Title: Encoding One Logical Qubit Into Six Physical Qubits

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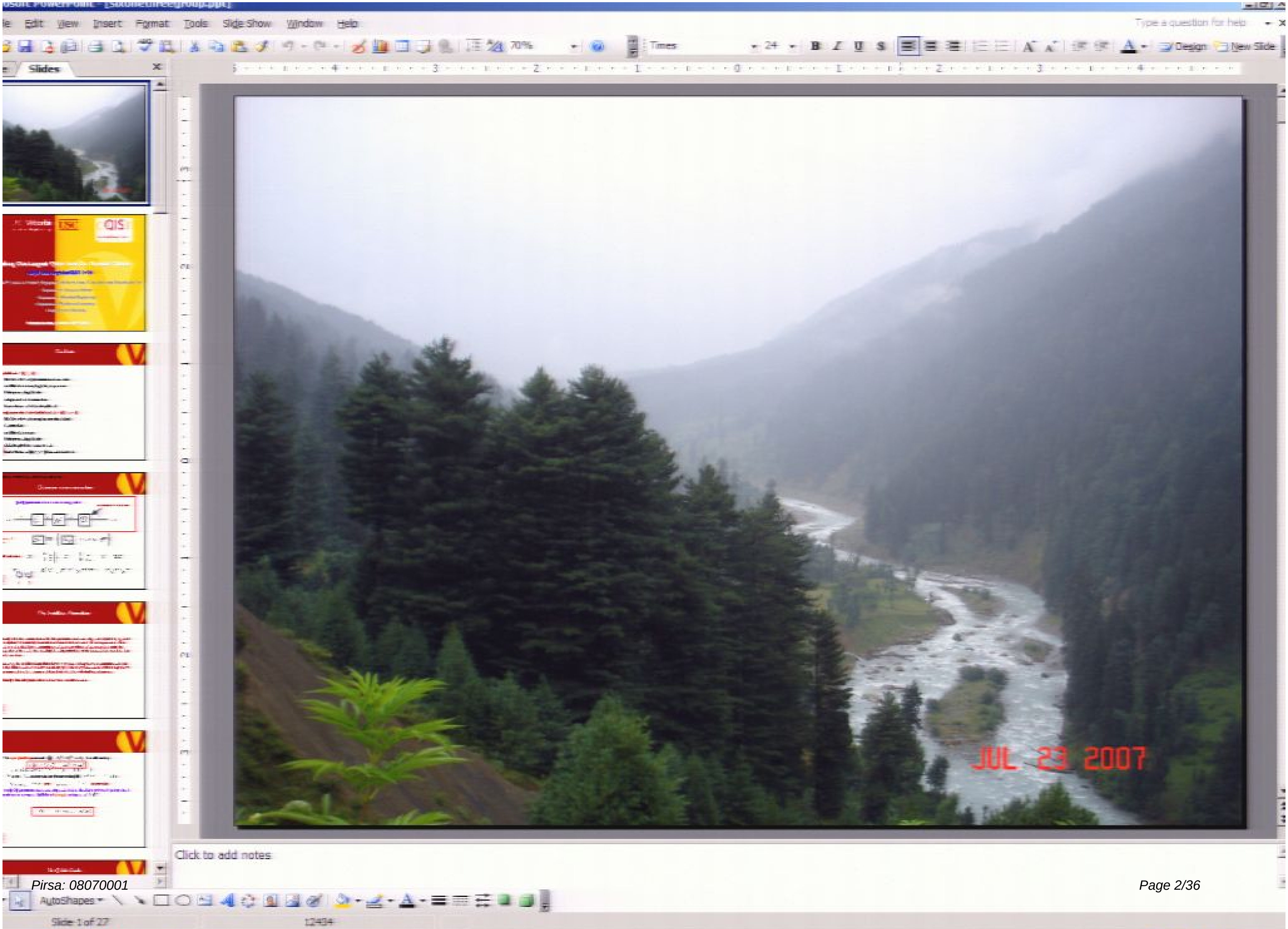
Abstract: We discuss two methods to encode one qubit into six physical qubits. Each of our two examples corrects an arbitrary single-qubit error. Our first example is a degenerate six-qubit quantum error-correcting code. We explicitly provide the stabilizer generators, encoding circuits, codewords, logical Pauli operators, and logical CNOT operator for this code. We also show how to convert this code into a non-trivial subsystem code that saturates the subsystem Singleton bound. We then prove that a six-qubit code without entanglement assistance cannot simultaneously possess a Calderbank-Shor-Steane (CSS) stabilizer and correct an arbitrary single-qubit error. A corollary of this result is that the Steane seven-qubit code is the smallest single-error correcting CSS code. Our second example is the construction of a non-degenerate six-qubit CSS entanglement-assisted code. This code uses one bit of entanglement (an ebit) shared between the sender and the receiver and corrects an arbitrary single-qubit error. The code we obtain is globally equivalent to the Steane seven-qubit code and thus corrects an arbitrary error on the receiver\'s half of the ebit as well. We prove that this code is the smallest code with a CSS structure that uses only one ebit and corrects an arbitrary single-qubit error on the sender\'s side. We discuss the advantages and disadvantages for each of the two codes.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Encoding One Logical Qubit Into Six Physical Qubits

http://arxiv.org/abs/0803.1495

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Perimeter Institute, Waterloo, July 9, 2008

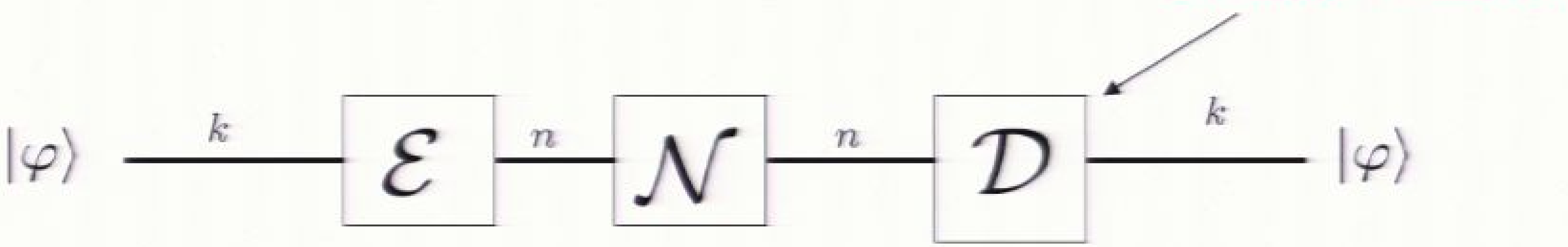
Outline

- Six-qubit Code – [[6, 1, 3]]
  - Brief Overview of Quantum Error Correction
  - Stabilizer Generators, logical X, Z operators
  - Unitary Encoding Circuit
  - Subsystem Code Construction
  - Non-existence of a CSS six-qubit code
- Entanglement-assisted Six-Qubit CSS Code – [[6, 1, 3; 1]]
  - Brief Overview of Entanglement-Assisted QEC
  - Construction
  - Stabilizer Generators
  - Unitary Encoding Circuit
  - Global Equivalence to Steane code
  - Non-existence of [[n, 1, 3; 1]] CSS codes for n < 6

or '95; Gottesman '96; Calderbank, Rains, Shor, Sloane '96

Quantum error correction

[[n,k]] quantum error correcting code



Discretization of errors

$$\boxed{\mathcal{N}} = \left\{ \boxed{N_u} : u \in S \subset \mathbb{Z}_2^{2n} \right\}$$

Pauli unitaries $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = iZX$

$$N_{\underbrace{011}_{Z}|\underbrace{110}_{X}} = Z^0X^1 \otimes Z^1X^1 \otimes Z^1X^0 := X \otimes Y \otimes Z$$

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The Stabilizer Formalism

Shortly after the construction of the first quantum error-correcting codes (QECCs), a general description was found by Gottesman of a broad class of codes: the *stabilizer* codes. These codes are described by a commuting set of operators; this set of operators determine the properties of the code, the encoding/decoding unitaries, and the protocol for error detection and correction.

Moreover, the stabilizer formalism allows a very close analogy between quantum codes and classical linear codes over GF(4) (or strictly, by classical *symplectic* codes). This analogy allows quantum codes to be constructed from classical codes, with similar performance.

Virtually all useful QECCs that are known are stabilizer codes.

• The symplectic product $\odot$ $: Z_2^{2n} \times Z_2^{2n} \to Z_2$ is defined by

$$(z|x) \odot (z'|x')^T = zx'^T + xz'^T$$

e.g. $(010|001) \odot (101|111)^T = 1 + 1 = 0$

USC

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- N_u and N_v commute (anti-commute) iff $u \odot v^T = 0$ (1)

$N_{(010|001)} = IZX$ and $N_{(101|111)} = YXY$ commute

- An [[*n*,*k*]] quantum error correcting code is described by a *(n-k) x 2n* parity check matrix H. Its rowspace *B(H)* is an isotropic subspace of $\mathbb{Z}_2^{2n}$

$$u \odot v^T = 0, \ \forall u, v \in B(H)$$

Six-Qubit Code
Stabilizer Generators

Commuting Stabilizer Generators

h_1	Y	I	Z	X	X	Y
h_2	Z	X	I	I	X	Z
h_3	I	Z	X	X	X	X
h_4	I	I	I	Z	I	Z
h_5	Z	Z	Z	I	Z	I
$\overline{X}$	Z	I	X	I	X	I
$\overline{Z}$	I	Z	I	I	Z	Z

$$\left(\begin{array}{cccccc|cccccc} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Z **X**

$n = 6, k = 1$

$$C = B^{\perp} = \{u : u \odot v^T = 0,\ \forall v \in B\}$$

Calderbank, Rains, Shor, Sloane, IEEE Trans. Inf. Theory 44, 1369 (1998)

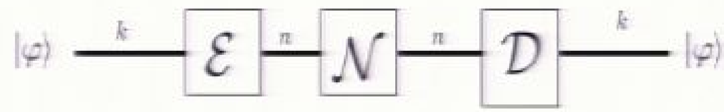
Gottesman Thesis:

arXiv:quant-ph/9705052

Quantum Stabilizer Codes

• The code space $\mathcal{E}(\mathcal{H}_2^{\otimes k}) \subset \mathcal{H}_2^{\otimes n}$ is defined as the simultaneous +1 eigenspace of the stabilizer operators $\{N_u : u \in C^\perp\}$

• The correctable error set S is defined by:



If $u, u' \in S$ and $u \neq u'$, then at least one of the two conditions hold:

1) $u - u' \notin C \;\Leftrightarrow\; H \odot u^T \neq H \odot u'^T$ distinct error syndromes

2) $u - u' \in C^\perp$ degenerate code

$u = (0\ 0\ 0\ 1\ 0\ 0 | 0\ 0\ 0\ 1\ 0\ 0)^T$ Y error on 4[th] q-bit

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \odot (0\ 0\ 0\ 1\ 0\ 0 | 0\ 0\ 0\ 1\ 0\ 0)^T = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Quantum Stabilizer Codes

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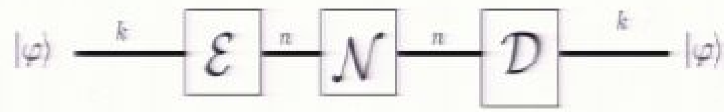

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$$u = (0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0 \,|\, 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)^T$$

Y error on 4[th] q-bit

$$\left(\begin{array}{cccccc|cccccc} 1&0&1&0&0&1&1&0&0&1&1&1\\ 1&0&0&0&0&1&0&1&0&0&1&0\\ 0&1&0&0&0&0&0&0&1&1&1&1\\ 0&0&0&1&0&1&0&0&0&0&0&0\\ 1&1&1&0&1&0&0&0&0&0&0&0 \end{array}\right) \odot (0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0 \,|\, 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0)^T = \begin{pmatrix}1\\0\\1\\1\\0\end{pmatrix}$$

- Decoding involves measuring the "error syndrome" (i.e. the simultaneous eigenvector of the stabilizer generators) . $H \odot u^T$

Properties of Stabilizer Codes

We can see that stabilizer codes have the following properties:

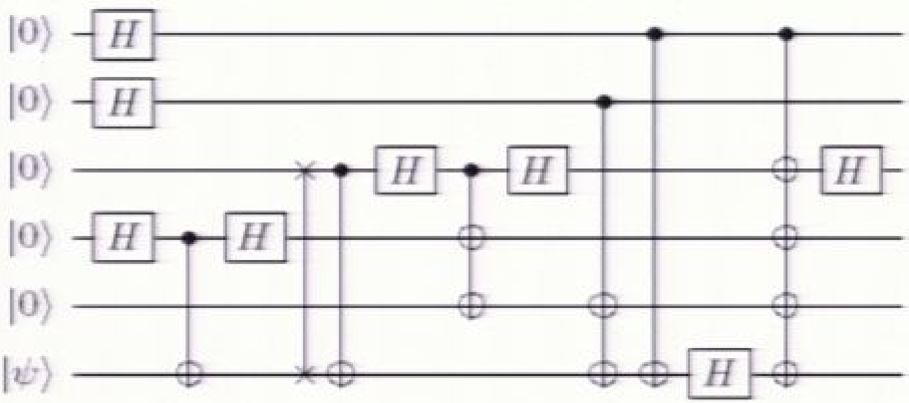
The code corresponds to an isotropic (that is, dual-containing) classical code over a symplectic space.

The error correcting conditions are *almost* the same as classical (except for the existence of *degenerate* quantum codes, in which distinct errors share the same error syndrome).

Correction consists of measuring an error syndrome and performing an appropriate correcting action (a unitary).

Six-Qubit Code

Unitary Encoding Circuit



A CNOT from qubit i qubit j adds column i to column j in the X submatrix, and column j to column i in the Z submatrix.
A Hadamard on qubit i swaps column i in the Z submatrix with column i in the X submatrix.
A Phase gate on qubit i adds a column i in the X submatrix to column i in the Z submatrix.
A SWAP gate applied to qubit i and j exchanges the respective columns in the X and Z submatrices.

$$\left(\begin{array}{cccccc|cccccc} Z & I & I & I & I & I & I & I & I & I & I & I \\ I & Z & I & I & I & I & I & I & I & I & I & I \\ I & I & Z & I & I & I & I & I & I & I & I & I \\ I & I & I & Z & I & I & I & I & I & I & I & I \\ I & I & I & I & Z & I & I & I & I & I & I & I \end{array}\right)$$

$$S = \begin{pmatrix} Y & I & Z & X & X & Y \\ Z & X & I & I & X & Z \\ I & Z & X & X & X & X \\ I & I & I & Z & I & Z \\ Z & Z & Z & I & Z & I \end{pmatrix}$$

$$\left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xleftarrow{\text{Matrix Reduction}} \left(\begin{array}{cccccc|cccccc} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Z **X**

USC

Six-Qubit Code

Subsystem Code Constructions

h_1	Y	I	Z	X	X	Y
h_2	Z	X	I	I	X	Z
h_3	I	Z	X	X	X	X
h_5	Z	Z	Z	I	Z	I
H_X	I	I	I	X	I	I
H_Z	I	I	I	Z	I	Z
$\overline{X}$	Z	I	X	I	X	I
$\overline{Z}$	I	Z	I	I	Z	Z

The encoding unitary transforms the unencoded sixth qubit into subsystem H_A

Convert the fourth unecoded qubit into a gauge qubit, considering it noisy. The errors X_4, Z_4 have no effect on the quantum information.The encoding unitary transforms these errors into

$X_4, Z_4 Z_6$ which generates the gauge subgroup $H_G = \langle X_4, Z_4 Z_6 \rangle$

The most general way to decompose a Hilbert space is

$$H = (H_A \otimes H_B) \oplus H_C$$

Subsystem Singleton Bound**

$$n - k - r \geq 2(d-1)$$

$$n = 6; k = 1; r = 1; d = 3$$

Salah A. Aly - arXiv:quant-ph/0802.4270

**A. Klappenecker and P.K. Sarvepalli, arXiv: quant-ph/0703213 (2007)

Six-Qubit Code

Non-existence of a Six-Qubit CSS Code

Proposition: There is no six-qubit code that encodes one qubit, possesses the CSS structure, and corrects an arbitrary single-qubit error.

Six-Qubit Code
Non-existence of a Six-Qubit CSS Code

$$\begin{pmatrix} X & I & X & X & I & X \\ X & X & X & I & I & X \\ X & X & I & X & X & I \\ I & I & I & X & X & X \\ X & I & I & X & X & X \end{pmatrix}$$

Case 1: All generators of type X or Z

Can't correct single-qubit errors of type X or Z

$$\begin{pmatrix} I & I & I & X & X & X \\ Z & Z & Z & I & Z & I \\ Z & Z & Z & I & I & Z \\ Z & I & I & Z & Z & Z \\ Z & I & I & I & Z & Z \end{pmatrix}$$

Case 2(a): First generator of type X, with I on qubit 1; the rest type Z

Can't correct single-qubit error Z1, unless Z1 in stabilizer, implies trivial code |0>

$$\begin{pmatrix} X & X & X & X & X & X \\ Z & I & I & Z & Z & Z \\ Z & I & Z & Z & I & Z \\ Z & I & I & Z & Z & Z \\ I & Z & Z & I & Z & Z \end{pmatrix}$$

Case 2(b): Two-qubit Z errors commute. Must belong to stabilizer. Have five independent such pairs.

Z1Z2, Z1Z3, Z1Z4, Z1Z5, Z1Z6

$$\begin{pmatrix} X & I & X & X & X & X \\ X & I & I & I & X & X \\ Z & Z & I & Z & Z & Z \\ Z & I & Z & Z & Z & Z \\ I & I & Z & Z & I & I \end{pmatrix}$$

Case 3: Two generators of type X, and three of type Z. Cannot have I, appear in the same column of g1, g2 twice. E.g. Z2, not correctible

Six-Qubit Code

Non-existence of a Six-Qubit CSS Code

ach column appears twice.

$$\begin{matrix} I & X & X \\ X & I & X \end{matrix}$$

$$\begin{aligned} g_1 &= X\ X\ I\ I\ X\ X \\ g_2 &= I\ I\ X\ X\ X\ X \end{aligned}$$

-errors that commute with
e Stabilizer

1Z2, Z3Z4, Z5Z6

an be part of Stabilizer

wo qubit X errors:

1X2, X3X4, X5X6

ommute

One column appears three times, another column appears twice, and the third column appears once.

$$\begin{aligned} g_1 &= X\ X\ X\ I\ I\ X \\ g_2 &= I\ I\ I\ X\ X\ X \end{aligned}$$

Z-errors that commute with the Stabilizer:

Z1Z2, Z1Z3, Z2Z3, Z4Z5

Only three of the four are independent. But all of these generators have an I acting on the sixth qubit and so errors of type X6 go undetected. If we include X6 in the Stabilizer, this will again make for a trivial code.

One column appears three times, and another appears thrice.

$$\begin{aligned} g_1 &= I\ I\ I\ X\ X\ X \\ g_2 &= X\ X\ X\ I\ I\ I \end{aligned}$$

Z-errors that commute with the Stabilizer:

Z1Z2, Z1Z3, Z2Z3, Z4Z5, Z4Z6, Z5Z6

Only four of which are independent. They cannot all belong to the Stabilizer because there are only three generators of the Z-type.

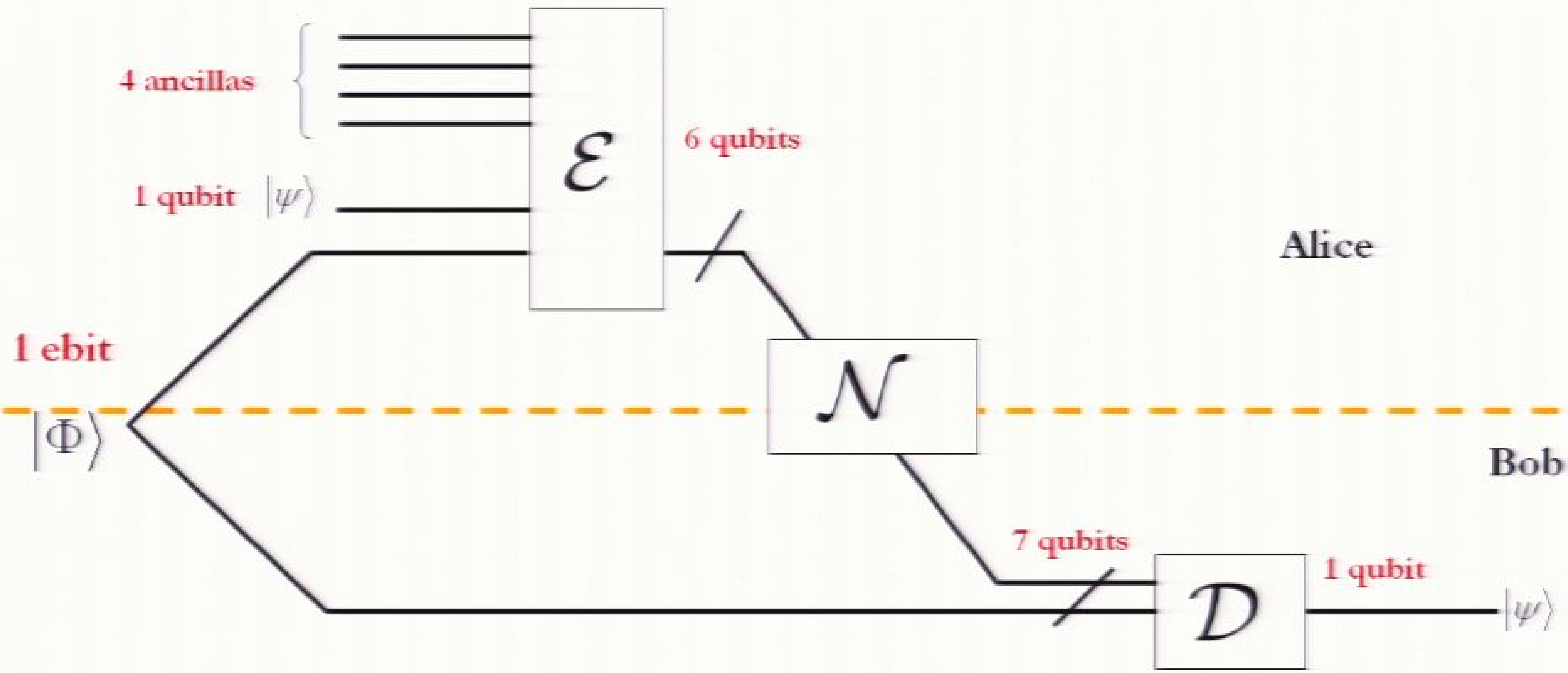
At least one column appears more than three times.

$$\begin{aligned} g_1 &= I\ I\ I\ I\ X\ X \\ g_2 &= X\ X\ X\ X\ I\ I \end{aligned}$$

Three or more independent pairs of Z-errors commuting with the stabilizer, e.g. Z1Z2, Z1Z3, Z1Z4. Independent pairs more than three. If exactly three, take as generators, but then X-errors go undetected.

Entanglement-Assisted Six-Qubit CSS Code

[[6,1,3;1]]



$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^A|0\rangle^B + |1\rangle^A|1\rangle^B\right)$$

1 ebit

Brun, Devetak, Hsieh, Science **314**, pp. 436 (2006)

Entanglement-assisted Stabilizer Formalism

It turns out that we can establish a simple extension of the usual stabilizer formalism to describe entanglement-assisted codes. We again establish a "stabilizer" which is a subgroup of the Pauli group on *n* q-bits; but we no longer require this subgroup to be Abelian. For such a subgroup, we can find a set of generators which fall into two groups:

Isotropic generators, which commute with all other generators; and

Symplectic generators, which come in anticommuting pairs; each pair commutes with all other generators.

Entanglement-Assisted Six-Qubit CSS Code

Construction

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \longrightarrow H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \left(\begin{array}{c|c} H & 0 \\ 0 & H \end{array} \right)$$

Consider a [[7,4,3]] Hamming code parity check matrix

CSS Construction

$$\left(\begin{array}{c|c} H & 0 \\ 0 & H \end{array} \right) \longrightarrow \begin{matrix} Z & I & I & Z & I & Z \\ I & Z & I & Z & Z & I \\ I & I & Z & I & Z & Z \\ X & I & I & X & I & X \\ I & X & I & X & X & I \\ I & I & X & I & X & X \end{matrix} \longrightarrow \begin{matrix} Z & I & I & Z & I & Z \\ X & I & I & X & I & X \\ I & Z & I & Z & Z & I \\ I & X & I & X & X & I \\ I & I & Z & I & Z & Z \\ I & I & X & I & X & X \end{matrix}$$

Entanglement-Assisted Six-Qubit CSS Code Construction

$$\begin{matrix} Z & I & I & Z & I & Z \\ X & I & I & X & I & X \\ I & Z & I & Z & Z & I \\ I & X & I & X & X & I \\ I & I & Z & I & Z & Z \\ I & I & X & I & X & X \end{matrix} \xrightarrow{\text{Gram-Schmidt Orthogonalization}} \begin{matrix} g_1 & = & Z & I & I & Z & I & Z \\ g_2 & = & X & I & I & X & I & X \\ g_3 & = & Z & Z & I & I & Z & Z \\ g_4 & = & X & X & I & I & X & X \\ g_5 & = & Z & I & Z & Z & Z & I \\ g_6 & = & X & I & X & X & X & I \end{matrix}$$

$$g_j = g_j . g_1^{f(g_2,g_j)} . g_2^{f(g_1,g_j)} \quad \forall j \in \{3,4,5,6\}$$

where f is the symplectic product

minimum # of ebits $= rank(HH^T)$ CSS Code

minimum # of ebits $= rank(H_1 H_2^T)$ non-CSS Code

USC

- An [[n,k;c]] EA quantum error correcting code is described by a (n-k) x $2n$ parity check matrix H. B = rowspace(H). Again, $C = B^{\perp}$

- Take a general symplectic matrix H. Its rowspace B can be written as

$$B = \mathrm{iso}(B) \oplus \underbrace{\mathrm{symp}(B)}_{\bigoplus_{i=1}^{c} \mathrm{span}\{e_i, f_i\}}$$

symplectic pairs

$$e_i \odot f_i = 1$$

$$e_i, f_i \odot u = 0, \text{ otherwise}$$

$$H = \left(\begin{array}{cccccc|cccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{pmatrix} I & Z & I & I & I & I \\ I & I & Z & I & I & I \\ Z & I & I & I & I & I \\ I & I & I & Z & I & I \\ I & I & I & I & Z & I \\ X & I & I & I & I & I \end{pmatrix}$$

Entanglement-Assisted Six-Qubit CSS Code Construction

	Bob	Alice					
g_1'	I	I	Z	I	I	I	I
g_2'	I	I	I	Z	I	I	I
g_3'	Z	Z	I	I	I	I	I
g_4'	I	I	I	I	Z	I	I
g_5'	I	I	I	I	I	Z	I
g_6'	X	X	I	I	I	I	I
$\overline{X}'$	I	I	I	I	I	I	X
$\overline{Z}'$	I	I	I	I	I	I	Z

UNENCODED STABILIZER

ENCODING CIRCUIT

symplectic pair

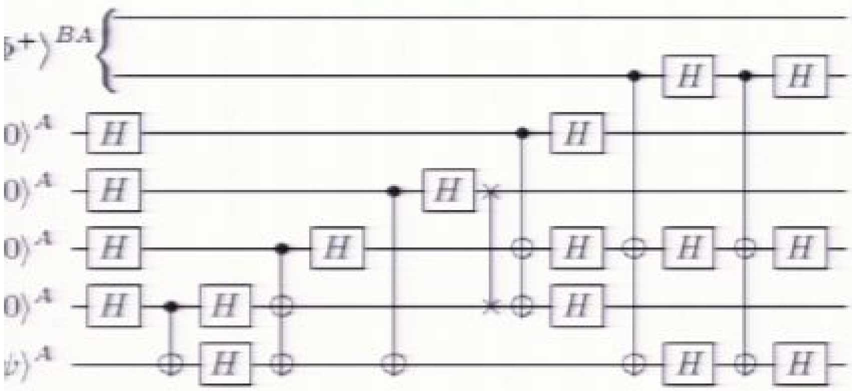
symplectic subspace

	Bob	Alice					
g_1	I	Z	I	Z	Z	Z	I
g_2	I	Z	Z	I	I	Z	Z
g_3	Z	Z	I	I	Z	I	Z
g_4	I	X	X	I	I	X	X
g_5	I	I	X	X	X	I	X
g_6	X	X	I	I	X	I	X
$\overline{X}$	I	I	I	I	X	X	X
$\overline{Z}$	I	I	Z	Z	I	Z	I

ENCODED STABILIZER

Entanglement-Assisted Six-Qubit CSS Code

Unitary Encoding Circuit



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A Phase gate on qubit i adds a column i in the X submatrix to column i in the Z submatrix

A SWAP gate applied to qubit i and j exchanges the respective columns in the X and Z submatrices.

$$\begin{pmatrix} Z & I & I & I & I & I \\ X & I & I & I & I & I \\ I & Z & I & I & I & I \\ I & I & Z & I & I & I \\ I & I & I & Z & I & I \\ I & I & I & I & Z & I \end{pmatrix} \qquad \begin{pmatrix} Z & I & I & Z & I & Z \\ I & Z & I & Z & Z & I \\ I & I & Z & I & Z & Z \\ X & I & I & X & I & X \\ I & X & I & X & X & I \\ I & I & X & I & X & X \end{pmatrix}$$

$$\left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xleftarrow{\text{Matrix Reduction}} \left(\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Z **X**

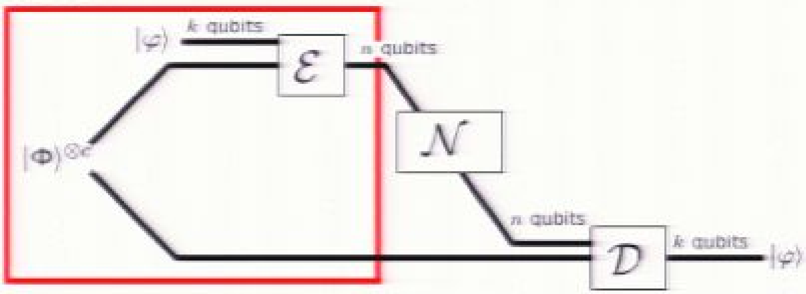
Entanglement-Assisted Codes

- The correctable error set S is defined by:

  If $u, u' \in S$ and $u \neq u'$, then at least one of the two conditions hold:

  1) $u - u' \notin C$

  2) $u - u' \in \text{iso}(C^{\perp})$ degenerate code



- The code space $\mathcal{E}(\mathcal{H}_2^{\otimes k})$ is defined as the simultaneous +1 eigenspace of the stabilizer generators

$$\{\underbrace{N_u}_{n} \otimes \underbrace{I^{\otimes c}}_{c} : u \in \text{iso}(C^{\perp})\} \cup \bigcup_{i=1}^{c} \{\underbrace{N_{e_i}}_{n} \otimes \underbrace{Z_i}_{c}, \underbrace{N_{f_i}}_{n} \otimes \underbrace{X_i}_{c}\}$$

- Decoding involves measuring the "error syndrome" (i.e. the simultaneous eigenvector of the stabilizer generators) , $H \odot u^T$

Properties of the Entanglement-Assisted Stabilizer Formalism

We can now compare the properties of an EAQECC to those of an ordinary QECC:

The code corresponds to a classical code over a symplectic space. (No longer needs to be dual-containing!)

The error correcting conditions are *almost* the same as classical (except for the existence of *degenerate* quantum codes, in which distinct errors share the same error syndrome).

Correction consists of measuring an error syndrome and performing an appropriate correcting action (a unitary).

Entanglement-Assisted Six-Qubit CSS Code

Equivalence to Steane Code

- The entanglement-assisted six-qubit code can correct for errors on Bob's side as well. It turns out that [[6, 1, 3; 1]] codes are in fact equivalent to the well-known seven-qubit Steane code.

$$
\begin{pmatrix}
g_1 & I & Z & I & Z & Z & Z & I \\
g_2 & I & Z & Z & I & I & Z & Z \\
g_3 & Z & Z & I & I & Z & I & Z \\
g_4 & I & X & X & I & I & X & X \\
g_5 & I & I & X & X & X & I & X \\
g_6 & X & X & I & I & X & I & X
\end{pmatrix}
\xrightarrow[g_5 \leftrightarrow g_5 g_6]{g_1 \leftrightarrow g_1 g_2 g_3}
\begin{pmatrix}
g_1 & Z & Z & Z & Z & I & I & I \\
g_2 & I & Z & Z & I & I & Z & Z \\
g_3 & Z & Z & I & I & Z & I & Z \\
g_4 & I & X & X & I & I & X & X \\
g_5 & X & X & X & X & I & I & I \\
g_6 & X & X & I & I & X & I & X
\end{pmatrix}
\xrightarrow[c_2 \leftrightarrow c_3;\, c_1 \leftrightarrow c_5]{g_4 \leftrightarrow g_5}
$$

$$
\rightarrow
\begin{pmatrix}
g_1 & I & Z & Z & Z & Z & I & I \\
g_2 & I & Z & Z & I & I & Z & Z \\
g_3 & Z & I & Z & I & Z & I & Z \\
g_4 & I & X & X & I & I & X & X \\
g_5 & I & X & X & X & X & I & I \\
g_6 & X & I & X & I & X & I & X
\end{pmatrix}
\xrightarrow{\text{Cyclically permute columns once}}
\begin{pmatrix}
g_1 & I & I & Z & Z & Z & Z & I \\
g_2 & Z & I & Z & Z & I & I & Z \\
g_3 & Z & Z & I & Z & I & Z & I \\
g_4 & X & I & X & X & I & I & X \\
g_5 & I & I & X & X & X & X & I \\
g_6 & X & X & I & X & I & X & I
\end{pmatrix}
$$

Steane Code Stabilizer

Entanglement-Assisted Six-Qubit CSS Code

Proposition: There does not exist an [[n, 1, 3; 1]] entanglement-assisted CSS code for $n < 6$ that corrects an arbitrary single-qubit error on Alice's side.

Conclusion

- We gave a proof as to why the six-qubit code does not have a CSS structure.
- Can the above be generalized to higher-distance CSS codes?
- We gave an example of the smallest subsystem code that uses just one gauge qubit.
- We gave an example of the smallest entanglement-assisted CSS code.
- Can the proof-technique for the non-existence of entanglement-assisted CSS codes be generalized to higher distance EA codes, or to EA-codes that use more than one e-bit?

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$$\begin{pmatrix} g_1 & I & Z & I & Z & Z & Z & I \\ g_2 & I & Z & Z & I & I & Z & Z \\ g_3 & Z & Z & I & I & Z & I & Z \\ g_4 & I & X & X & I & I & X & X \\ g_5 & I & I & X & X & X & I & X \\ g_6 & X & X & I & I & X & I & X \end{pmatrix} \xrightarrow[g_5 \leftrightarrow g_5 g_6]{g_1 \leftrightarrow g_1 g_2 g_3} \begin{pmatrix} g_1 & Z & Z & Z & Z & I & I & I \\ g_2 & I & Z & Z & I & I & Z & Z \\ g_3 & Z & Z & I & I & Z & I & Z \\ g_4 & I & X & X & I & I & X & X \\ g_5 & X & X & X & X & I & I & I \\ g_6 & X & X & I & I & X & I & X \end{pmatrix} \xrightarrow[c_2 \leftrightarrow c_3;\, c_1 \leftrightarrow c_5]{g_4 \leftrightarrow g_5}$$

$$\longrightarrow \begin{pmatrix} g_1 & I & Z & Z & Z & Z & I & I \\ g_2 & I & Z & Z & I & I & Z & Z \\ g_3 & Z & I & Z & I & Z & I & Z \\ g_4 & I & X & X & I & I & X & X \\ g_5 & I & X & X & X & X & I & I \\ g_6 & X & I & X & I & X & I & X \end{pmatrix} \xrightarrow{\text{Cyclically permute columns once}} \begin{pmatrix} g_1 & I & I & Z & Z & Z & Z & I \\ g_2 & Z & I & Z & Z & I & I & Z \\ g_3 & Z & Z & I & Z & I & Z & I \\ g_4 & X & I & X & X & I & I & X \\ g_5 & I & I & X & X & X & X & I \\ g_6 & X & X & I & X & I & X & I \end{pmatrix}$$

Steane Code Stabilizer

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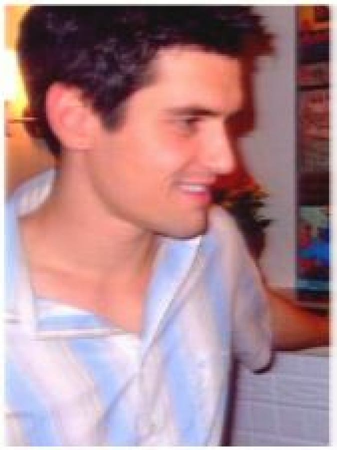
The Usual Suspects



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Mark

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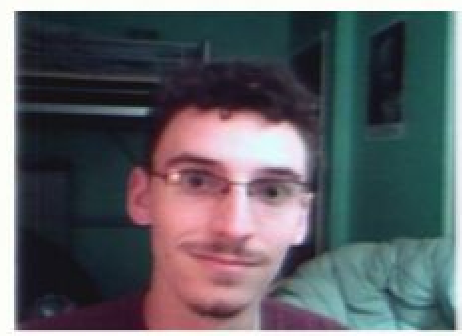
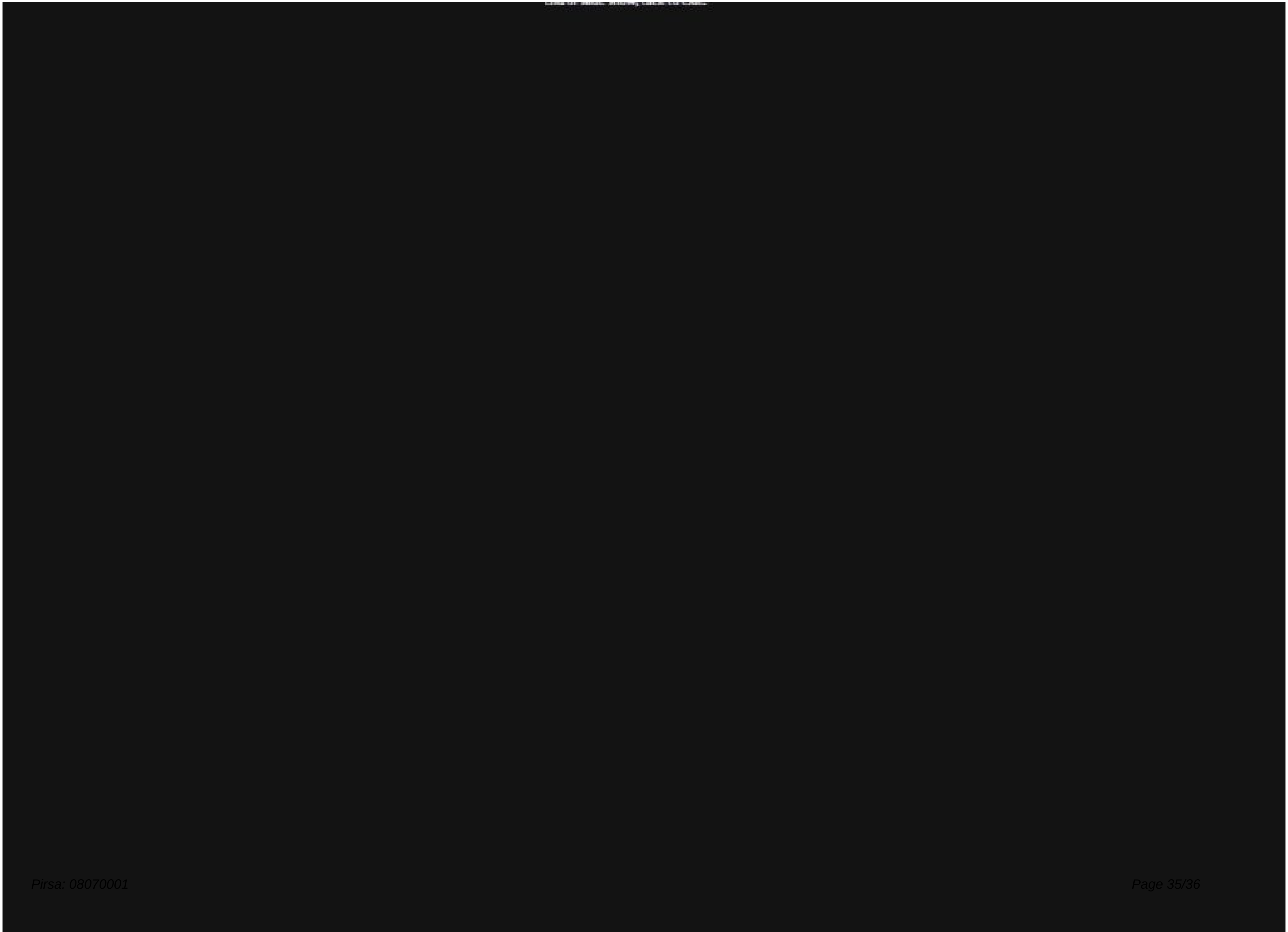
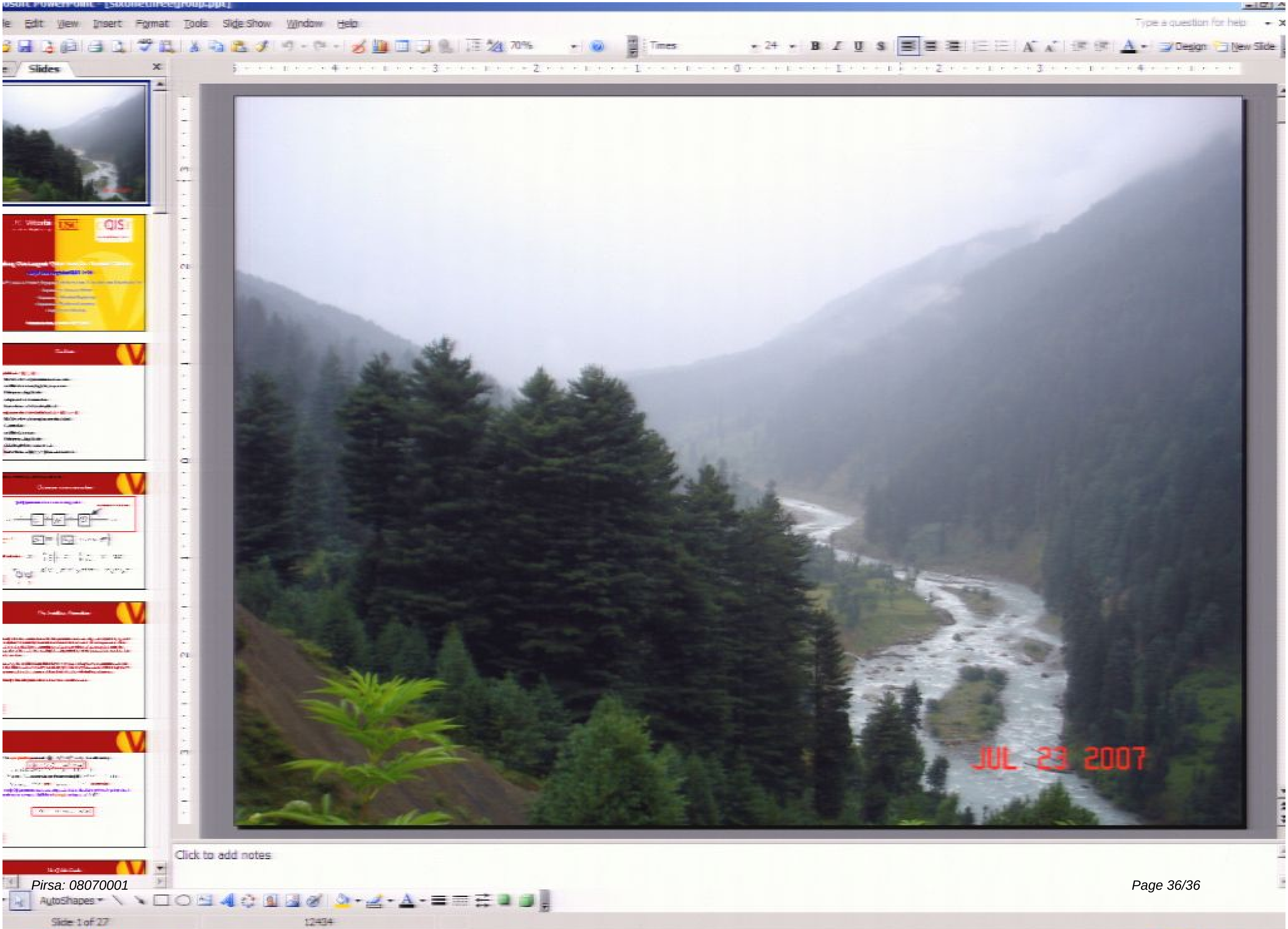


Isaac

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