Title: The intersection of general relativity and quantum mechanics

Date: Aug 12, 2008 04:00 PM

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Abstract: Domains were introduced in computer science in the late 1960\'s by Dana Scott to provide a semantics for the lambda calculus (the lambda calculus is the basic prototype for a functional programming language i.e. ML). The study of domains with measurements was initiated in the speaker\'s thesis: a domain provides a qualitative view of information expressed in part by an \'information order\' and a measurement on a domain expresses a quantitative view of information with respect to the underlying qualitative aspect. The theory of domains and measurements was initially introduced to provide a first order model of computation, one in which a computation is viewed as a process that evolves in a space of informatic objects, where processes have informatic rates of change determined by the manner in which they manipulate information. There is a domain of binary channels with capacity as a measurement. There is a domain of finite probability distributions with entropy as a measurement. There is a domain of quantum mixed states with entropy as a measurement. There is a domain of spacetime intervals with global time as a measurement. In this setting, similarities between QM and GR emerge, but also some important differences. In a domain, if we write x <= y, then it means that x carries information about y, while x << y is a stronger relation that means x carries *essential* information about y. In GR, the domain theoretic relation << can be proven to be timelike causality. It possesses stronger mathematical properties than << does in QM. However, by an application of the maximum entropy principle, we can restrict the mixed states in consideration and this difference is removed: the domains of events and mixed states are both globally hyperbolic -- where globally hyperbolic is a purely order theoretic idea that just happens to coincide with the usual notion in the case of GR. Along the way, we will see domain theoretic ways of distinguishing between the Newtonian and relativistic notions of time, how to reconstruct the topology and geometry of spacetime in a purely order theoretic manner beginning from only a countable set, see that the Holevo capacity of a unital qubit channel is determined by the largest value of its informatic derivative and have reason to wonder if distance can be defined as the amount of information (capacity) that can be transmitted between two points.

$\mathrm{GR} \cap \mathrm{QM} \neq \emptyset$

(*one's not half two. It's two are halves of one: – e.e cummings*)

Keye Martin

Naval Research Laboratory
Washington DC

Overview

- An attempt to steal postdocs from PI
- Brief outline of domain theory and measurement
- Domain theory in general relativity
- Domain theory in quantum mechanics
- The intersection

How to betray your friends while making them laugh

- We have an informatic phenomena group at NRL with broad interests
- ONR recently started a 6M dollar program on quantum information
- I am looking for researchers/students that have a healthy disgregard for convention
- I know some people that are influential in the right way

Any interested parties should send email to

keye.martin@nrl.navy.mil

Domain theory and measurement in physics (2003)

- One of the things people with no emotions do: Science.
- One of the areas of science that isn't really science: Computer Science.
- The part of any science no one respects: Theory.
- The part of theory that the theorists don't respect: Track B.
- The part of Track B near extinction: Semantics.
- The part of semantics that is extinct: Domain theory.

Domain theory and measurement in physics (2003)

- The fringe element in Domain theory: Measurement.
- The part of measurement for people who need medication: "Analysis of informatic phenomena."
- The community has about 5 people, but only 1 will admit it in public. Their consensus on the current work:

  As I read this paper, an excerpt from Blake's Auguries of Innocence came to mind: "To see a world in a grain of sand." Except without the world, and not quite a complete grain of sand, but only part of one.
  The part of a grain that is not interesting.

- **The silver lining**: I'm a postdoc.

Domains and measurement

Domains are partially ordered sets $(D, \sqsubseteq)$ with an unusual amount of structure:

- The partial order $\sqsubseteq$ is an 'information order' which reflects a relationship among the informative objects in D:

  $$x \sqsubseteq y \Rightarrow y \text{ is more informative than } x$$

  or

  $$x \sqsubseteq y \Rightarrow x \text{ carries information about } y$$

- Some elements in D are 'total' in that they are maximally informative:

  $$\max(D) = \{x \in D : (\forall y)\ x \sqsubseteq y \Rightarrow x = y\}$$

  others are 'partial' such as $\perp \sqsubseteq x$ for all x.

Domains and measurement

- Domains have an intrinsic notion of completeness:

$$x_1 \sqsubseteq x_2 \sqsubseteq \ldots \Rightarrow \bigsqcup x_n \in D$$

- An intrinsic notion of approximation:

$$x \ll y \Rightarrow x \text{ approximates } y$$

or

$$x \ll y \Rightarrow x \text{ carries } \textit{essential} \text{ information about } y$$

- A quantitative aspect: to each informative object $x \in D$, a measurement μ assigns the information content μx of x.

A surprise: degree of approximation = information content, in the context of an information order.

A qualitative aspect: completeness

A *dcpo* is a poset in which every directed subset has a supremum.

Defn. A function $f : D \to E$ between dcpo's is Scott continuous if it is monotone

$$x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$$

and preserves directed suprema: for any directed $S \subseteq D$,

$$f(\bigsqcup S) = \bigsqcup f(S).$$

Theorem. Let D be a dcpo with a least element $\perp$. If $f : D \to D$ is Scott continuous, then

$$\bigsqcup_{n \geq 0} f^n(\perp)$$

is the least fixed point of f.

A qualitative aspect: approximation

Defn. For elements x, y of a poset P, write $x \ll y$ iff for all directed sets S with a supremum,

$$y \sqsubseteq \bigsqcup S \Rightarrow (\exists s \in S)\, x \sqsubseteq s.$$

We set $\mathord{\downarrow\!\!\downarrow} x = \{a \in P : a \ll x\}$ and $\mathord{\uparrow\!\!\uparrow} x = \{a \in P : x \ll a\}$.

Note: $x \ll y$ means all paths to y or beyond must pass through x.

Defn. A poset P is *continuous* if it has a *basis*: a subset $B \subseteq P$ for which $B \cap \mathord{\downarrow\!\!\downarrow} x$ is directed with supremum x for each $x \in P$.

A *domain* is a continuous dcpo.

Example: The interval domain

The collection of compact intervals of the real line

$$\mathbf{I}\mathbb{R} = \{[a,b] : a,b \in \mathbb{R} \ \& \ a \leq b\}$$

ordered under reverse inclusion

$$[a,b] \sqsubseteq [c,d] \Leftrightarrow [c,d] \subseteq [a,b]$$

is an ω-continuous dcpo:

- For directed $S \subseteq \mathbf{I}\mathbb{R}$, $\bigsqcup S = \bigcap S$,
- $[a,b] \ll [c,d] \Leftrightarrow [c,d] \subseteq (a,b)$, and
- $\{[p,q] : p,q \in \mathbb{Q} \ \& \ p \leq q\}$ is a countable basis for $\mathbf{I}\mathbb{R}$.

Example: Majorization

The majorization relation is defined on *monotone states*:

$$\Lambda^n := \{x \in \Delta^n : (\forall i < n)\, x_i \geq x_{i+1}\}$$

Definition. For $x, y \in \Lambda^n$,

$$x \leq y \equiv (\forall k)\, s^k x \leq s^k y,$$

$$s^k x := \sum_{i=1}^{k} x_i$$

for all $k \in \{0, \ldots, n\}$. Note that $s^0 x = 0$ for all $x \in \Lambda^n$.

Example: Majorization

Theorem. $(\Lambda^n, \leq)$ is a continuous dcpo with least element $\perp = (1/n, \ldots, 1/n)$.

Basic facts about majorization:

- Introduced by Muirhead in 1903,
- Has numerous applications in computer science, economics, physics; at least two books have been written about it,
- Gained notoriety mathematically as a technique in proving inequalities (Hardy, Littlewood, Polya),
- Numerous uses in quantum mechanics, including *entanglement transformation* and *the classification theorem for ensembles*.

The Scott topology: entanglement monotones

Defn. Let P be a continuous poset. The sets

$$\mathord{\uparrow\!\!\!\uparrow} x := \{y \in P : x \ll y\}$$

form a basis for a topology on P called *the Scott topology.*

Example. A basic open set in $\mathbf{I}\mathbb{R}$ is

$$\mathord{\uparrow\!\!\!\uparrow}[a,b] = \{x \in \mathbf{I}\mathbb{R} : x \subseteq (a,b)\}$$

Theorem. A function $f : D \to E$ is Scott continuous iff the inverse image of a Scott open set in E is Scott open in D.

Example. A Euclidean continuous function $\mu : \Lambda^n \to [0,\infty)^*$ is Scott continuous iff it is an *entanglement monotone.*

The quantitative aspect: measurement

A *measurement* $\mu : D \to [0, \infty)^*$ assigns to each $x \in D$ a number μx which measures the information content of the object x.

Measurements are always Scott continuous:

- For all $x, y \in D$, $x \sqsubseteq y \Rightarrow \mu x \geq \mu y$, and
- If (x_n) is an increasing sequence in D, then

$$\mu\left(\bigsqcup_{n\geq 1} x_n\right) = \lim_{n\to\infty} \mu x_n.$$

They also satisfy an additional property which ensures that they measure *information content*.

Definition

Defn. A Scott continuous $\mu : D \to [0,\infty)^*$ is said to *measure the content* of $x \in D$ if for all Scott open sets $U \subseteq D$,

$$x \in U \Rightarrow (\exists \varepsilon > 0)\, x \in \mu_\varepsilon(x) \subseteq U$$

where

$$\mu_\varepsilon(x) := \{y \in D : y \sqsubseteq x \;\&\; |\mu x - \mu y| < \varepsilon\}$$

are the ε–approximations of x.

Every observation about x is also an observation about its approximations, provided that these approximations are close enough to x in a sense *determined* by μ.

In this talk, a *measurement* μ is a function that measures D at each $x \in D$.

Properties of measurements

Theorem. Let $\mu : D \to [0,\infty)^*$ be a measurement.

- If $x \in \ker(\mu)$, then $x \in \max(D)$.
- For all $x, y \in D$,

$$x \sqsubseteq y \ \&\ \mu x = \mu y \Rightarrow x = y.$$

- The collection

$$\{\uparrow\mu_\varepsilon(x) \cap X : x \in X, \varepsilon > 0\}$$

is a basis for the Scott topology on D.

The second property helps us assign rates of change to processes $f : D \to D$ on domains as follows.

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The informatic derivative

Definition. If $f : D \to D$ is a function on a domain (D, μ) with a measurement, then

$$df_\mu(p) := \lim_{x \to p} \frac{\mu f(x) - \mu f(p)}{\mu x - \mu p}$$

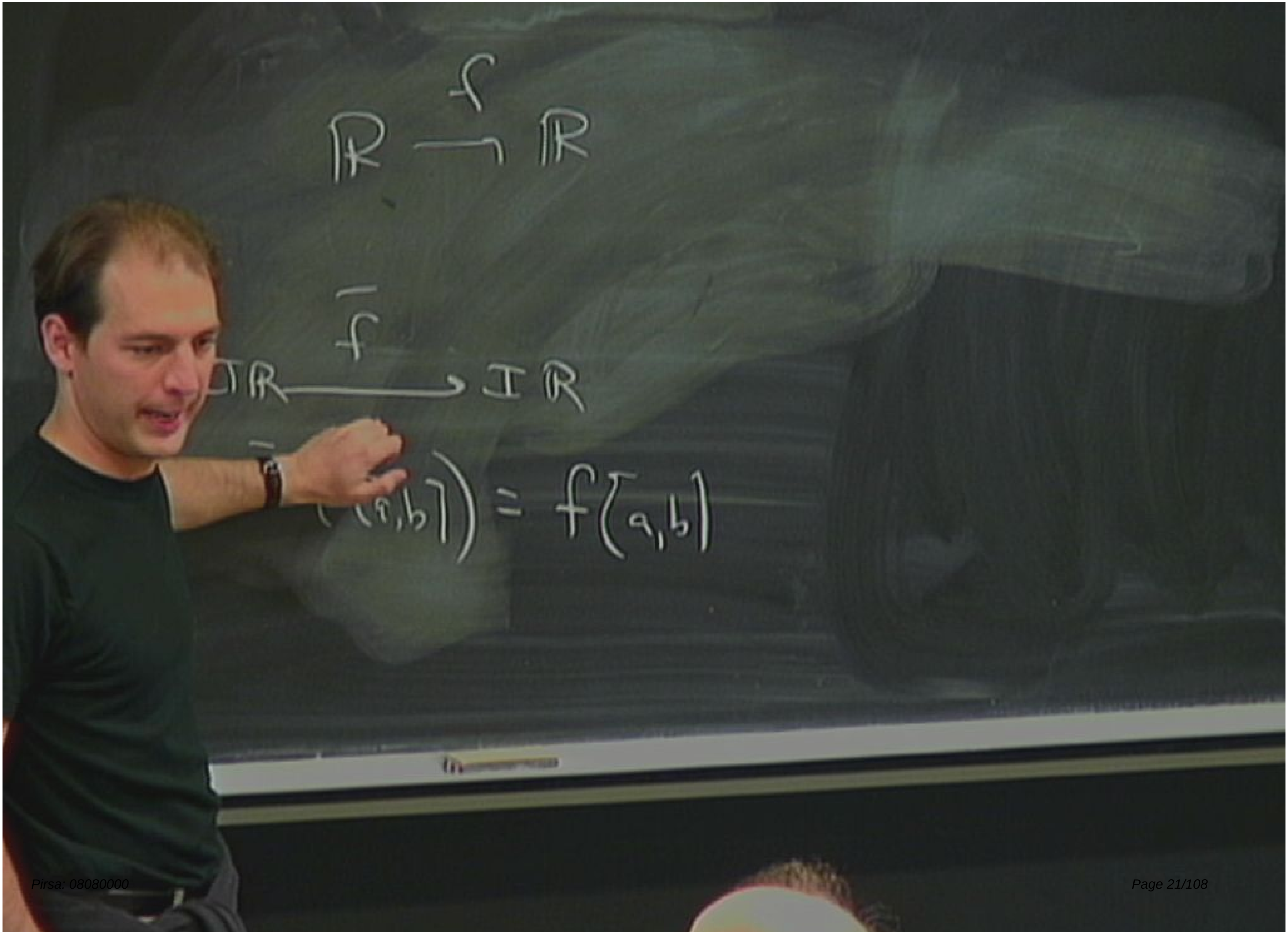
is called the *informatic derivative* of f at p with respect to μ.

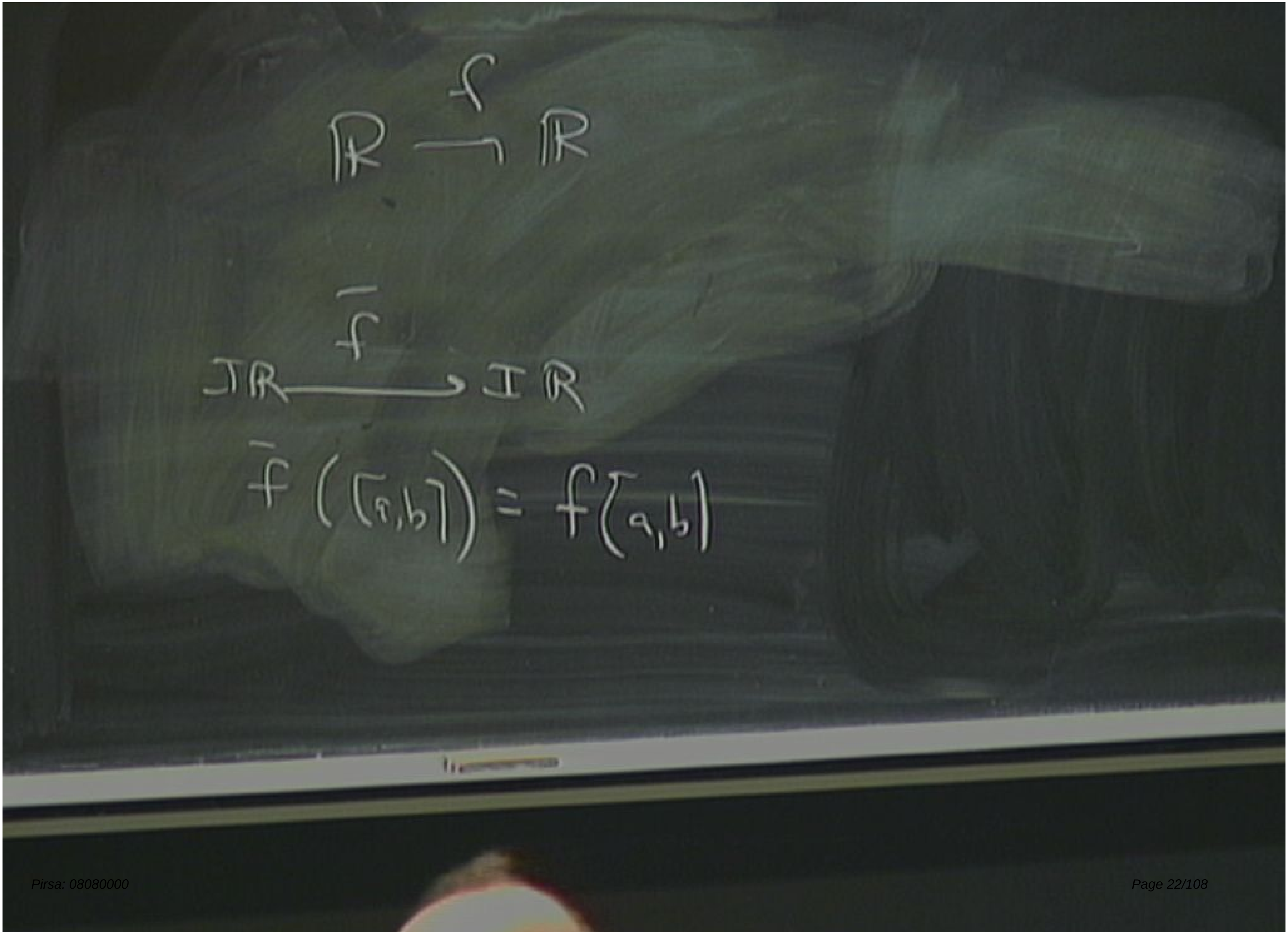
Example. If $f : \mathbb{R} \to \mathbb{R}$ and $f'(p)$ exists, then

$$d\bar{f}_\mu([p, p]) = |f'(p)|$$

where $\bar{f}$ is the extension of f to $\mathbf{I}\mathbb{R}$ and μ is the length measurement.

Many informatic derivatives have no classical counterpart – even in numerical analysis. There is also a 'discrete' derivative.


f
ℝ → ℝ
f̄
Iℝ → Iℝ
) = f[a,b]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A couple of things to do with a measurement

- Existence of fixed points $p = \bigsqcup f^n(\bot)$ for maps f whose *measures* $\mu \circ f$ are continuous,

- If $p = f(p)$ is a fixed point, then $df_\mu(p)$ measures the rate at which $f^n(\bot)$ converges to p.

- Each measurement μ gives rise to a 'metric'

$$d(\mu) : D^2 \to [0, \infty)$$

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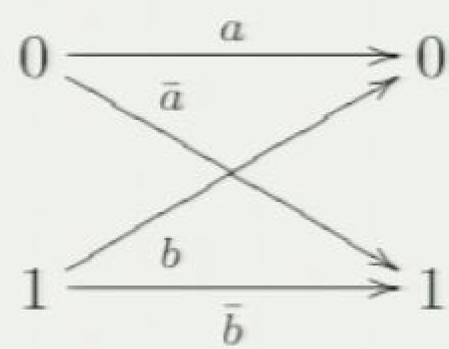
Example: the domain of binary channels

The effect of noise on data transmitted through a binary channel is modelled by a noise matrix $(a, b) \in [0, 1]^2$ of probabilities which describe the likelihood of receiving the symbol 0:



$$0 \to 0 \text{ with probability } a$$
$$0 \to 1 \text{ with probability } \bar{a} = 1 - a$$

$$1 \to 0 \text{ with probability } b$$
$$1 \to 1 \text{ with probability } \bar{b} = 1 - b$$

Capacity

The amount of information that may be sent through a channel (a, b) is given by its capacity

$$\mathrm{C}(a,b) = \sup_{x\in[0,1]} H((a-b)x+b) - xH(a) - (1-x)H(b)$$

This is explicitly given by

$$\mathrm{C}(a,b) = \log_2\left(2^{\frac{\bar{a}H(b)-\bar{b}H(a)}{a-b}} + 2^{\frac{bH(a)-aH(b)}{a-b}}\right)$$

where $\mathrm{C}(a,a) := 0$ and $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the base two entropy.

$$\mathrm{C}(a,b) = \mathrm{C}(b,a)$$

This means we can assume that all channels are nonnegative.

Again?

For a binary channel $x = (a, b)$, its probability of error is

$$e_x(t) = t\bar{a} + (1 - t)b$$

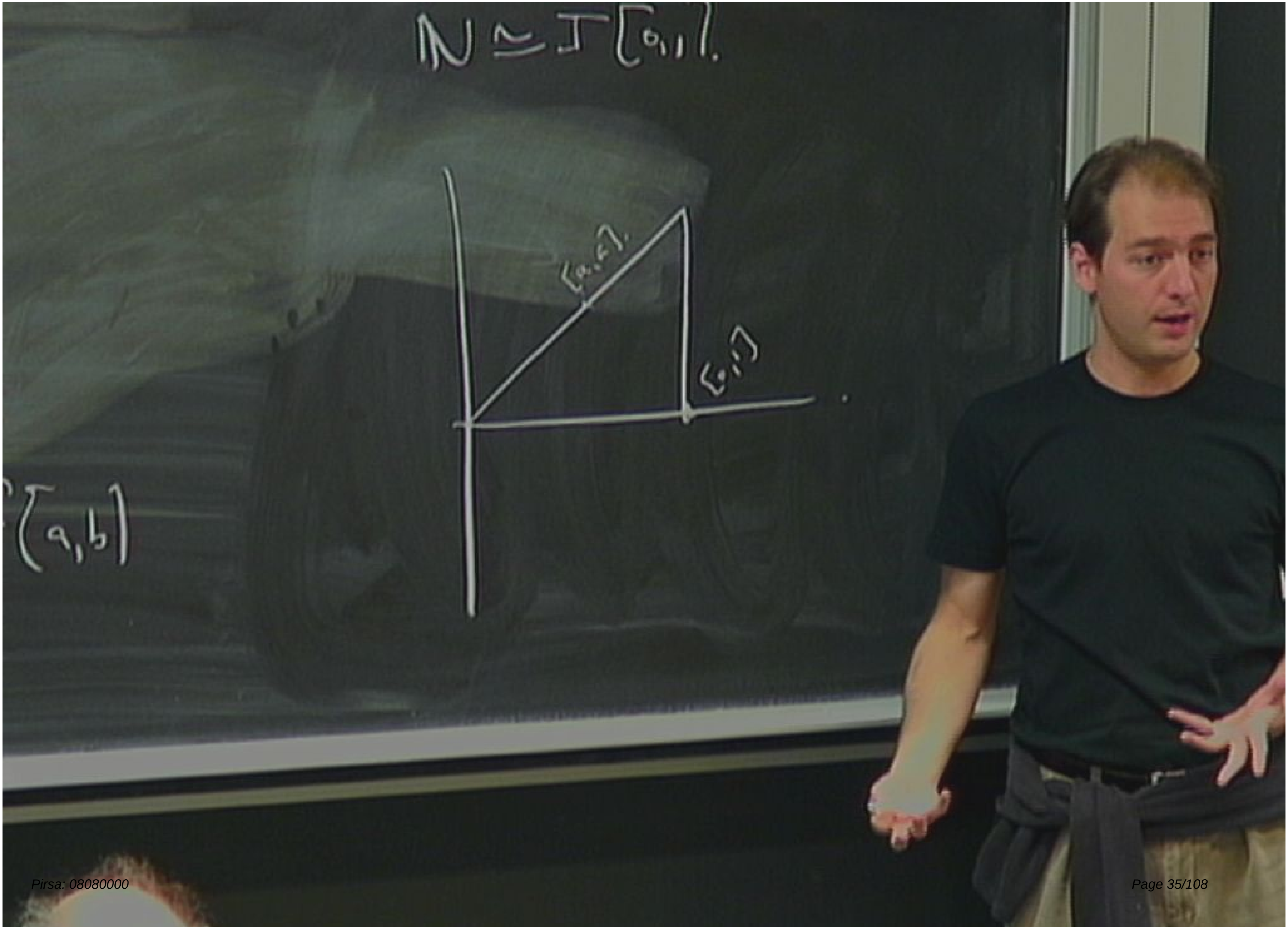
where t is the probability that '0' is sent through the channel. Order the nonnegative channels $\mathbb{N}$ by

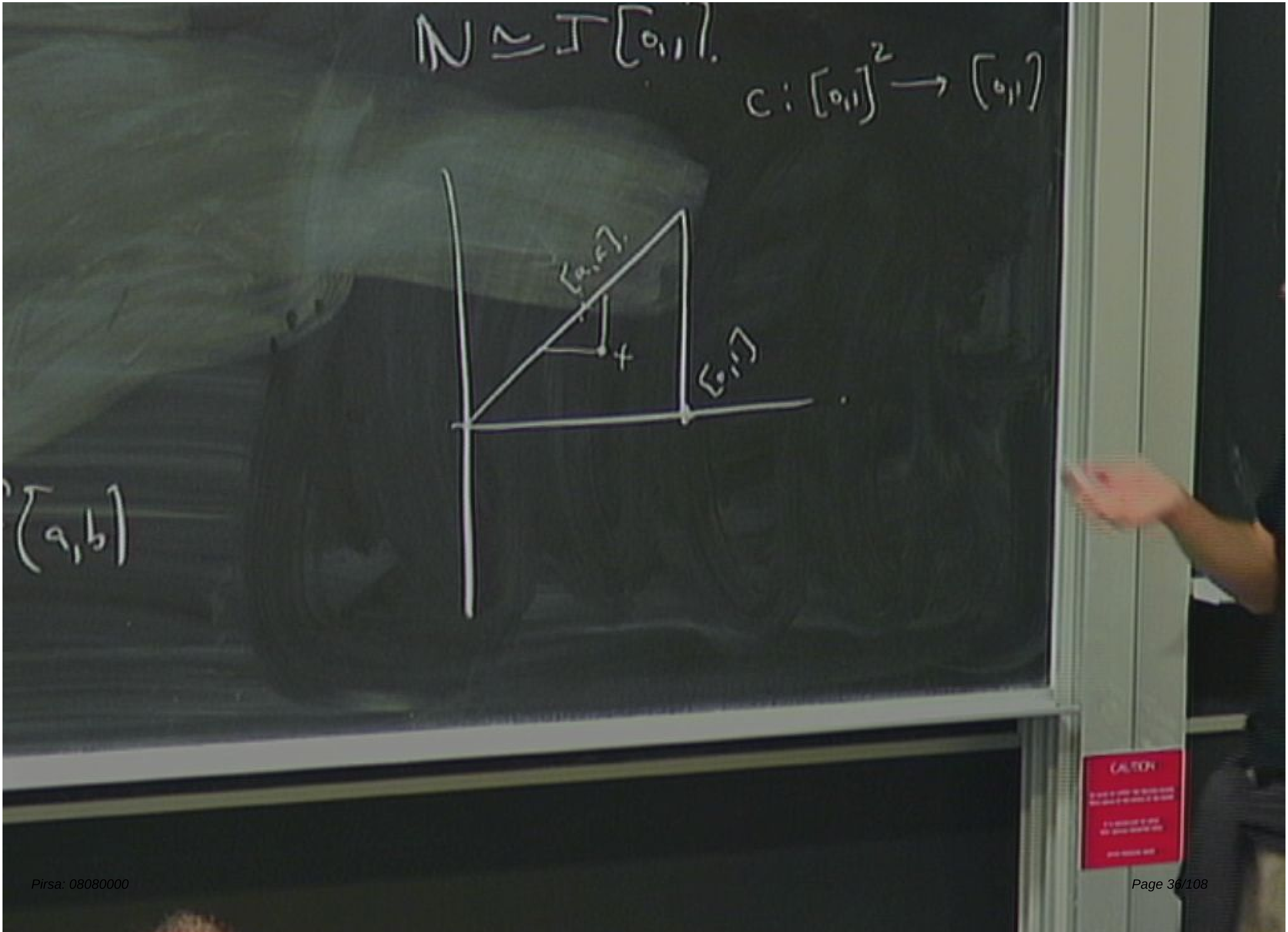
$$x \sqsubseteq y \Leftrightarrow (\forall t \in [0, 1])\ e_x(t) \leq e_y(t)$$

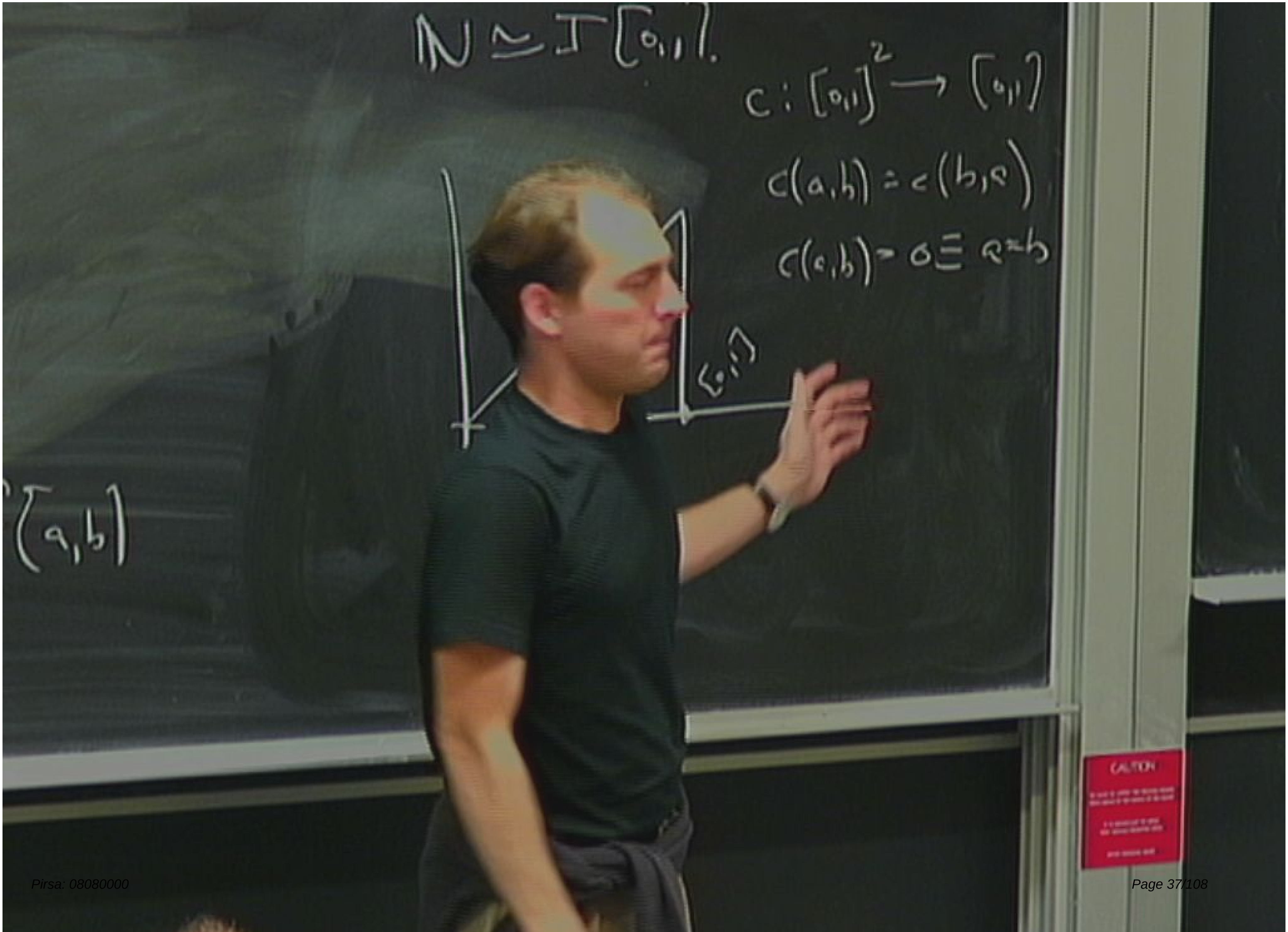
Theorem. The set $\mathbb{N}$ of nonnegative channels is a domain with capacity as a measurement. In fact, $\mathbb{N} \simeq \mathbf{I}[0, 1]$ and

$$\ker(\mathrm{C}) = \{(a, b) : a = b\} = \max(\mathbb{N}) \simeq [0, 1].$$

Since capacity must then yield the Euclidean topology on $[0, 1]$, we can define distance to be the *amount of information that can be transmitted between two points.*


N ≃ I[0,1].
[a,c]
[0,1]
[a,b]


N ≃ I [0,1].
c : [0,1]^2 → [0,1]
[0,c]
x
[0,1]
[a,b]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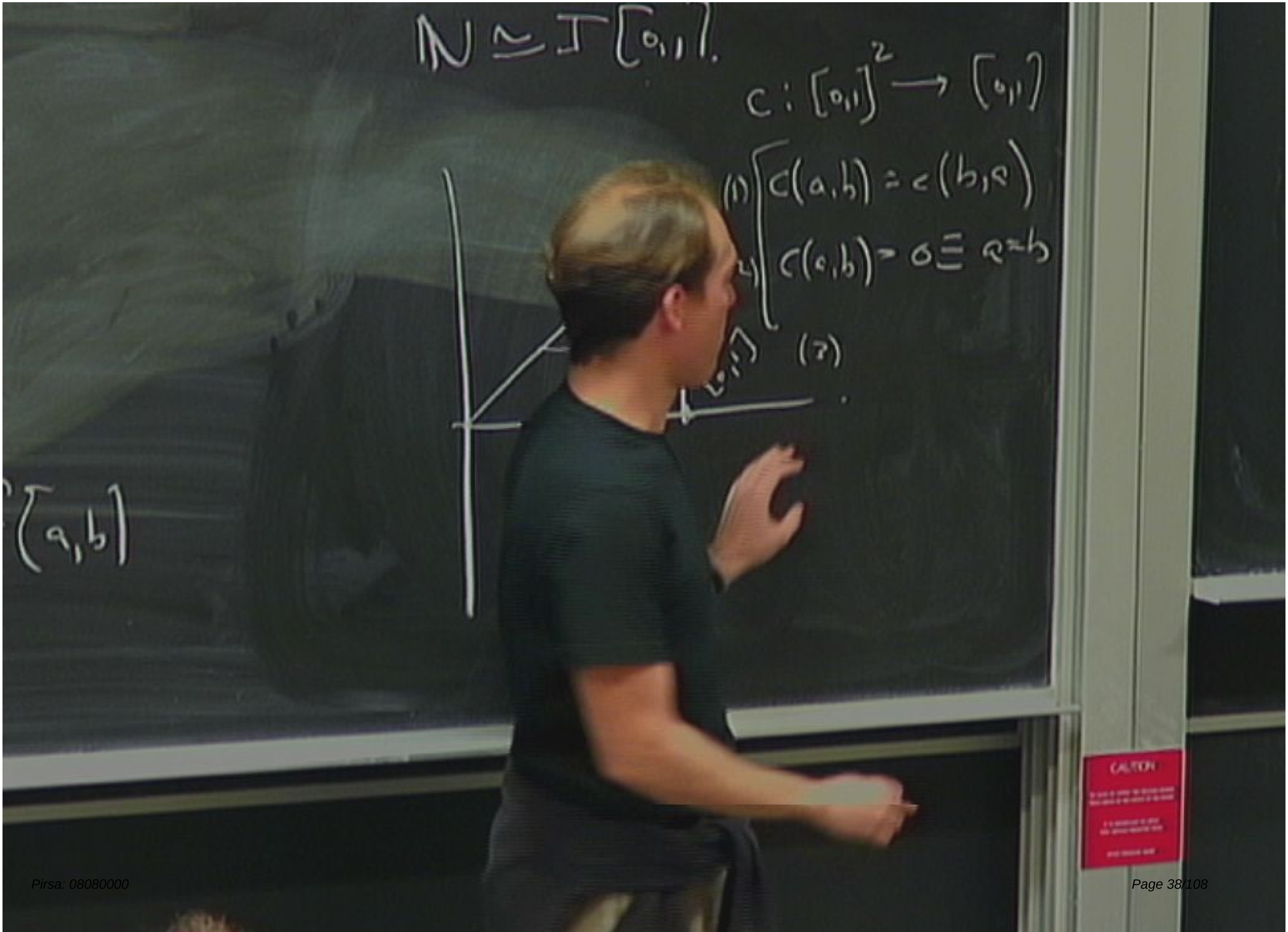
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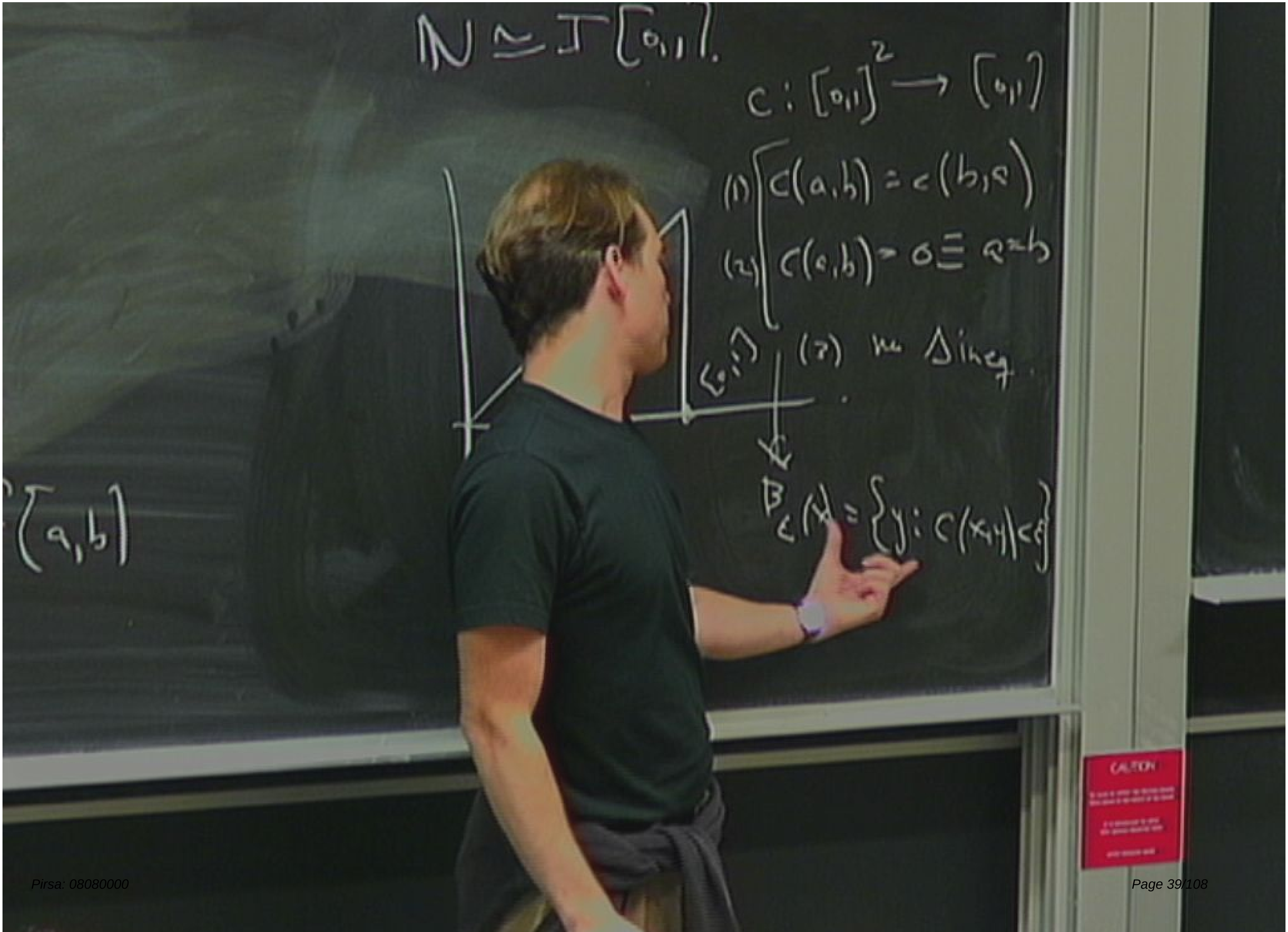
(3)
[a,b]
CAUTION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(3) ~ Δ ineq.
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[a,b]
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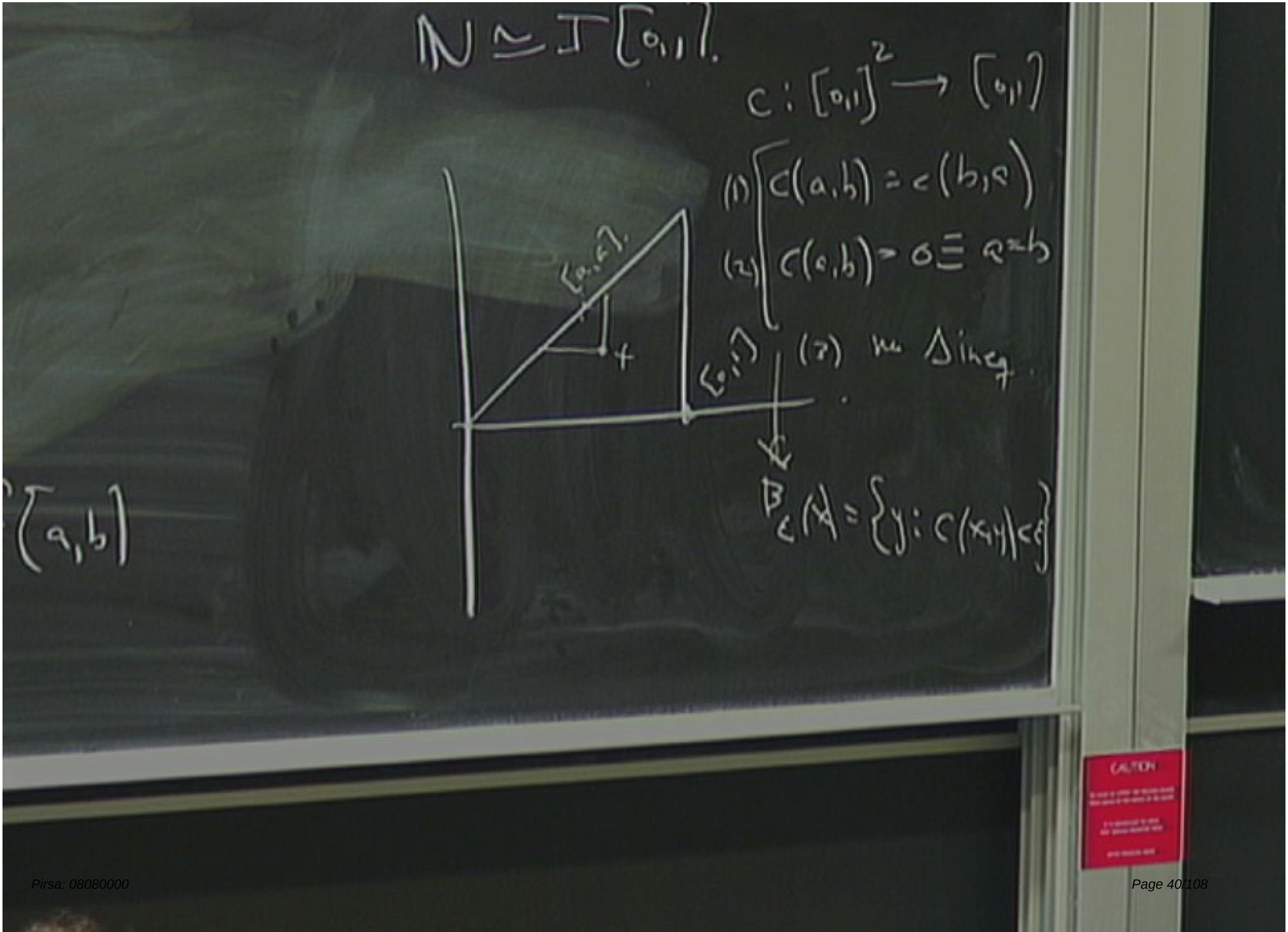
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Since capacity must then yield the Euclidean topology on $[0, 1]$, we can define distance to be the *amount of information that can be transmitted between two points.*

Where do interval domains come from?

The real line $(\mathbb{R}, \leq)$ is continuous with

$$x \ll y \equiv x < y$$

but is not a dcpo. However, it is closely related to a domain, the *interval domain*

$$\mathbf{I}\mathbb{R} = \{[a, b] : a \leq b \ \& \ a, b \in \mathbb{R}\}$$

ordered by reverse inclusion with $\mathbb{R} \simeq \max(\mathbf{I}\mathbb{R})$.

In general, what type of structure must a poset $(X, \leq)$ have to allow the construction of an 'interval domain' $\mathbf{I}X$?

The interval domain of $(X, \leq)$

Defn. A continuous poset $(X, \leq)$ is *bicontinuous* if

- For all $x, y \in X$, $x \ll y$ iff for all filtered $S \subseteq X$ with an infimum,
$$\bigwedge S \leq x \Rightarrow (\exists s \in S)\, s \leq y,$$
- For each $x \in X$, $\uparrow\!x$ is filtered with infimum x.

Defn. For $(X, \leq)$ bicontinuous, the sets

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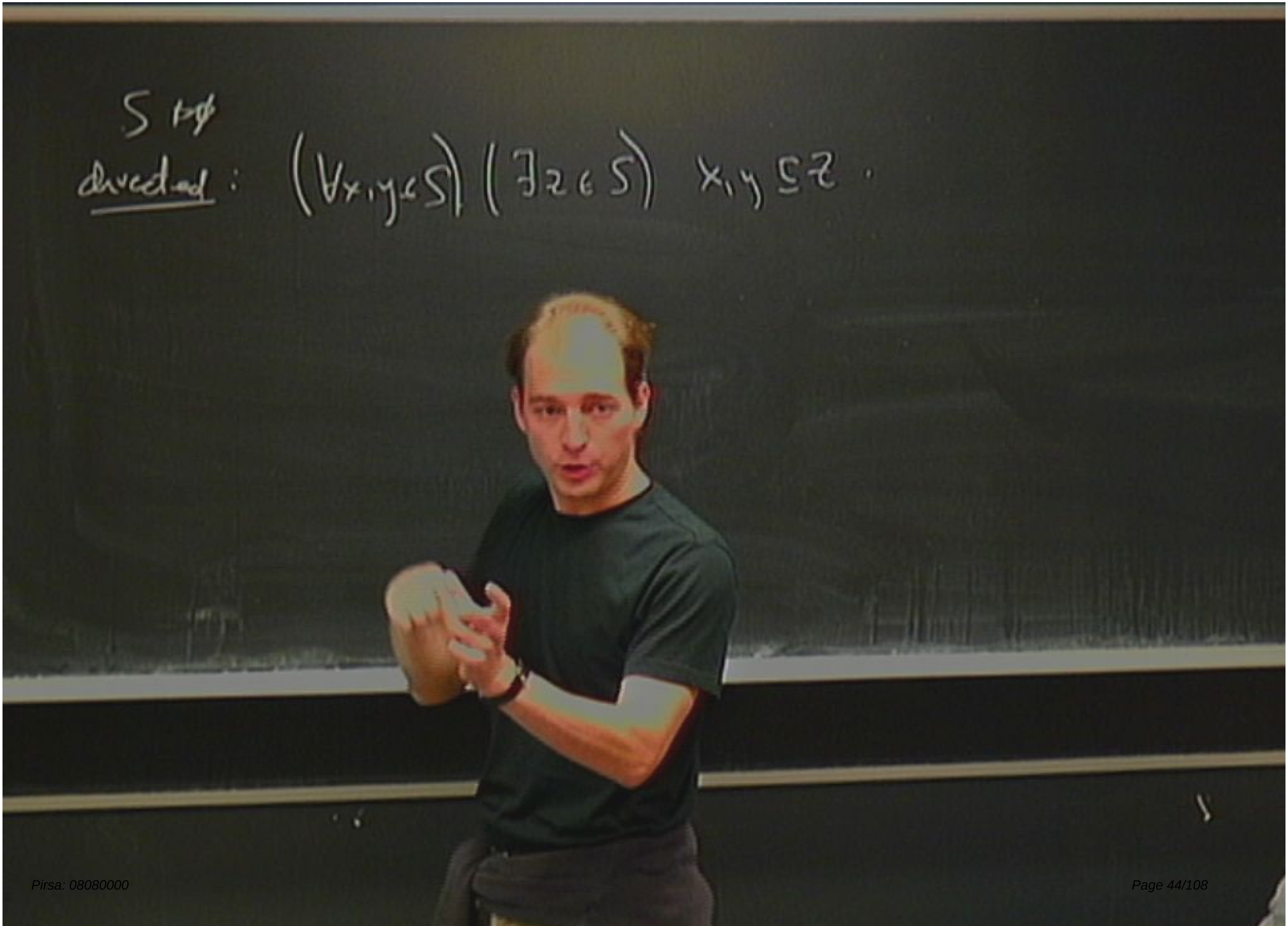
form a basis for a topology called *the interval topology*.


S ≠ ∅
directed: (∀x,y∈S)(∃z∈S) x,y ⊑ z .

$S \neq \emptyset$

<u>directed</u>: $(\forall x, y \in S)(\exists z \in S)\ x, y \sqsubseteq z$.

<u>filtered</u> $(\forall x, y \in S)(\exists z \in S)\ z \sqsubseteq x, y$.

$\cdots \sqsubseteq x_3 \sqsubseteq x_2 \sqsubseteq x_1$

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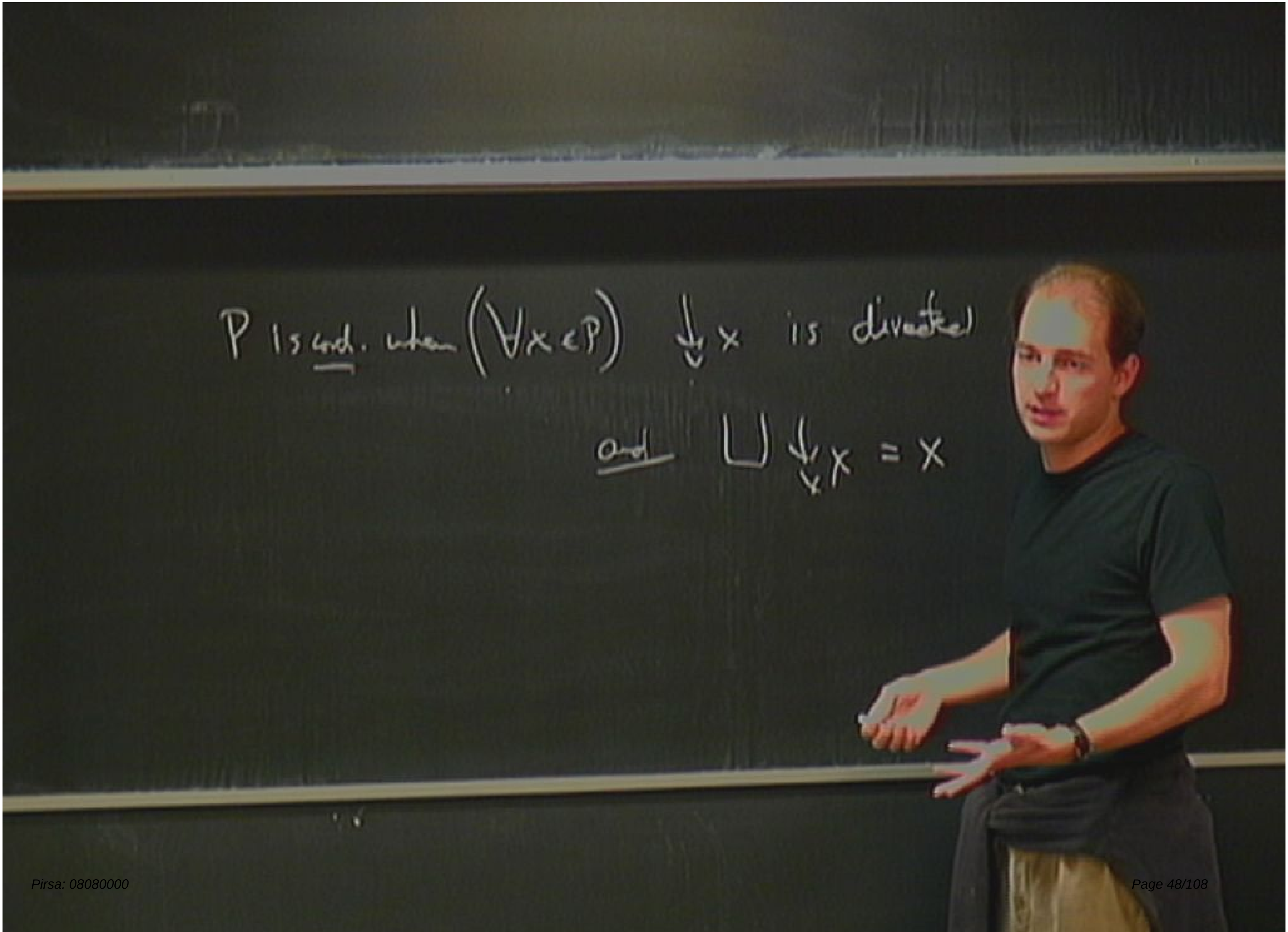
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Theorem. If $(X, \leq)$ is a globally hyperbolic poset, then

$$\mathbf{I}X = \{[a, b] : a \leq b \;\&\; a, b \in X\}$$

is a continuous dcpo when ordered by reverse inclusion with

$$X \simeq \max(\mathbf{I}X)$$

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Spacetime

Let $(\mathcal{M}, g_{ab})$ be a time orientable spacetime satisfying the usual chronology conditions and strong causality.

The *timelike future* of an event p is

$$I^+(p) = \{q \in \mathcal{M} : \exists \text{ f.d. timelike } \pi, \pi(0) = p, \pi(1) = q\}$$

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Intuition: On the real line,

- $x < y \equiv (\exists \pi)\ \pi(0) = x,\ \pi(1) = y,\ \dot{\pi} > 0$ (timelike)
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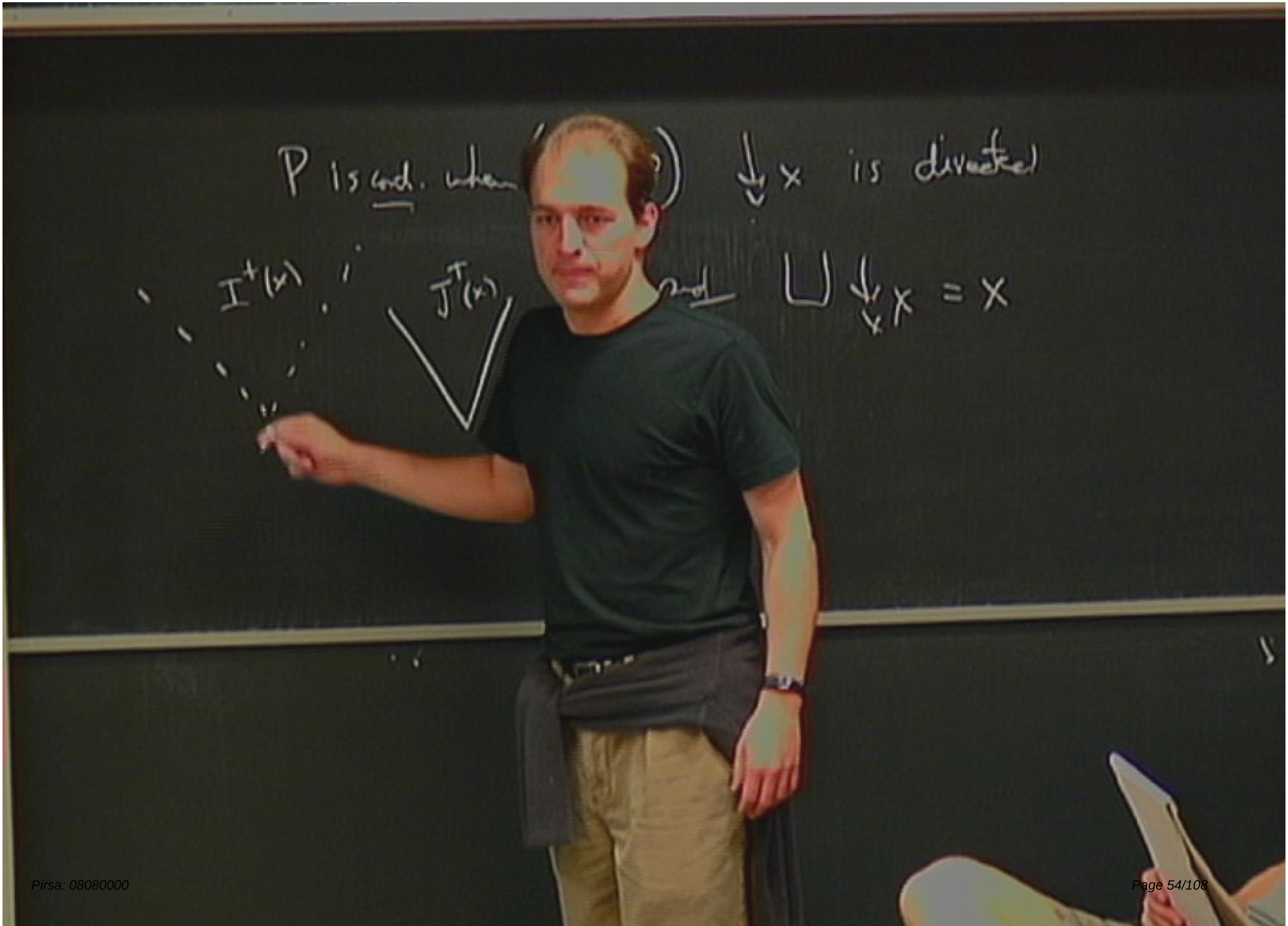
P is
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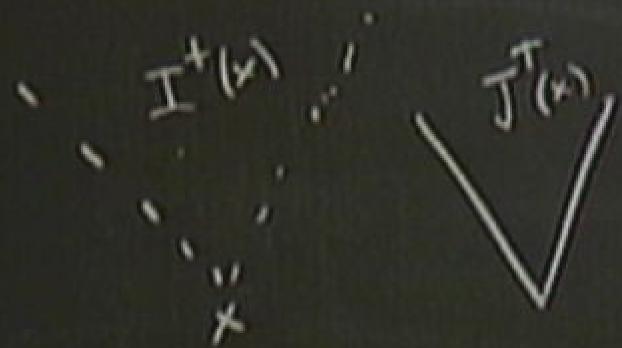

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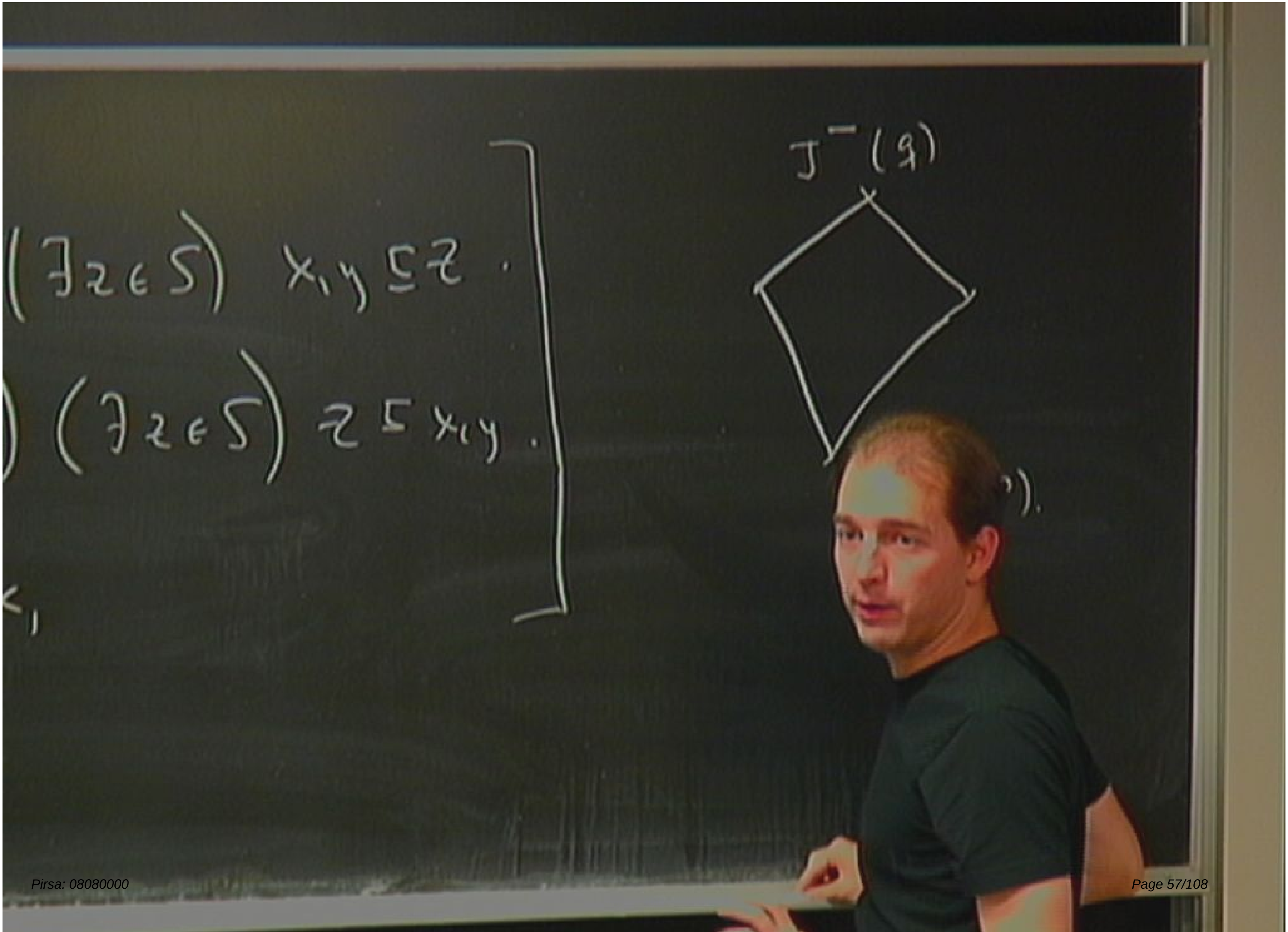
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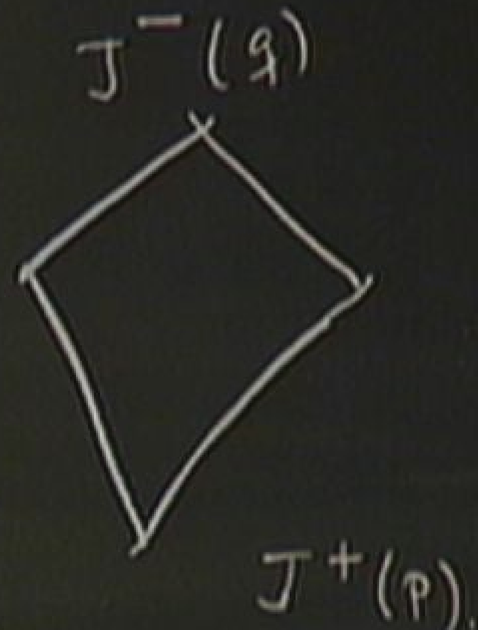

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Global hyperbolicity of spacetime

Write

$$x \leq y \equiv y \in J^+(x)$$

for the causality relation on a spacetime.

Theorem. If $(\mathcal{M}, g_{ab})$ is a spacetime with causal relation $\leq$, then

- $(\mathcal{M}, \leq)$ is a globally hyperbolic poset.
- For $x, y \in \mathcal{M}$, $x \ll y \equiv y \in I^+(x)$.
- The interval topology of $(\mathcal{M}, \leq)$ is the manifold topology.

Thus, $\mathbf{I}\mathcal{M}$ is a continuous dcpo, and *so* spacetime can be reconstructed in a purely order theoretic manner from only a countable set of events.

Reconstruction of spacetime

Given a countable dense set $(C, \ll)$, we form an abstract basis

$$\mathrm{int}(C) = \{(a, b) : a \ll b\} = \ll \subseteq C^2$$

whose relation is

$$(a, b) \ll (c, d) \equiv a \ll c \;\&\; d \ll b.$$

and then take its rounded ideal completion $\mathbf{I}(C)$. We are then able to recover the spacetime manifold $\mathcal{M}$ as

$$\max(\mathbf{I}C) \simeq \mathcal{M}$$

where the set of maximal elements have the Scott topology.

Question: But does global hyperbolicity abstractly capture some essential aspect of causality?

The space of causal curves

A subset $\pi \subseteq \mathcal{M}$ of spacetime is the image of a causal curve iff it is compact connected and linearly ordered by causality.

Let $(X, \leq)$ be a globally hyperbolic poset.

Defn. A subset $\pi \subseteq X$ is a *causal curve* if it is compact, connected and linearly ordered. We define

$$\pi(0) := \bot \text{ and } \pi(1) := \top$$

where $\bot$ and $\top$ are the least/greatest elements of π.

Events belong to $\max(\mathbf{I}X)$, causal curves belong to $\max(\mathbf{C}(\mathbf{I}X))$.

For $P, Q \subseteq X$,

$$C(P,Q) := \{\pi : \pi \text{ causal curve}, \pi(0) \in P, \pi(1) \in Q\}$$

is *the space of causal curves* between P and Q.

Theorem. The space $C(P,Q)$ is compact in the Vietoris topology when $P, Q \subseteq X$ are compact.

Applications:

- Existence of maximum length geodesics (Hawking),
- Positive mass theorem (Penrose, Sorkin, Woolgar),
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What is global hyperbolicity?

An *interval domain* is a continuous dcpo D with two maps

$$\text{left} : D \to \max(D) \quad \& \quad \text{right} : D \to \max(D)$$

satisfying axioms to be expected of intervals in a space.

Theorem. The category of globally hyperbolic posets is naturally isomorphic to the category of interval domains.

Proof: $X \to (\mathbf{I}X, \sqsubseteq, \text{left}, \text{right}) \to (\max(\mathbf{I}X), \leq)$

This is the underlying reason for why it is possible to causally reconstruct spacetime.

Global time as a measurement

A global time function $t : \mathcal{M} \to \mathbb{R}$ on a globally hyperbolic spacetime $\mathcal{M}$ is a continuous function such that

$$x < y \Rightarrow t(x) < t(y)$$

and such that $t^{-1}(r) = \Sigma$ is a Cauchy surface for $\mathcal{M}$, for all $r \in \mathbb{R}$.

Theorem. For any global time function $t : \mathcal{M} \to \mathbb{R}$ on a globally hyperbolic spacetime, the function $\Delta t : \mathbf{I}(\mathcal{M}) \to [0, \infty)^*$ given by

$$\Delta t[a, b] = t(b) - t(a)$$

measures $\mathbf{I}(\mathcal{M})$. It is a measurement with $\ker(\Delta t) = \max(\mathbf{I}(\mathcal{M}))$.

Lorentz distance

Let $d : \mathbf{I}(\mathcal{M}) \to [0, \infty)^*$ denote the Lorentz distance on a globally hyperbolic spacetime

$$d[a, b] = \sup_{\pi_{ab}} \mathrm{len}(\pi_{ab})$$

where the sup is taken over all causal curves that join a to b.

Theorem. The Lorentz distance $d : \mathbf{I}(\mathcal{M}) \to [0, \infty)^*$ is Scott continuous.

Consequence: Now we can reconstruct its geometry too.

Reconstruction of spacetime geometry

If we also have a countable collection of numbers l_{ab} chosen for each $(a, b) \in \mathrm{int}(C)$ in such a way that the map

$$\mathrm{int}(C) \to [0, \infty)^* :: (a, b) \mapsto l_{ab}$$

is monotone, then $l : \mathbf{I}C \to [0, \infty)^*$ given by

$$l(x) = \inf\{l_{ab} : (a, b) \ll x\}$$

is Scott continuous. If the countable number of l_{ab} chosen are the Lorentz distances $l_{ab} = d[a, b]$, then $l = d$.

Point: From a countable set of events and a countable set of distances, we can reconstruct the spacetime manifold together with its geometry in a purely order theoretic (or *causal*) manner.

Information content: perimeter not area

Lemma. The Lorentz distance d is not a measurement.

Why?

Because a clock travelling at the speed of light records no time as having elapsed:

$$\ker(d) \setminus \max(\mathbf{I}(\mathcal{M})) \neq \emptyset$$

We can say more than this though.

Topological distinction between notions of time

Defn. The *interval topology* on a continuous poset P exists when sets of the form

$$(a, b) = \{x \in P : a \ll x \ll b\} \;\; \& \;\; \Uparrow x = \{y \in P : x \ll y\}$$

form a basis for a topology on P.

Theorem.

- Δt and d are Scott continuous.
- The Lorentz distance d is interval continuous, Δt is not.
- Global time Δt is a measurement, d is not.
- No interval continuous function $\mu : \mathbf{I}(\mathcal{M}) \to [0, \infty)^*$ is a measurement: $\mu x = 0$ for any x with $\Uparrow x = \emptyset$.

Things we learned

- Causality can be studied using only domain theoretic ideas: geometry and differentiable structure are not needed to understand or prove many of its important aspects.
- We can reconstruct the spacetime manifold with its geometry from a countable set in a purely order theoretic manner.
- The difference between the Newtonian and relativistic notions of time can be described topologically.
- The difference between the topology and geometry can be described topologically – using a topology derived from causal relationships!
- A step toward a general definition of Lorentz invariance: Lorentz invariant quantities are interval continuous.

The Bayesian order

If we know $x \in \Delta^{n+1}$ and by some means determine outcome i is not possible, our knowledge improves to

$$p_i(x) = \frac{1}{1 - x_i}(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \in \Delta^n$$

where $p_i(x)$ is obtained by first removing x_i from x and then renormalizing.

The partial mappings which result

$$p_i : \Delta^{n+1} \rightharpoonup \Delta^n$$

with $\mathrm{dom}(p_i) = \Delta^{n+1} \setminus \{e_i\}$, are called the *Bayesian projections*.

The Bayesian projections suggest the following relation on Δ^n.

Definition. For $x, y \in \Delta^{n+1}$,

$$x \sqsubseteq y \equiv (\forall i)(x, y \in \mathrm{dom}(p_i) \Rightarrow p_i(x) \sqsubseteq p_i(y)).$$

For $x, y \in \Delta^2$,

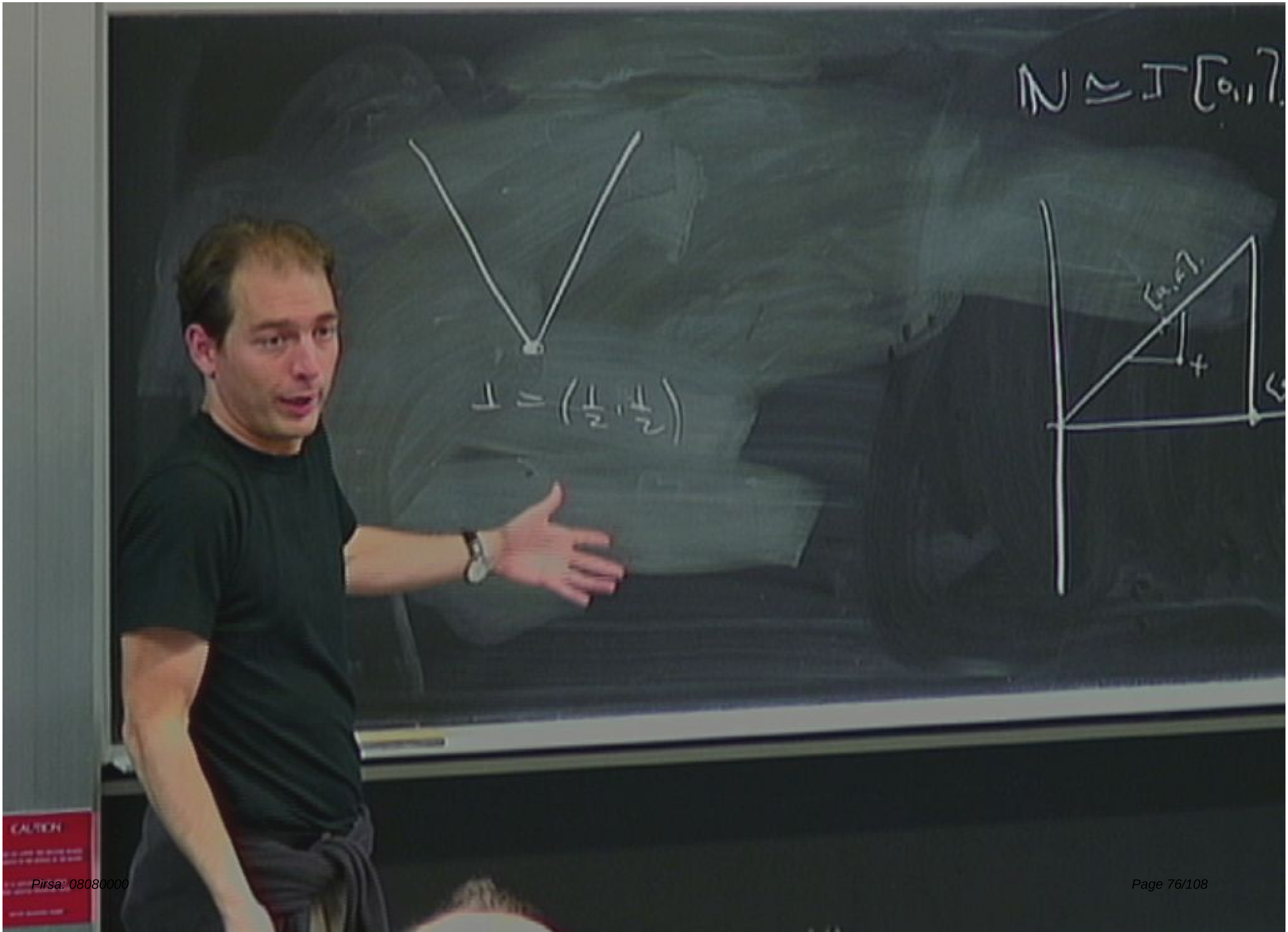
$$x \sqsubseteq y \equiv (y_1 \leq x_1 \leq 1/2) \text{ or } (1/2 \leq x_1 \leq y_1)\,.$$

The relation $\sqsubseteq$ on Δ^n is called the *Bayesian order*.

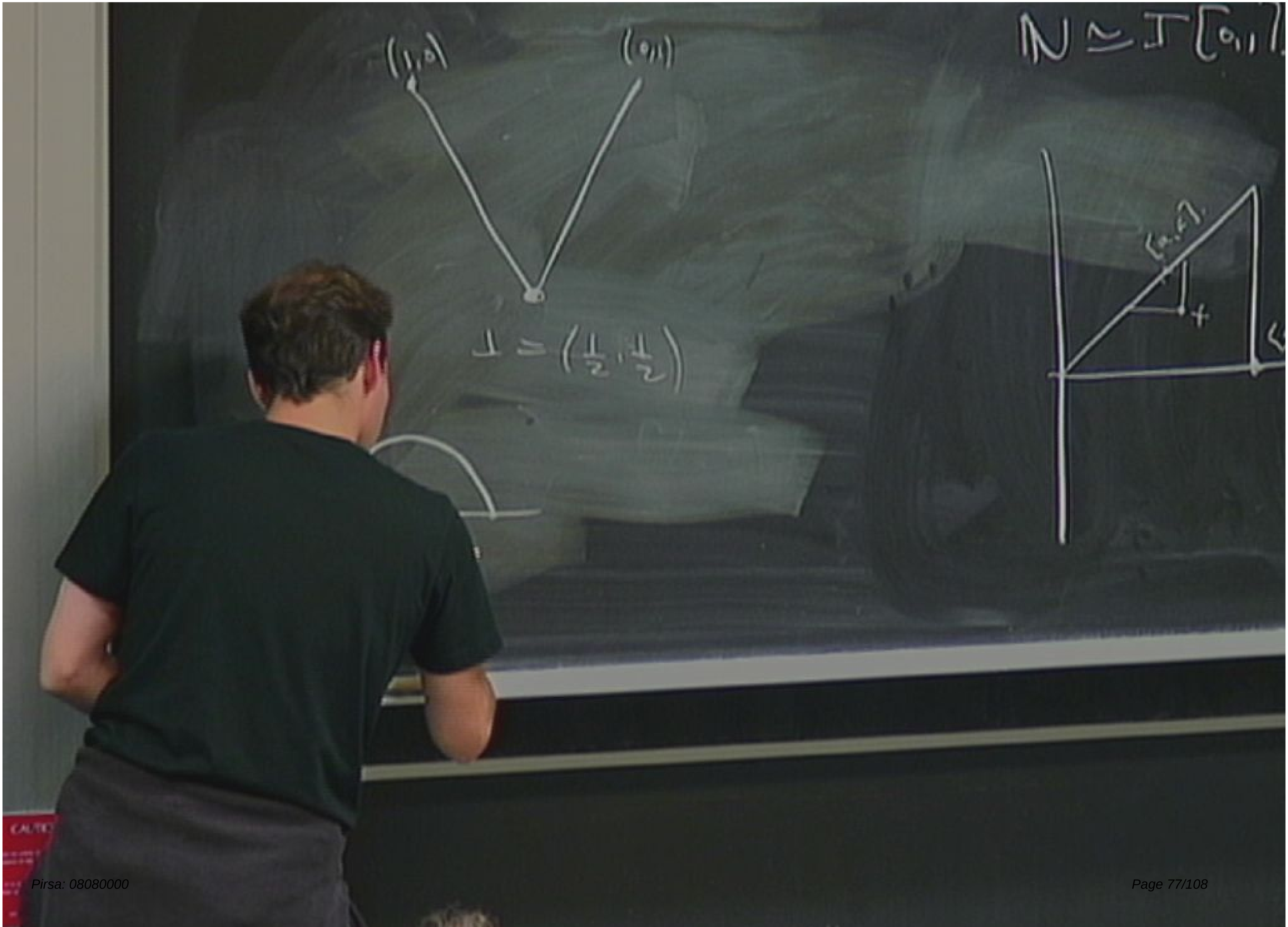
Theorem. There is a unique partial order on Δ^2 which has $\perp := (1/2, 1/2)$ and satisfies the mixing law

$$x \sqsubseteq y \text{ and } p \in [0,1] \;\Rightarrow\; x \sqsubseteq (1-p)x + py \sqsubseteq y\,.$$

It is the Bayesian order on classical two states.



CAUTION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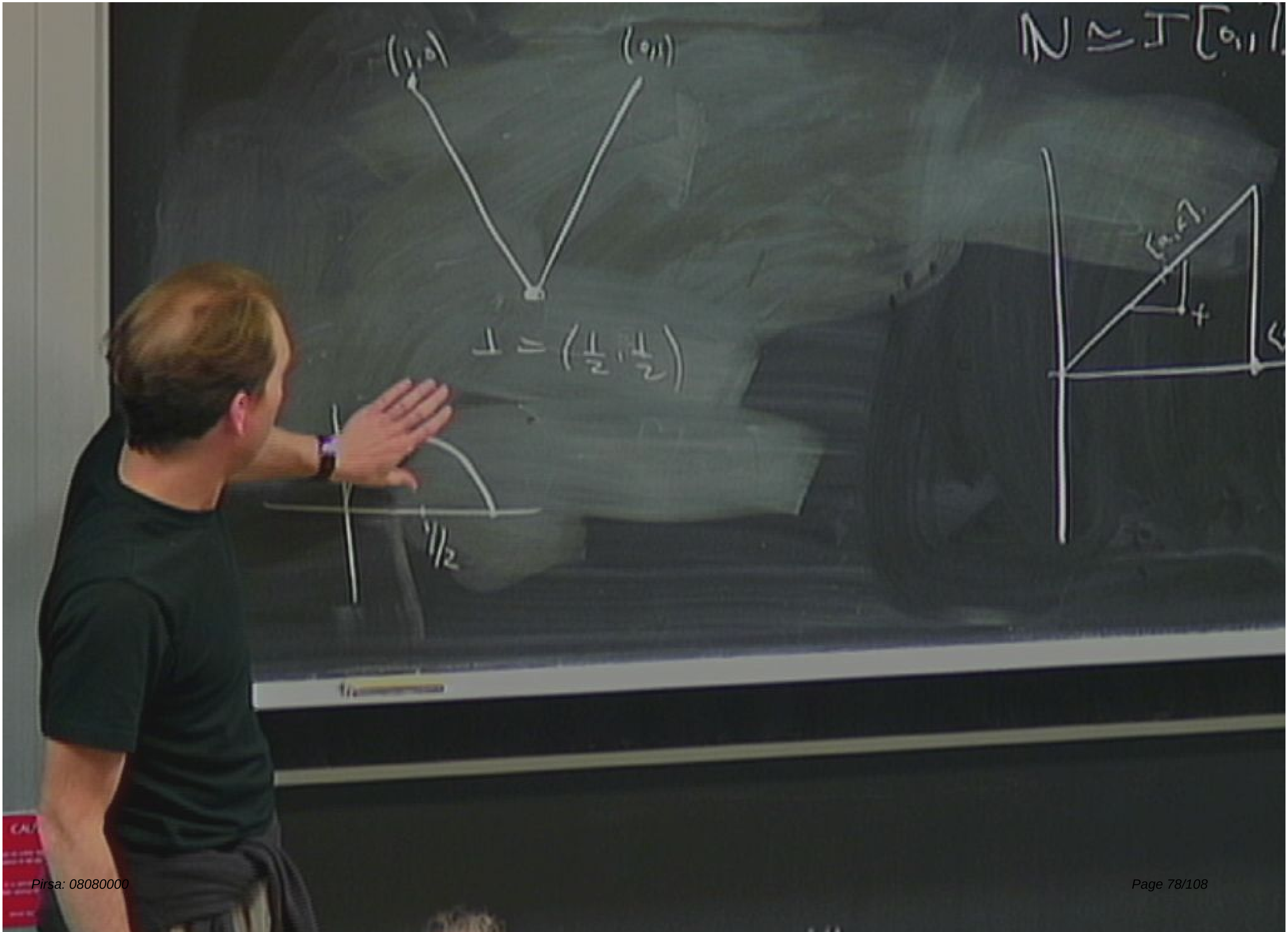
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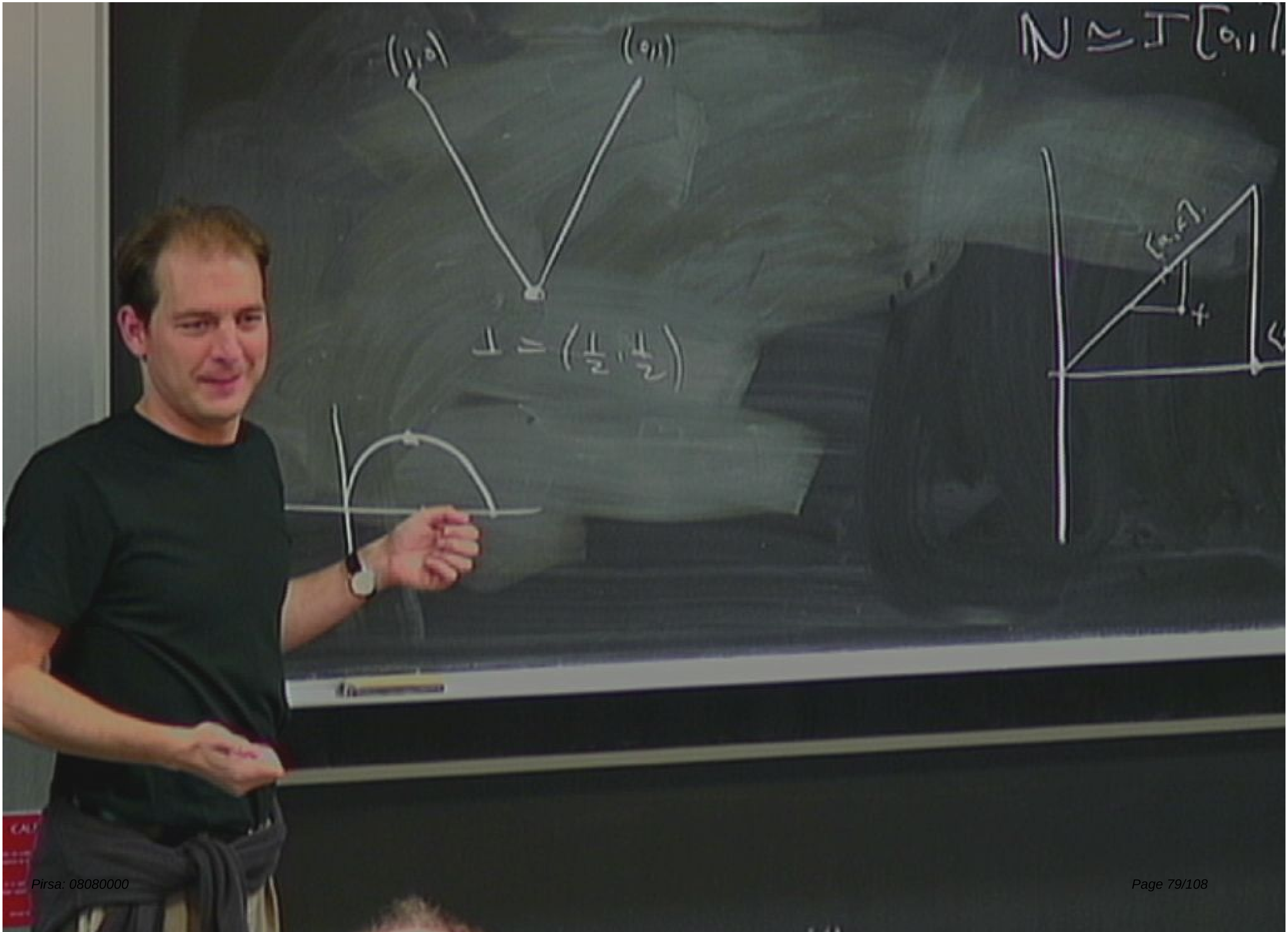
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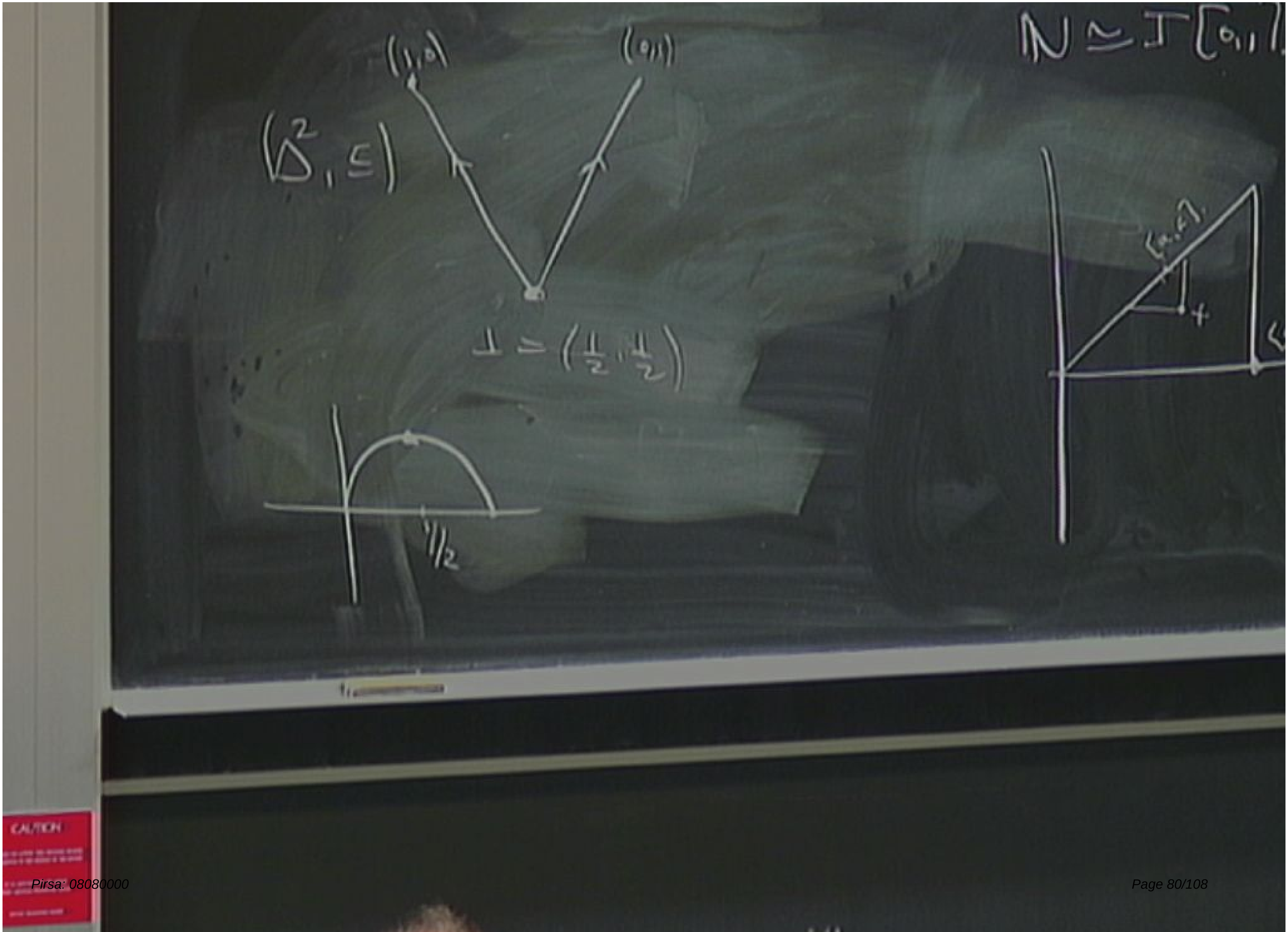
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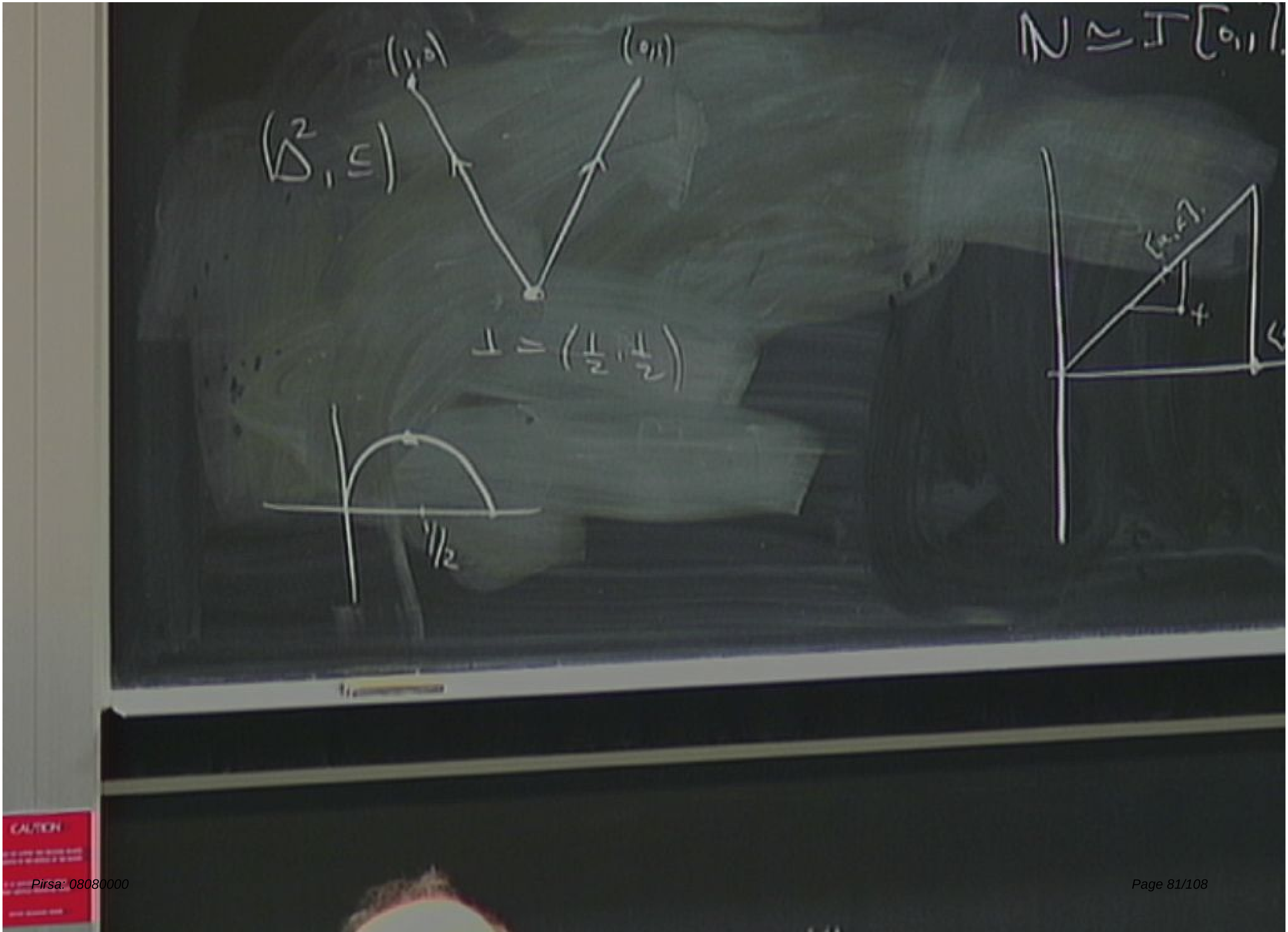
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It is the Bayesian order on classical two states.

Let e_i be the state x with $x_i = 1$.

Theorem. $(\Delta^n, \sqsubseteq)$ is an exact dcpo with maximal elements

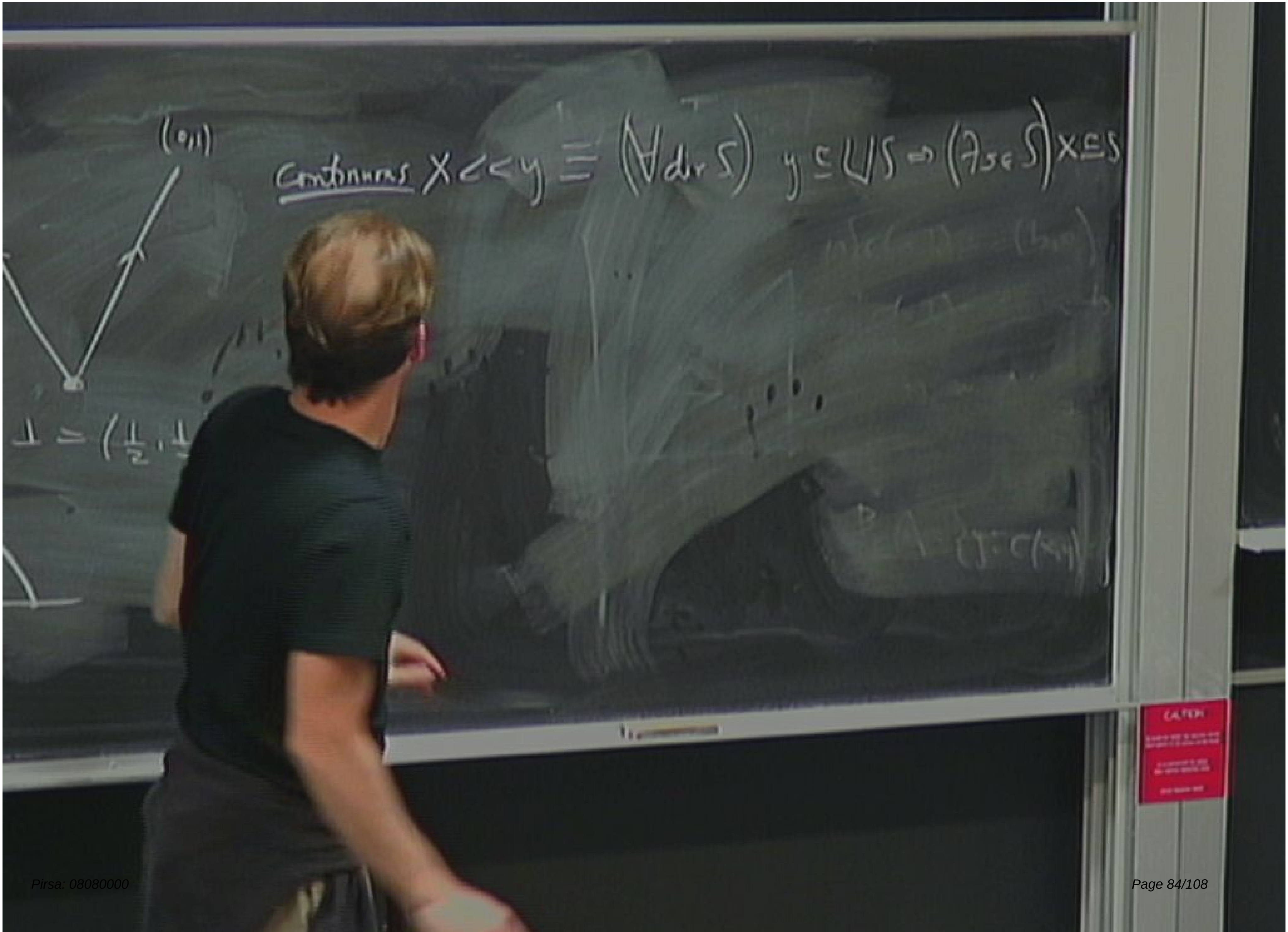
$$\max(\Delta^n) = \{e_i : 1 \leq i \leq n\}$$

and least element $\perp := (1/n, \ldots, 1/n)$. It satisfies the mixing law and has Shannon entropy as a measurement.

Example. Because (Δ^n) satisfies the mixing law, the *depolarization channel*

$$dx = p\perp + (1-p)x$$

satisfies $dx \sqsubseteq x$. Noise increases uncertainty.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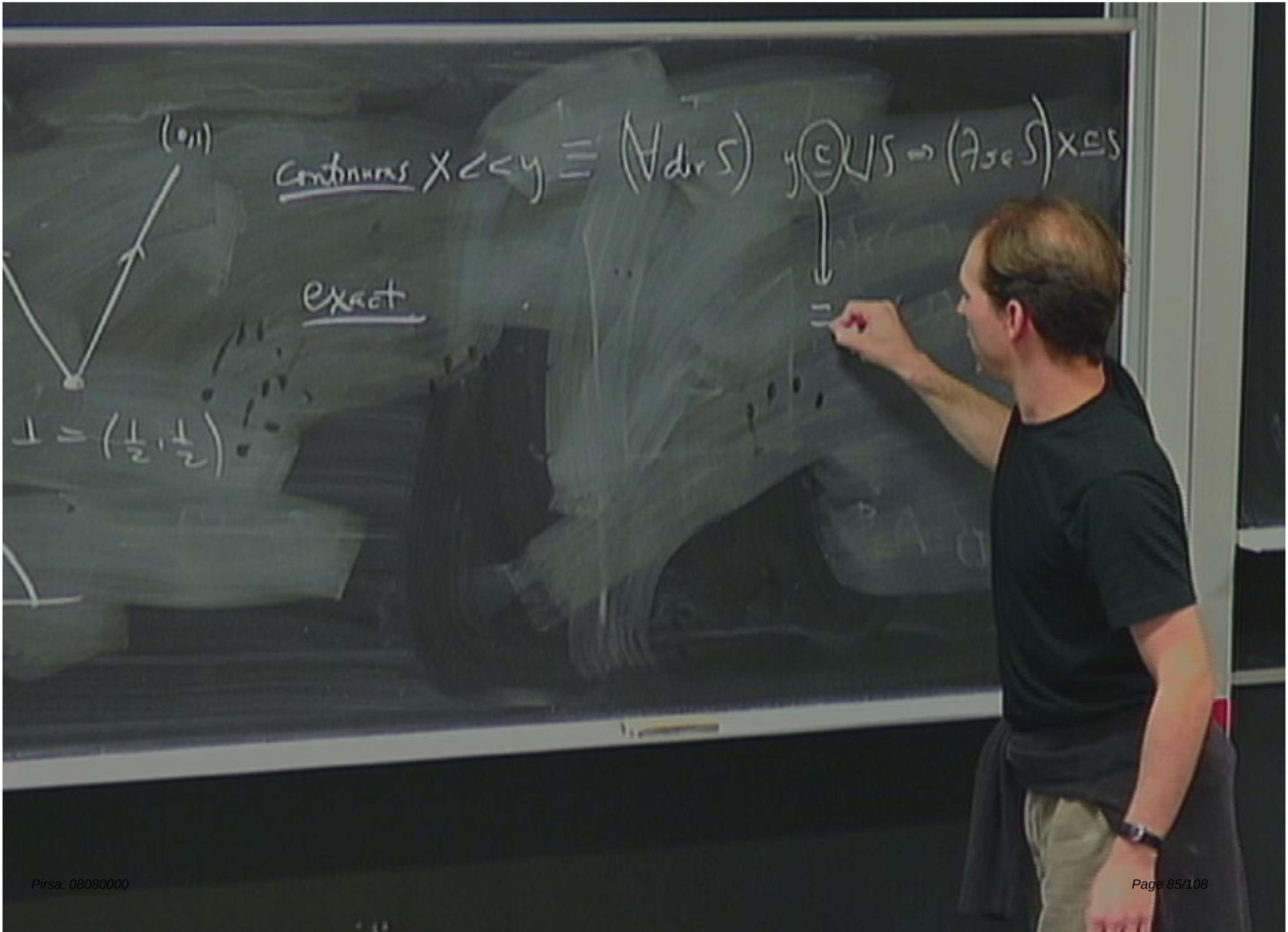
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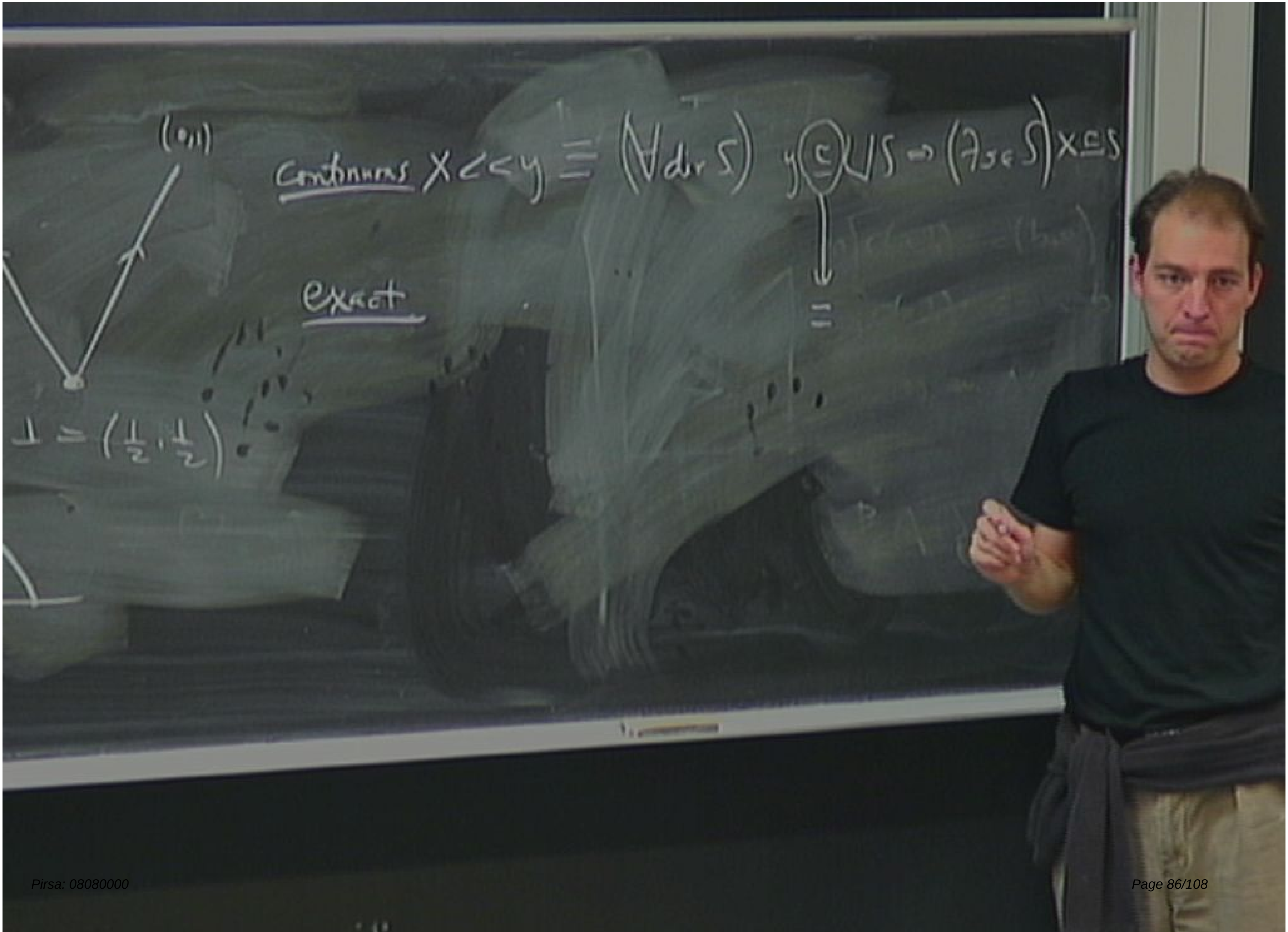
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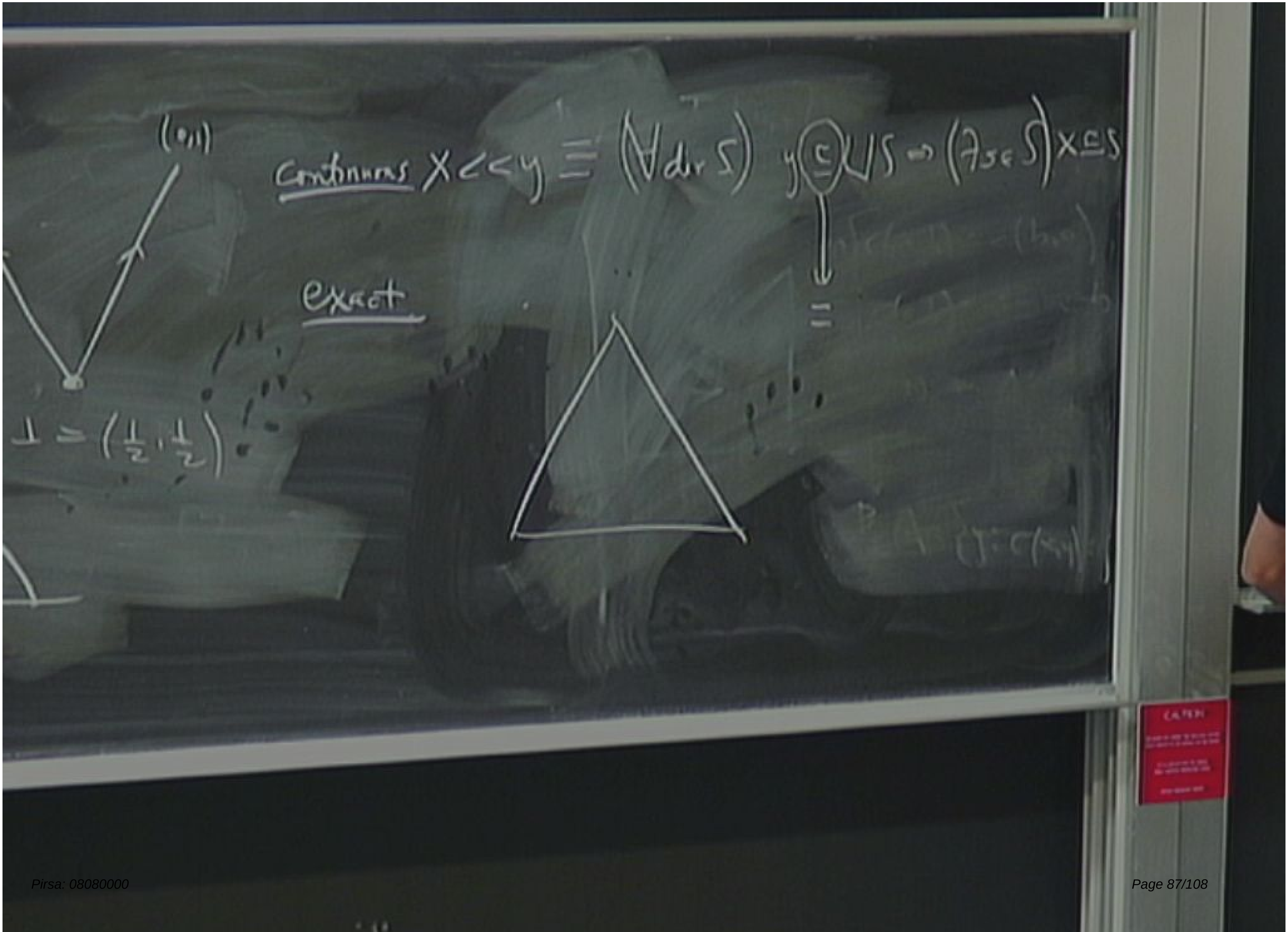
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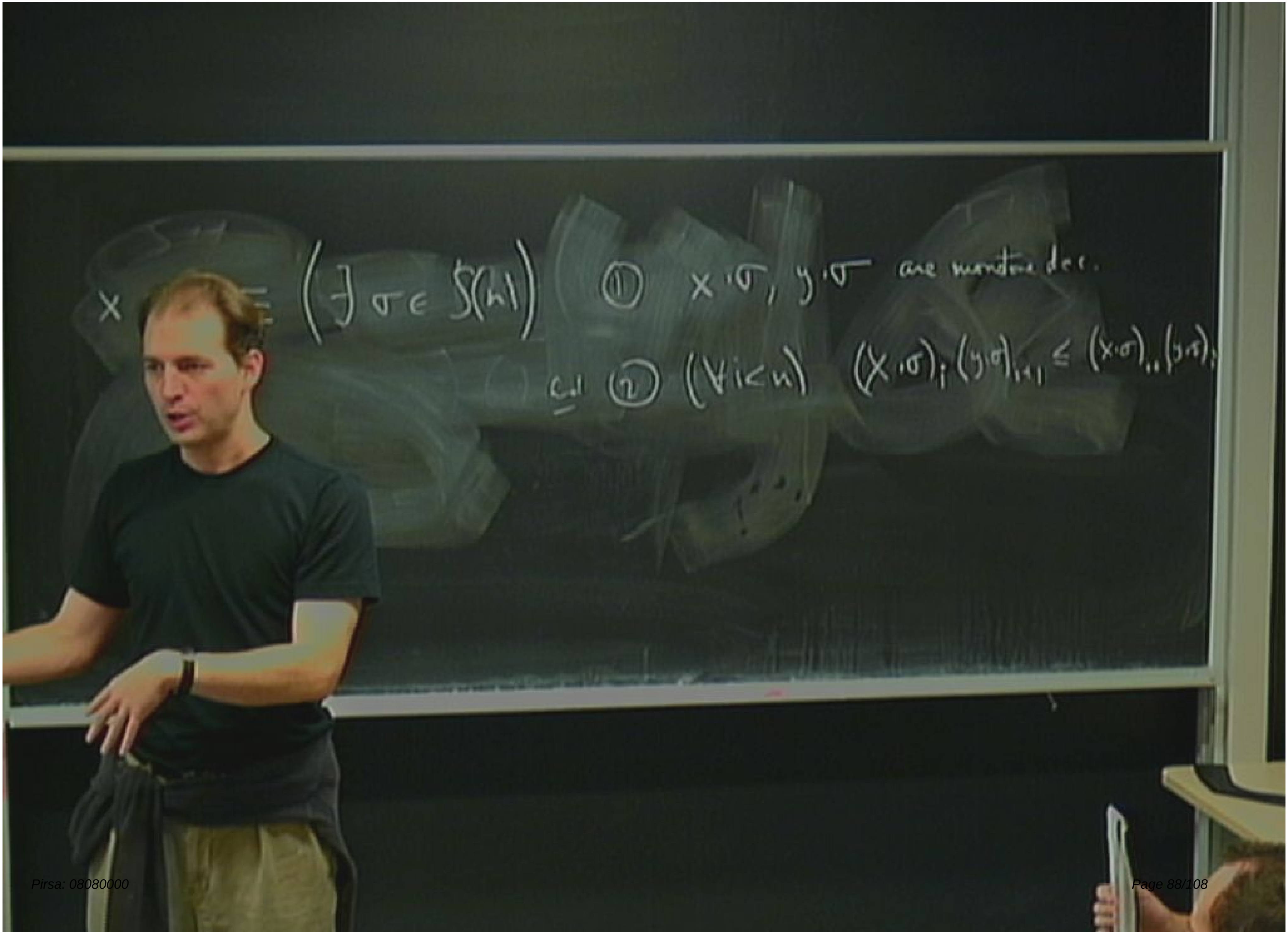
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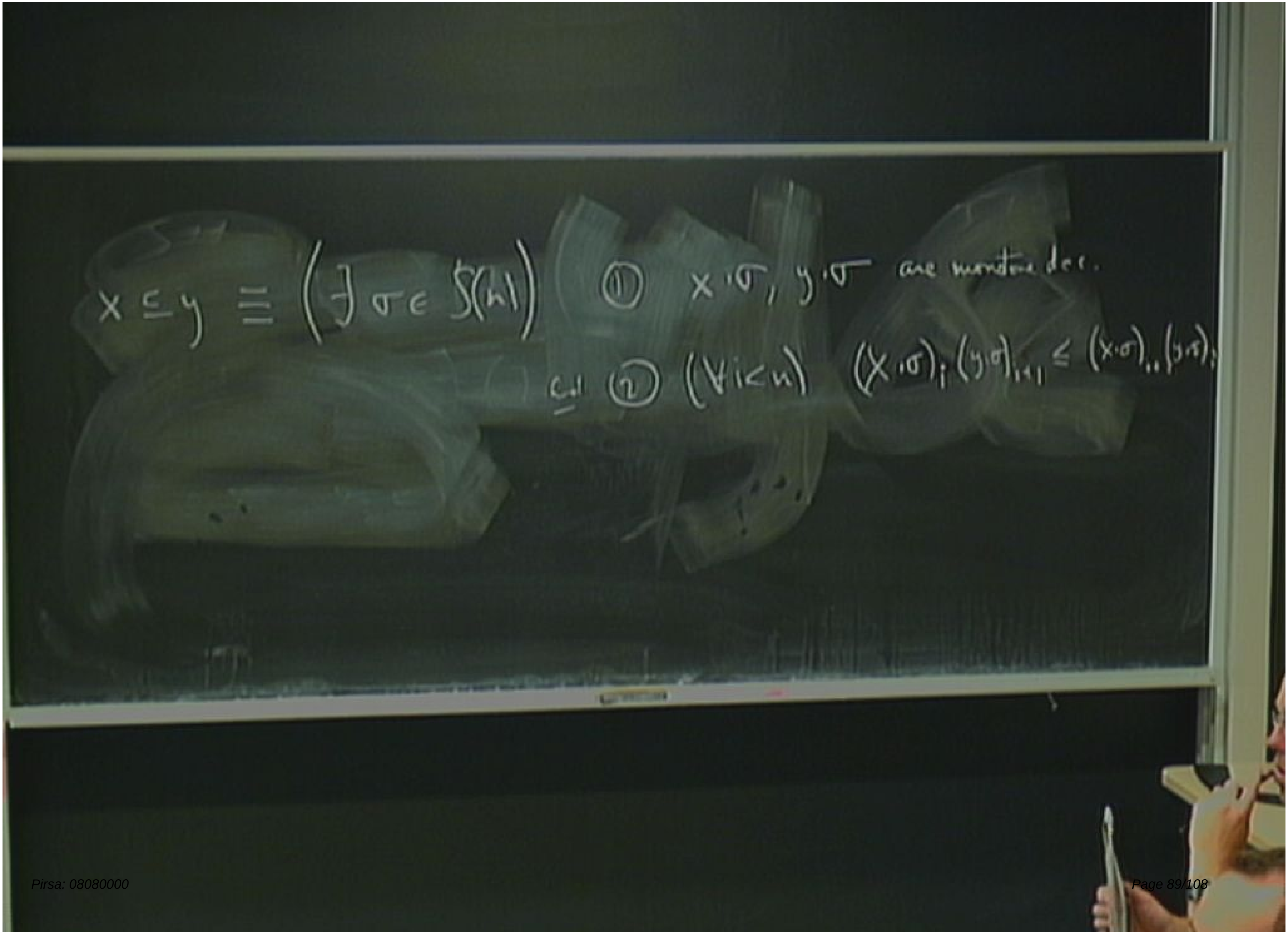
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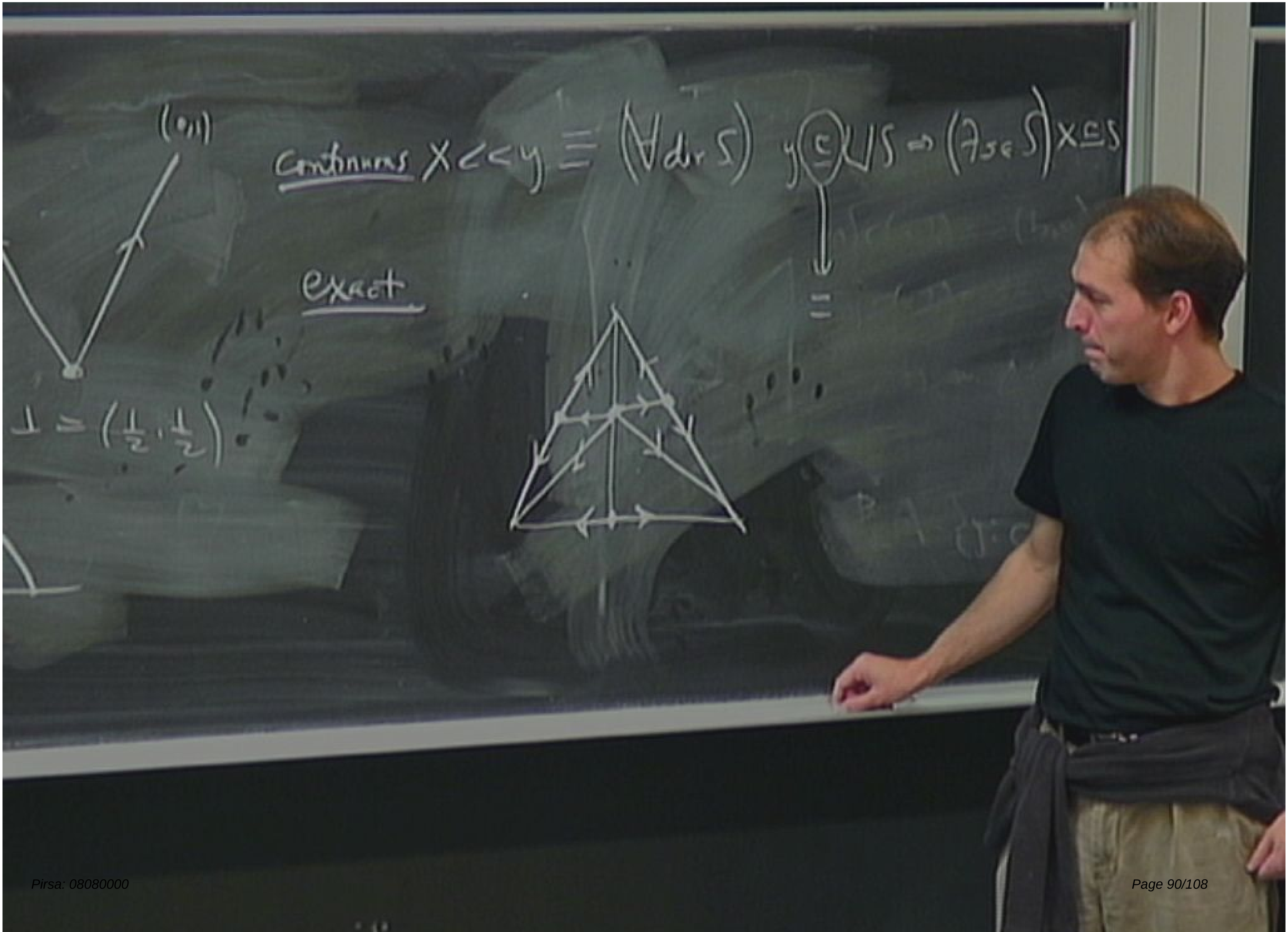
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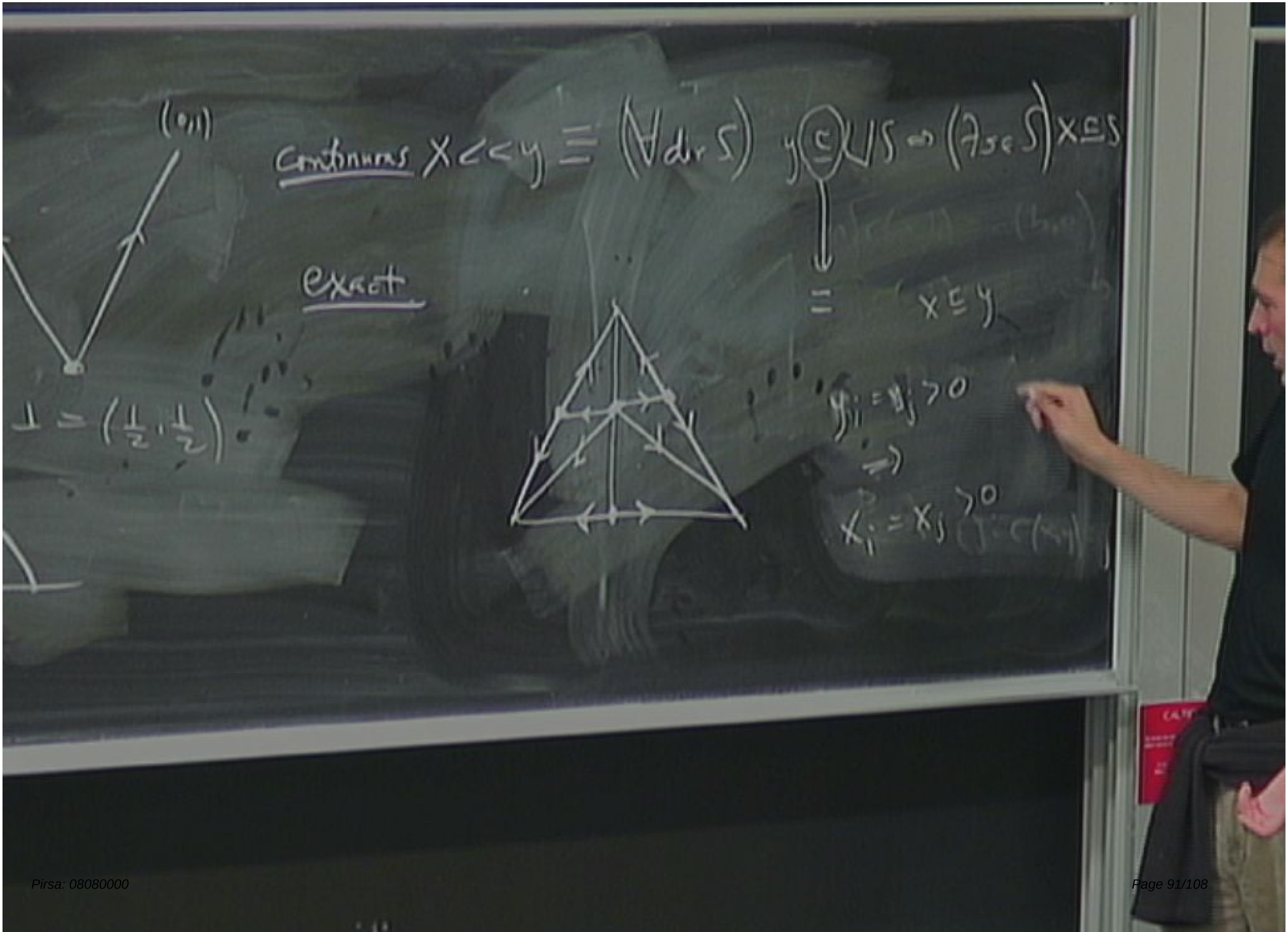
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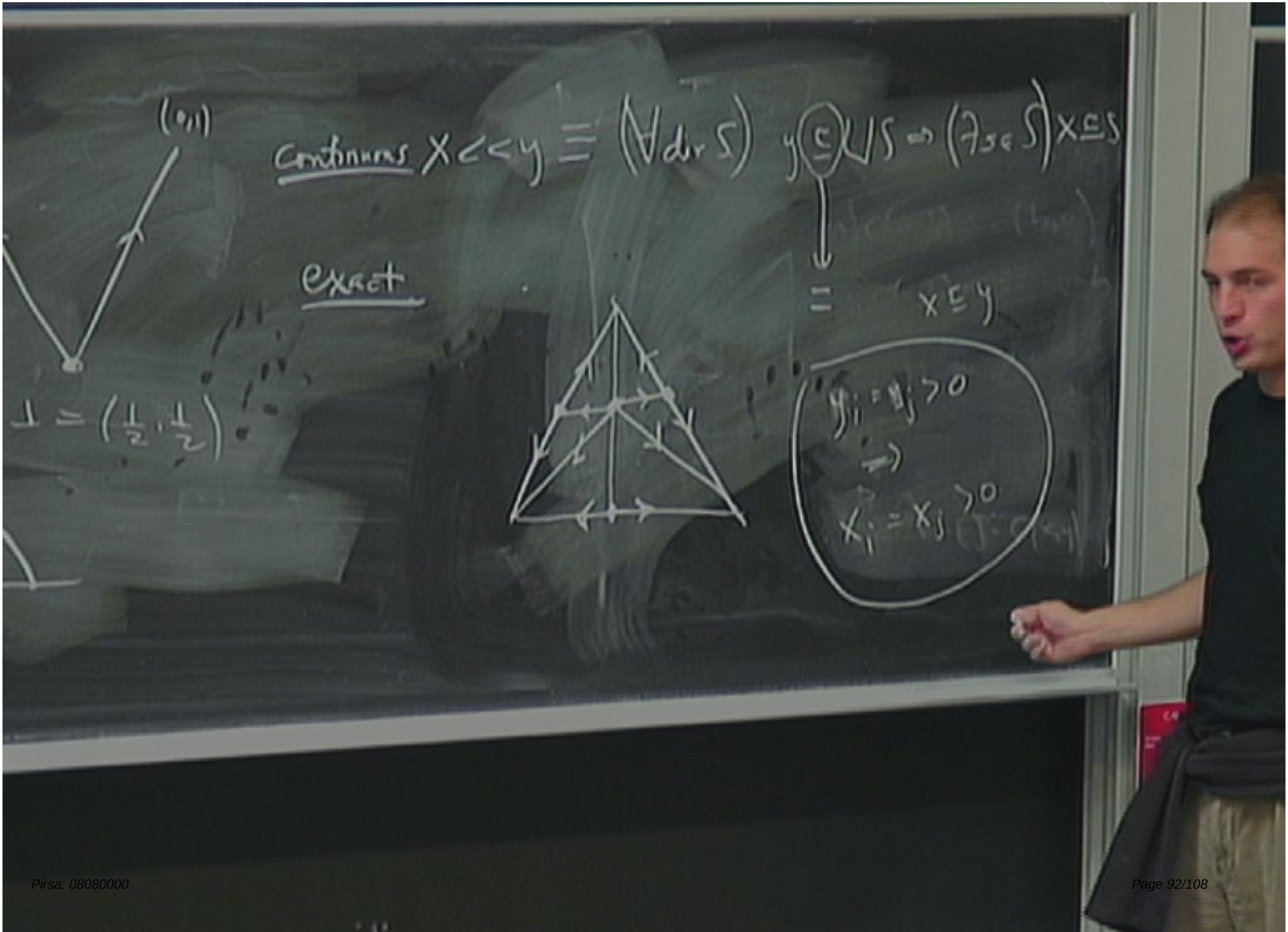
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⇒
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The spectral order

Definition. For a quantum state $\rho \in \Omega^n$,

$$\mathrm{spec}(\rho|e) := (\mathrm{pr}(\rho \to e_1), \ldots, \mathrm{pr}(\rho \to e_n)) \in \Delta^n.$$

Thus, $\mathrm{spec}(\rho|e)$ determines our ability to *predict* the result of experiment e, when the density operator for the system is ρ.

Definition. Let $n \geq 2$. For quantum states $\rho, \sigma \in \Omega^n$, we have $\rho \sqsubseteq \sigma$ iff there is an observable $e : \mathcal{H}^n \to \mathcal{H}^n$ such that $[\rho, e] = [\sigma, e] = 0$ and $\mathrm{spec}(\rho|e) \sqsubseteq \mathrm{spec}(\sigma|e)$ in Δ^n.

This is called the *spectral order* on quantum states.

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$$\mathrm{spec}(\rho|e) := (\mathrm{pr}(\rho \to e_1), \ldots, \mathrm{pr}(\rho \to e_n)) \in \Delta^n.$$

Thus, $\mathrm{spec}(\rho|e)$ determines our ability to *predict* the result of experiment e, when the density operator for the system is ρ.

Definition. Let $n \geq 2$. For quantum states $\rho, \sigma \in \Omega^n$, we have $\rho \sqsubseteq \sigma$ iff there is an observable $e : \mathcal{H}^n \to \mathcal{H}^n$ such that $[\rho, e] = [\sigma, e] = 0$ and $\mathrm{spec}(\rho|e) \sqsubseteq \mathrm{spec}(\sigma|e)$ in Δ^n.

This is called the *spectral order* on quantum states.

A quantum state $\rho \in \Omega^n$ is *pure* if

$$\mathrm{spec}(\rho) \subseteq \{0, 1\}.$$

The set of pure states is denoted Σ^n.

Theorem. $(\Omega^n, \sqsubseteq)$ is an exact dcpo with maximal elements

$$\max(\Omega^n) = \Sigma^n$$

and least element $\perp = I/n$, where I is the identity matrix. It satisfies the mixing law and has von Neumann entropy as a measurement.

Unitary operators on $\mathcal{H}^n$ induce order isomorphisms on Ω^n.

Classical and quantum logic

Definition. An element x of a dcpo D is *irreducible* when

$$\bigwedge(\uparrow x \cap \max(D)) = x$$

The set of irreducible elements in D is written $\mathrm{Ir}(D)$.

The order dual of a poset $(D, \sqsubseteq_D)$ is written D^*.

Theorem. For $n \geq 2$, the classical lattices arise as

$$\mathrm{Ir}(\Delta^n)^* \simeq \mathcal{P}\{1, \ldots, n\} \setminus \{\emptyset\},$$

and the quantum lattices arise as

$$\mathrm{Ir}(\Omega^n)^* \simeq \mathbb{L}^n \setminus \{0\}.$$

Both logics are derived from the same method.

Binary classical and quantum channels

Theorem.

- A classical channel $f : \Delta^2 \to \Delta^2$ is *binary symmetric* iff it is Scott continuous and its set of fixed points is Scott closed.
- A quantum channel $f : \Omega^2 \to \Omega^2$ is *unital* if and only if it is Scott continuous and its set of fixed points is Scott closed.

More: the spectral order on Ω^2 is the *unique* partial order that has $\perp = I/2$, satisfies the mixing law and makes all unital channels Scott continuous with a Scott closed set of fixed points.

Scott continuity of unital channels has an interesting consequence.

Capacity from the informatic derivative

Theorem. Let $\mu(x) = 1 - |x|$ denote the standard measurement on Ω^2. For any unital channel f and any $p \in \Omega^2$ different from $\perp$,

$$df_\mu(p) = \frac{|f(p)|}{|p|}$$

and its Holevo capacity $C(f)$ is

$$\mathrm{C}(f) = 1 - H\left(\frac{1}{2} + \frac{1}{2} \sup_{x \in \ker(\mu)} df_\mu(x)\right)$$

Thus, if f is symmetric,

$$\mathrm{C}(f) = 1 - H\left(\frac{1}{2} + \frac{1}{2} \max_{1 \leq i \leq 3} |\lambda_i|\right)$$

A ~~million~~ 925,000 dollar question

Is there a way to reconcile the continuous domains of GR with the exact domains of QM?

A plan:

- Just as we considered the 'physically reasonable' spacetimes, we will now consider the 'physically realizable' quantum states,
- Thermodynamics and the maximum entropy principle.

Coincidence: There is something 'unusually thermodynamical' about GR.

The Boltzmann distribution

Imagine a gas composed of a number of identical largely uninteracting molecules.

- Each molecule can have one of several different energy levels $a_1 < a_2 < \ldots < a_n$.
- We measure the temperature of the gas and in the process observe the average energy E.
- What is the probability p_i that a given molecule is in state a_i?
- The set of distributions $p = (p_1, \ldots, p_n)$ are the *physically realizable* states, the only ones which can refer to a possible state of the molecule.

Time to calculate p.

The maximum entropy principle

Assume throughout $a_1 < E < a_n$. Otherwise the problem is easy.

Theorem. Entropy has a maximum value on

$$\left\{ x \in \Delta^n : \sum_{i=1}^{n} a_i x_i = E \right\}$$

the set of distributions which achieve the observed average E. This maximum is achieved at a *unique* point.

This unique point is called *the maximum entropy state.*

How do we calculate it?

The maximum entropy state

Define

$$f(x) = \frac{\sum_{i=1}^{n} a_i e^{x a_i}}{\sum_{i=1}^{n} e^{x a_i}} - E$$

$$I_f(x) = x - \frac{f(x)}{(a_n - a_1)^2}$$

for any $x \in \mathbb{R}$.

Define $\lambda : \Delta^n \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\lambda(x) = \begin{cases} \frac{\log\left(\frac{\mathrm{sort}(x)_1}{\mathrm{sort}(x)_2}\right)}{a_n - a_{n-1}} & \text{if } I_f(0) > 0; \\ \frac{\log\left(\frac{\mathrm{sort}(x)_1}{\mathrm{sort}(x)_2}\right)}{a_1 - a_2} & \text{otherwise.} \end{cases}$$

with the understanding for pure states that $\lambda x = \infty$ in the first case and $\lambda x = -\infty$ in the other. The map sort puts states into

A least fixed point

Theorem. The map $\phi : \Delta^n \to \Delta^n$ given by

$$\phi(x) = (e^{I_f(\lambda x)a_1}, \ldots, e^{I_f(\lambda x)a_n}) \cdot \frac{1}{Z(x)}$$

$$Z(x) = \sum_{i=1}^{n} e^{I_f(\lambda x)a_i}$$

is Scott continuous in the Bayesian order. Its least fixed point

$$\text{fix}(\phi) := \bigsqcup_{n \geq 0} \phi^n(\bot)$$

is the maximum entropy state.

The physically realizable states

Thus, the maximum entropy state has the form

$$\operatorname{fix}(\phi) = \frac{1}{Z}(e^{\lambda a_1}, \ldots, e^{\lambda a_n})$$

where λ is a real number. Then either

$$\lambda = 0 \text{ or } \lambda \neq 0$$

so the distribution x that describes the state of a molecule is either

- $x = \perp = (1/n, \ldots, 1/n)$, or has the form
- $(\forall i, j)\, x_i = x_j \Rightarrow i = j$ and $(\forall i) x_i > 0$

We denote these *physically realizable* states by $\mathbb{G}^n$.

$GR \cap QM$

Of the physically realizable states $\mathbb{G}^n$, one of them does not teach us anything about the system:

$$\perp = (1/n, \ldots, 1/n)$$

Theorem. $\mathbb{G}^n \setminus \{\perp\}$ is globally hyperbolic in the Bayesian order.

For the quantum case, we consider copies of largely uninteracting harmonic oscillators. Same result:

Theorem. $\{\rho \in \Omega^n : \mathrm{spec}(\rho) \in \mathbb{G}^n \setminus \{\perp\}\}$ is globally hyperbolic in the spectral order.

Limitations imposed by physical law

Quantum mechanics: limits our ability to extract information from a system. Partial information in the form of density operators leads to natural domain theoretic structure.

General relativity: limits our ability to transmit information between systems. Partial information in the form of spacetime intervals leads to natural domain theoretic structure.

A principle?

- Ask: 'what happens when?'
- Physical law limits what is possible and at times provides *partial information* that can be used to explain what will happen.
- This partial information seems to necessitate domain theoretic structure.

What type of structure, specifically?

- Physically reasonable spacetimes lead to global hyperbolicity.
- Physically realizable states lead to global hyperbolicity.

What we learned

- GR and QM have some extremely nontrivial domain theoretic structure in common (global hyperbolicity),
- In each case, global hyperbolicity is uncovered by 'thinking physically' i.e. restricting attention to objects that are physically possible in a certain context,
- Limitations imposed by physical law seem to necessitate domain theoretic structure whenever they are expressible in terms of partial information.
- Applicability: the physics of secure communication.