Title: Entanglement-Breaking Channels in Infinite Dimensions

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Abstract:

ENTANGLEMENT - BREAKING CHAHHELS

in ∞-dimensions

quant-ph/0802.0235

A. S. Holevo
Steklov Mathematical Institute

1. Definition and properties of EB-channels
2. Structural theorems for EB-channels
3. Gaussian EB-channels
4. One mode

Prelude: finite dimensions

(Horodecki – Shor - Ruskai : quant-ph/0302031)

$$\Phi[\rho] = \sum_x \rho_B^x \operatorname{Tr} \rho M_A^x$$

M_A^x – POVM = observable

$$= \sum_y |b_y\rangle\langle a_y|\rho|a_y\rangle\langle b_y|$$

- Fulfill additivity conjecture, $C(\Phi) = C_\chi(\Phi)$
- Anti-degradable, $Q(\Phi) = 0$

$$(\Phi \otimes Id_R)[\rho_{AR}] = \sum_j \pi_j \rho_B^j \otimes \rho_R^j \quad \text{– separable}$$

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$$(\Phi \otimes Id_R)[\rho_{AR}] = \sum_j \pi_j \rho_B^j \otimes \rho_R^j$$ – separable

$\mathcal{H}$ – separable Hilbert space, $\dim \mathcal{H} \leq \infty$

$\mathfrak{S}(\mathcal{H})$ – density operators
(complete separable metric space)

$\mathcal{T}(\mathcal{H})$ – trace class operators

$\mathcal{L}(\mathcal{H})$ – bounded operators

Def $\rho \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ separable $\Leftrightarrow$
$\rho \in$ conv. closure of product states.

Holevo – Shirokov – Werner : quant-ph/0504204

Thm ρ – separable $\Leftrightarrow \rho = \int_{\mathcal{X}} \rho_1(x) \otimes \rho_2(x) \pi(dx)$

There are separable states $\neq \sum_j \pi_j \rho_1^j \otimes \rho_2^j$

Def Channel $A \to B$ $\Leftrightarrow$
bounded linear cptp map $\Phi : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$

EB-channel $\Leftrightarrow (\Phi \otimes Id_R)[\rho_{AR}]$ separable $\forall \rho_{AR}$

There are EB-channels which has no representation:

$$\sum_y |\psi_B^y\rangle\langle\varphi_A^y| \rho |\varphi_A^y\rangle\langle\psi_B^y|$$

„ Kraus rank-1 decomposition "

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1. EB-channels fulfill the strongest form of the additivity conjecture:

$$C_\chi(\Phi_1 \otimes \Phi_2, \mathcal{A}_1 \otimes \mathcal{A}_2) = C_\chi(\Phi_1, \mathcal{A}_1) + C_\chi(\Phi_2, \mathcal{A}_2)$$

$$\forall \Phi_2; \mathcal{A}_1, \mathcal{A}_2 \subset \mathfrak{S}(\mathcal{H})$$

$$C_\chi(\Phi, \mathcal{A}) = \sup_{\pi:\, \bar{\rho}_\pi \in \mathcal{A}} \underbrace{\int_{\mathfrak{S}(\mathcal{H})} H(\Phi[\rho]; \Phi[\bar{\rho}_\pi])\, \pi(d\rho)}$$

generalized ensemble, $\bar{\rho}_\pi = \int_{\mathfrak{S}(\mathcal{H})} \rho\, \pi(d\rho)$

$$H(\Phi[\bar{\rho}_\pi]) - \int H(\Phi[\rho])\, \pi(d\rho) \text{ —}$$

requires finiteness of the entropy

Energy constraint:

$$\mathcal{A} = \{\rho : \operatorname{Tr} \rho H \leq E\}$$

Constrained classical capacity:

$$C(\Phi, H, E) = C_\chi(\Phi, H, E) \text{ — additive}$$

2. EB-channels are anti-degradable

requires Kraus rank-1 decomposition!

Thm 1 (HSW) Φ is EB-channel $\Leftrightarrow$

$$\Phi[\rho] = \int_{\mathcal{X}} \underbrace{\rho_B(x)}_{\text{preparation}} \, \underbrace{m_\rho(dx)}_{\text{measurement}}$$

$$m_\rho(S) = \mathrm{Tr}\, \rho\, \underbrace{M_A(S)}_{\text{POVM = observable in } \mathcal{H}_A}; \quad S \subset \mathcal{X}$$

c-q channels $x \to \rho_B(x) \Leftrightarrow M_A$ – sharp observable (projection-valued)

no q-c channels $\Rightarrow$ observables $\rho \to m_\rho(dx)$

* * *

Generalized Kraus rank-1 decomposition.

Let $\mathcal{D} \subset \mathcal{H}$ – dense domain

$a: \psi \to \langle a|\psi\rangle$ – linear function on $\mathcal{D}$

Example: $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{D} = C(\mathbb{R}) \cap L^2(\mathbb{R})$

$\langle a|\psi\rangle = \psi(0)$ (Dirac's delta)

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M_A — POVM observable in $\mathcal{H}_A$

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$\uparrow$ POVM = [illegible]servable in $\mathcal{H}_A$

c-q channels $x \to \rho_B(x)$ [illegible] M_A – sharp observable (projection-valued)

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<u>Generalized</u> Krau[illegible] rank-1 decomposition.

Let $\mathcal{D} \subset$ [illegible] – dense domain

$a:\ \psi \to$ [illegible]$\psi\rangle$ – linear function on $\mathcal{D}$

<u>Example:</u> $\mathcal{H}$ [illegible] $L^2(\mathbb{R})$, $\mathcal{D} = C(\mathbb{R}) \cap L^2(\mathbb{R})$

[illegible]$|\psi\rangle = \psi(0)$ (Dirac's delta)

<u>Thm 2</u> [illegible] is EB-channel $\Leftrightarrow$

$\exists$ [illegible], ν, $\mathcal{D} \subset \mathcal{H}_A$, $a(y): \mathcal{Y} \to \mathcal{D}'$, $b(y): \mathcal{Y} \to \mathcal{H}_B^1$

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Ex[illegible]ple: $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{D} = C(\mathbb{R}) \cap L^2(\mathbb{R})$

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* * *

<u>Generalized</u> Kraus rank-1 decomposition.

Let $\mathcal{D} \subset \mathcal{H}$ – dense domain

$a: \psi \to \langle a|\psi\rangle$ – linear function on $\mathcal{D}$

<u>Example:</u> $\mathcal{H} = L^2(\mathbb{R})$, $\mathcal{D} = C(\mathbb{R}) \cap L^2(\mathbb{R})$

$\langle a|\psi\rangle = \psi(0)$ (Dirac's delta)

<u>Thm 2</u> Φ is EB-channel $\Leftrightarrow$

$\exists\, \mathcal{Y}, \nu, \mathcal{D} \subset \mathcal{H}_A$, $a(y): \mathcal{Y} \to \mathcal{D}'$, $b(y): \mathcal{Y} \to \mathcal{H}_B$

$\|b(y)\| \equiv 1$

$$\Phi[\rho] = \int_{\mathcal{Y}} |b(y)\rangle\langle b(y)|\, |\langle a(y)|\psi\rangle|^2 \nu(dy)$$

for $\rho = |\psi\rangle\langle\psi|$, $\psi \in \mathcal{D}$

<u>Cor</u> -channel $\Rightarrow$ anti-deg.: $\Phi = T \circ \tilde{\Phi}$

Thm 1 (HSW) Φ is EB-channel $\Leftrightarrow$

$$\Phi[\rho] = \int_{\mathcal{X}} \underbrace{\rho_B(x)}_{\text{preparation}} \underbrace{m_\rho(dx)}_{\text{measurement}}$$

$$m_\rho(S) = \mathrm{Tr}\,\rho\, M_A(S); \quad S \subset \mathcal{X}$$

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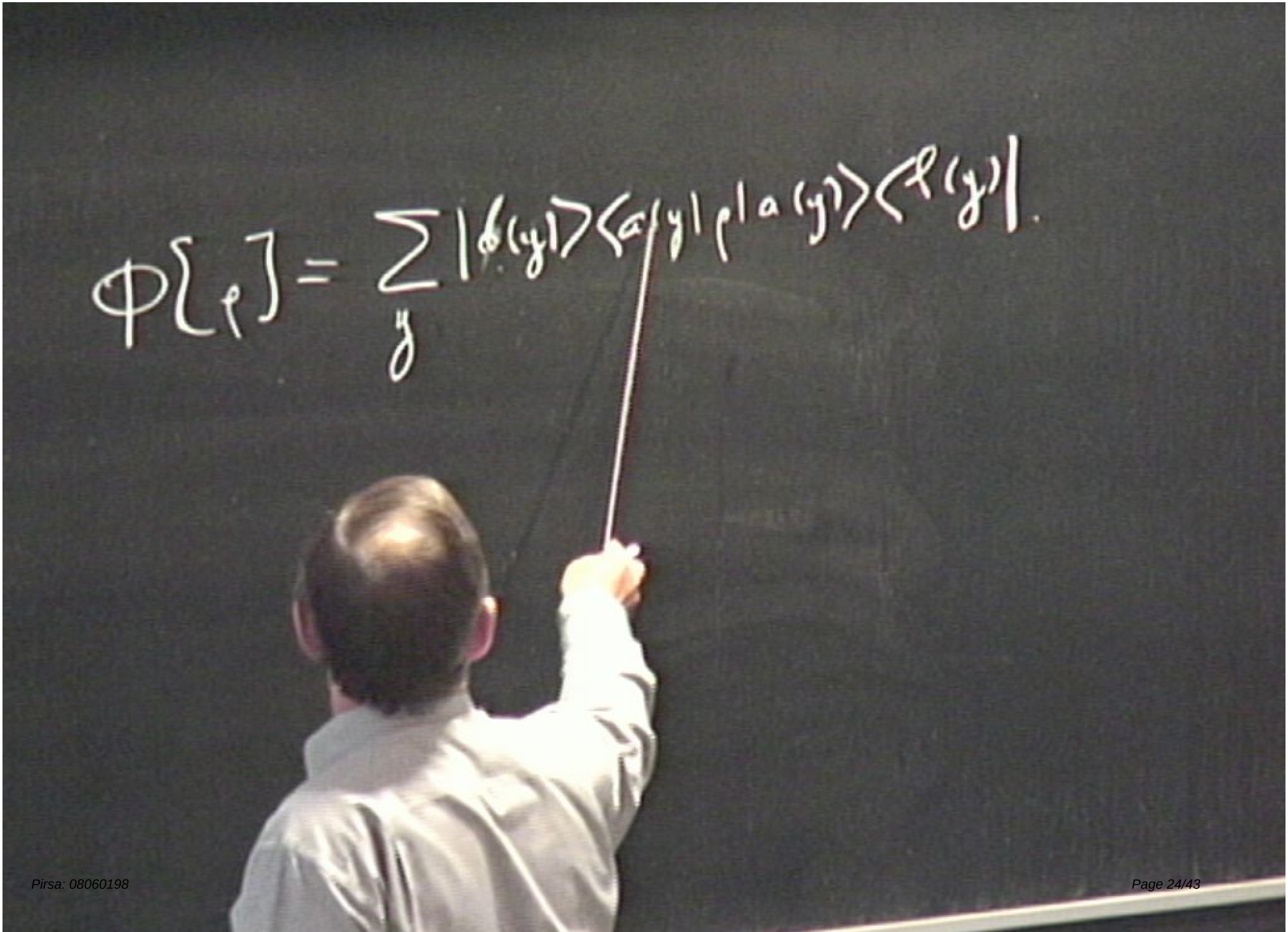
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Gaussian channels

Bosonic open system:

$$\Phi: \left.\begin{matrix}\rho_D \\ \otimes \\ \rho_A\end{matrix}\right\} \xrightarrow{U_T} \left\{\begin{matrix}\rho_E' \\ \\ \rho_B'\end{matrix}\right.$$

$$Z_A \oplus Z_D \xrightarrow{T} Z_B \oplus Z_E$$

$R = (q, p)$ – canonical observables

$$\begin{cases} R_B' = U_T R_B U_T^* = R_A K + R_D K_D \\ R_E' = U_T R_E U_T^* = R_A L + R_D L_D \end{cases}$$

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Weyl operators $W(z) = e^{iR\cdot z}, \; z \in Z$

Linear Bosonic channel Φ:

$$\begin{aligned}\phi_B(z_B) &\equiv \mathrm{Tr}\, \Phi[\rho_A] W(z_B) \\ &= \mathrm{Tr}\,(\rho_A \otimes \rho_D)\, e^{i(R_A K + R_D K_D)\cdot z_B} \\ &= \phi_A(K z_B) \cdot \underbrace{\phi_D(K_D z_B)}_{f(z_B)}\end{aligned}$$

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$$\Phi^*[W(z_B)] = W(K z_B)\, f(z_B)$$

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$$\Phi^*[W(z_B)] = W(Kz_B)\exp\left(i\ell z_B - \frac{1}{2} z_B^T \alpha z_B\right)$$

(K, ℓ, α) – parameters (w.l.o.g. $\ell = 0$)

$$\boxed{\alpha \geq \frac{i}{2}\left(\Delta_B - K^T \Delta_A K\right)} \quad (*)$$

$\Delta = \mathrm{diag}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ – commutation matrix

Thm 3 $(*) \Leftrightarrow$ Gaussian channel $\Leftrightarrow$ Gaussian open system

(Caruso–Giovannetti–Eisert–Holevo: q-p/0804.0511)

Thm 4 Gaussian channel Φ is EB $\Leftrightarrow$

$\alpha = \nu + \mu$, where $\nu \geq \frac{i}{2}\Delta_B$

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σ_B – Gaussian state $(0, \nu)$

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Gaussian observable

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Gaussian observable

$$\int e^{iw^T\Delta z} M_A(dz) = \exp\left(iR_A K w - \tfrac{1}{2} w^T \mu w\right)$$

sharp observable $\Leftrightarrow$ $\mu = 0$ (c-q channel)

One mode (quant-ph/0607051)

$$z=(x,y)\,,\quad W(z)=e^{i(xq+yp)}\,,\quad \Delta=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

Canonical form of Gaussian channel:

$$\Phi'[\rho]=U_{T_2}\,\Phi[U_{T_1}\rho\,U_{T_1}^{*}]\,U_{T_2}^{*}$$

Five classes: $\left(I=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ k>0\right)$

A) $K=\begin{bmatrix}k&0\\0&0\end{bmatrix}$, $\alpha=(N+\frac{1}{2})I$

$q_B=kq_A+q_D$, $p_B=p_D$

B_1) $K=I$, $\alpha=\begin{bmatrix}0&0\\0&1/2\end{bmatrix}$

$q_B=q_A$, $p_B=p_A+\eta$

B_2) $K=I$, $\alpha=N\cdot I$

additive classical noise

$q_B=q_A+\xi$, $p_B=p_A+\eta$

C) $K=k\cdot I\ (k\neq 1)$, $\alpha=\left(N+\frac{|1-k^2|}{2}\right)I$

attenuator/amplifier

$$\begin{cases}q_B=kq_A+\sqrt{1-k^2}\,q_D\\ p_B=kp_B+\sqrt{1-k^2}\,p_D\end{cases}\quad k<1$$

$\cdots\cdots\cdots\cdots$ $k>1$

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B$_1$) $K = I, \quad \alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$

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$q_B = k q_A + q_D,\quad p_B = p_D$ } EB, in fact c–q

B$_1$) $K = I,\quad \alpha = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$

$q_B = q_A,\quad p_B = p_A + \eta$ } non–EB

B$_2$) $K = I,\quad \alpha = N \cdot I$

additive classical noise

$q_B = q_A + \xi,\quad p_B = p_A + \eta$

C) $K = k \cdot I \ (k \neq 1),\quad \alpha = (N + \frac{|1-k^2|}{2}) I$

attenuator/amplifier

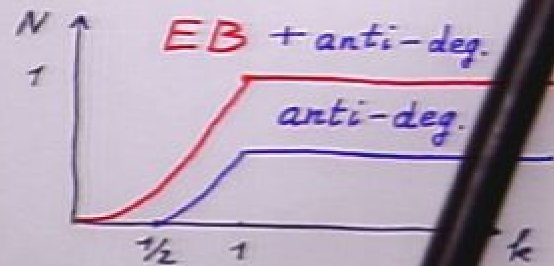
$$\begin{cases} q_B = k q_A + \sqrt{1-k^2}\, q_D \\ p_B = k p_B + \sqrt{1-k^2}\, p_D \end{cases} \quad k < 1$$

. . . . $k > 1$

D) $K = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix},\quad \alpha = (N + \frac{1+k^2}{2}) I$

$$\begin{cases} q_B = k q_A + \sqrt{1+k^2}\, q_D \\ p_B = -k p_A + \sqrt{1+k^2}\, p_D \end{cases}$$ } EB

B_2, C) EB $\Leftrightarrow$ $N \geq \min(1, k^2)$



Anti-deg.: Caruso-Giova[illegible]tti-Holevo: q-p/0609013

Constrained classical [illegible]acity

$H = a^\dagger a = \frac{1}{2}(q^2 + [illegible] - 1)$

$C(\Phi, H, E) = C_\chi([illegible], H, E) \overset{?}{=} C_{Gauss}(\Phi, H, E)$

a[illegible]tive ?

$C_G(\Phi, H, E) = [illegible]{}^2 E + N_0) - g(N_0)$

$N_0 = \begin{cases} [illegible] - 1)_+ + N, & \text{case } B_2, C \\ [illegible]{}^2 + N, & \text{case } D \end{cases}$

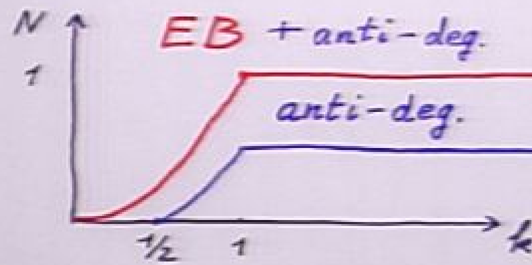
$g(x) = (x + [illegible] \log(x+1) - x \log x$

$C_\chi = C_{Gauss}$ [illegible] proved for pure-loss channel

[illegible]ase C with $N = 0, k < 1$)

Giovannett[illegible] Guha, Lloyd, Maccone, Shapiro, Yuen:
quant-p[illegible]308012

B_2, C) EB $\Leftrightarrow$ $N \geq \min(1, k^2)$



Anti-deg.: Caruso-Giovannetti-Holevo: q-p/0609013

Constrained classical capacity

$$H = a^\dagger a = \frac{1}{2}(q^2 + p^2 - 1)$$

$$C(\Phi, H, E) = \underbrace{C_\chi(\Phi, H, E)}_{\text{additive ?}} \overset{?}{=} C_{Gauss}(\Phi, H, E)$$

$$C_G(\Phi, H, E) = g(k^2 E + N_0) - g(N_0)$$

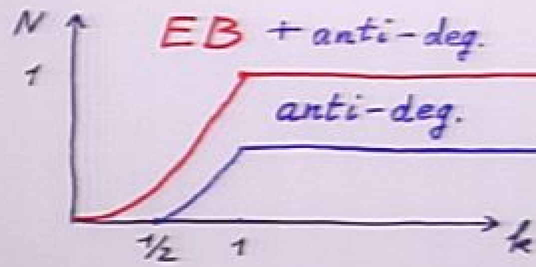
$$N_0 = \begin{cases} (k^2-1)_+ + N, & \text{case } B_2, C \\ k^2 + N, & \text{case } D \end{cases}$$

$$g(x) = (x+1)\log(x+1) - x \log x$$

$C_\chi = C_{Gauss}$ is proved for pure-loss channel (case C with $N = 0$, $k < 1$)

Giovannetti, Guha, Lloyd, Maccone, Shapiro, Yuen: quant-ph/0308012

$B_2, C)$ EB $\Leftrightarrow N \geq \min(1, k^2)$



Anti-deg.: Caruso-Giovannetti-Holevo: q-p/0609013

<u>Constrained</u> <u>classical</u> <u>capacity</u>

$$H = a^\dagger a = \tfrac{1}{2}(q^2 + p^2 - 1)$$

$$C(\Phi, H, E) = \underbrace{C_\chi(\Phi, H, E)}_{\text{additive?}} \stackrel{?}{=} C_{Gauss}(\Phi, H, E)$$

$$C_G(\Phi, H, E) = g(k^2 E + N_0) - g(N_0)$$

$$N_0 = \begin{cases} (k^2 - 1)_+ + N, & \text{case } B_2, C \\ k^2 + N, & \text{case } D \end{cases}$$

$$g(x) = (x+1)\log(x+1) - x\log x$$

$C_\chi = C_{Gauss}$ is proved for <u>pure-loss</u> <u>channel</u>

(case C with $N = 0, k < 1$)

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