Title: Uncertainty Principle, Conservation Laws, and the Size of Quantum Controllers

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Abstract:

Osamu Hirota, A True Quantum Communications Channel

Perimeter Institute, June 26, 2008

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Nagoya University

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1 Measuring Process

$(\mathcal{K}, \sigma, U, M)$: a measuring process $\Leftrightarrow$

$$\begin{aligned}
\mathcal{K} &= \text{a Hilbert space} \\
\sigma &= \text{a density operator on } \mathcal{K} \\
U &= \text{a unitary operator on } \mathcal{H} \otimes \mathcal{K} \\
M &= \text{a self-adjoint operator on } \mathcal{K}
\end{aligned}$$

Define POVM: $\Pi(m) = \mathrm{Tr}_{\mathcal{K}}[U^{\dagger}(I \otimes E^{M}(m))U(I \otimes \sigma)]$.

Define TPCP map: $T\rho = \mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes \sigma)U^{\dagger}]$.

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2 Noise

$$\begin{aligned} \text{Noise operator: } N(A) &= U^\dagger(I \otimes M)U - (A \otimes I) \\ \text{Reduced noise operator: } n(A) &= \mathrm{Tr}[N(A)(I \otimes \sigma)] \\ \text{Root-mean-square noise: } \epsilon(A) &= \mathrm{Tr}[N(A)^2(\rho \otimes \sigma)]^{1/2} \end{aligned}$$

Theorem: $n(A)$ and $\epsilon(A)$ are determined by the POVM Π, i.e.,

$$\begin{aligned} n(A) &= O(\Pi) - A, \\ \epsilon(A) &= \mathrm{Tr}[(O^{(2)}(\Pi) - O(\Pi)A - AO(\Pi) + A^2)\rho], \end{aligned}$$

where $O(\Pi) = \sum m\Pi(m)$, $O^{(2)}(\Pi) = \sum m^2\Pi(m)$.

3 Disturbance

$$\text{Disturbance operator: } D(B) = U^{\dagger}(B \otimes I)U - (B \otimes I)$$

$$\text{Reduced disturbance operator: } d(B) = \mathrm{Tr}[D(B)(I \otimes \sigma)]$$

$$\text{Root-mean-square disturbance: } \eta(B) = \mathrm{Tr}[D(B)^2(\rho \otimes \sigma)]^{1/2}$$

Theorem: $d(B)$ and $\eta(B)$ are determined by the CPTP map T, i.e.,

$$d(B) = T^*(B) - B,$$

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4 Two Formulations of the Uncertainty Principle

A. The Heisenberg inequality for noise ϵ and disturbance η:

$$\epsilon(A)\eta(B) \geq \frac{1}{2}|\mathrm{Tr}[[A, B]\rho]|$$

B. The Kennard-Robertson inequality for standard deviation σ:

$$\sigma(A)\sigma(B) \geq \frac{1}{2}|\mathrm{Tr}[[A, B]\rho]|$$

5 Wigner-Araki-Yanase (WAY) Theorem:

Any observable which does not commute with an additively conserved quantity cannot be measured with absolute precision.

CAVEAT: The WAY theorem

1. does not conclude a superselection rule (unmeasurability of observables non-commuting with a conserved quantity).
2. merely sets the accuracy limit of the measurement with size limited apparatus in the presence of conserved quantities.

6 Yanase's Bound:

If a measuring process $(\mathcal{K}, \sigma, U, M)$ for $\mathcal{H} = \mathbf{C}^2$ satisfies

(i) $[U, S_x + L_x] = 0$,

(ii) $[M, L_x] = 0$,

then we have

$$\max_{\rho} \epsilon(S_z)^2 \geq\sim \frac{\hbar^4}{16\langle L_x^2 \rangle},$$

where S_j is the j-component of the object spin and L_j is the j-component of the angular momentum of the apparatus.

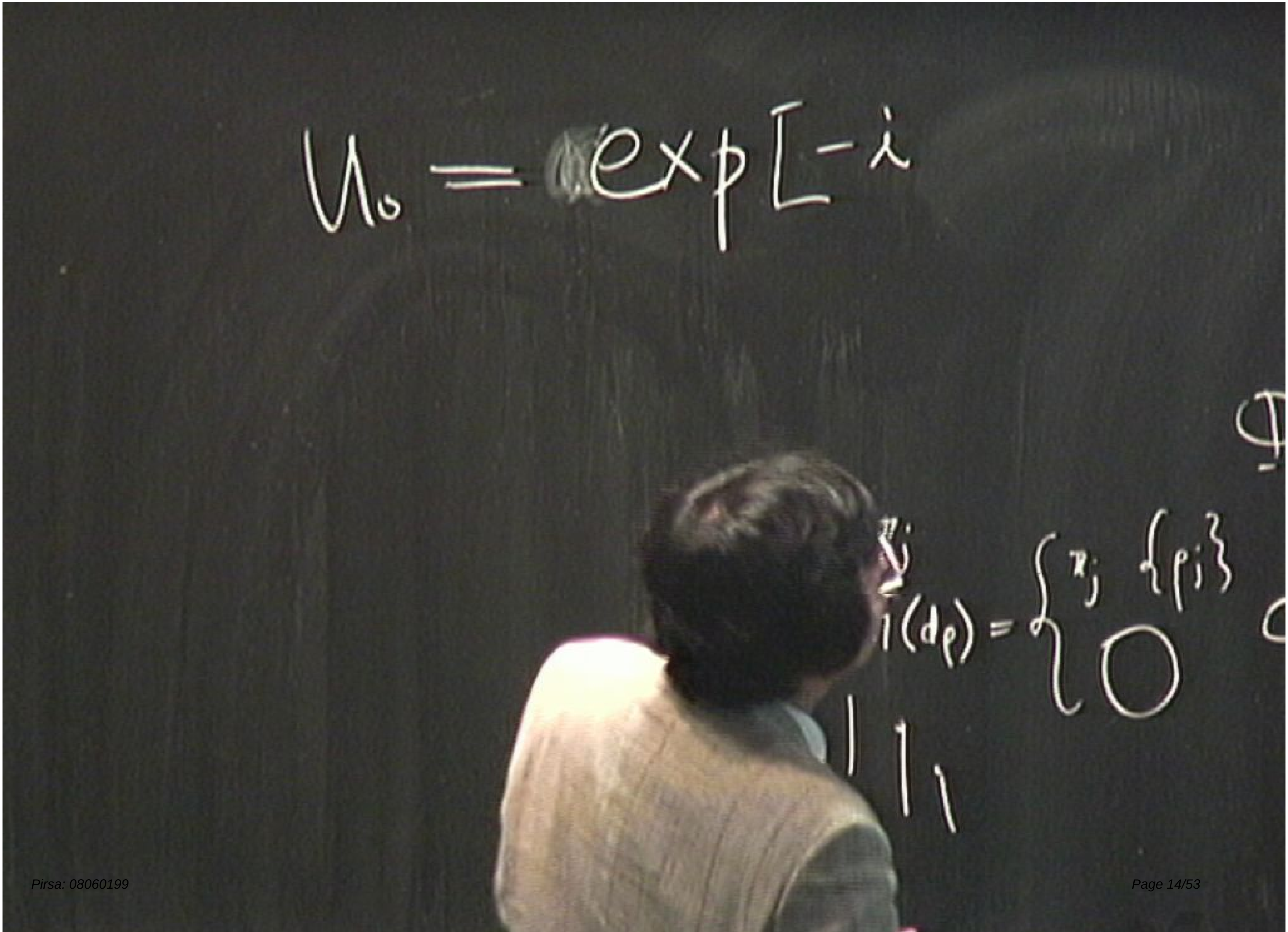
7 WAY Theorem vs Uncertainty Principle

Let $(\mathcal{K}_0, \sigma_0, U_0, M_0)$ be a Lüders measurement of M on $\mathcal{K}$. We can assume $[U_0, L_x \otimes I] = 0$. Then,

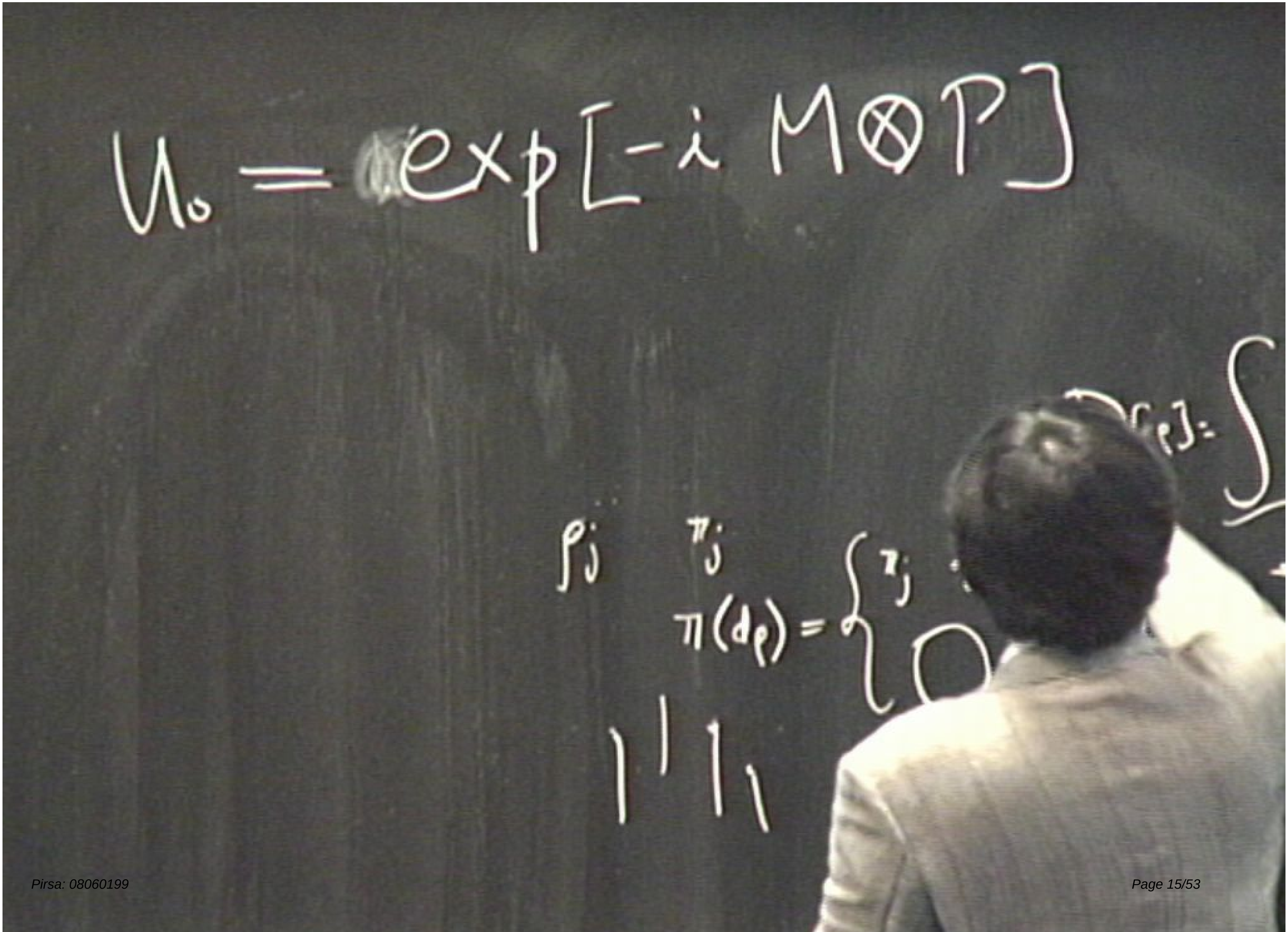
$$(\mathcal{K}_0, \sigma_0, (I \otimes U_0)(U \otimes I), M_0)$$

is a measuring process for $\mathcal{H} \otimes \mathcal{K}$ with the same output as $(\mathcal{K}, \sigma, U, M)$, and we have $\eta(S_x + L_x) = 0$.

We shall show the Heisenberg inequality contradicts Yanase's bound.



$U_0 = \exp[-i$


U0 = exp[-i M⊗P]
ρj πj
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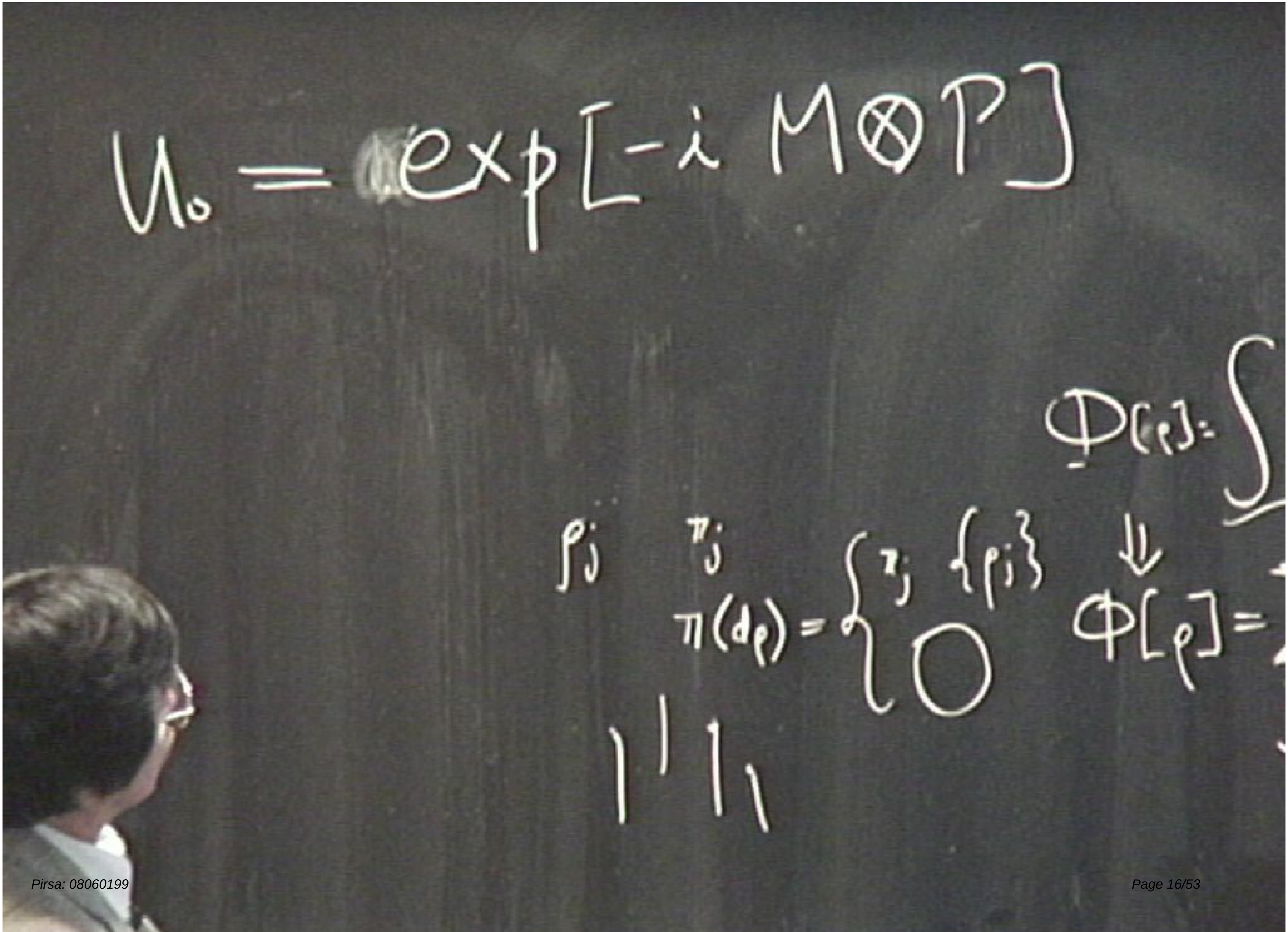
0


$U_0 = \exp[-i\, M \otimes P]$
$\Phi[\rho] := \int$
ρ_j π_j
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$\Downarrow$
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Suppose Heisenberg's inequality holds. $\Rightarrow$

$$\epsilon(S_z)\eta(S_x + L_x) \geq \frac{|\langle [S_z, S_x] \rangle|}{2} = \frac{\hbar^2}{4},$$

for the S_y eigenstate. $\Rightarrow$

$$\eta(S_x + L_x) = 0 \quad \text{implies} \quad \epsilon(S_z) = \infty$$

$\Rightarrow$ This contradicts Yanase's bound

$$\max_\rho \epsilon(S_z)^2 \sim \frac{\hbar^2}{16\langle L_x^2 \rangle}$$

if $\langle L_x^2 \rangle$ is large.

8 What Is the Correct Noise-Disturbance Relation?

Theorem: For every measuring process, we have

$$\begin{aligned} & \epsilon(A)\eta(B) + \epsilon(A)\,\sigma(B) + \sigma(A)\,\eta(B) \\ \geq\ & \epsilon(A)\eta(B) + \frac{1}{2}|\langle[n(A), B]\rangle| + \frac{1}{2}|\langle[A, d(B)]\rangle| \\ \geq\ & \frac{1}{2}|\langle[A, B]\rangle|. \end{aligned}$$

If the noise and disturbance are statistically independent from the object, i.e., $n(A)$ and $d(B)$ are constant operator, then we have

$$\epsilon(A)\eta(B) \geq \frac{1}{2}|\langle[A, B]\rangle|.$$

9 Measurements beyond Heisenberg's inequality

1. (Type I violations) The case where $\eta(B) = 0$:

$$\epsilon(A)\,\sigma(B) \geq \frac{1}{2}|\langle[A, B]\rangle|$$

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10 Quantitative Generalization of WAY Theorem

Theorem: If a measuring process $(\mathcal{K}, \sigma, U, M)$ for $\mathcal{H}$ satisfies

(i) $[U, L_1 + L_2] = 0$,

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then we have

$$\epsilon(A)^2 \geq \frac{|\langle [A, L_1] \rangle|^2}{4\sigma(L_1)^2 + 4\sigma(L_2)^2}.$$

$$L_1 = \tilde{L}_1 \otimes I, \quad L_2 = I \otimes \tilde{L}_2$$

$$U_0 = \exp[-i\, M \otimes P]$$

$$\Rightarrow [U_0, M] = 0$$

$$[U_0, L_x] = 0$$

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11 Proof

From $[U, L_1 + L_2] = 0$ and $[M, L_2] = 0$, we have $\eta(L_1 + L_2) = 0$. The type I violation formula implies

$$\epsilon(A)^2 \sigma(L_1 + L_2)^2 \geq \frac{1}{4} |\langle [A, L_1 + L_2] \rangle|^2$$

Here, $[A, L_1 + L_2] = [A, L_1]$ and $\sigma(L_1 + L_2)^2 = \sigma(L_1)^2 + \sigma(L_2)^2$, so that we have

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QED

12 Spin measurement

Let $A = S_z,\ L_1 = S_x$, and $L_2 = L_x$. We have

$$\epsilon(S_z)^2 \geq \frac{|\langle [S_z, S_x] \rangle|^2}{4\sigma(S_x)^2 + 4\sigma(L_x)^2}.$$

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Thus, we have

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13 Implementations of Quantum Gates

Let G be a quantum gate on $\mathcal{H} = \mathbf{C}^n$. An implementation of G is a triple $\alpha = (\mathcal{K}, U, \xi)$ such that

(i) $\mathcal{K}$ is a Hilbert space,

(ii) U is a unitary operator on $\mathcal{H} \otimes \mathcal{K}$,

(iii) ξ is a unit vector in $\mathcal{K}$.

An implemetation $\alpha = (\mathcal{K}, U, \xi)$ defines an operation $\mathcal{E}_\alpha$ by

$$\mathcal{E}_\alpha(\rho) = \mathrm{Tr}_{\mathcal{K}}[U^\dagger(\rho \otimes |\alpha\rangle\langle\alpha|)U].$$

The gate fidelity of α is defined by

$$F(\mathcal{E}_\alpha, G) = \inf_\psi \langle\psi|G^\dagger \mathcal{E}_\alpha(|\psi\rangle\langle\psi|)G|\psi\rangle.$$

The error probability of α is defined by

$$P_e = 1 - F(\mathcal{E}_\alpha, G)^2.$$

The conserved quantitiy of α is $L_1 \otimes I + I \otimes L_2$ such that

$$[U, L_1 \otimes I + I \otimes L_2] = 0.$$

14 Limitation on Quantum State Control due to Conservation Laws

Theorem: The error probability of any implementation of the Hadamard gate H on a spin qubit satisfying the angular momentum conservation law with n qubit separable controller is no less than $1/4(n+1)$ and the one with entangled controller is no less than $1/4(n^2+1)$.

Remark: Any implementations of self-adjoint gates (Hadamard gate, NOT gate), CNOT gate, Fredkin gate, and Toffoli gate have similar error probability. However, SWAP gate has no contraint.

Proof.

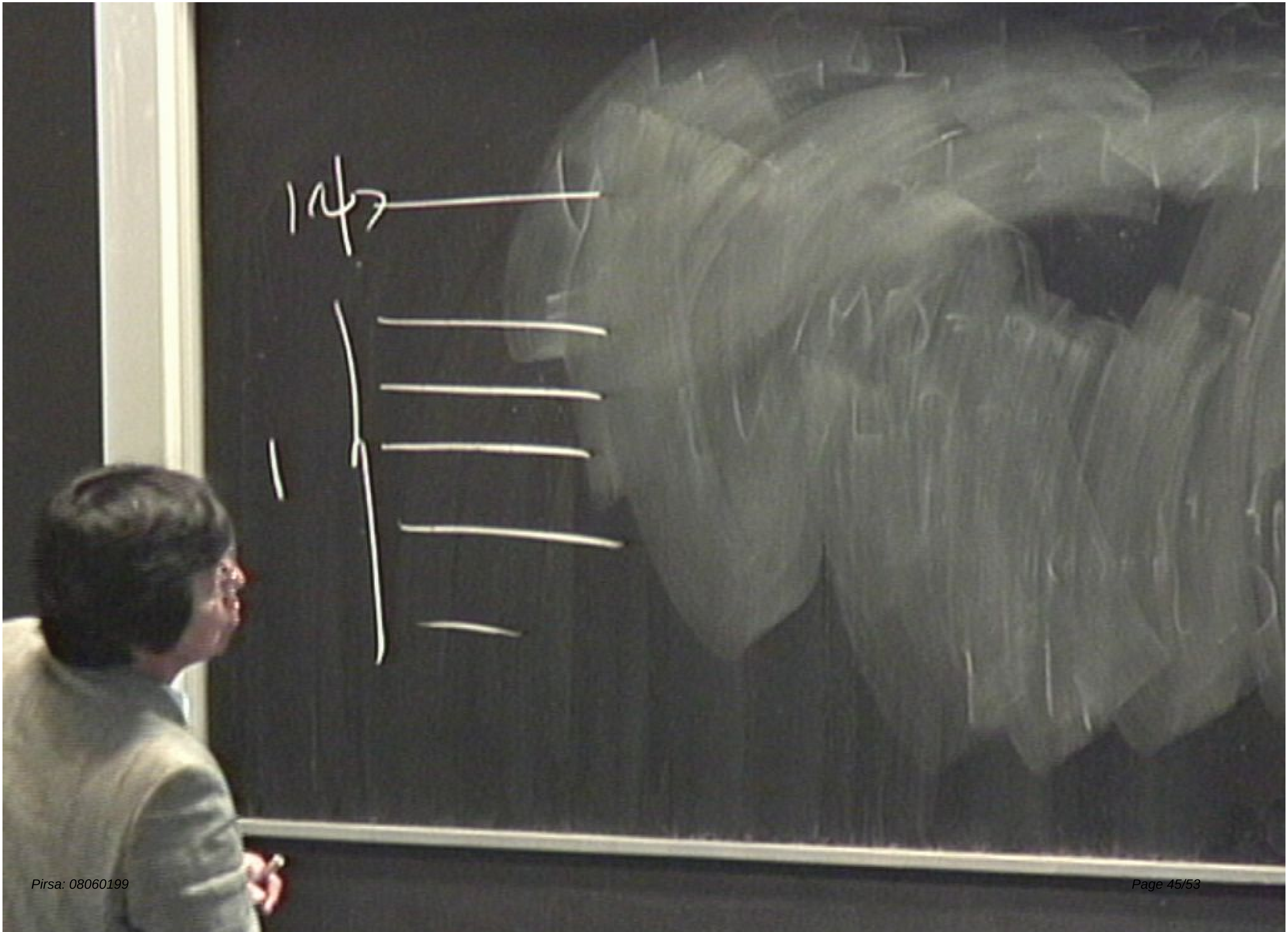
Let $\alpha = (\mathcal{K}, U, \xi)$ be an implementation of the Hadamard gate satisfying the angular momentum conservation law with n qubit controller. Consider the measuring process $(\mathcal{K}, U, \xi, X)$:

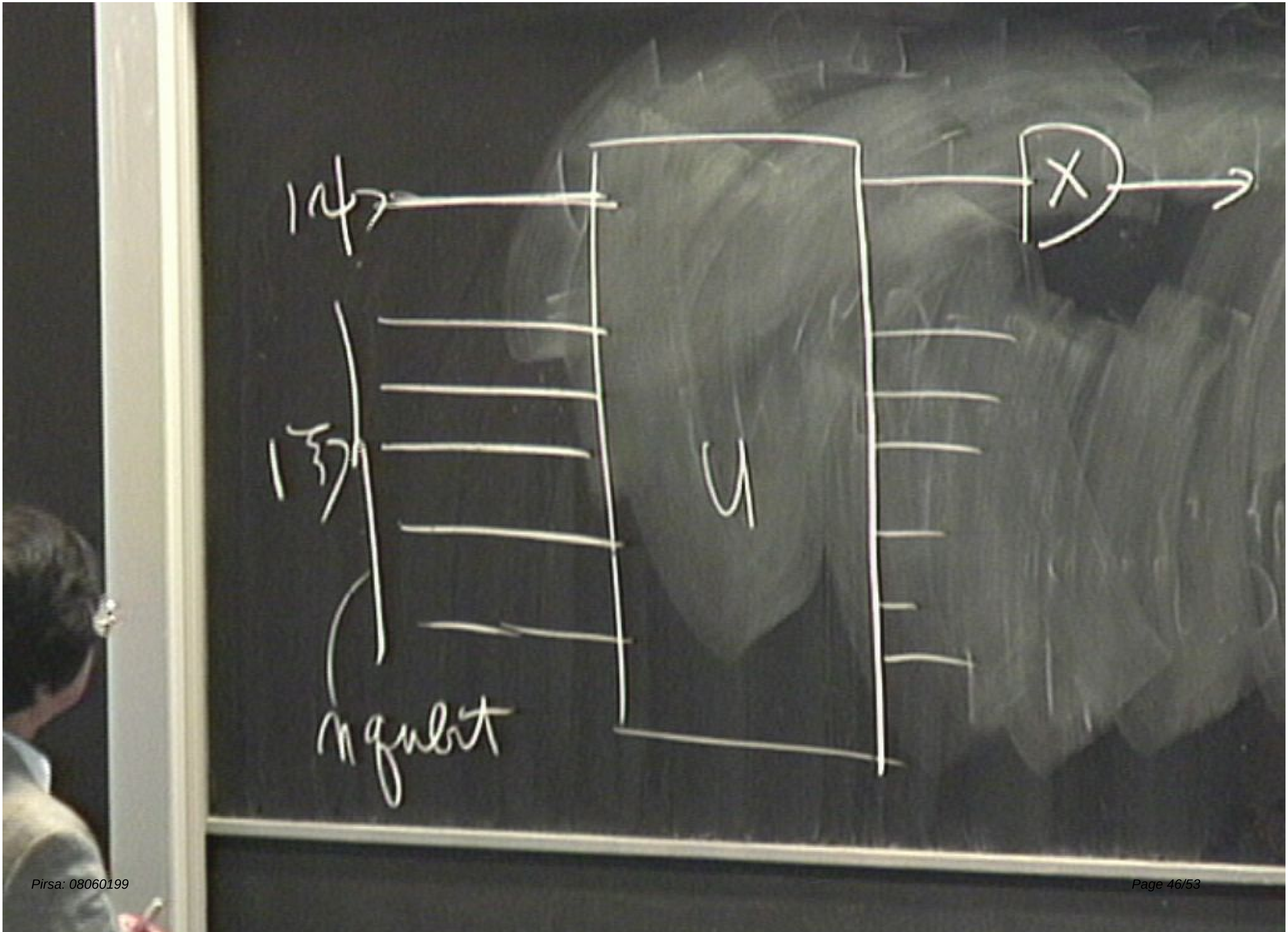
$|\psi, \xi\rangle \longrightarrow U \longrightarrow$ (measurement of $X \otimes I \otimes \cdots \otimes I$) $\longrightarrow$

This can be regarded as an approximate measurement of $Z = H^{-1}XH$ with $[U, X + X_1 + \cdots + X_n] = 0$ and $[M, X] = 0$ with $M = X$. Thus, we have

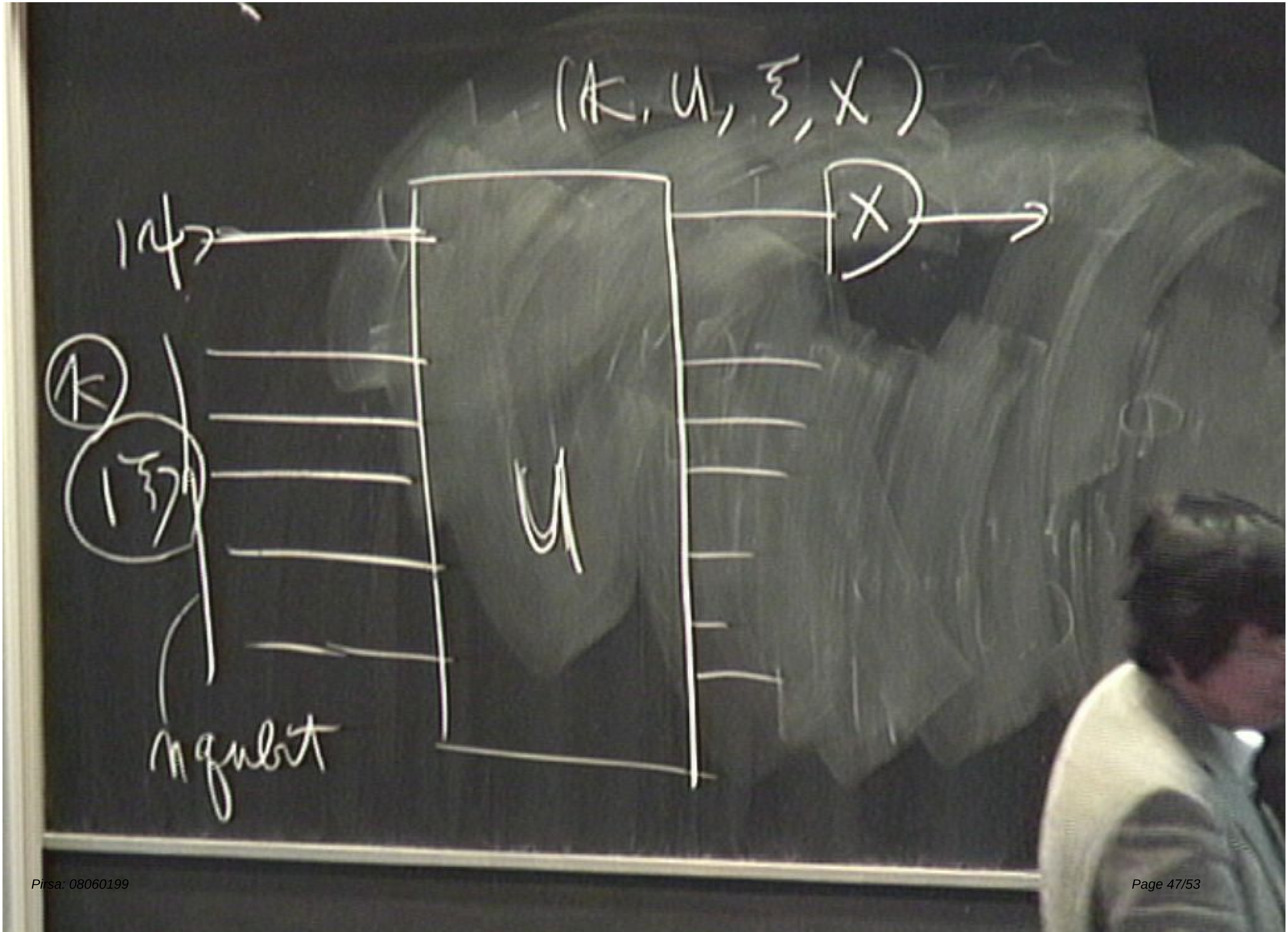
$$\epsilon(Z)^2 \geq \frac{|\langle\psi|[Z, X]|\psi\rangle|^2}{4(\Delta X)^2 + 4\Delta(X_1 + \cdots + X_n)^2}.$$

For $\psi = |Y = 1\rangle$, we have $|\langle\psi|[Z, X]|\psi\rangle| = 2$, and the error probability P_e satisfies





|ψ⟩
|ξ⟩
n qubit
U
X



(K, U, ξ, X)
|ψ⟩
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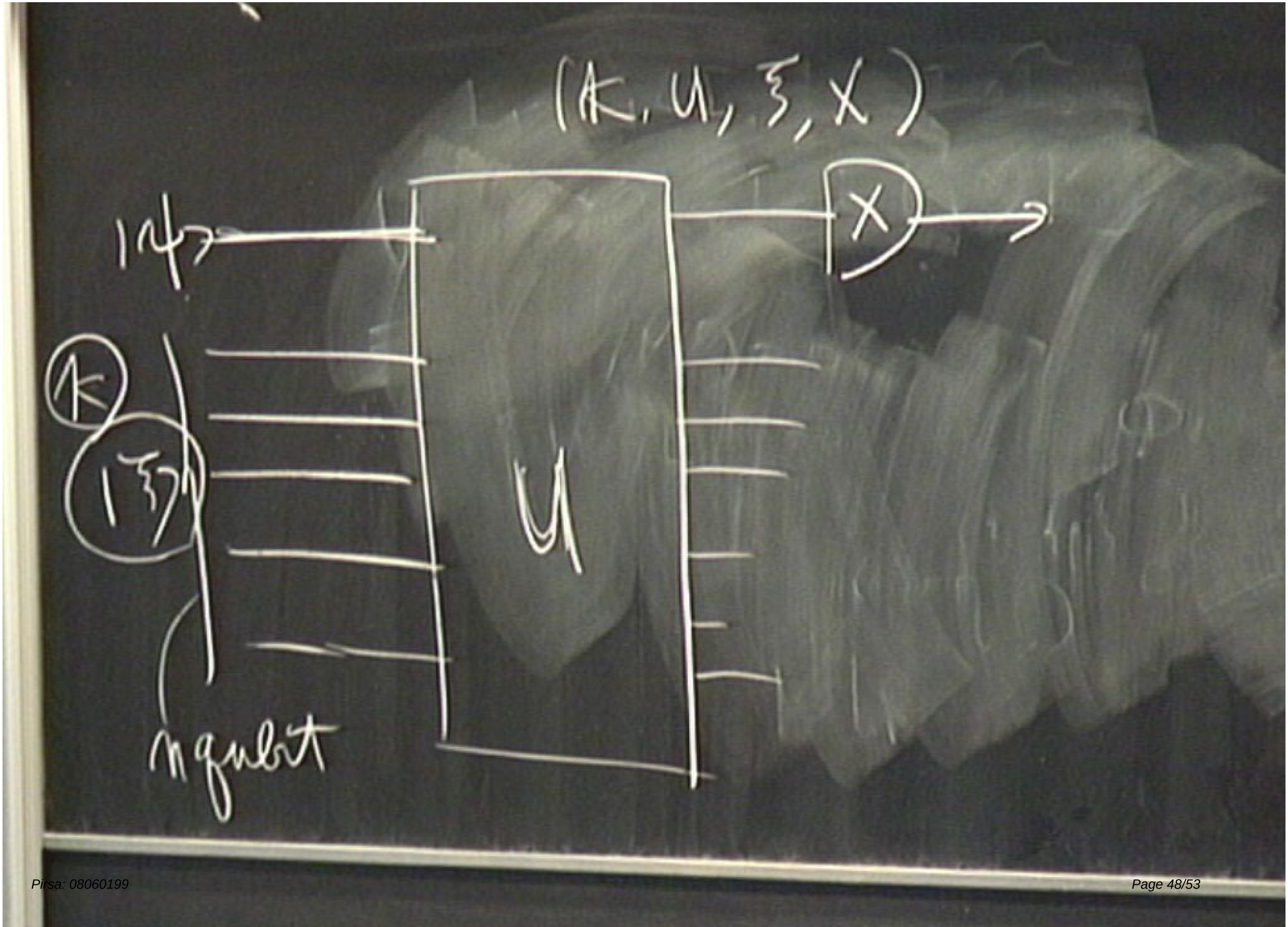
U
X
n qubit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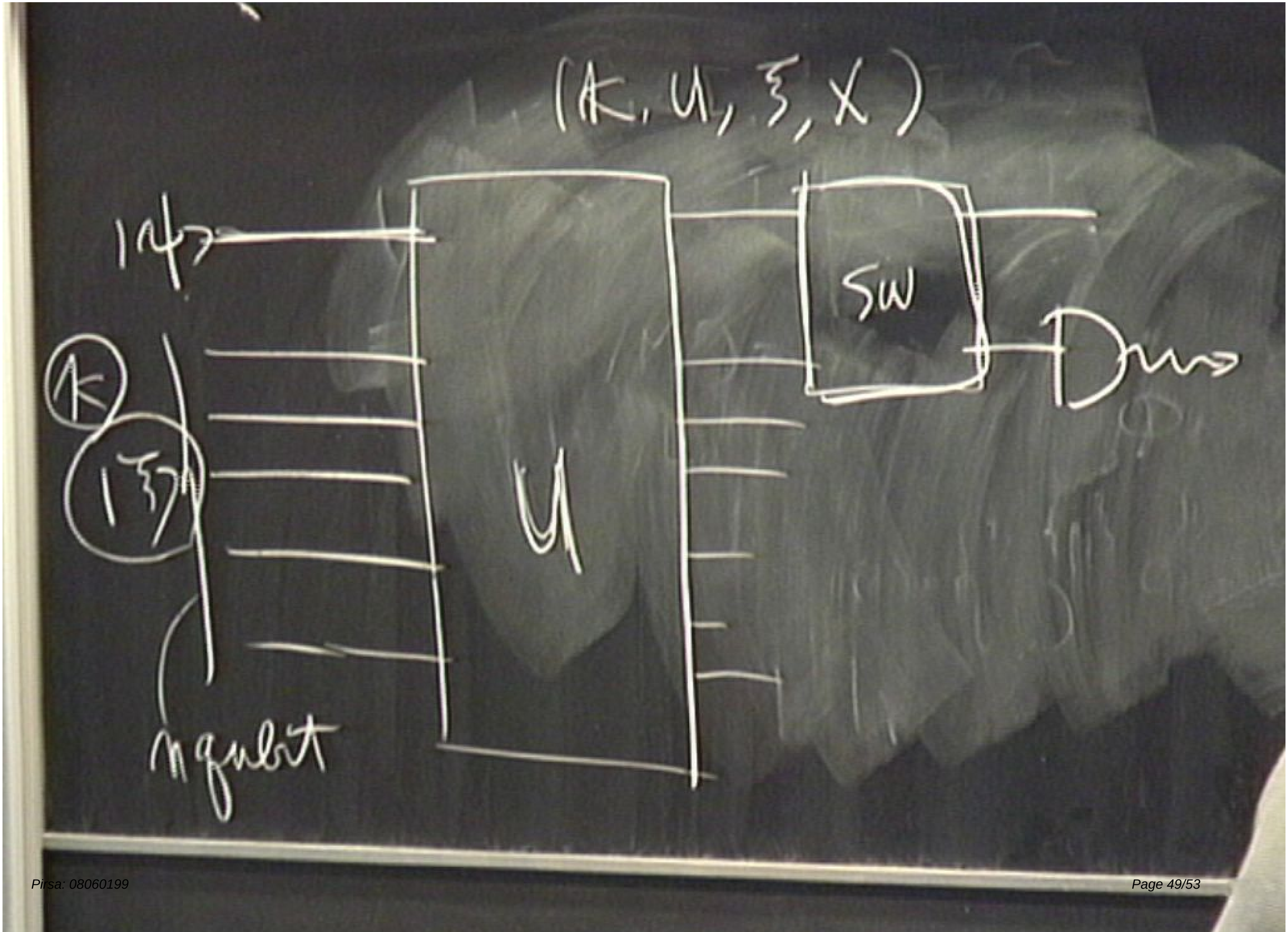
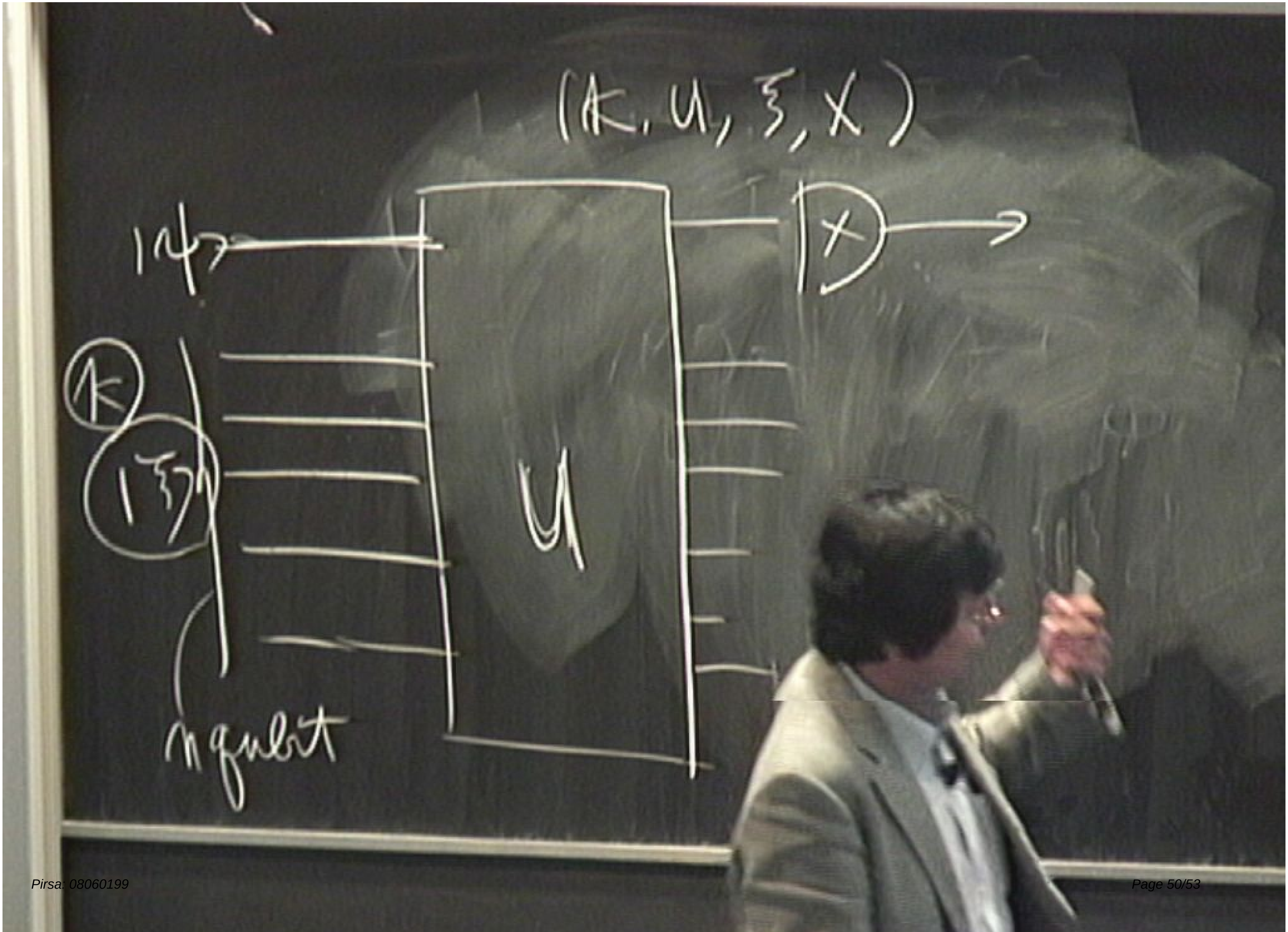

(K, U, ξ, X)
|ψ⟩
K
|ξ⟩
n qubit
U
SW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$$P_e \geq \frac{1}{4(n+1)}.$$

— A realization of 1-qubit Hadamard needs a 100-qubit entangled ancilla or 10000-qubit separable ancilla to clear $P_e \sim 10^{-5}$.

However, you can encode the logical qubit into 3 physical qubits to make rotationally invariant interactions universal [DiVincenzo, Bacon, Kempe, Burkard, and Whaley Nature 408 (2000) 339].

Thus, the proper choice of encoding is an important problem.

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