Title: Continuity of various capacities of a quantum-channel

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Abstract:

Continuity of channel capacities

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arXiv.org:08??.soon

Thanks: Aram Harrow, John Smolin, and IBM group

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$MITACS, NSERC, CIFAR, CRC, CFI, ORF, ARO$
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Continuity -- why it matters

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- it's a fundamental question

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- it's a fundamental question

- continuity implies less broken
  (& perhaps more useful) intuition

Continuity -- excerpt of a chat @IBM Apr08

You're working on THIS 'cause there's nothing better to do?

Wanna do something useful, huh? OK, why don't we bound the classical capacity of the amplitude damping channel ?

[surprised] there's no upper bound ?????

No, the same old additivity problem ...
Perhaps we can look at the capacity of some nearby depolarizing channel [whose capacity we know] ?

Who said the capacity is continuous?

Crap :-(

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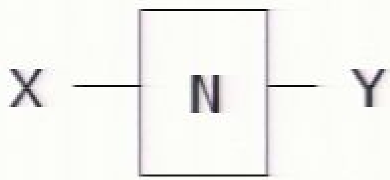
Who said the capacity is continuous?

Crap :-(Well, we should prove continuity then.

Does it work classically? How?

Continuity : classical capacity of classical channel N (iid)



N: pr(y|x), say Y=N(X)

$$C(N) = \max_X I(X:N(X))$$
$$= \max_X H(X)+H(N(X))-H(XN(X))$$

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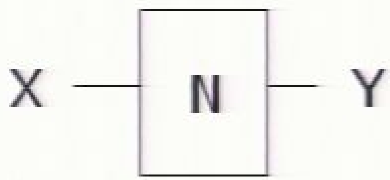

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$$C(N) = \max_X I(X:N(X))$$
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For 2 channels N_1 and N_2,

$$C(N_i) = \max_{X_i} H(X_i)+H(N_i(X_i))-H(X_iN_i(X_i))$$

When comparing $C(N_1)$ & $C(N_2)$, the difference is caused by difference in N_1, N_2, and also that in optimal X_1, X_2 .

Proof ideas for continuity of C(N)

$$C(N_i) = \max_{X_i} \boxed{H(X_i)+H(N_i(X_i))-H(X_iN_i(X_i))} \longleftarrow \text{call this } f(X_i,N_i)$$

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Let X_i^{op} be optimal input distribution for N_i:

$C(N_1) =$

$f(X_1^{op},N_1)$ $f(X_1^{op},N_2)$

$f(X_2^{op},N_1)$ $f(X_2^{op},N_2)$

$= C(N_2)$

stick each X_i^{op} in both channel capacity expression anyways

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$f(X_1^{op},N_1) \longleftarrow$ if $N_1 \approx N_2$ $X_1^{op}N_1(X_1^{op}) \approx X_1^{op}N_2(X_1^{op})$ & $f(X_1^{op},N_1) \approx f(X_1^{op},N_2)$ $\rightarrow f(X_1^{op},N_2)$

$\geq$ $\leq$

$f(X_2^{op},N_1)$ $\qquad$ $f(X_2^{op},N_2)$

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$C(N_1) =$

if $N_1 \approx N_2$

$f(X_1^{op},N_1) \longleftarrow X_1^{op}N_1(X_1^{op}) \approx X_1^{op}N_2(X_1^{op}) \longrightarrow f(X_1^{op},N_2)$

$\& \; f(X_1^{op},N_1) \approx f(X_1^{op},N_2)$

$\geq \qquad\qquad \leq$

Similarly

$f(X_2^{op},N_1) \longleftarrow f(X_2^{op},N_1) \approx f(X_2^{op},N_2) \longrightarrow f(X_2^{op},N_2)$

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Let X_i^{op} be optimal input distribution for N_i:

$\therefore C(N_1) \leq C(N_2)+\text{gap}$

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$C(N_1) \approx C(N_2)$ up to change in f due to diff in $XN_i(X)$ [same X]

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The gap: how to make $N_1 \approx N_2$ imply

(a) $X_1^{op}N_1(X_1^{op}) \approx X_1^{op}N_2(X_1^{op})$ and (b) $f(X_1^{op},N_1) \approx f(X_1^{op},N_2)$??

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(a) measure of proximity of N_1, N_2 (need $\forall X\ XN_1(X) \approx XN_2(X)$)
take $||N_1-N_2||=\max_X||XN_1(X)-XN_2(X)||_{tr}$

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$\Delta f \leq |\ H(N_1(X))-H(N_2(X))\ | + |\ H(XN_1(X))-H(XN_2(X))\ |$

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$\leq ||N_1(X)-N_2(X)||_{tr} \log d_{out} + ||XN_1(X)-XN_2(X)||_{tr} \log d_{in}d_{out}$

+ ... Thanks to Fannes73

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$+ \ldots$ Thanks to Fannes73

$\leq 3\ ||N_1-N_2|| \log d + \ldots$ where $d = \max(d_{in,out})$

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$f(X_1^{op},N_1) \longleftarrow$ differ by at most $3\,||N_1-N_2||\log d + \ldots \longrightarrow f(X_1^{op},N_2)$

$\geq \qquad\qquad\qquad\qquad \leq$

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$C(N_1) - 3\ ||N_1-N_2||\ \log d \leq C(N_2)$

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$$C(N_1) \geq -\,3\,||N_1-N_2||\log d + C(N_2)$$

Thus $|C(N_1)-C(N_2)| \leq ||N_1-N_2||\;3\log d + \ldots$

What about quantum channels?

Each channel N takes A_i to B_i

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- Holevo-Schumacher-Westmoreland:

$$C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n \; I(X:B_1B_2 \cdots B_n)$$

$$\text{evaluated on } \sum_x p_x \underbrace{|x\rangle\langle x|}_{X} \otimes \underbrace{N^{\otimes n}(\rho_x)}_{B_1B_2 \cdots B_n}$$

What about quantum channels?

Each channel N takes A_i to B_i

$I(K:L) = S(K) + S(L) - S(KL)$

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- Lloyd-Shor-Devetak:

$$Q(N) = \lim_{n\to\infty} \max_{|\psi\rangle} 1/n \; I^{coh}(R>B_1B_2\cdots B_n)$$

evaluated on $I\otimes N^{\otimes n}|\psi\rangle_{RA_1A_2\cdots A_n}$

What about quantum channels?

Each channel N takes A_i to B_i

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$I(K{:}L) = S(K) + S(L) - S(KL)$

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- Lloyd-Shor-Devetak:

$I^{coh}(K>L) = S(L) - S(KL)$

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e.g. $C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n \; I(X:B_1 B_2 \cdots B_n)$

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Try nimic continuity argument in classical setting despite the n-use capacity expression and bound the gap:

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Try nimic continuity argument in classical setting despite the n-use capacity expression and bound the gap:

(1) For the same input, even if

$||N_1-N_2||_\diamond := \max_\rho || I\otimes N_1(\rho) - I\otimes N_2(\rho) ||_{tr}$ small

$\sum_x p_x|x\rangle\langle x| \otimes N_1^{\otimes n}(\rho_x)$ and $\sum_x p_x|x\rangle\langle x| \otimes N_2^{\otimes n}(\rho_x)$

can be "$n||N_1-N_2||$" apart.

What about quantum channels?

e.g. $C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n \; I(X:B_1B_2 \cdots B_n)$

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$\sum_x p_x|x\rangle\langle x| \otimes N_1^{\otimes n}(\rho_x)$ and $\sum_x p_x|x\rangle\langle x| \otimes N_2^{\otimes n}(\rho_x)$ can be "$n||N_1-N_2||$" apart.

(2) "log dim" dependence in Fannes ineq grows as n.

What about quantum channels?

e.g. $C(N) = \lim_{n\to\infty} \max_{p_x,\rho_x} 1/n \; I(X:B_1B_2 \cdots B_n)$

evaluated on $\sum_x p_x|x\rangle\langle x| \otimes N^{\otimes n}(\rho_x)$

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So, gap $\sim 1/n * n^2 * ||N_1\text{-}N_2||$ ... :-(

Similar problem for
(1) classical capacity (unknown additivity)
(2) quantum capacity (known nonadditivity)
(3) private classical capacity (unknown additivity)

All 3 capacities are sup over some output entropies.

Need tighter bound concerning the entropy difference between two n-use output states.

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same state — only 1 factor of n

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For any input ensemble, resulting outputs have $1/n\ I(X:B_1B_2 \ldots B_n)$ differing by no more than $8 \varepsilon \log d + 4 H(\varepsilon)$ (the gap). Now, we can repeat earlier argument for classical capacity.

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Again, rewrite I^{coh} as entropies and apply lemma.

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Corollary 3: If $\| N_1 - N_2 \|_\diamond \leq \varepsilon$, then

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private classical capacity

Now apply it to N_1 = amplitude damping channel and closest generalized Pauli channel (N_2) we obtained an upper of $C(N_1) < 1+O(\varepsilon)$:p

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C_p: private classical capacity

If d is unbounded, there are

(a) families N_1^k, N_2^k s.t.

as $k \to \infty$, $\| N_1^k - N_2^k \|_\diamond \to 0$, but $|C(N_1^k)-C(N_2^k)|=1$

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The chat @IBM continued ...

Do you believe the proof?

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but now it's too simple to be not proved before.
The Alicki-Fannes inequality is 4 years old.
Are you sure it's not done already?

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At least not according to Werner's homepage on open problems ...

Prior work we found on the web:

Keyl & Werner 2002: Q(N) lower semicontinuous

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Incidentally, Alicki and Fannes 2004 proved their nice ineq to show continuity of squash entanglement.

Subsequently, Harrow, L, Shor used it to prove continuity of the one-way-[entanglement-assisted-classical-capacity] of quantum two-way channels.

The chat @IBM continued ...

Now what about Q_2 [quantum capacity assisted by free 2-way classical communication]?

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is continuous ... but doesn't mean anything here.

So, Guifre Vidal's proof idea for continuity of distillable entanglement adapts wholesale!!

Guifre's idea: if two distillable states ρ_1, ρ_2 are similar, n copies of one can be converted into $\approx$ n copies of the other with free LOCC.

If ρ_1 has much higher distillation entanglement than ρ_2, we can distill ρ_2 by first converting into ρ_1 leading to a contradiction.

Such conversion works for channels too!

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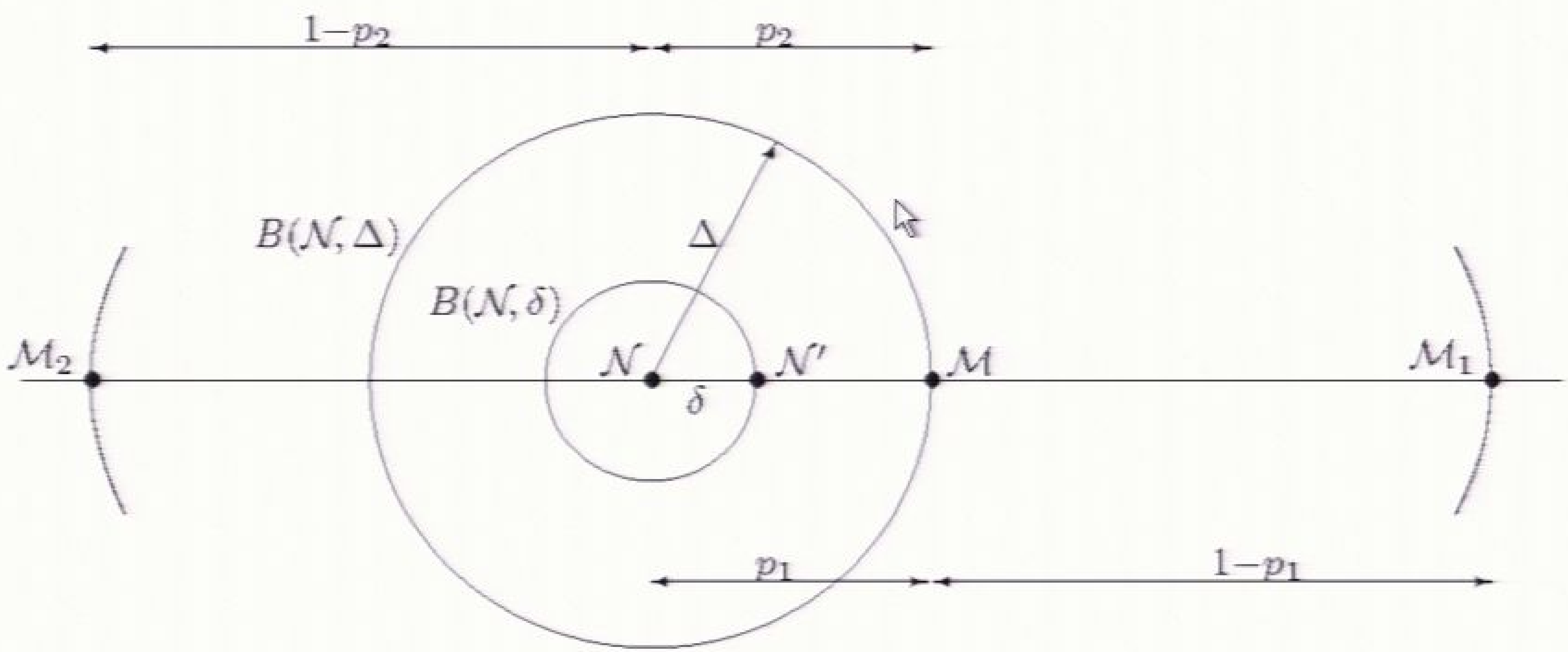

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Replace M by $N'=(1-q)N + qM$ for $q \to 0$

$p_1, p_2 \to q$ in the above argument:

$0 \leftarrow q \log d \geq |Q_2(N') - Q_2(N)|$

Same argument holds for $Q_B(N)$.
In particular, Q_B of the erasure channel is continuous in the erasure probability.

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Open question: is $Q_2(N)$ continuous on the boundary?