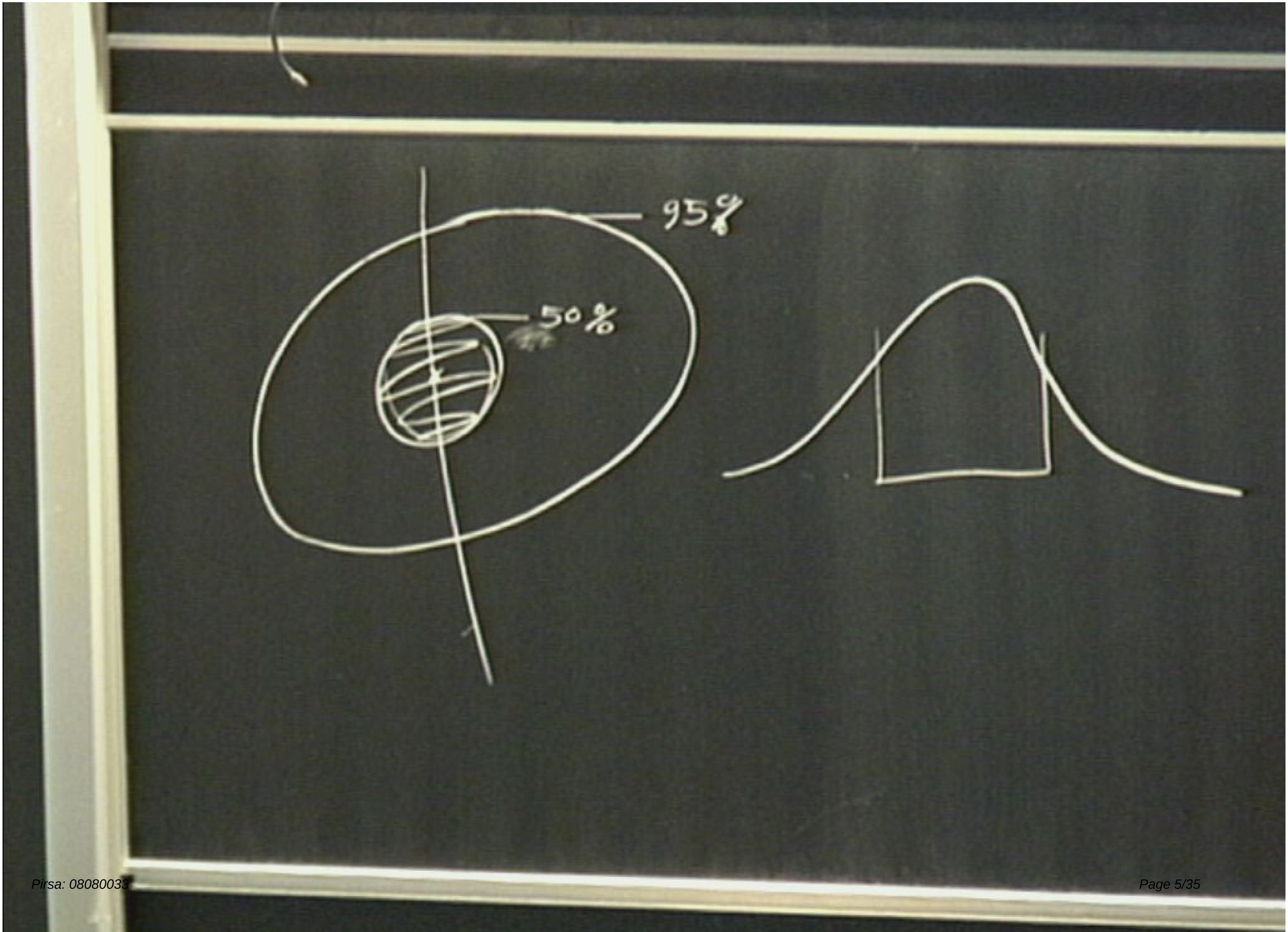
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Methods

- Proper Bayesian inference
- Approximation of tomographic process by a linear-Gaussian model,
- and use the Kalman filter update equations.
- What it gives us is NOT the maximum likelihood solution...
- ... but a confidence region centered around a mean value.
- The MaxLik solution is in the CR, but is not the mean value.

The tomographic process

- Consider state tomography: a state is measured N times using some POVM $\Pi^{(k)}$
- State $\rho \longrightarrow$ probabilities $\boldsymbol{p} \longrightarrow$ frequencies $\boldsymbol{f}$
- First step is linear: $p_k = \mathrm{Tr}[\rho\Pi^{(k)}]$
- Second step is the quantum noise process: depending on setup a multinomial, or a Poisson process $\boldsymbol{F} \sim \mathrm{Mtn}(N, \boldsymbol{p})$.
- This process is performed for different POVMs (hence: tomography)

Bayesian Inference

- From the "forward" process $\rho \longrightarrow \boldsymbol{p} \longrightarrow \boldsymbol{f}$,
- infer the statistics "backwards": $\boldsymbol{f} \longrightarrow \boldsymbol{p} \longrightarrow \rho$
- using Bayes' rule:

$$\underset{\text{posterior PDF}}{f(\rho|\boldsymbol{f})} \propto \underset{\text{Likelihood function}}{f(\boldsymbol{f}|\rho)} \times \underset{\text{prior PDF}}{f(\rho)}$$

Proportionality factor follows by normalising the posterior

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Bayesian Inference

- For the first tomographic "slice" (first POVM), assume a flat prior (unknown state):

posterior PDF Likelihood function

$$f(\rho|\boldsymbol{f}) \propto f(\boldsymbol{f}|\rho)$$

- For the next slices (next POVMs), posterior is prior for next slice

$$f(\rho|\boldsymbol{f}^{(i+1)}, \boldsymbol{f}^{(i)}, \ldots) \propto f(\boldsymbol{f}^{(i+1)}|\rho) \quad \times \quad f(\rho|\boldsymbol{f}^{(i)}, \ldots)$$

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In practice...

- Bayesian inference is simple in theory, but practice tells otherwise
- We will proceed in two steps:
- First Bayesian inversion of the quantum noise process $p \longrightarrow f$
- Then Bayesian inversion of the linear (POVM) mapping $\rho \longrightarrow p$

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First Bayesian Inversion Step

- Here we want to avoid the "zero-eigenvalue problem":
- From $f_i = 0$ we shouldn't conclude $p_i = 0$.
- Having made N out of N safe airplane journeys does not prevent your next flight from crashing
- One of the first statistics problems to be treated properly, by...

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Pierre-Simon, Marquis de Laplace

1774: "Mémoire sur la probabilité des causes par les évènements"

Laplace's rule of succession

- Suppose f out of N flights crash, what is the probability that your next flight crashes?
- ... and based on that information alone
- Laplace's answer: "rule of succession"

$$p = \frac{f+1}{N+2}$$

- Both outcomes, crash and no crash, get one additional count: a pseudocount

First Bayesian Inversion Step

- In modern Bayesian language:
- Forward process: $\boldsymbol{p} \longrightarrow \boldsymbol{f}$: $\boldsymbol{F} \sim \text{Mtn}(N, \boldsymbol{p})$
- The likelihood function is $f(\boldsymbol{f}|\boldsymbol{p}) = \binom{N}{\boldsymbol{f}} p_1^{f_1} \dots p_d^{f_d}$
- With a flat prior, this gives a posterior PDF $f(\boldsymbol{p}|\boldsymbol{f}) \propto p_1^{f_1} \dots p_d^{f_d}$
- Normalise over the simplex $p_i \geq 0$, $\sum_i p_i = 1$: multiply by $(N + d - 1)!$.
- Yields the Dirichlet distribution: $\boldsymbol{P}|\boldsymbol{F} \sim \text{Dirichlet}(\boldsymbol{f})$.
- Higher-dimensional generalisation of the Beta distribution.
- Its mean: $\mu(P_i|\boldsymbol{F}) = \frac{f_i+1}{N+d}$ as found by Laplace!

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First Bayesian Inversion Step

- Note: the mode (position of maximum = MaxLik solution) of the Dirichlet is at f/N.
- We prefer the mean over the mode because

  "the mean of a posterior can be thought of as being more representative [than its mode] as it takes into account the skewness of the PDF."
  — D. Sivia

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Second Bayesian Inversion Step

- Inversion of linear part of the measurement process $\rho \longrightarrow \mathbf{p}$
- Problems: ρ may be high-dimensional, and how do you efficiently represent products of Dirichlets over quantum state space?
- Our proposal: approximate the Dirichlet by a Gaussian, with same mean and covariance.
- This turns the tomography process into a linear-Gaussian model, allowing the use of Kalman filtering

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R.E. Kalman

1960: "A New Approach to Linear Filtering and Prediction Problems"

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Linear Gaussian Models

- Consider the position of Apollo-13 in space...
- Dynamic model: state $\boldsymbol{x}(t)$, obeys a linear ODE, with an additional Gaussian perturbation term

$$\boldsymbol{x}(t+\Delta t) = U\boldsymbol{x}(t) + \mathcal{N}(0, \Sigma)$$

- Measurement of state (telemetry) also linear, and also with a Gaussian perturbation term

$$\boldsymbol{z}(t) = H\boldsymbol{x}(t) + \mathcal{N}(0, \Theta)$$

- We only need the measurement part, as the state is (supposedly) static
- Dynamical part becomes necessary in case of drift (work in progress).

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Kalman filter update equations

- For linear-Gaussian models, all PDFs are Gaussians, determined by their first and second order moments
- Bayesian inference equations simplify, giving the Kalman filter update equations:

$$\begin{aligned} K &= \Sigma H^*(H\Sigma H^* + \Theta)^{-1} \\ \mu' &= \mu + K(z - H\mu) \\ \Sigma' &= \Sigma - KH\Sigma. \end{aligned}$$

- Here, K is the so-called Kalman gain factor, μ and Σ are the prior moments of the state PDF, and μ' and Σ' are the moments of the updated, posterior PDF.
- The inputs are the measurement data: the moments z and Θ,
- while H defines the measurement itself.

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KF Tomographic Reconstruction

- In our setting, z is given by the mean of Dirichlet(f) and Θ by its covariance matrix.
- The state ρ is represented by a Hilbert space representative
- The measurement matrix H is the matrix representing the linear mapping $\rho \mapsto p_k = \mathrm{Tr}[\rho\Pi^{(k)}]$.
- As prior PDF, take a Gaussian with large covariances, approximating a flat prior.
- The measurement results f for each POVM (tomographic slice) can then be incorporated by applying the Kalman update equations with the appropriate moments.

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And, does this work in practice?

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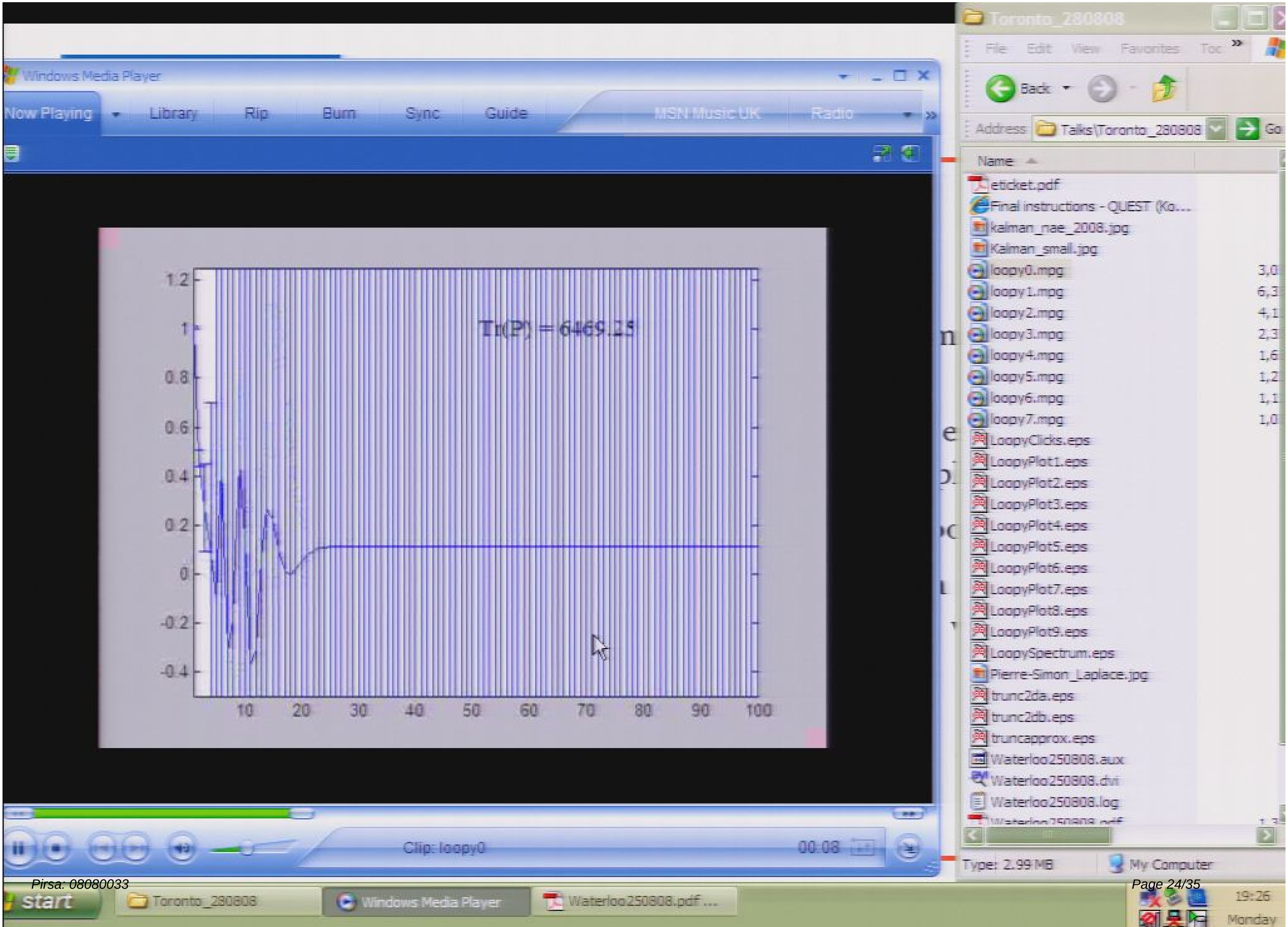
A Demo

- We have implemented the KF method for reconstructing tomography data of an optical POVM (data courtesy I. Walmsley)
- The POVM consists of 9 elements $\Pi^{(k)}$, each element represented by a diagonal matrix of size $d = 100$. The matrix elements correspond to photon numbers.
- Tomography input states are coherent states with varying power $|\alpha|^2$.
- Per power setting, $N = 38084$ light pulses were sent through the POVM, and the frequencies of the outcomes were recorded: f_i, $i = 0, \ldots, 8$, with $\sum_i f_i = N$.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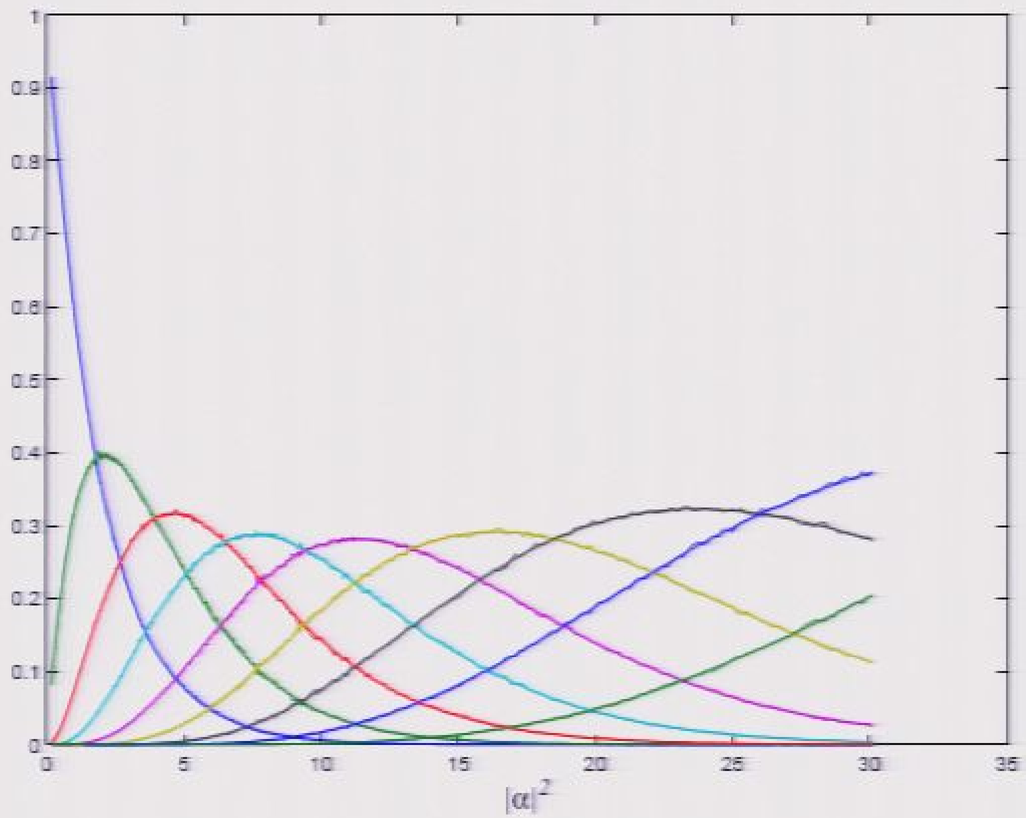
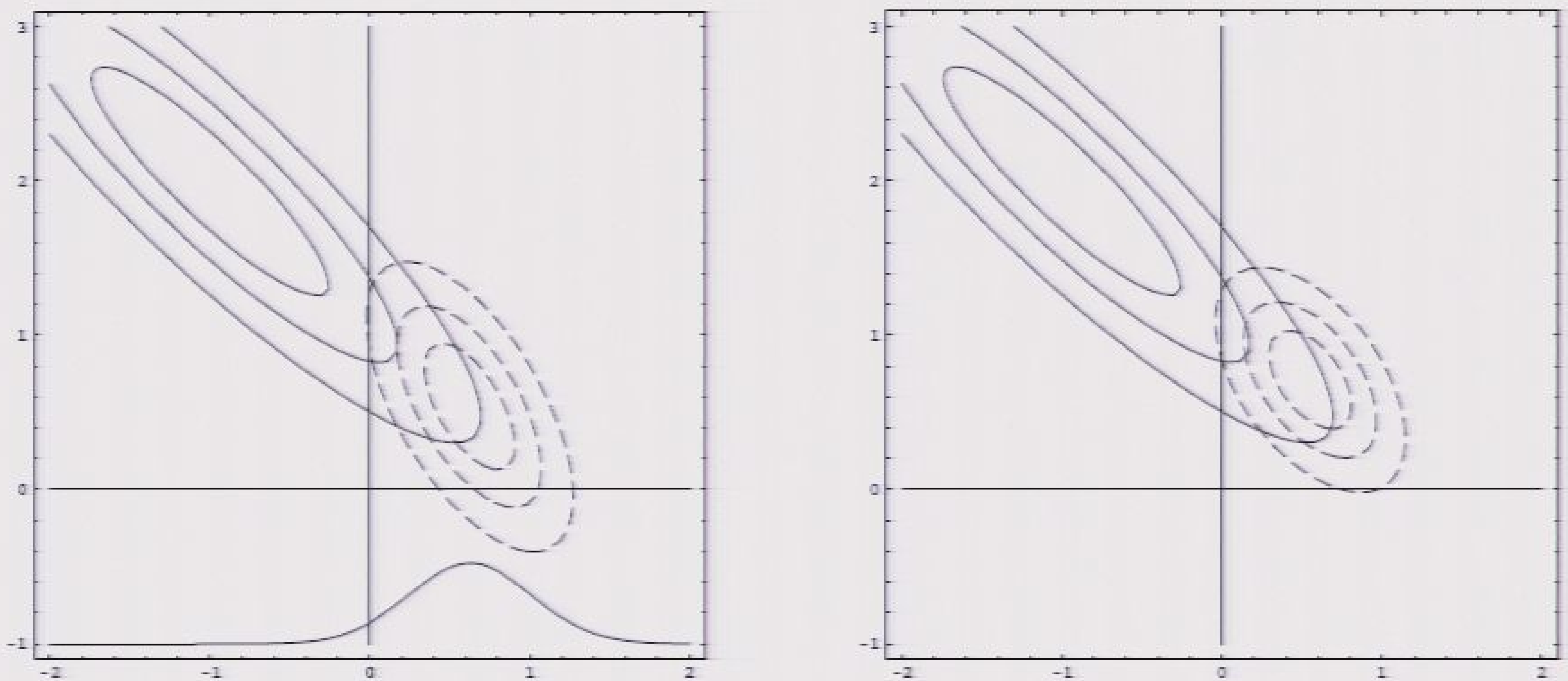
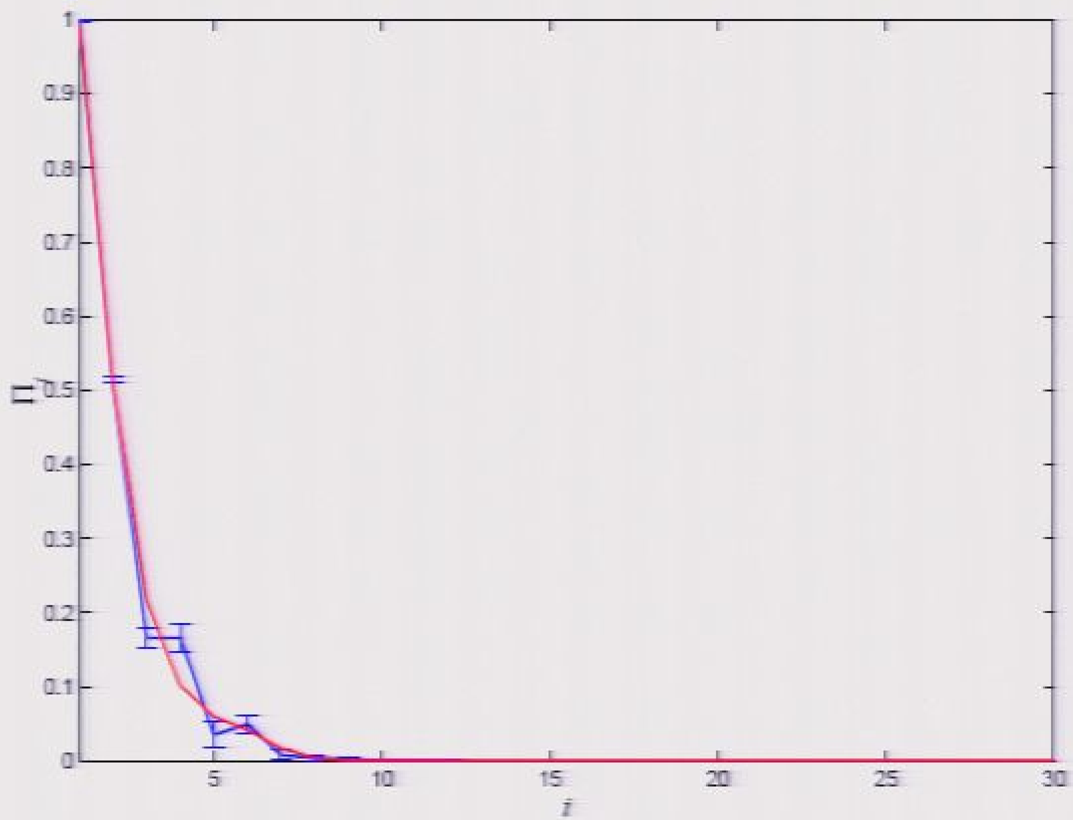
Frequencies



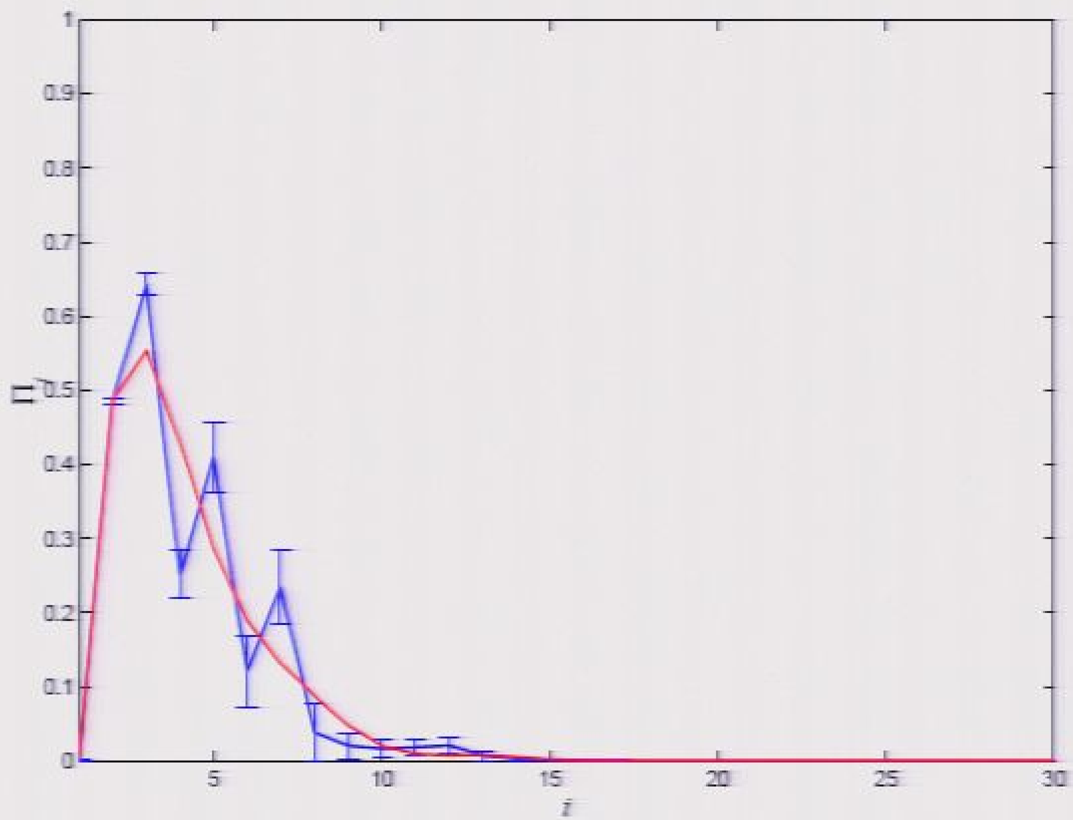
Restriction Procedure (2D)



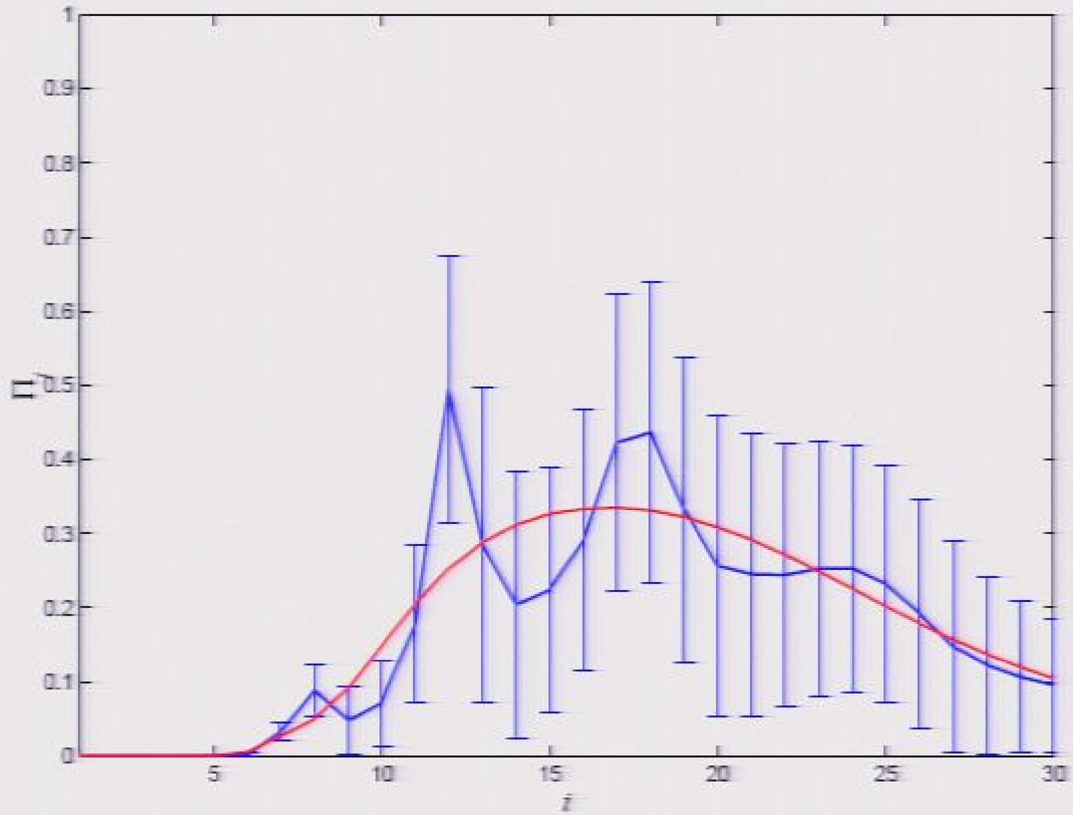
Final POVM reconstruction



Final POVM reconstruction



Final POVM reconstruction



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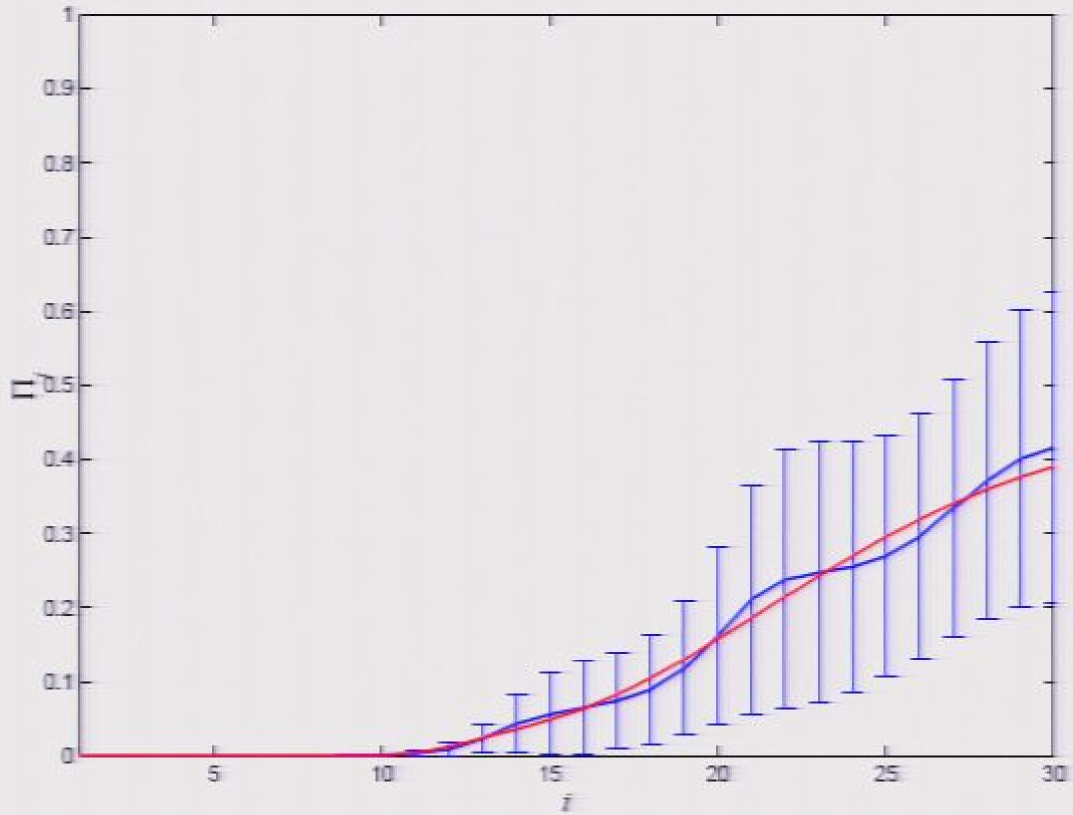
Final POVM reconstruction



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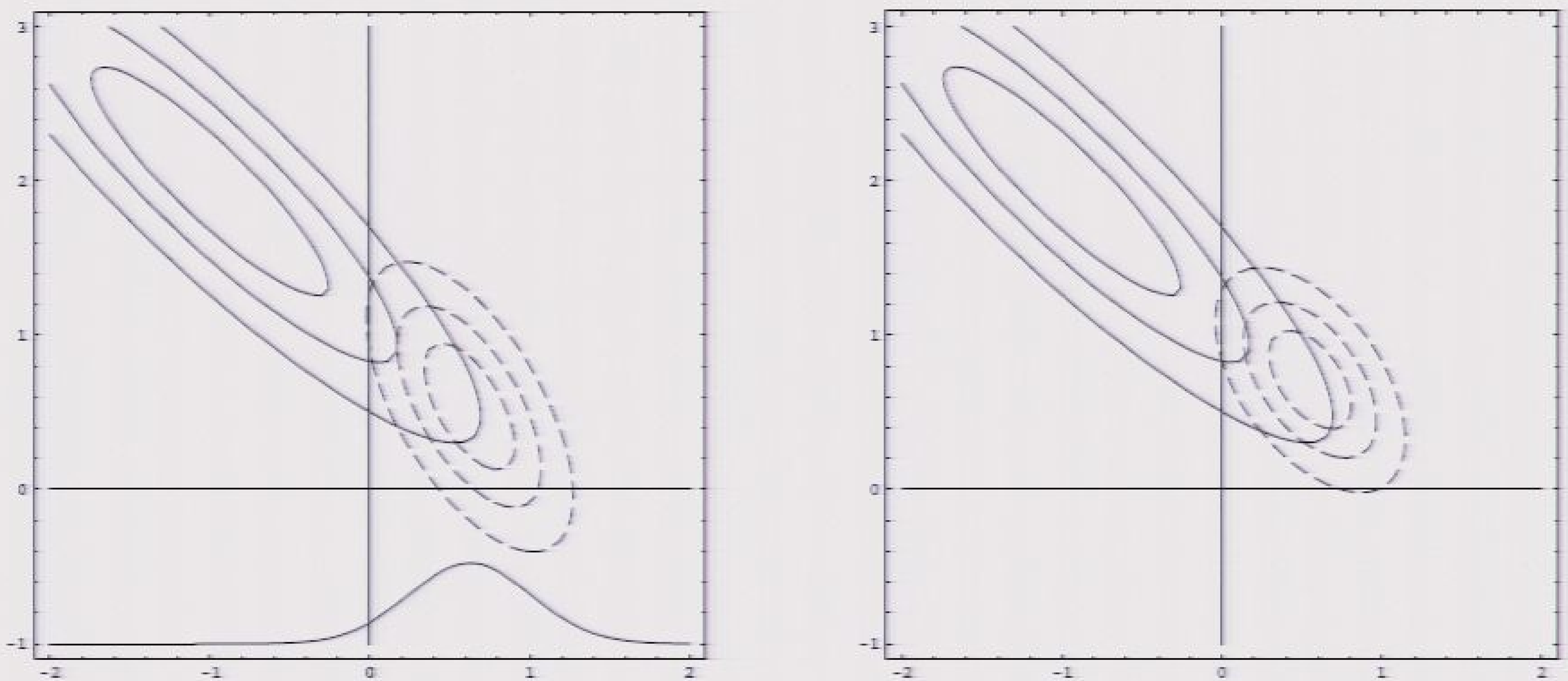

KF Demo

- Kalman Filtering yields a Gaussian PDF over "POVM space", with a mean value μ and a covariance matrix Σ.
- We'll plot the mean value for POVM element $\Pi^{(0)}$, in function of photon number,
- and the diagonal elements of the covariance matrix for that POVM element, as error bars.
- These diagonal elements are the variances of the marginal PDFs of $\Pi^{(0)}_{jj}$, thus we neglect the correlations.
- We'll show this after every KF update step.
- You will note the unphysicality of the solution; pls suspend your disbelief for a minute

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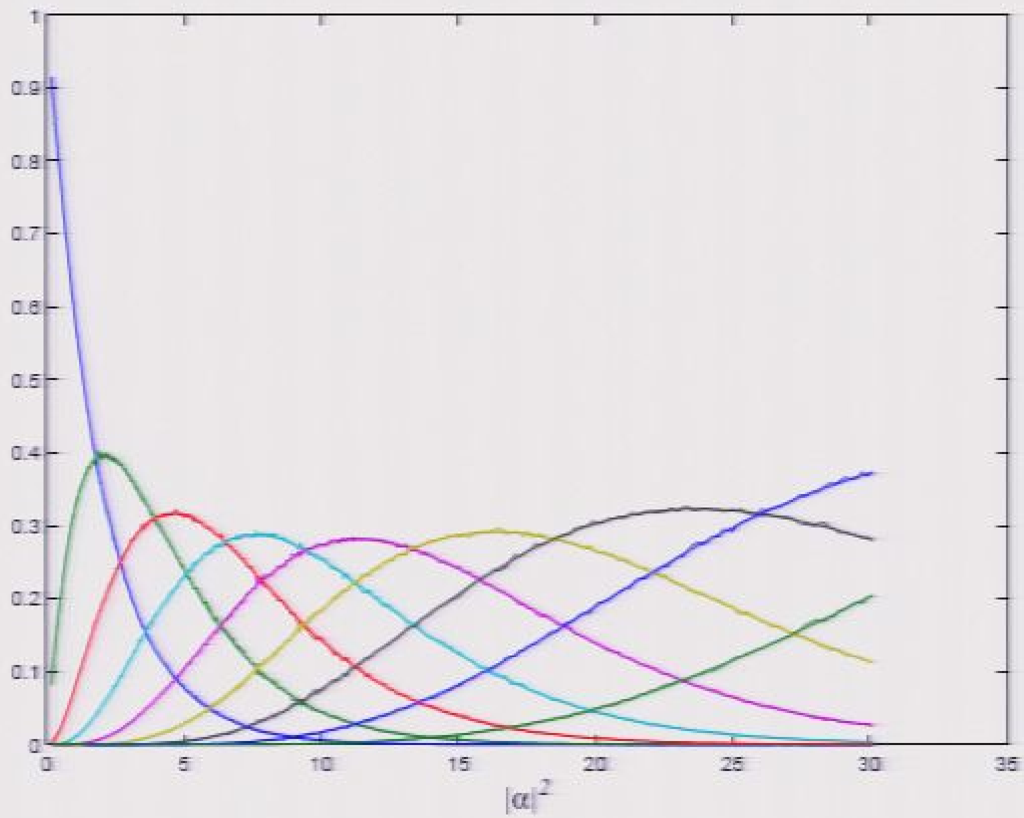
Frequencies



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Conclusion

Let's fight Global Warming with clever use of Kalman Filtering!