Title: SICkening Axioms for Quantum Mechanics

Date: Jun 25, 2008 02:30 PM

URL: http://pirsa.org/08060197

Abstract:

# SIC World (1999)

"symmetric informationally complete"

For $|\psi_i\rangle \in \mathcal{H}_d$ (d-dimensional)

find $d^2$ projections $\Pi_i = |\psi_i\rangle\langle\psi_i|$

such that

$$\mathrm{tr}\,\Pi_i \Pi_j = \text{constant}$$

for $i \neq j$.

Caves
Zauner

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## Inequivalent SIC Sets

Let $d = 3$ , $\omega = e^{\frac{2\pi i}{3}}$ .

$$
\text{Set 1} \quad
\begin{matrix}
\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ -\omega \\ \bar{\omega} \end{bmatrix} & \begin{bmatrix} 0 \\ -\bar{\omega} \\ \omega \end{bmatrix} \\
\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ \bar{\omega} \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ -\bar{\omega} \end{bmatrix} \\
\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ \omega \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ \bar{\omega} \\ 0 \end{bmatrix}
\end{matrix}
$$

# Inequivalent SIC Sets

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**Set 1**

$$\begin{bmatrix}0\\-1\\1\end{bmatrix} \quad \begin{bmatrix}0\\-\omega\\\bar{\omega}\end{bmatrix} \quad \begin{bmatrix}0\\-\bar{\omega}\\\omega\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\-1\end{bmatrix} \quad \begin{bmatrix}1\\0\\\bar{\omega}\end{bmatrix} \quad \begin{bmatrix}1\\0\\-\bar{\omega}\end{bmatrix}$$

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} \quad \begin{bmatrix}-1\\\omega\\0\end{bmatrix} \quad \begin{bmatrix}-1\\\bar{\omega}\\0\end{bmatrix}$$

---

**Set 2**

$$\begin{bmatrix}-2\\1\\1\end{bmatrix} \quad \begin{bmatrix}-2\\\omega\\\bar{\omega}\end{bmatrix} \quad \begin{bmatrix}-2\\\bar{\omega}\\\omega\end{bmatrix}$$

$$\begin{bmatrix}1\\-2\\1\end{bmatrix} \quad \begin{bmatrix}1\\-2\omega\\\bar{\omega}\end{bmatrix} \quad \begin{bmatrix}1\\-2\bar{\omega}\\\omega\end{bmatrix}$$

$$\begin{bmatrix}1\\1\\-2\end{bmatrix} \quad \begin{bmatrix}1\\\omega\\-2\bar{\omega}\end{bmatrix} \quad \begin{bmatrix}1\\\bar{\omega}\\-2\omega\end{bmatrix}$$

# Evidence for Existence

Analytical Constructions

$d = 2-13, 15, 19$

Numerical ($\epsilon \leq 10^{-11}$)

$d = 2-47$

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# Existence

- Klappenecker et al, quant-ph/0503239
  On approximately symmetric informationally ...
- Klappenecker et al, quant-ph/0502031
  Mutually unbiased bases are complex projective ...
- Appleby, quant-ph/0412001
  SIC-POVMs and the extended Clifford group
- Grassl, quant-ph/0406175
  On SIC-POVMs and MUBs in Dimension 6
- Wootters, quant-ph/0406032
  Quantum measurements and finite geometry
- Renes et al, quant-ph/0310075
  Symmetric informationally complete quantum ...

⋮

- Lemmens + Seidel, J. Alg. 24, p. 494 (1973)
  Equiangular Lines

# Activity at & through PI

**In-house:**

- Blume-Kohout
- Flammia
- Fuchs

**Recent Visitors:**

- Appleby
- Hughston
- Larsson

**Previous Visitors:**

- Barnum
- Caves
- Roetteler
- Scott
- Wootters
- ⋮

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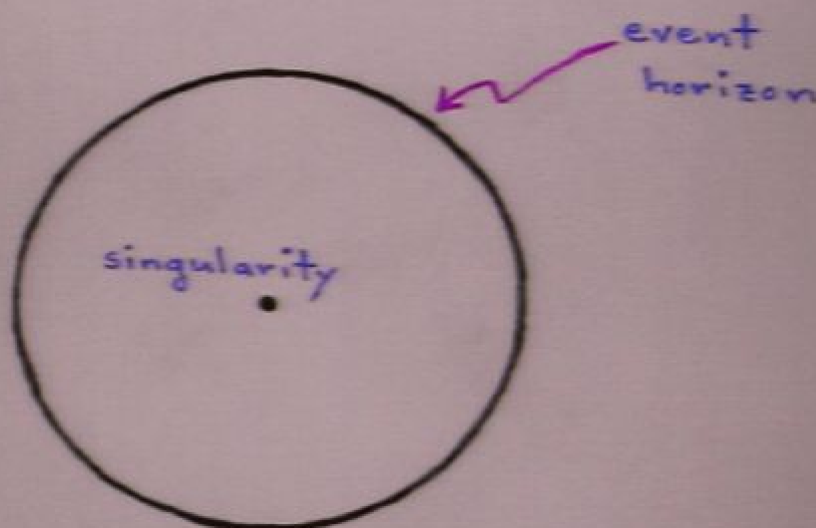
Zauner

# Black Hole



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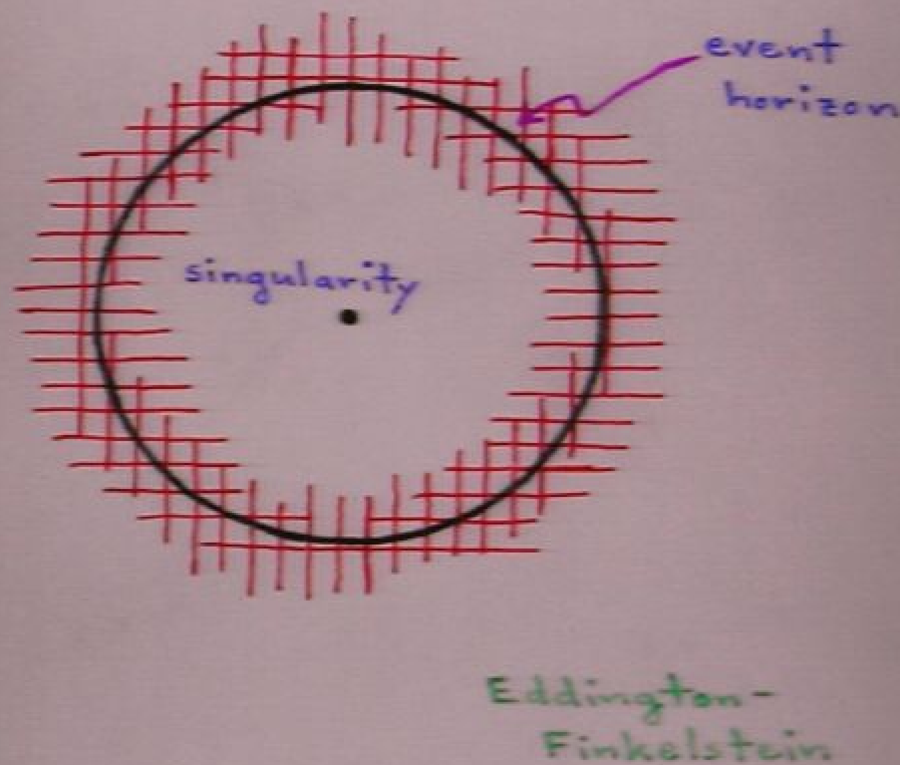

$$|\psi\rangle$$

# Density Operators

linear operators

$$\rho \in \mathcal{L}(\mathcal{H}_d)$$

complex vector space

catalog of uncertainties

1) $\rho^{\dagger} = \rho$
2) $\mathrm{tr}\,\rho = 1$
3) $\lambda_i(\rho) \geq 0$

convex hull of the set $\{ |\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H}_d \}$

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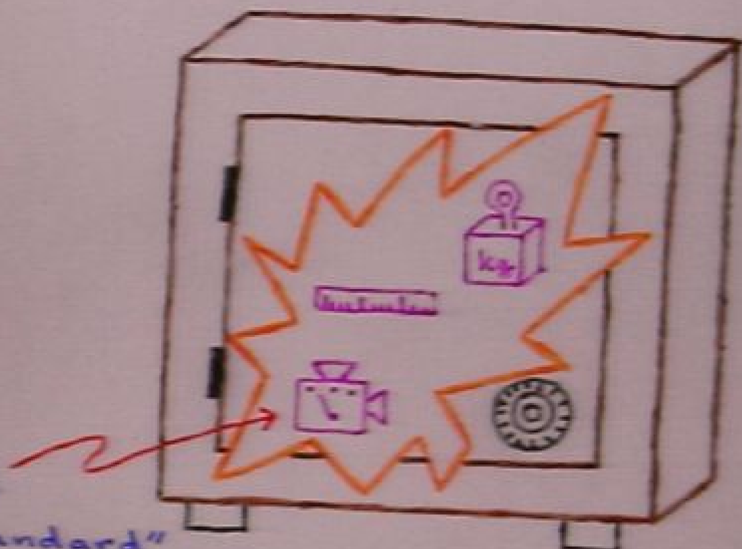
eigenvalues

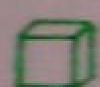
convex hull of the set $\{ |\psi\rangle\langle\psi| : |\psi\rangle \in \mathcal{H}_d \}$

$$\rho \longleftrightarrow p(h)$$

# Bureau of Standards




the
"standard"
quantum measurement



$\rho$

# Quantum Probability

Given a state $\rho$,

and an observable

$$H = \sum_i \alpha_i \Pi_i \quad \text{(eigendecomp)},$$

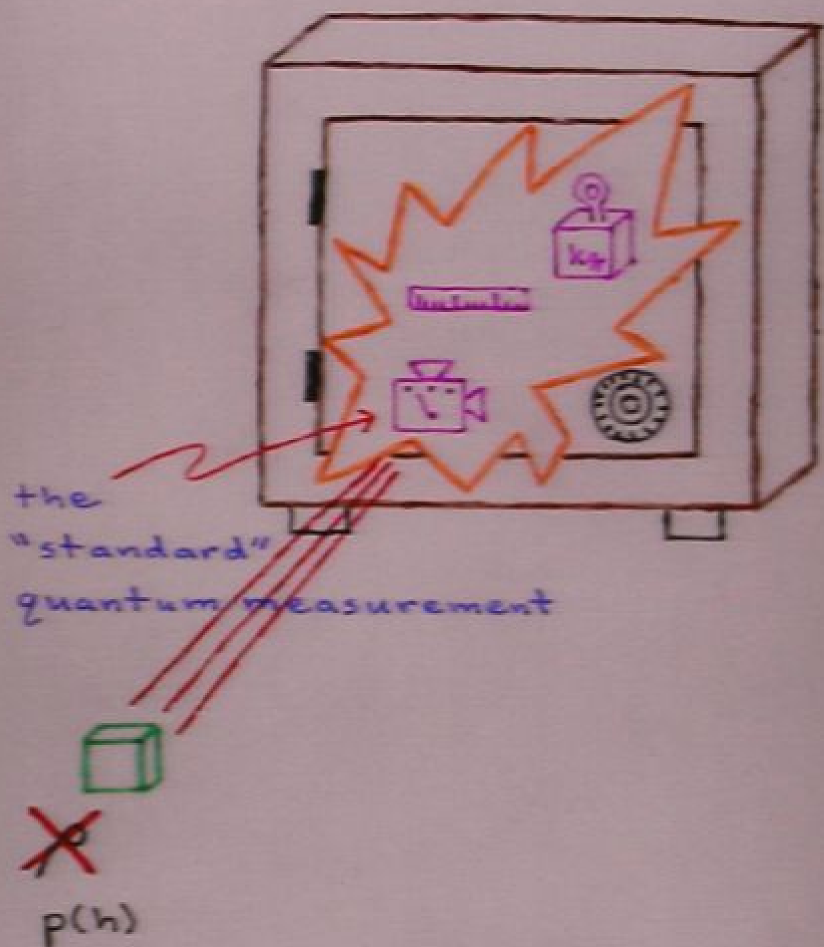
the probability of outcomes is

$$p_i = \operatorname{tr} \rho \Pi_i \, .$$

Why this rule and not some other ?

Or simply, how to understand it?

# Bureau of Standards



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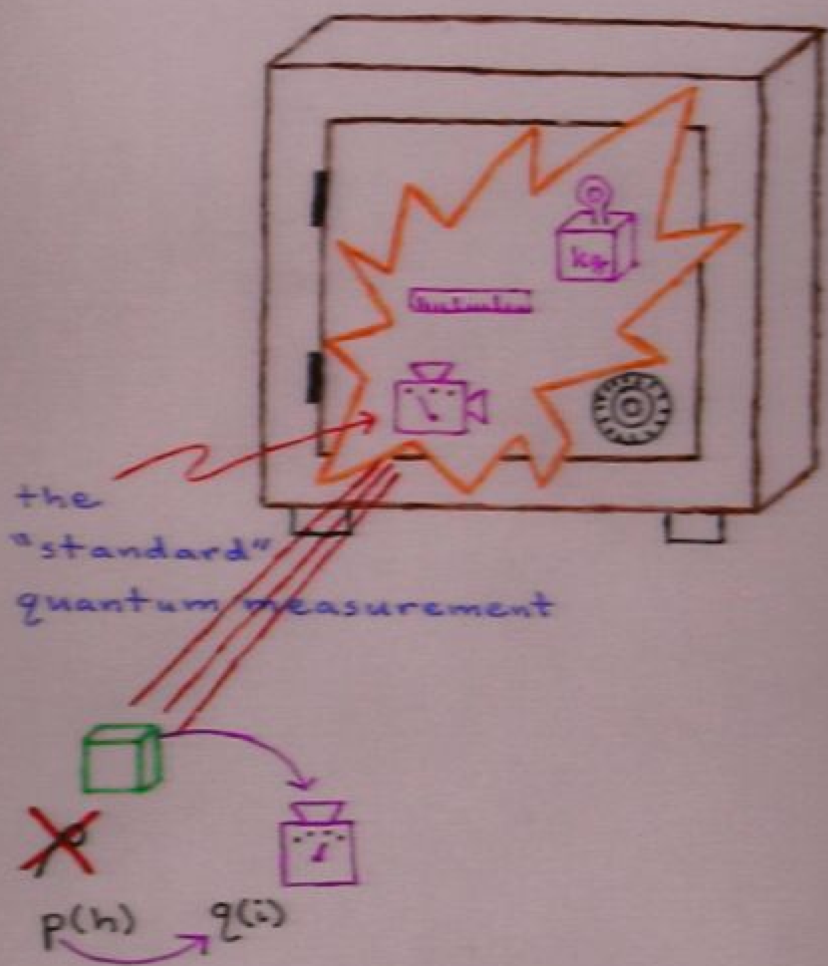

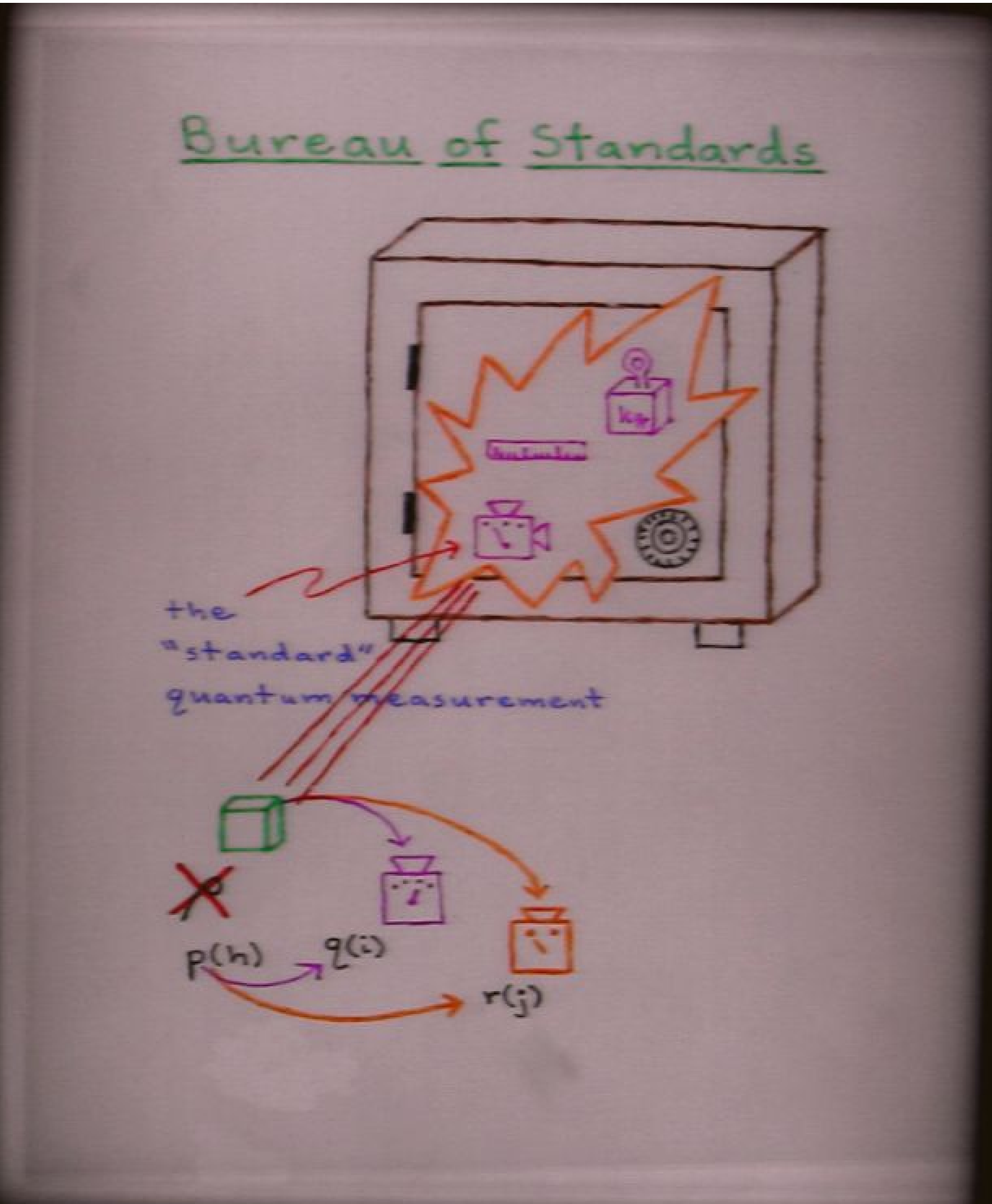
Bureau of Standards
kg
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p(h)
q(i)
r(j)

# Informational Completeness

quantum states

$\rho \in \mathcal{L}(\mathcal{H}_D)$ — $D^2$-dimensional vector space

Choose POVM $\{E_h\}$, $h = 1, \ldots, D^2$, with $E_h$ all linearly independent. (Can be done.)

$D^2$ numbers $p(h) = \mathrm{tr}\,\rho E_h$ determine $\rho$.

Because $(A,B) = \mathrm{tr}\,A^\dagger B$ is an inner product

projection of $\rho$ onto $E_h$

Any such $\{E_h\}$ can be the standard quantum measurement.

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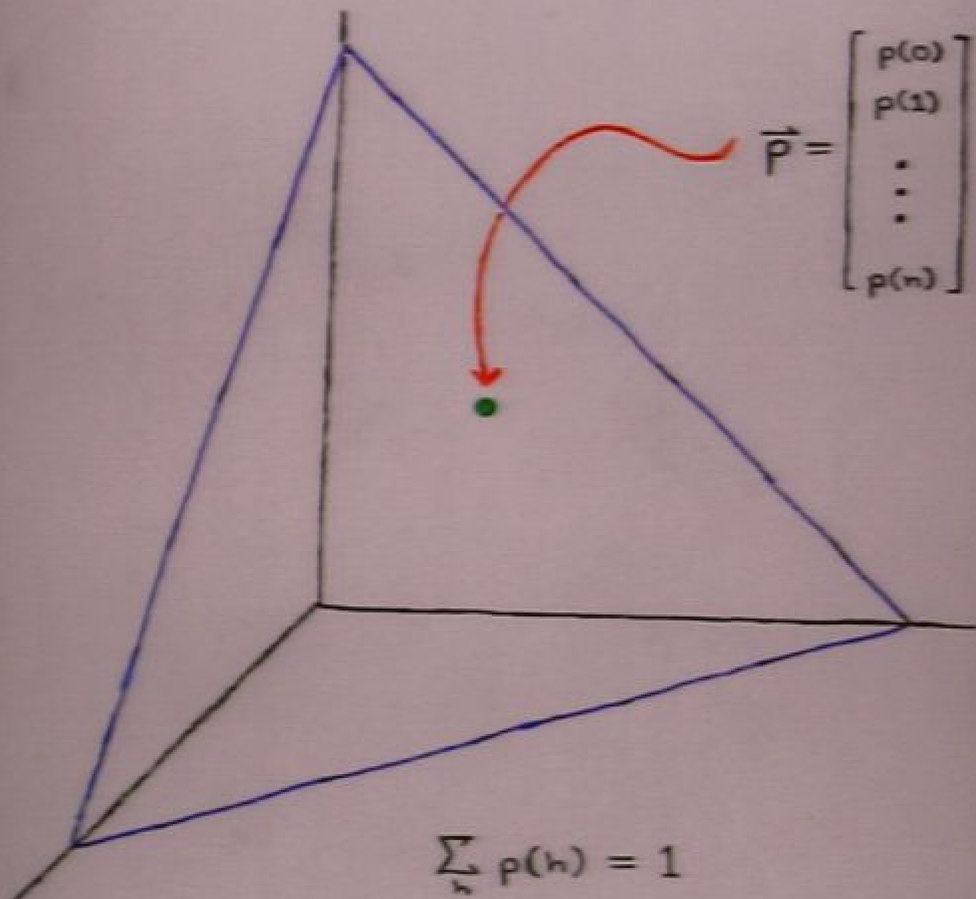
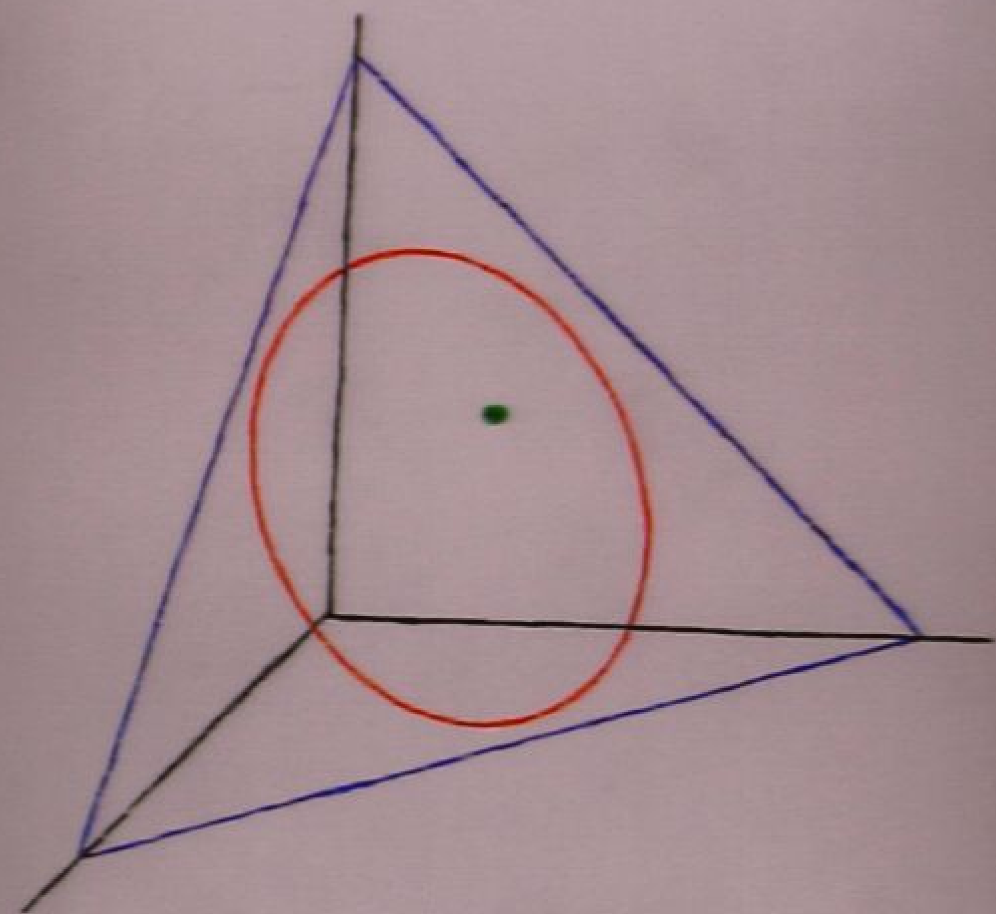
# Probability Simplex



$$\vec{P} = \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(n) \end{bmatrix}$$

$$\sum_h p(h) = 1$$

$$p(h) \geq 0 \quad \forall h$$



# Path Back to Density Ops

Suppose $\{E_j\}$, $j=1,\ldots,d^2$, is ICP.

Then $p(j)$ determines $\rho$.

But also $\rho = \sum_j \alpha_j E_j$ for some $\alpha_j$'s.

Thus

$$p(j) = \mathrm{tr}\,\rho E_j = \sum_k \alpha_k \,\mathrm{tr}\, E_j E_k$$

i.e.

$\vec{p} = M\vec{\alpha}$ where $M = [\mathrm{tr}\, E_j E_k]$

nonnegative matrix

and so

$$\boxed{\vec{\alpha} = M^{-1}\vec{p}}$$

Prettiest when $M_{jk} = a + b\,\delta_{jk}$.

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# Example of Pretty Ones

Let $H(\mathcal{H}_d)$ be the real vector spac
of Hermitian operators.

Consider $d^2-1$ dimensional subspace
$T(\mathcal{H}_d)$ of trace-free operators.

Choose $B_i \in T(\mathcal{H}_d)$, $i=1,\ldots,d^2$, so tha

1) they lie on unit sphere
$\operatorname{tr} B^2 = 1$,

2) they form a regular simplex
$\operatorname{tr} B_i B_j = -1/(d^2-1)$.

Then can choose $\epsilon > 0$ so that
$$E_i = \frac{1}{d^2}(I + \epsilon B_i)$$
are all positive semi-definite.

It follows that 1) $E_i$ are ICP,
2) $M_{jk} = a + b\,\delta_{jk}$.

# A Very Fundamental Mmt ?

Caves, 1999

Suppose $d^2$ projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|$ satisfying

$$\mathrm{tr}\,\Pi_i \Pi_j = \frac{1}{d+1} \quad , \quad i \neq j$$

exist.

Can prove:

1) the $\Pi_i$ linearly independent

2) $\sum_i \frac{1}{d} \Pi_i = I$

So good for Bureau of Standards.

Also

$$p(i) = \frac{1}{d}\,\mathrm{tr}\,\rho \Pi_i$$

$$\boxed{\rho = \sum_i \left[ (d+1) p(i) - \frac{1}{d} \right] \Pi_i}$$

## Analogy

$$|\psi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$$

$$\rho = \sum_{i=1}^{d^2} \alpha_i A_i$$

operators

What properties should we demand of these?

orthonormal? $\quad (A,B) = \operatorname{tr} A^{\dagger} B$

density operators themselves?

measurement outcomes?

# Measure of Orthonormality

ADF, 0707.2071

Suppose $A_i$, $i = 1, \ldots, d^2$

positive semi-definite.

And $\mathrm{tr}\, A_i^2 = 1$.

"Orthonormality"

$$K = \sum_{i \neq j} (\mathrm{tr}\, A_i A_j)^2$$

smaller the better

Can prove

$$K \geq \frac{d^2(d-1)}{d+1}$$ with = iff

$$\mathrm{tr}\, A_i A_j = \frac{1}{d+1} \quad \forall i \neq j$$

$A_i$ — rank-1

# If They Exist ...

1) the $|\psi_i\rangle$ form a set of states maximally sensitive to eavesdropping in quantum crypto settings

CAF, quant-ph/0404122

2) are optimal for some natural cases of quantum tomography

A. J. Scott, quant-ph/0604049

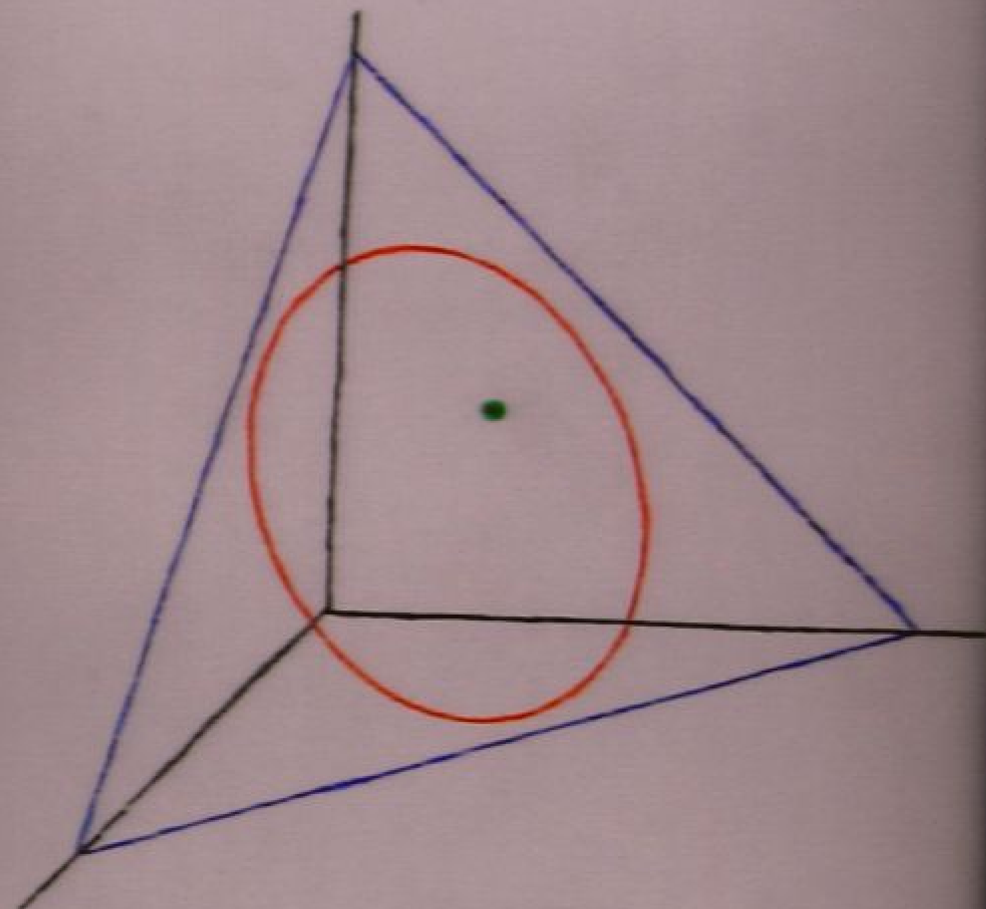
3) in prime d form "minimum uncertainty" states, in analogy to coherent states, for complete sets of mutually unbiased bases

Appleby, Dang, CAF, 0707.2071

But do such sets of states

EXIST

?



# Extreme Points

## Characterization 1:

$\rho = |\psi\rangle\langle\psi|$

## Characterization 2:

$\rho$ is hermitian

$\rho^2 = \rho$

$\mathrm{tr}\,\rho = 1$

## Characterization 3:

$\rho$ is positive semi-definite

$\mathrm{tr}\,\rho = 1$

$\mathrm{tr}\,\rho^2 = 1$

# Remarkable Theorem

Jones & Linden, PRA 71 (2005)
Flammia, (unpub, 2004)

Let A be Hermitian, $A^{\dagger} = A$.

Then, $A = |\psi\rangle\langle\psi|$ if and only if

$$\mathrm{tr}\, A^2 = \mathrm{tr}\, A^3 = 1 .$$

## Proof:

$a_i$ — eigenvalues of $A$

$\mathrm{tr} A^2 = \sum_i a_i^2 = 1 \quad \Rightarrow \quad |a_i| \leq 1$

$1 - a_i \geq 0$

$0 = \mathrm{tr} A^2 - \mathrm{tr} A^3 = \sum_i a_i^2 (1 - a_i)$

$\Rightarrow \quad a_i = 0$ or $1 - a_i = 0$

$\mathrm{tr} A^2 = 1 \quad \Rightarrow \quad a_i = 1$ for one and only one $i$.

QED

# Pure States in SIC Language

Conditions

$$\rho^+ = \rho \quad , \quad \mathrm{tr}\,\rho^2 = \mathrm{tr}\,\rho^3 = 1$$

translate to

$$\sum_i p(i)^2 = \frac{2}{d(d+1)}$$

and

$$\sum_{jkl} c_{jkl}\, p(j)\, p(k)\, p(l) = \frac{d+7}{(d+1)^3}$$

where

$$c_{jkl} = \mathrm{Re}\ \mathrm{tr}\, \Pi_j \Pi_k \Pi_l$$

Could these be independently motivatable physical constants?

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Could t

motivat

# Building Hilbert Space

Given $c_{ijk}$, prove representation theorem!

For instance, can define a Jordan product:

$$\Pi_i \bullet \Pi_j \equiv \sum_k \alpha_{ijk} \Pi_k$$

with

$$\alpha_{ijk} = \frac{1}{d}(d+1)\left[ c_{ijk} - \frac{d\delta_{ij}+1}{(d+1)^2} \right].$$

## Unitarity

$$
\begin{tikzcd}
\rho \arrow[r] \arrow[d] & U\rho U^{\dagger} \arrow[d] \\
p(i) \arrow[r] & q(j)
\end{tikzcd}
$$

## Unitarity

$$\begin{tikzcd} \rho \arrow[r] \arrow[d] & U\rho U^{+} \arrow[d] \\ p(i) \arrow[r] & q(j) \end{tikzcd}$$

# Unitarity

$$\begin{tikzcd}
\rho \arrow[r] \arrow[d] & U\rho U^{\dagger} \arrow[d] \\
p(i) \arrow[r] & q(j)
\end{tikzcd}$$

Define $t(j|i) = \frac{1}{d}\,\mathrm{tr}\, U\Pi_i U^{\dagger}\Pi_j$

doubly stochastic matrix

Then

$$\boxed{q(j) = (d+1)\sum_i p(i)\,t(j|i) - \frac{1}{d}}$$

Could hardly be simpler.

# Role of the Born Rule

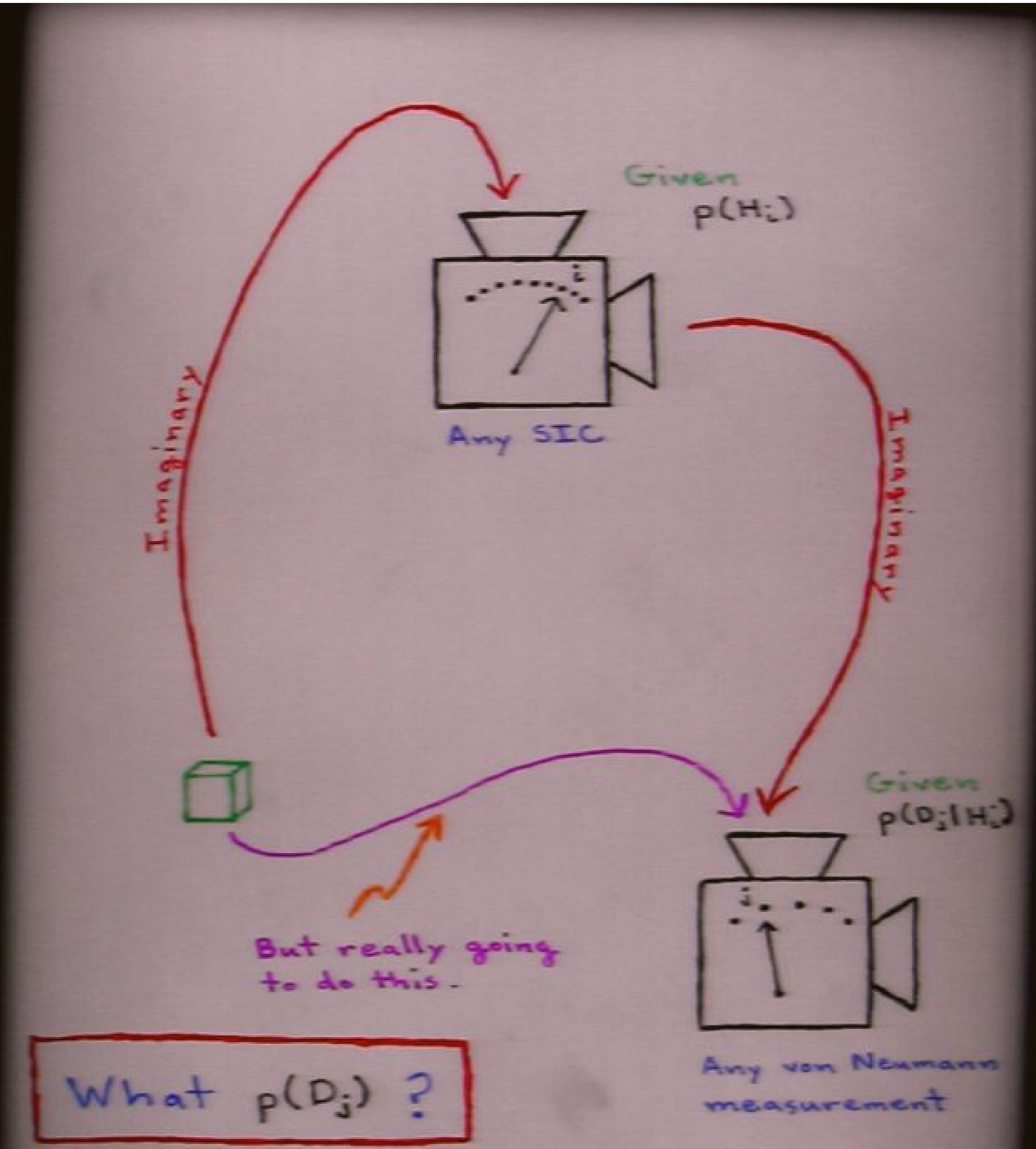
quantum state $\rho$

measurement $\{E_h\}$

$$q(h) = \mathrm{tr}\,\rho E_h$$

To transform or relate probabilities!

$$q(h) = \sum_i \left[(d+1)p(i) - \frac{1}{d}\right] \mathrm{tr}\,\Pi_i E_h$$


Given
p(H_i)
Imaginary
Any SIC
Imaginary
Given
p(D_j|H_i)
But really going
to do this.
What p(D_j) ?
Any von Neumann
measurement

## Laws of Probability

$H_i$ – various hypotheses one might have

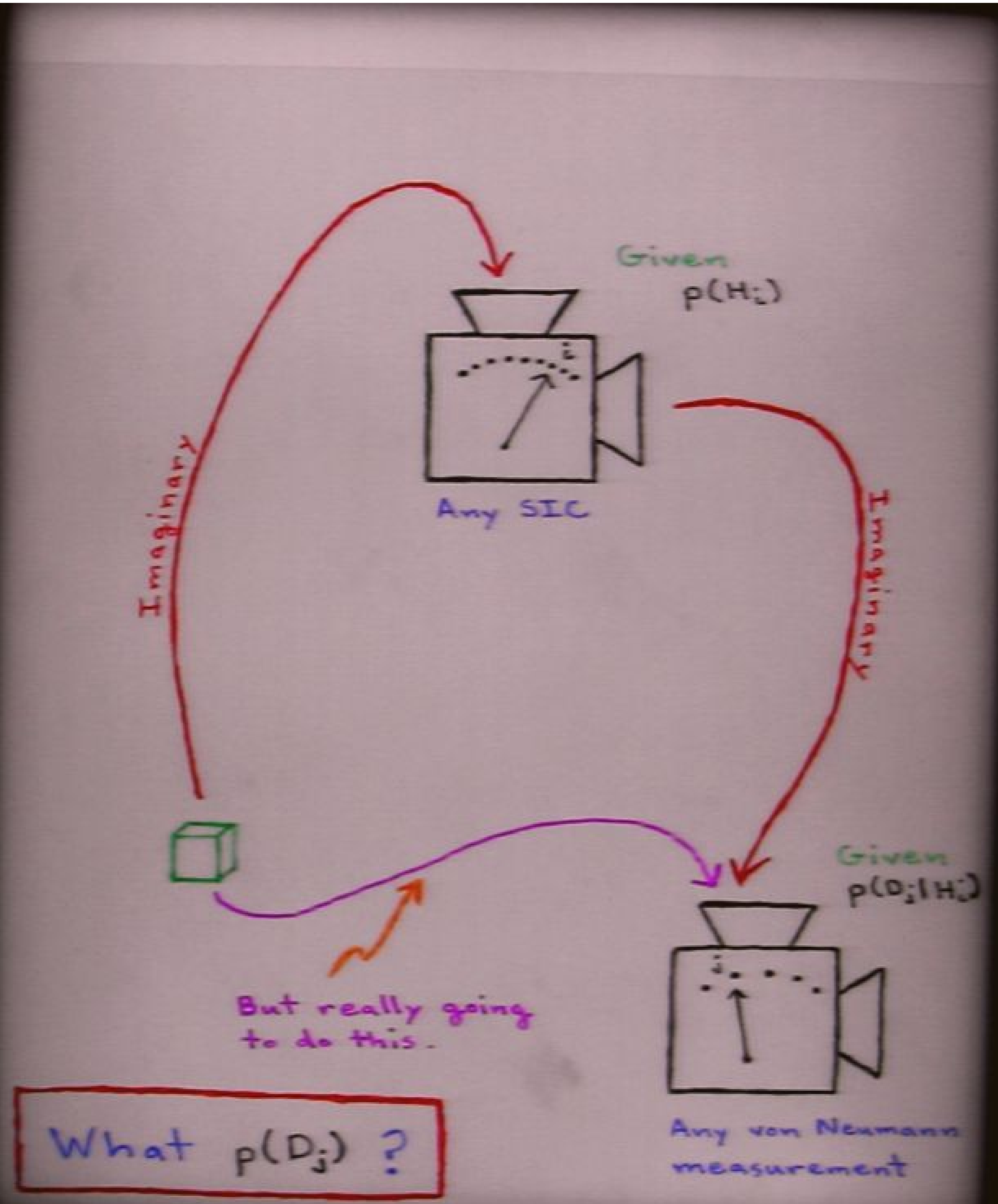
$D_j$ – data values one might gather

**Given:** $p(D_j | H_i)$ ← expectations for data given hypothesis

$p(H_i)$ ← expectations for hypotheses themselves

**Question:** What expectations should one have for the $D_j$ ?

**Answer:** $P(D_j) = \sum_i p(H_i)\, p(D_j | H_i)$



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What p(D_j) ?

$$\underbrace{p(D_j)}_{\text{Quantum}} = (d+1)\underbrace{\sum_i p(H_i)\,p(D_j|H_i)}_{\text{(Usual) Bayesian}} - 1$$

Magic!

## "Measurement"

Does it reveal a pre-existing, but unknown, value?

or

Does it in some sense go toward creating the very value?

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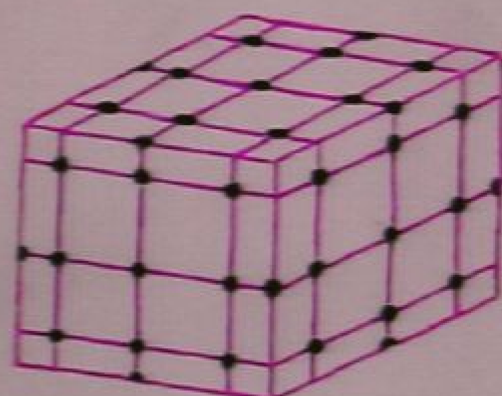
# Kochen - Specker

Cannot be colored:



33 rays, Peres

(when completed into full triads, consists of 40 triads made from 57 rays)

$$\underbrace{p(D_j)}_{\text{Quantum}} = (d+1) \underbrace{\sum_i p(H_i)\, p(D_j|H_i)}_{\text{(Usual) Bayesian}} - 1$$

Magic!

Think SIC thoughts!

... and maybe by way of it we'll come to understand quantum mechanics a little better.