Title: Additive Extensions of a Quantum Channel

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Abstract:

Additive extensions of a quantum channel

Graeme Smith and John A Smolin

arXiv:0712.2471

Additive Extensions of a Quantum Channel

- Quantum Channels
- Capacities and Additivity
- Known Bounds
- Additive and Degradable Extensions
- New Bounds on quantum and private capacity
- Relation to Symmetric Side Channels (SSW 2006)

Shannon 1948

Mutual Information

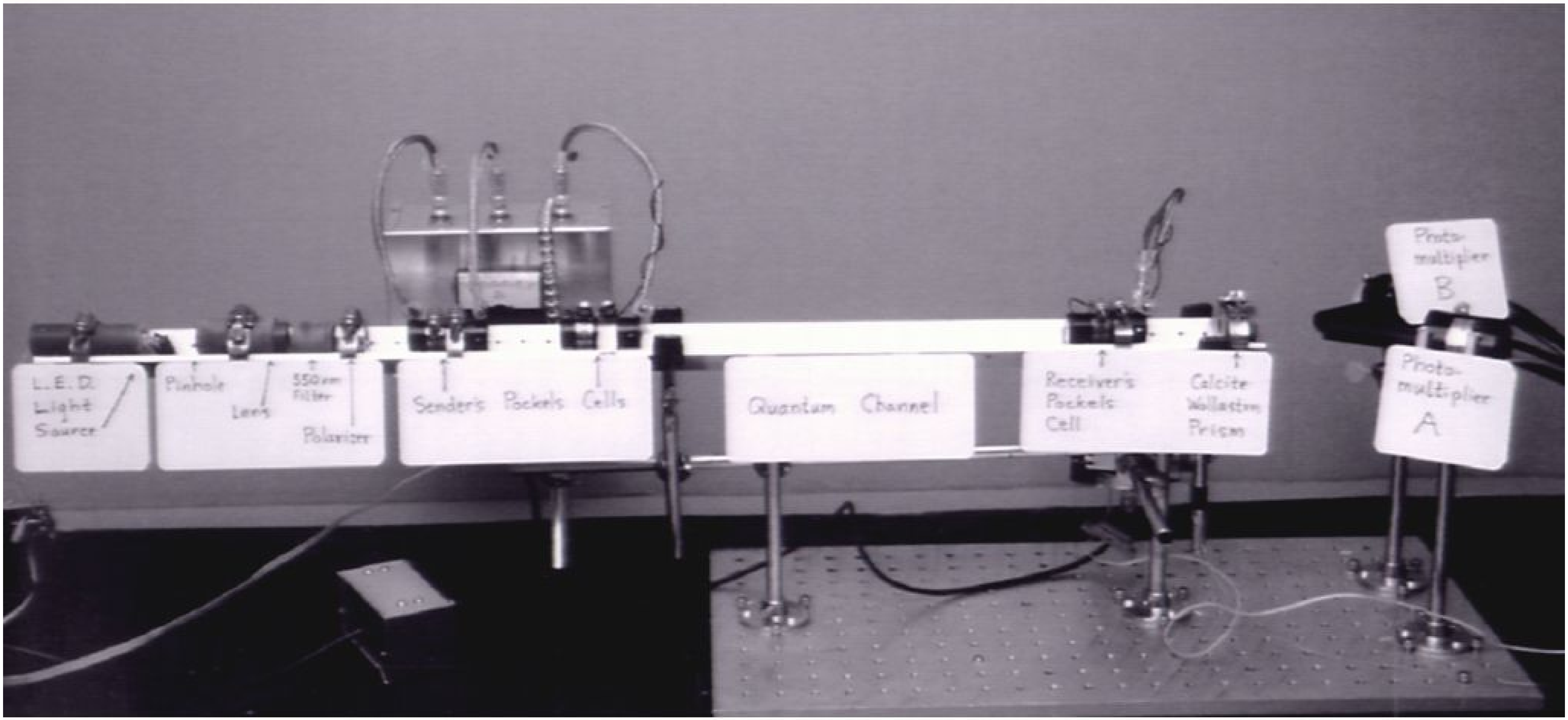
$$I(A{:}B) = H(A) + H(B) - H(A,B)$$

Channel maps X to B

Classical capacity $C = \mathrm{Max}_X\, I(A{:}B)$

Remarkable that you can achieve this with random coding
And that it is a single-letter formula

A Quantum Channel



Quantum channels

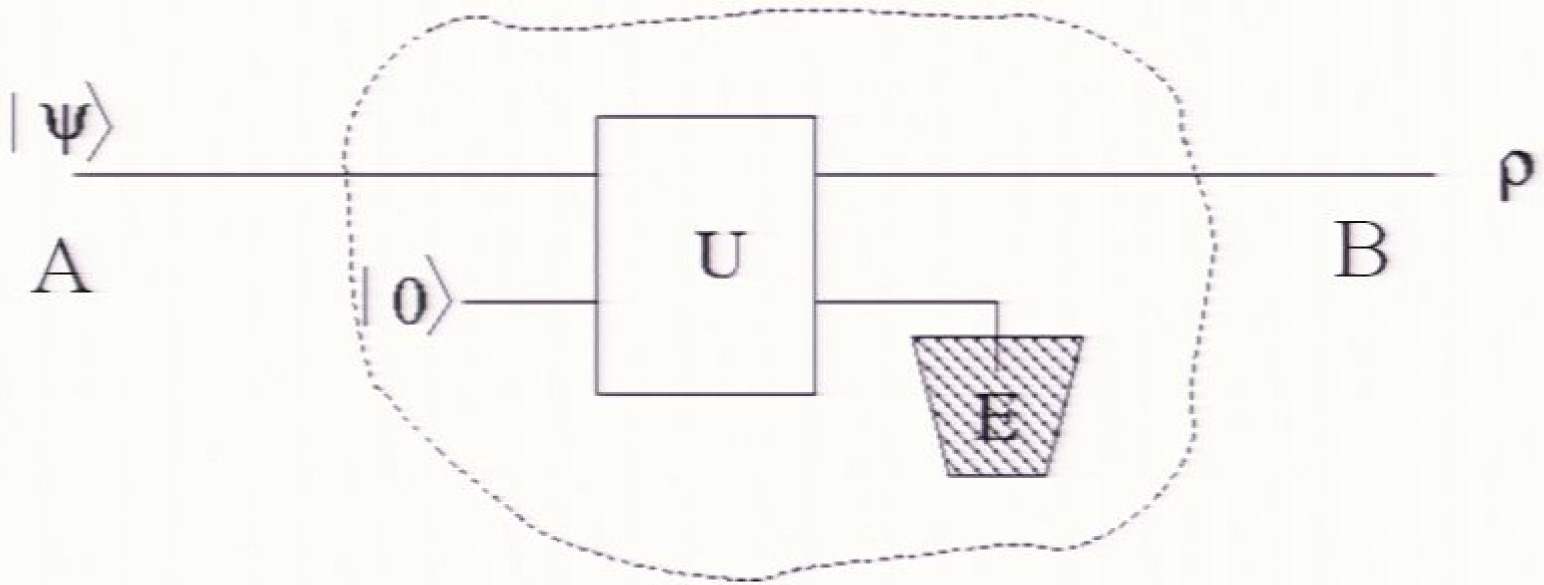
A quantum channel is represented by a trace-preserving, completely positive (TCP) map from density operators to density operators.

$$\rho \rightarrow \sum_i A_i \rho A_i^\dagger$$

$$\sum_i A_i^\dagger A_i = 1$$

Isometry

A-->BE



Coherent Information

$$\text{Coherent Information} : I^c(\rho_{AB}) = S(\rho_B) - S(\rho_{AB})$$

$$\text{of a channel and state)}\ I^c(\mathcal{N},\rho) = I^c\left(I \otimes \mathcal{N}(\phi^{AB})\right)$$

$$\text{where } \mathrm{Tr}_A|\phi^{AB}\rangle\langle\phi^{AB}| = \rho$$

$$\boxed{Q_1 = \max{}_\rho I^c(\mathcal{N},\rho)}$$

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Looks like Shannon formula

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Not the capacity SS96

Capacity Theorem

Have to regularize the one-shot capacity

Quantum Capacity : $\boxed{Q = \lim_{n\to\infty} \frac{1}{n} \max_{\rho_n} I^c\left(\mathcal{N}^{\otimes n}, \phi_n\right.}$

$$= \lim_{n\to\infty} \frac{1}{n} Q_1(\mathcal{N}^{\otimes n})$$

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* Ugly
* Usually impossible to evaluate

Additivity Questions Abound

Entanglement of formation
Holevo classical capacity
Minimum output entropy
Output rank

Quantum capacity is simply not single-letter
Bad: Hard to calculate
Good: Lots to work on

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Depolarizing channel

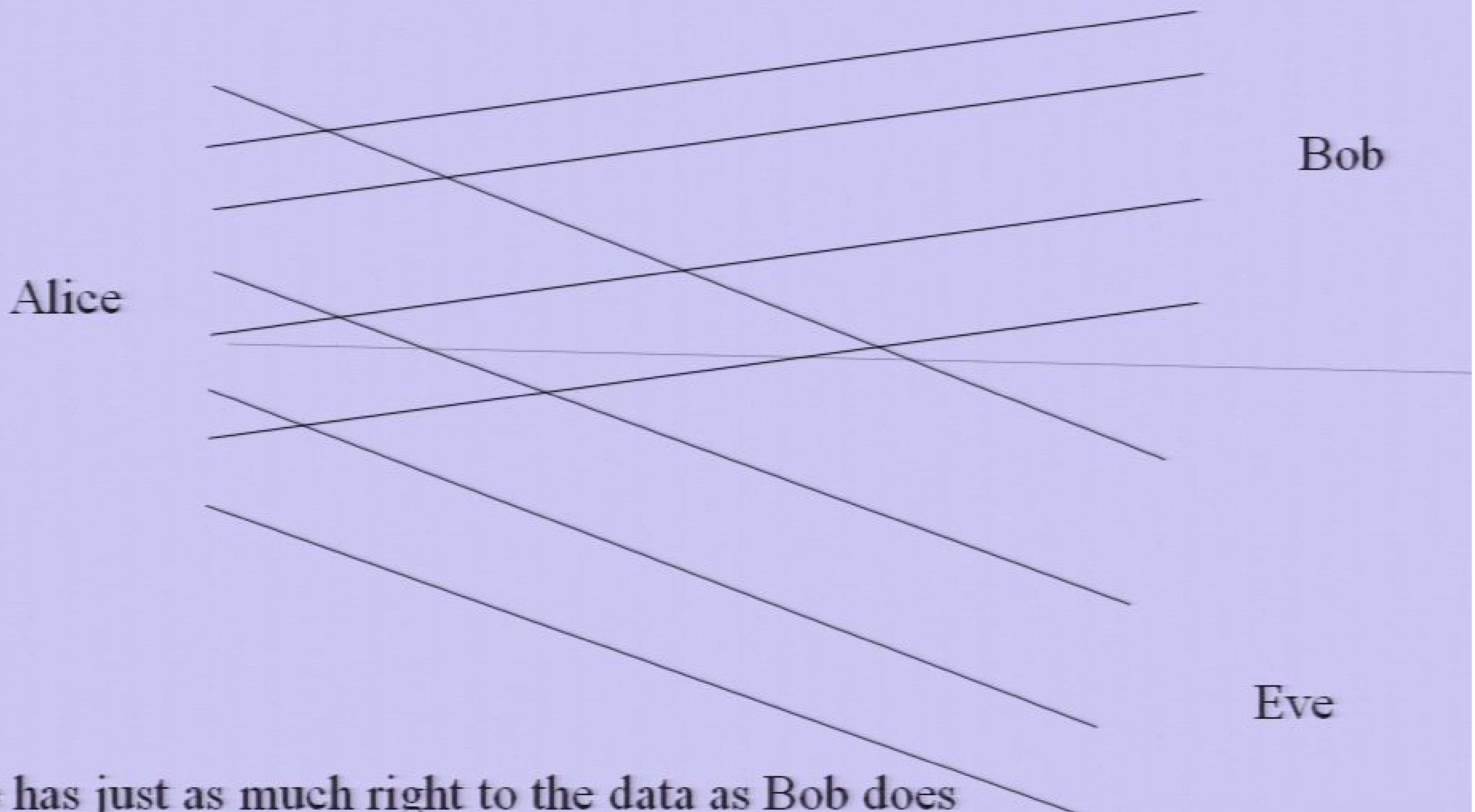
$$\mathcal{N}_p(\rho) = (1-p)\rho + \tfrac{p}{3}X\rho X + \tfrac{p}{3}Y\rho Y + \tfrac{p}{3}Z\rho Z$$

$$\mathcal{N}_x(\rho) = (1-x)\rho + x\tfrac{I}{2} = (1-x)\rho + \tfrac{x}{4}I\rho I + \tfrac{x}{4}X\rho X + \tfrac{x}{4}Y\rho Y + \tfrac{x}{4}Z\rho Z$$

$$p = \tfrac{3}{4}x$$

$$\mathcal{N}_p(U\rho U^\dagger) = U\mathcal{N}_p(\rho)U^\dagger$$

50% depolarizing probability



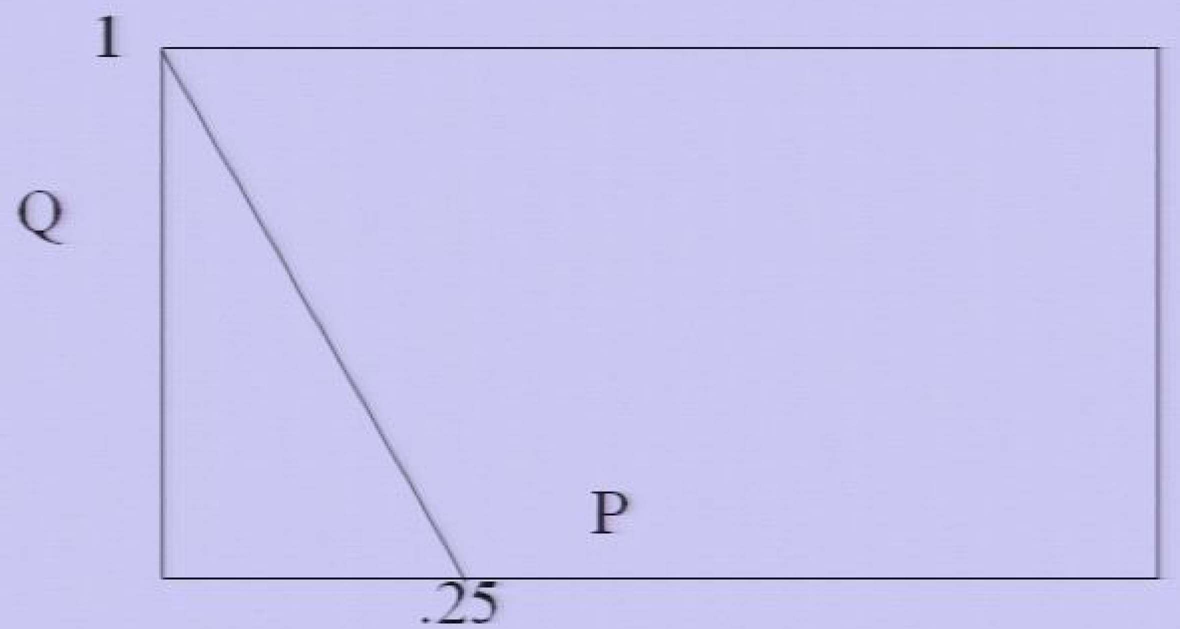
Eve has just as much right to the data as Bob does
But quantum information cannot be cloned <Dolly></Dolly>

No-cloning bound

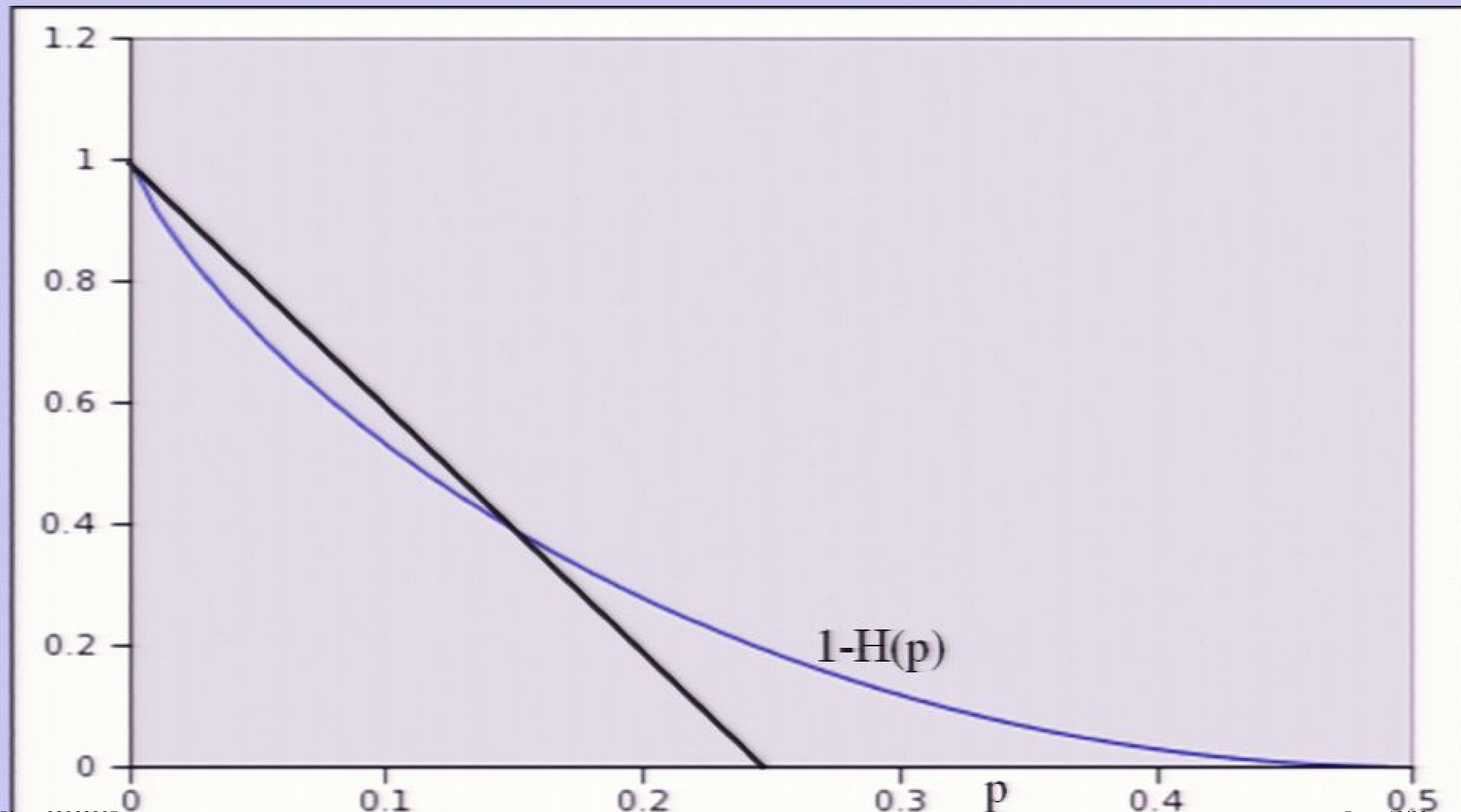
50% depolarlizing probability corresponds to $p=3/8$

Surprisingly, the approximate cloner of the previous slide is not optimal. Need ancilla to find optimal approximate cloning channel

Can achieve $p=1/4$ with optimal approximate cloner

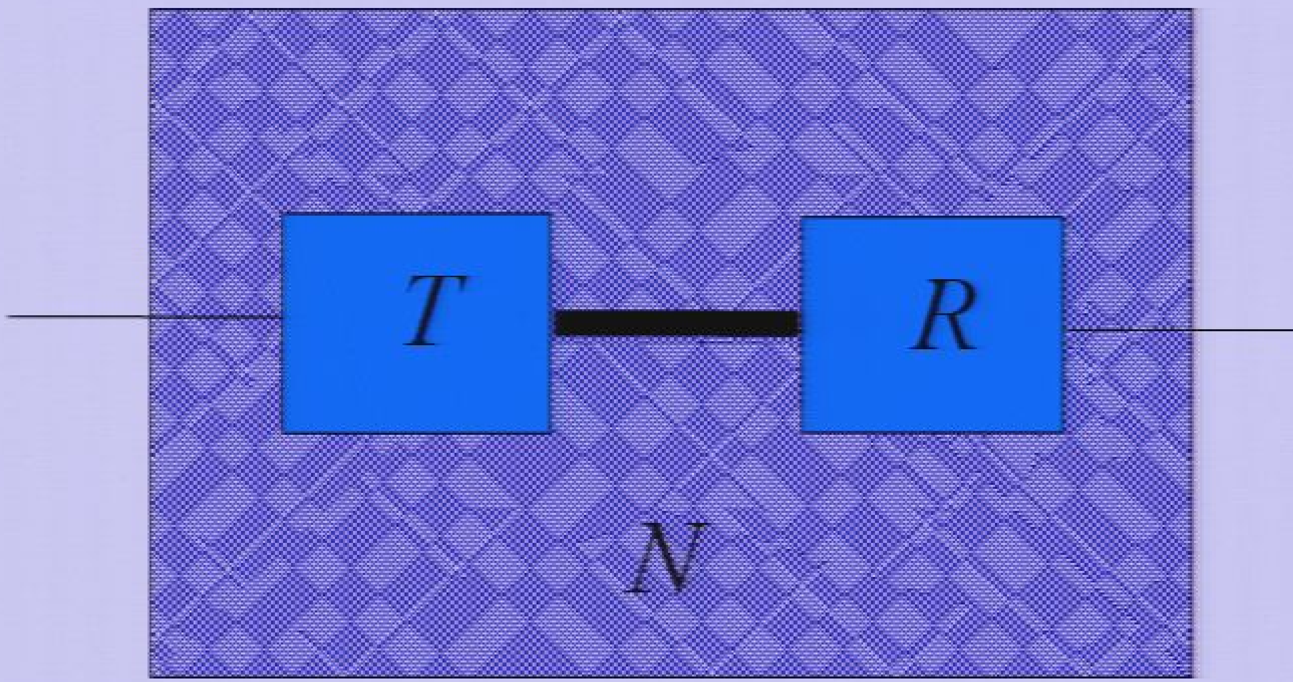


Rains bound



Additive Extension

Definition: T is an additive extension of a quantum channel $\mathcal{N}$ if there i
another channel $\mathcal{R}$ such that $\mathcal{N} = \mathcal{R} \circ \mathcal{T}$ and $Q(\mathcal{T}) = Q_1(\mathcal{T})$.



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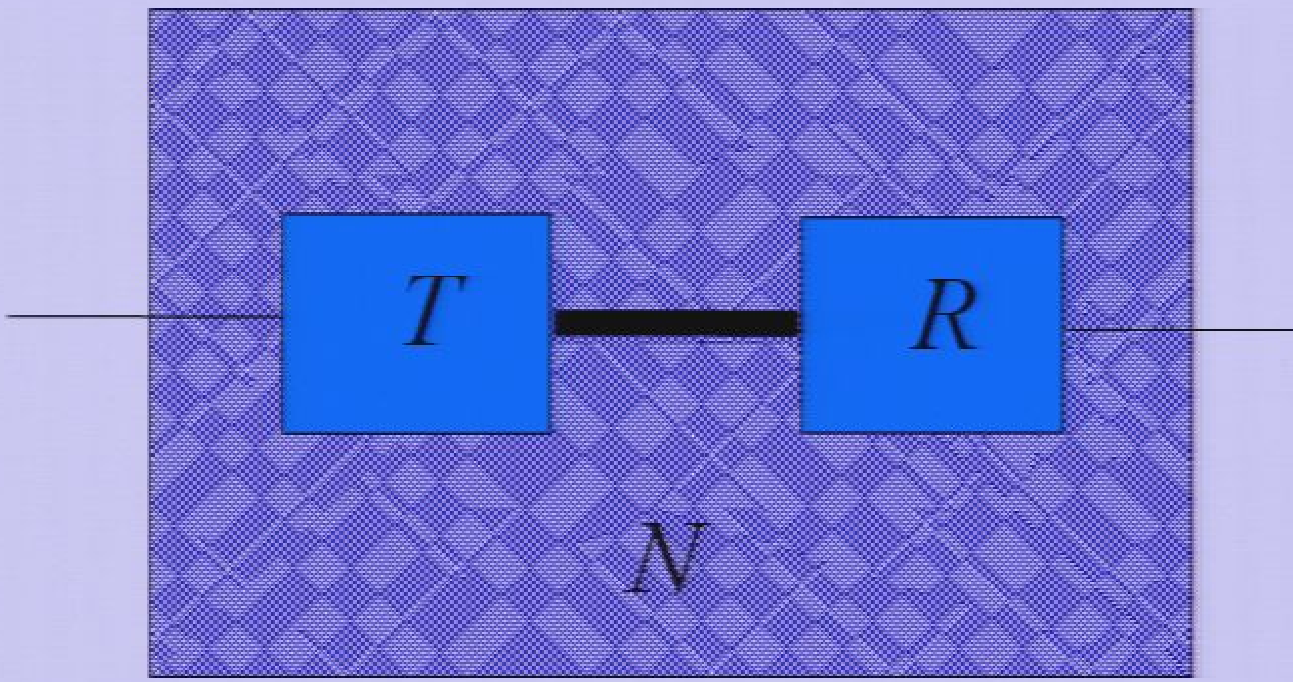

A degradable extension is an additive extension that is also degradable

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$\hat{}(\rho) = \mathrm{Tr}_B U \rho U^\dagger$. $\hat{\mathcal{N}}$ is called the *complementary channel* of $\mathcal{N}$.

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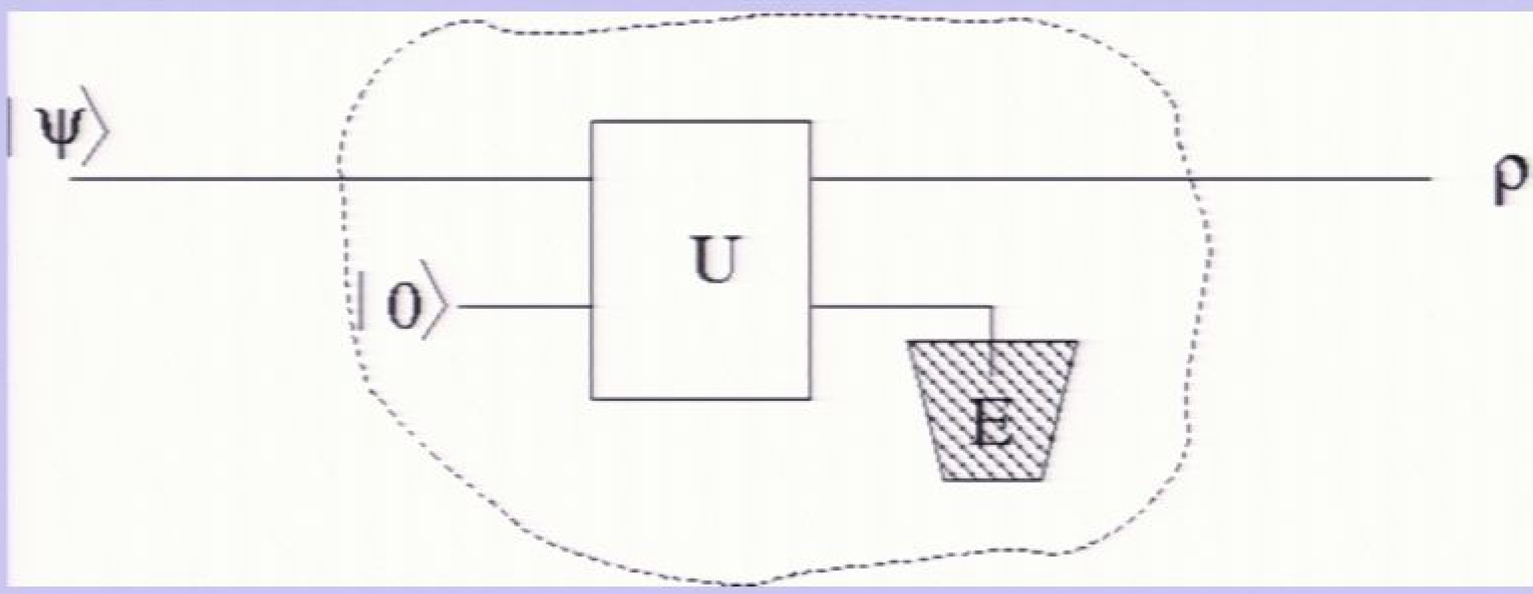

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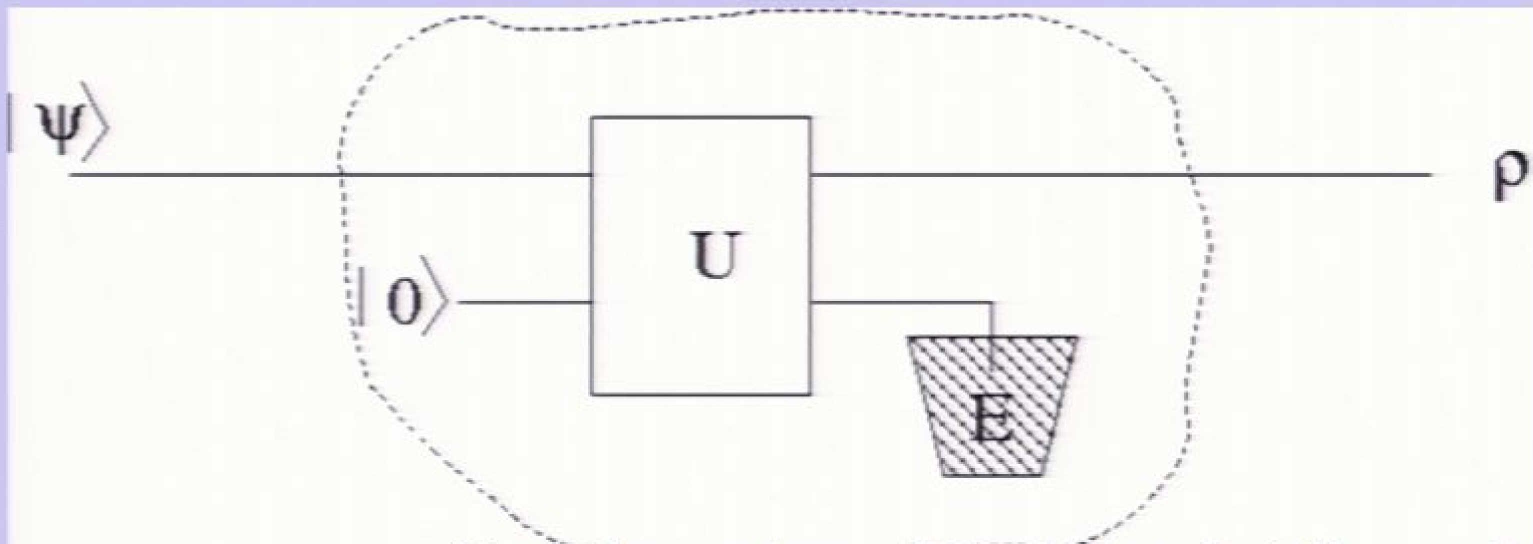

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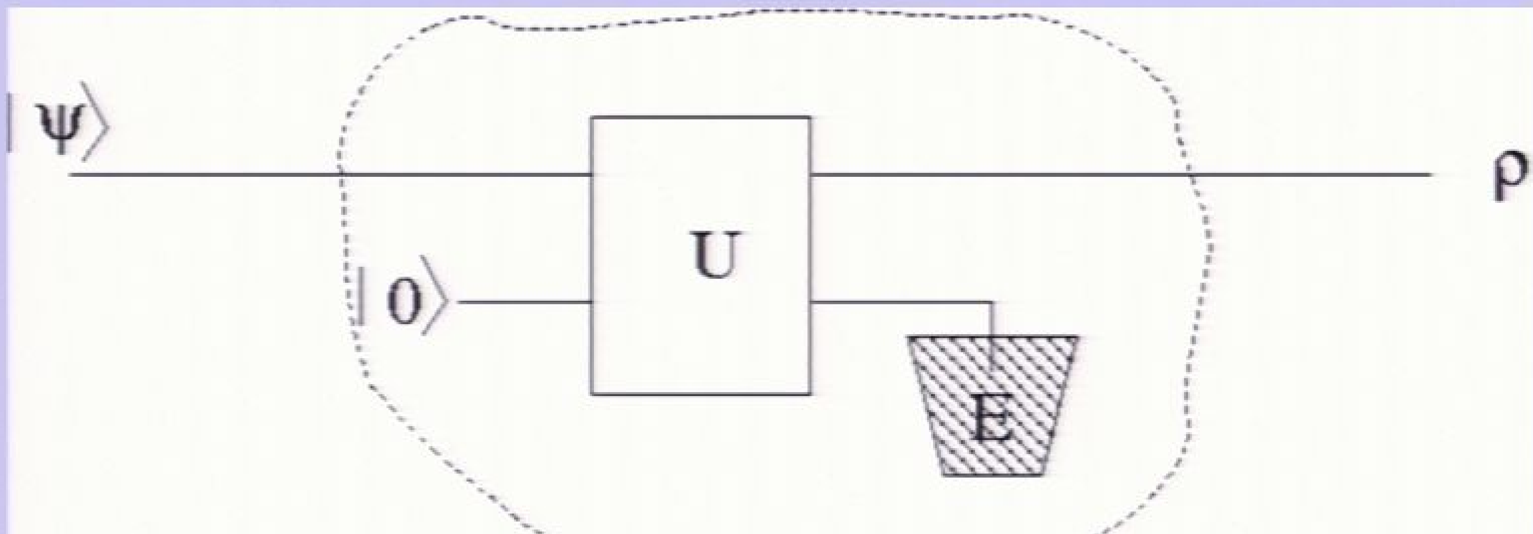

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Degradable channels have single-letter capacity   Devetak-Shor 2003
This is huge, since hardly anything is known to be single-letter!

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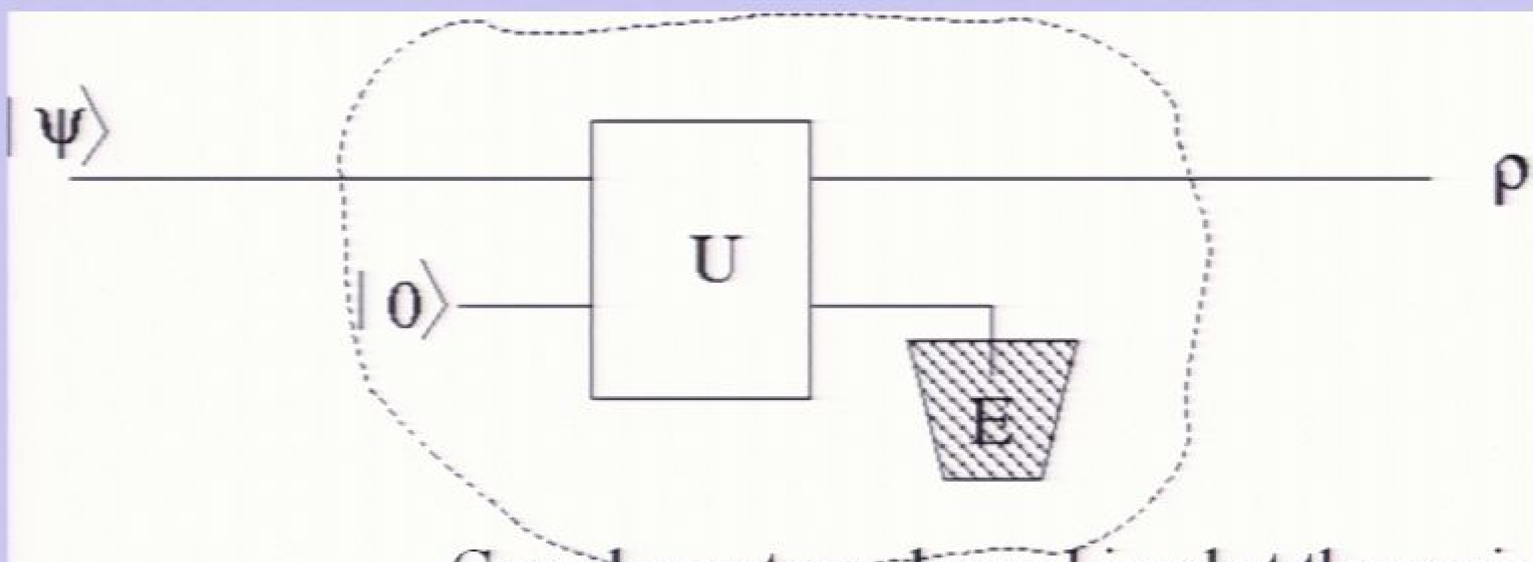

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Unfortunately, the depolarizing channel is not degradable

Good Properties of Additive Extensions

- Simple to understand
- Additive (single-letter) bounds on Q
- Includes all previous bounds on Q
- Can get Rains bound without my having to understand his paper
- Also bounds private capacity when degradable
- Good convexity properties
- Fixes argument for no-cloning straight line
- Yields new, tighter bounds

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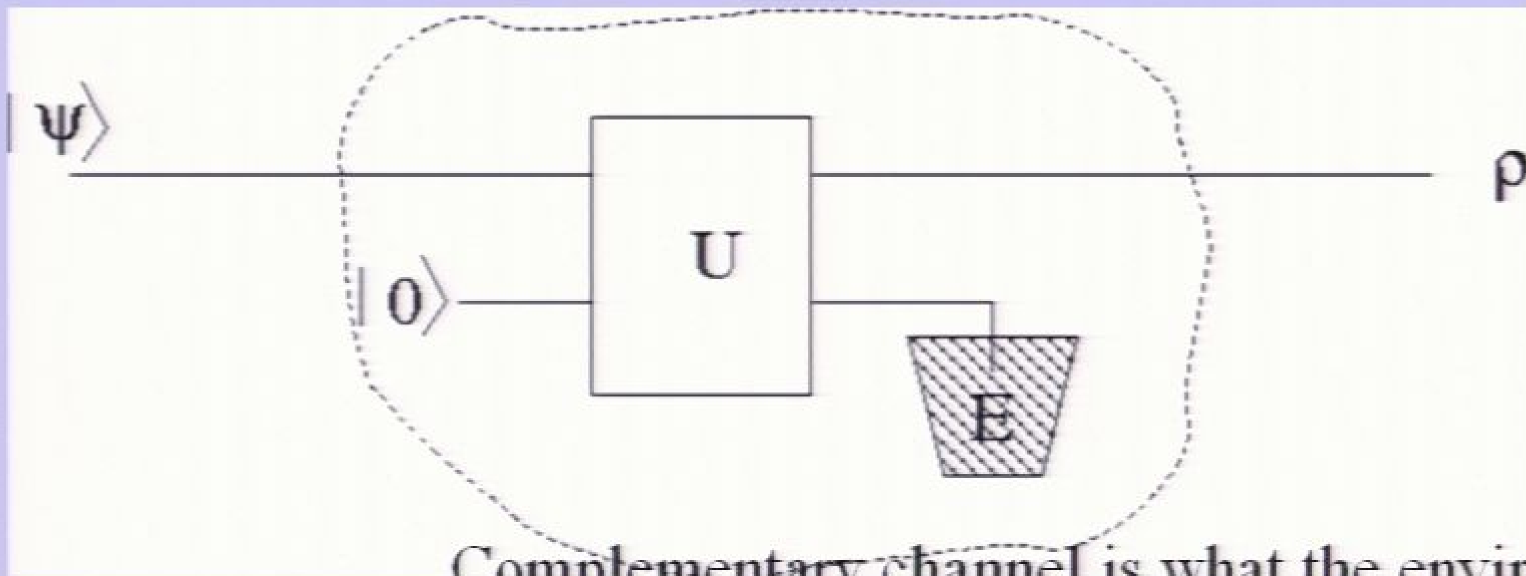

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Embarrassingly Simple Theorem

$$\boxed{\begin{array}{l} Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{T}) \\ \\ C_p(\mathcal{N}) \leq Q^{(1)}(\mathcal{T}) \end{array}}$$

$\mathcal{T}$ is an additive extension of $\mathcal{N}$

when $\mathcal{T}$ is also degradable

Proof:

Embarrassingly Simple Theorem

$Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{T})$ $\mathcal{T}$ is an additive extension of $\mathcal{N}$

$C_p(\mathcal{N}) \leq Q^{(1)}(\mathcal{T})$ when $\mathcal{T}$ is also degradable

Proof:

$Q^{(1)}(\mathcal{T}) = Q(\mathcal{T})$ by definition of additive extension

$Q(\mathcal{N}) \leq Q(\mathcal{T})$ can obtain $\mathcal{N}$ from $\mathcal{T}$ using $\mathcal{R}$

Similarly for C_p using $C_p(\mathcal{T}) = Q^{(1)}(\mathcal{T})$ (Smith07)

Flagged-Degradable Extensions

Lemma:

Suppose we have $\mathcal{N} = \sum_i p_i \mathcal{N}_i$ with $\mathcal{N}_i$ degradable

$\mathcal{T} = \sum_i p_i \mathcal{N}_i \otimes |i\rangle\langle i|$ is a degradable extension of $\mathcal{N}$

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$\mathcal{N} = \mathcal{R} \circ \mathcal{T}$: trace out flag system

$\mathcal{T}$ Degradable : read flag and apply $\mathcal{D}_\mathrm{i}$

Convex combination bound

If $\mathcal{T}$ is a flagged-degradable extension of $\mathcal{N}$ then

$$\boxed{Q(\mathcal{N}) \leq \sum_i p_i Q_1(\mathcal{N}_i)}$$

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Let ϕ be the optimal input state for $\mathcal{T}$:

$$Q_1(\mathcal{T}) = S\left(\sum_i p_i \mathcal{N}_i(\phi) \otimes |i\rangle\langle i|\right) - S\left(\sum_i p_i \hat{\mathcal{N}}_i(\phi) \otimes |i\rangle\langle i|\right)$$

$$= \sum_i p_i \left(S\left(\mathcal{N}_i(\phi)\right) - S\left(\hat{\mathcal{N}}_i(\phi)\right)\right)$$

$$\leq \sum_i p_i Q_1(\mathcal{N}_i)$$

also $Q(\mathcal{N}) \leq Q_1(\mathcal{T})$ **QED**

No cloning as special case

To show off how general Additive extensions are
To let us use convexity results on no-cloning bounds

$$\mathcal{N} \text{ Antidegradable} : \mathcal{D} \circ \hat{\mathcal{N}} = \mathcal{N}$$

$$\text{Isometry of } \mathcal{N} \text{ is } U : A \to BE$$

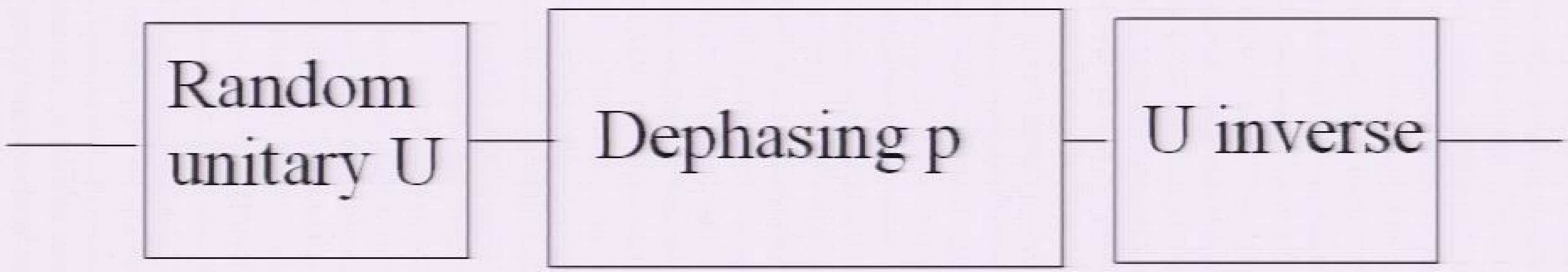
et $V|\phi\rangle = \frac{1}{\sqrt{2}} U|\phi\rangle|0\rangle_{f_1}|1\rangle_{f_2} + \frac{1}{\sqrt{2}} \mathrm{SWAP}_{BE} U|\phi\rangle|1\rangle_{f_1}|0\rangle_{f}$

$\mathcal{T}(\rho) = \mathrm{Tr}_{Ef_2} V \rho V^\dagger$ is a degradable extension of $\mathcal{N}$

Degradable, antidegradable and zero capacity by symmetry

Can get N by applying antidegrading map when necessary

Example: Rains bound



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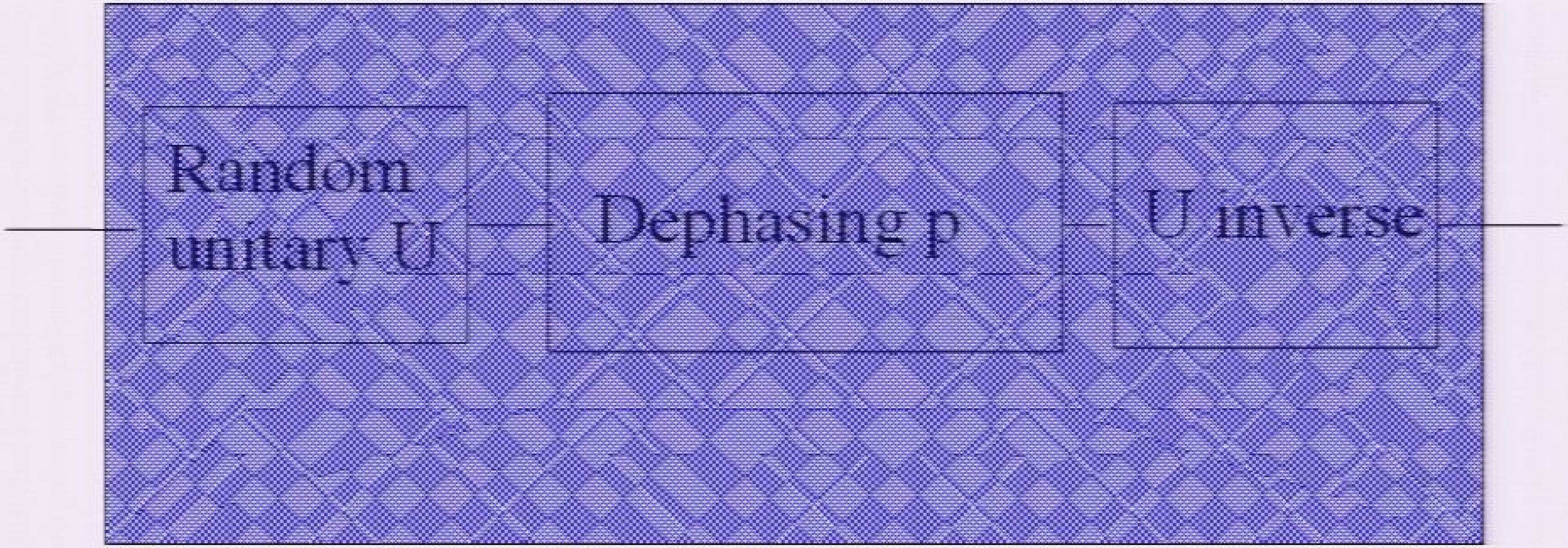

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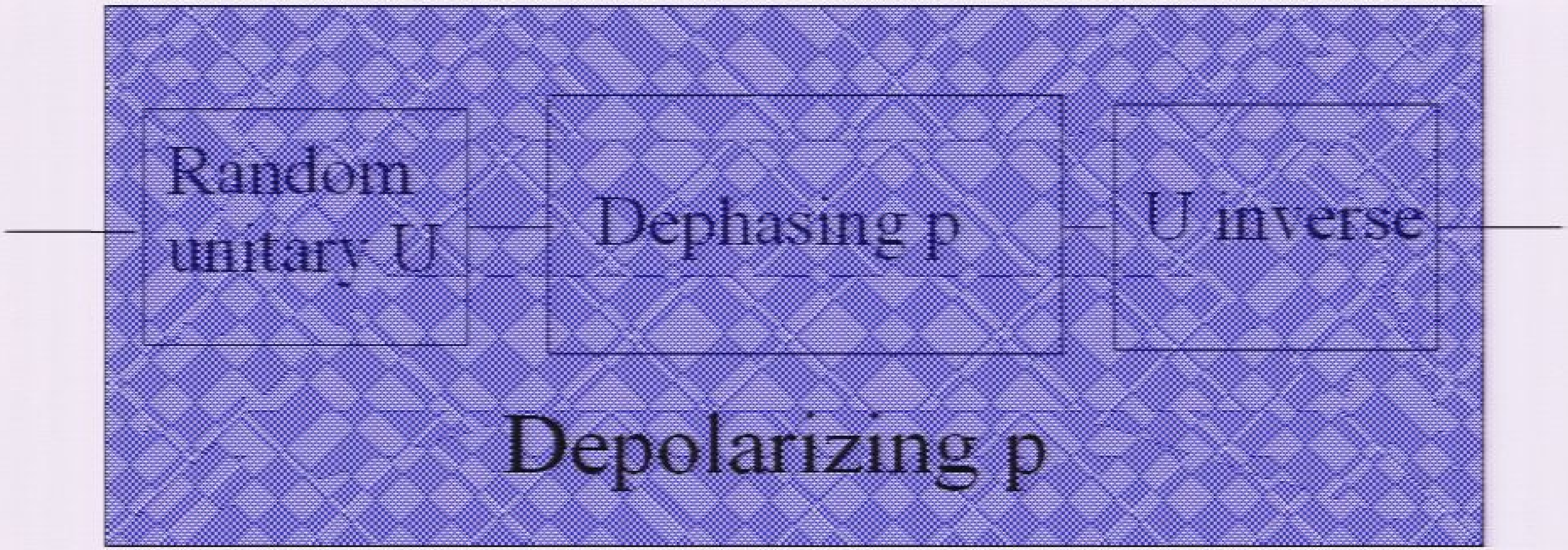

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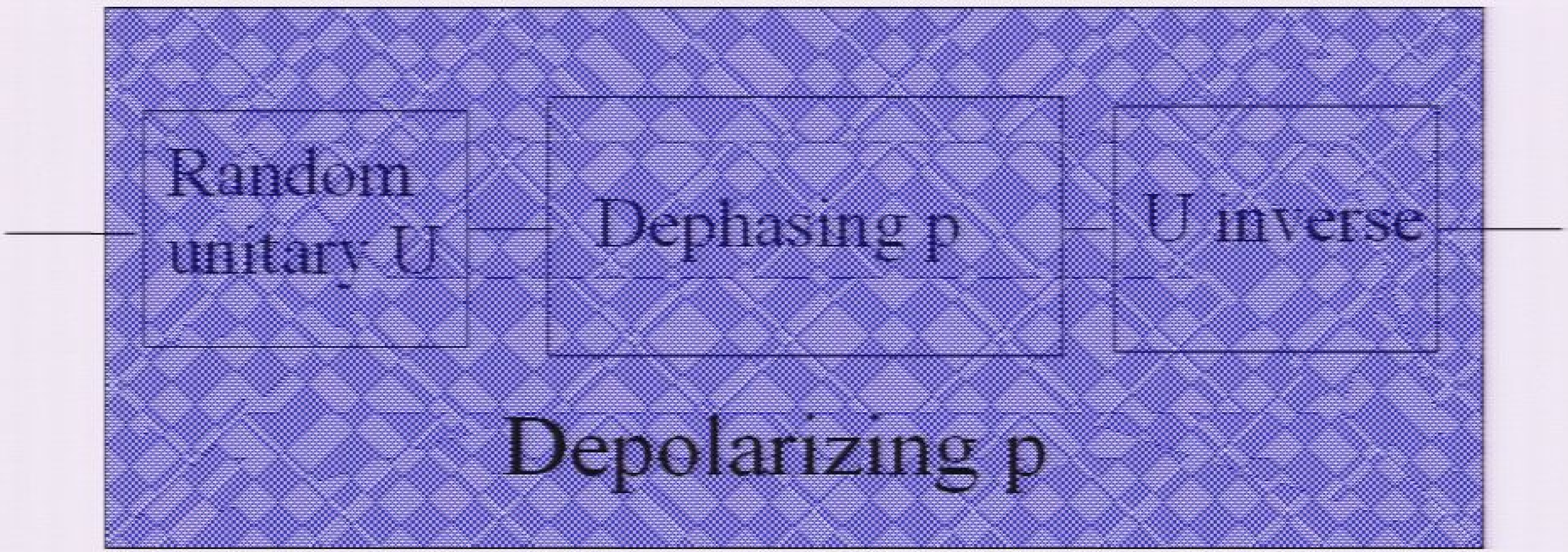

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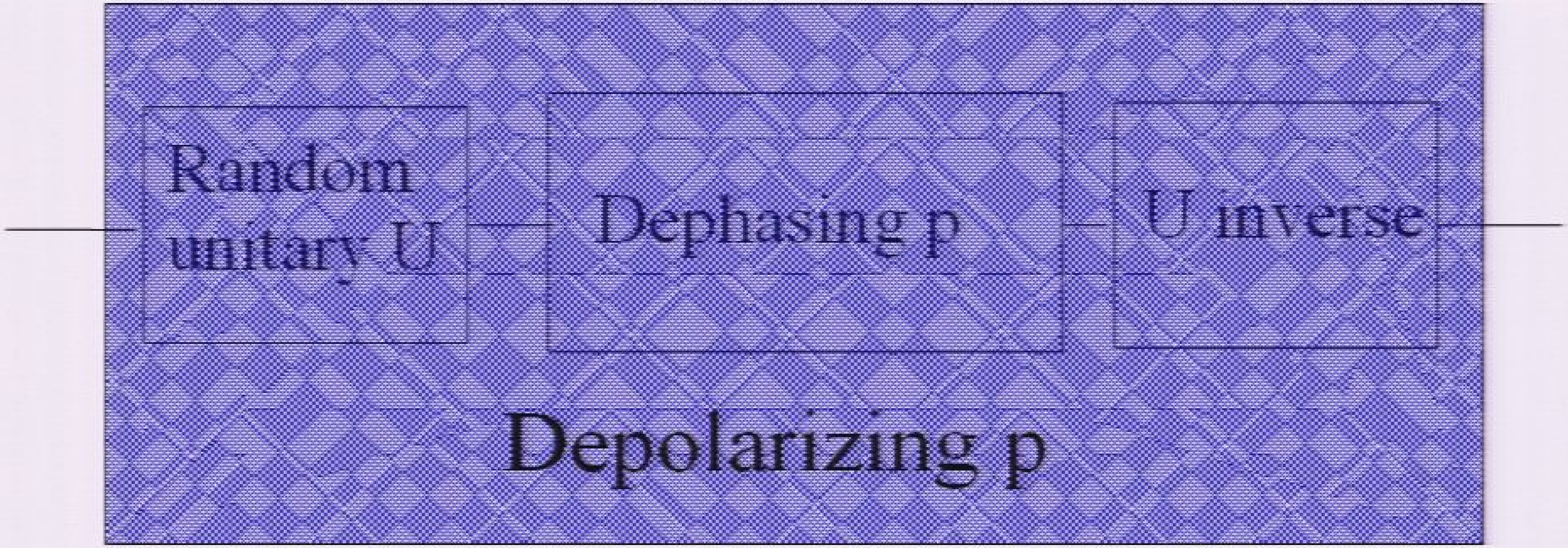

Depolarizing channels are convex mixtures of dephasing channels. Gives bound immediately

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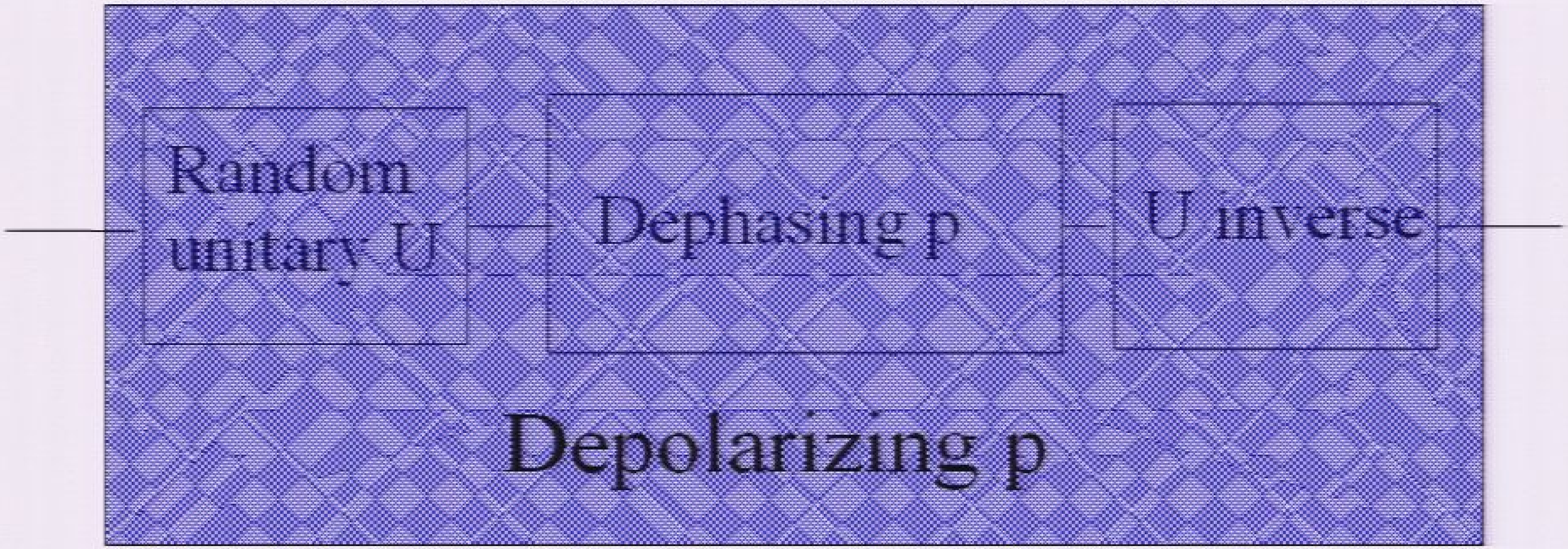

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Better than Convex

f $\mathcal{T}$ is a flagged-degradable extension of $\mathcal{N}$ then

$$\boxed{Q(\mathcal{N}) = \max_{\phi} \sum_i p_i \left[S\left(\mathcal{N}_i(\phi)\right) - S\left(\hat{\mathcal{N}}_i(\phi)\right) \right]}$$

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$$= \sum_i p_i \left(S\left(\mathcal{N}_i(\phi)\right) - S\left(\hat{\mathcal{N}}_i(\phi)\right)\right)$$

$$\leq \sum_i p_i Q_1(\mathcal{N}_i)$$

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also $Q(\mathcal{N}) \leq Q_1(\mathcal{T})$ **QED**

General Degradable Qubit channel

$$
\begin{aligned}
A_+ &= \begin{pmatrix} \cos(\frac{1}{2}(v-u)) & 0 \\ 0 & \cos(\frac{1}{2}(v+u)) \end{pmatrix} \\
A_- &= \begin{pmatrix} 0 & \sin(\frac{1}{2}(v+u)) \\ \sin(\frac{1}{2}(v-u)) & 0 \end{pmatrix}
\end{aligned}
$$

degradable when $|\sin v| \leq |\cos u|$

Wolf Perez-Garcia 2007

Cubitt, Ruskai, Smith 2008

General degradable-depolarizing mixture

$$\mathcal{T}^{\mathrm{dep}}_{(u,v)}(\rho) = \frac{1}{|\mathcal{C}|}\sum_{c\in\mathcal{C}} c^\dagger \mathcal{N}_{(u,v)}(c\rho c^\dagger)c \otimes |c\rangle\langle c|$$

$$\mathrm{Tr}_f \mathcal{T}^{\mathrm{dep}}_{(u,v)}(\rho) = \mathcal{N}_p(\rho)$$

$$p = 1 - \cos^2(u/2)\cos^2(v/2)$$

Finally, some results

$$Q(\mathcal{N}_p) \le \min\, H\left[\tfrac{1}{2}[1+\sin u \sin v]\right] - H\left[\tfrac{1}{2}[1+\cos u \cos v]\right]$$

$$\text{minimize over } (u,v) \text{ s.t. } \cos^2(u/2)\cos^2(v/2) = 1-p$$

$$Q(\mathcal{N}_p) \le \text{co}\left[1 - H(p), H\left(\tfrac{1-\gamma(p)}{2}\right) - H\left(\tfrac{\gamma(p)}{2}\right), 1-4p\right]$$

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minimize over (u,v) s.t. $\cos^2(u/2)\cos^2(v/2) = 1-p$

Rains

No cloning

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Rains — New bound — No cloning

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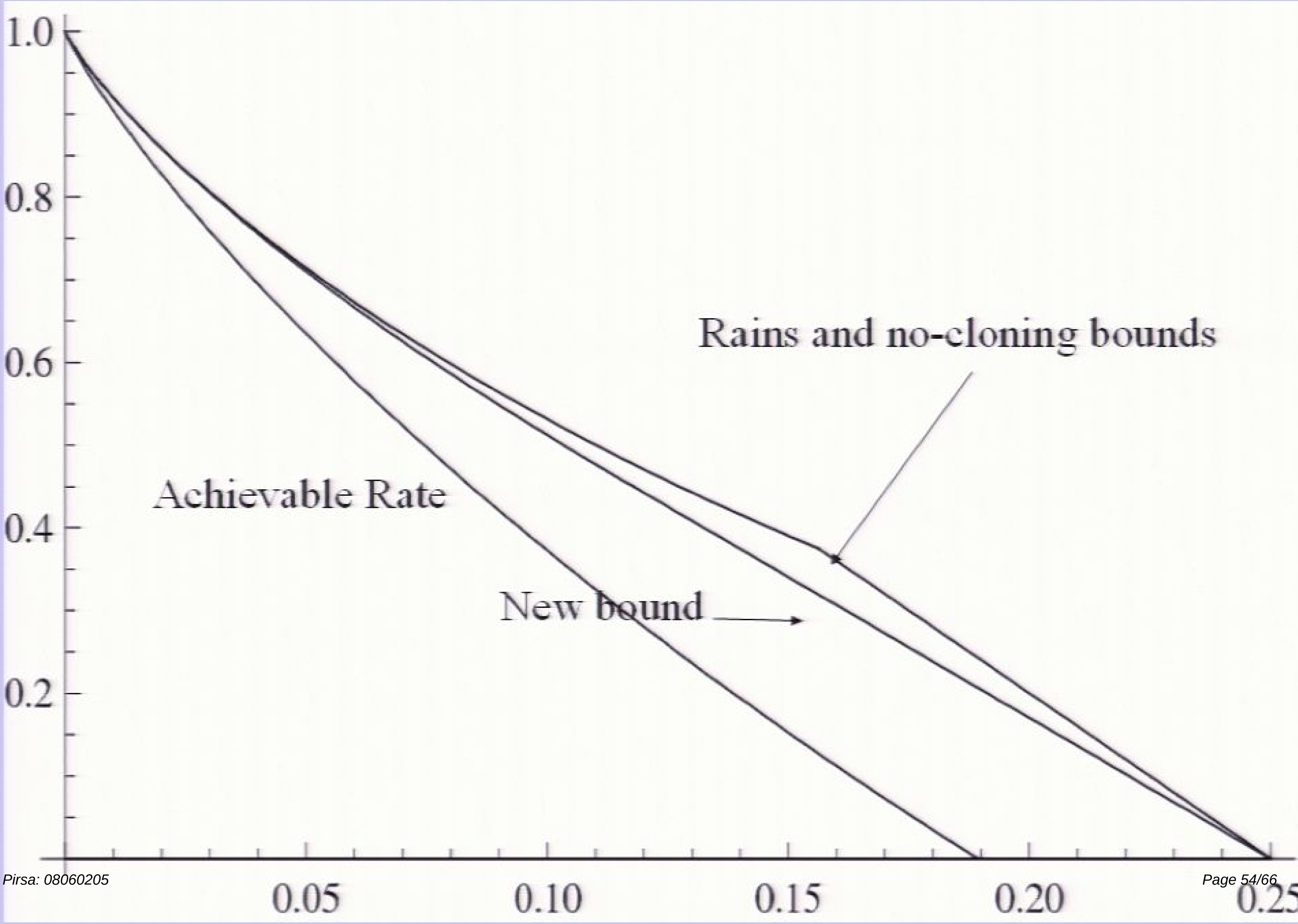
1.0
0.8
0.6
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0.10
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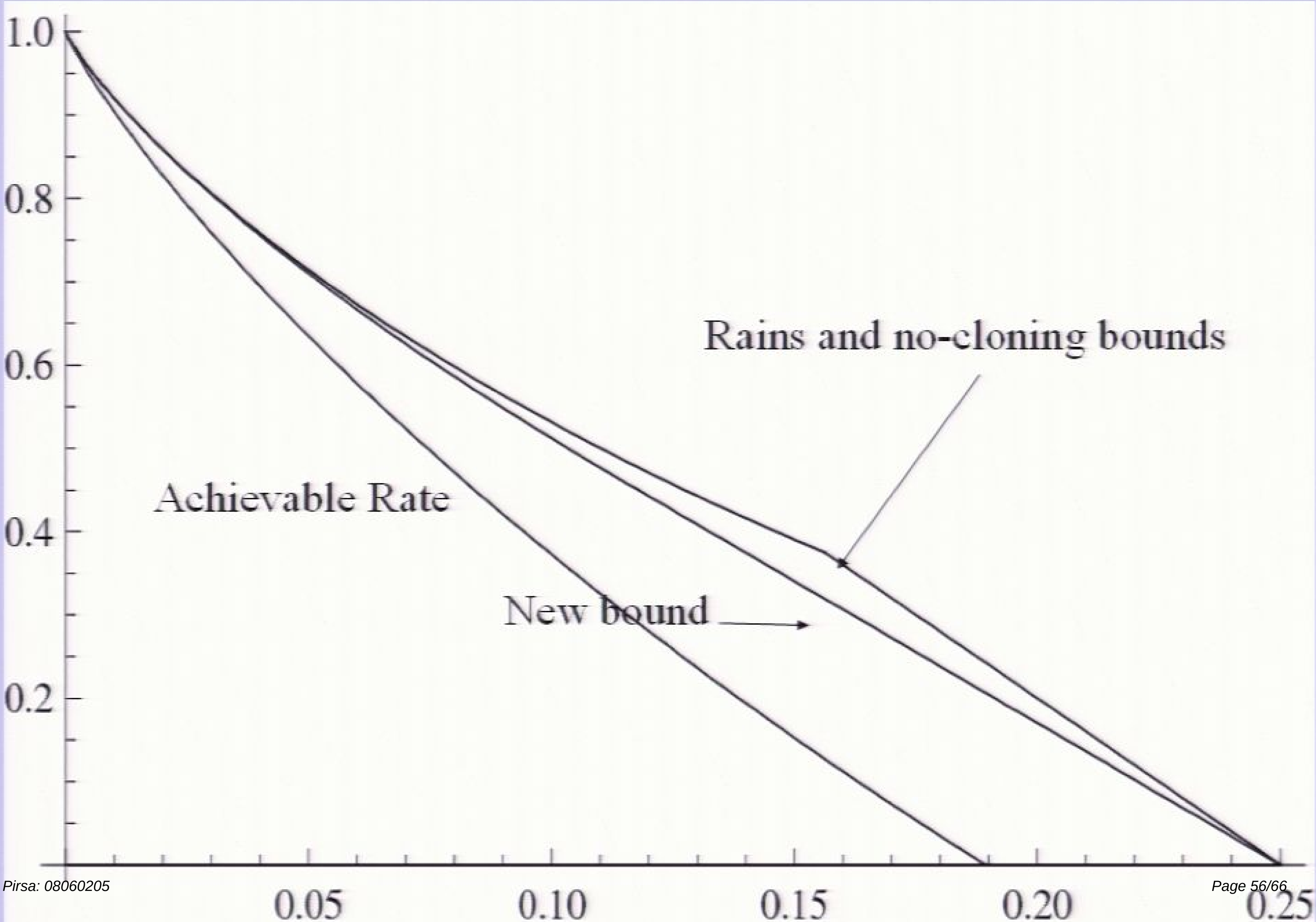
0.25
Rains and no-cloning bounds
Achievable Rate
New bound

Another Result: BB84

$$\mathcal{N}_q^{\mathrm{BB84}}(\rho) = (1-q)^2\rho + q(1-q)X\rho X + q(1-q)Z\rho Z + q^2 Y\rho Y$$


Rains and no-cloning bounds
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$$C_p(\mathcal{N}_q^{\mathrm{BB84}}) \leq H\left(\frac{1}{2} - 2q(1-q)\right) - H(2q(1-q))$$

Amplitude Damping

$$\mathcal{N}_\gamma^{\mathrm{ad}}: \qquad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

degradable when $\gamma \leq 1/2$

Giovanetti Fazio 2005

BB84 extension

$$\mathcal{T}_q^{\mathrm{BB84}} = \frac{1}{2}\mathcal{N}^{\mathrm{ad}}_{\gamma(q)}(\rho) \otimes |0\rangle\langle 0| + \frac{1}{2} Y \mathcal{N}^{\mathrm{ad}}_{\gamma(q)}(Y \rho Y) Y \otimes |1\rangle\langle 1$$

$$\gamma(q) = 4q(1-q)$$

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Nice because it has just two terms
We can fully calculate its Q_1

SS Capacity

Isometry : $V|(i,j)\rangle = \frac{1}{\sqrt{2}}(|i\rangle|j\rangle - |j\rangle|i\rangle)$

Channel : $\mathcal{A}(\rho) = \mathrm{Tr}_2 V\rho V^\dagger \quad Q(\mathcal{A}) = 0$

$$Q_{\mathrm{ss}}(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q_1(\mathcal{N}^{\otimes n} \otimes \mathcal{A})$$

$Q_{\mathrm{ss}}(\mathcal{N}) = Q_1(\mathcal{N} \otimes A)$ SSW2006

Want to show $\mathcal{N} \otimes \mathcal{A}$ is an additive extension of $\mathcal{N}$

with capacity $Q(\mathcal{N} \otimes \mathcal{A}) = Q_{\mathrm{ss}}(\mathcal{N})$

$$Q(\mathcal{N} \otimes \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} Q_1(\mathcal{N}^{\otimes n} \otimes \mathcal{A}^{\otimes n})$$

A is an additive extension

$$Q(\mathcal{N}\otimes\mathcal{A}) = \lim_{n\to\infty}\frac{1}{n}Q_1(\mathcal{N}^{\otimes n}\otimes\mathcal{A}^{\otimes n})$$

$$= \lim_{n\to\infty}\frac{1}{n}\left(Q_1(\mathcal{N}^{\otimes n}\otimes\mathcal{A}) + Q_1(\mathcal{A}^{\otimes(n-1)}\otimes\mathcal{A})\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}(nQ_1(\mathcal{N}\otimes\mathcal{A})) = Q(\mathcal{N}\otimes\mathcal{A}) = Q_1(\mathcal{N}\otimes A$$

End of slide show, click to exit.