Title: Extending Standard Quantum Interpretation by Quantum Set Theory

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Abstract: Set theory provides foundations of mathematics in the sense that all the mathematical notions like numbers, functions, relations, structures are defined in the axiomatic set theory called ZFC. Quantum set theory naturally extends ZFC to quantum logic. Hence, we can expect that quantum set theory provides mathematics based on quantum logic. In this talk, I will show a useful application of quantum set theory to quantum mechanics based on the fact that the real numbers constructed in quantum set theory exactly corresponds to the quantum observables. The standard formulation of quantum mechanics answers the question as to in what state an observable A has the value in an interval I. However, the question is not answered as to in what state two observables A and B have the same value. The notion of equality between the values of observables will play many important roles in foundations of quantum mechanics. The notion of measurement of an observable relies on the condition that the observable to be measured and the meter after the measurement should have the same value. We can define the notion of quantum disturbance through the condition whether the values of the given observable before and after the process is the same. It is shown that all the observational propositions on a quantum system corresponds to some propositions in quantum set theory and the equality relation naturally provides the proposition that two observables have the same value. It has been broadly accepted that we cannot speak of the values of quantum observables without assuming a hidden variable theory. However, quantum set theory enables us to do so without assuming hidden variables but alternatively under the consist use of quantum logic, which is more or less considered as logic of the superposition principle. [1] M. Ozawa, Transfer principle in quantum set theory, J. Symbolic Logic 72, 625-648 (2007), online preprint: http://arxiv.org/abs/math.LO/0604349. [2] M. Ozawa, Quantum perfect correlations, Ann. Phys. (N.Y.) 321, 744--769 (2006), online preprint: LANL quant-ph/0501081.

Quantum Foundations Seminar

Perimeter Institute, June 24, 2008

Extending Standard Quantum Interpretation by Quantum Set Theory

Masanao Ozawa

Nagoya University

1 Overview

- Quantum interpretation = establishing the concept of "the value of an observable"
- Problem: non-existence of hidden variables
- Method: establishing "quantum logical reality" instead of "hidden variable = classical logical reality"
- Problem of quantum logic as a propositional logic: unable to treat variables.
- Method: developing set theory based on quantum logic

1 Overview

- Quantum interpretation = establishing the concept of "the value of an observable"
- Problem: non-existence of hidden variables
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- Problem of quantum logic as a propositional logic: unable to treat variables.
- Method: developing set theory based on quantum logic

- Results
  - observables = real numbers constructed in quantum set theory
  - observational propositions $\subseteq$ propositions on real numbers
  - Extending observational propositions by propositions on real numbers:
    * introducing state-dependent simultaneous determinateness
    * introducing state-dependent equality between observables
    * definition of state-dependent values of observables
    * definition of state-dependent notion of disturbance

* definition of state-dependent notion of simultaneous measurability of values of observables

- Explanation of EPR paradox:
  1. successive measurement without disturbance $\rightarrow$ simultaneous measurement
  2. by correlations in a state, simultaneous measurement of non-commuting observables is possible
  3. Thus, two simultaneously non-determinate observables can be simultaneously measurable.
  4. Simultaneous determinateness and simultaneous measurability are different notions
  5. Simultaneous measurability doe not lead to simultaneous determinateness nor existence of hidden variables.

REFERENCES

M. Ozawa, Quantum perfect correlations, Ann. Phys. (N.Y.) 321, 744–769 (2006).

M. Ozawa, Transfer principle in quantum set theory, J. Symbolic Logic 72, 625–648 (2007).

M. Ozawa, Simultaneous Measurability of Non-Commuting Observables and the Universal Uncertainty Principle, Proc. 8th Int. Conf. on Quantum Communication, Measurement and Computing, (NICT Press, Tokyo, 2007); PP. 363–368.

2 Quantum equality — Significance

Problem: When two observables A and B are equal in a state ψ? When two observables A and B have the same value in a state ψ? We write this condition as $A \equiv_\psi B$.

Application 1: The condition that the value of an observables A does not change from time t_1 to t_2, if the system is originally in the state ψ, can be expressed by

$$A(t_1) \equiv_\psi A(t_2).$$

Application 2: When a measurement of an observable A in a state ψ is considered to be correct, or gives the correct value of A? The answer can be given by

$$A \otimes I \equiv_{\psi\otimes\xi} U^{\dagger}(I \otimes M)U,$$

where ξ is the apparatus initial state,

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3 Quantum equality — Difficulty 1

Special case1

According to the standard interpretation:

$A \equiv_\psi \alpha I$ (**A has the value α in ψ**) $\Leftrightarrow A\psi = \alpha\psi$.

Problem: $A \equiv_\psi B$ **if and only if** $A - B \equiv_\psi 0$**?**

Answer: NO.

Let $A(t_1)$ and $A(t_2)$ be such that

$$A(t_1) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A(t_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

with time evolution operator $U(t_2, t_1)$ and the state ψ such that

$$U(t_2, t_1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, we have $A(t_1)\psi = A(t_2)\psi$, i.e., $A(t_A) - A(t_2) \equiv_\psi 0$.

The first and the second moments of A are unchanged, i.e., $\langle\psi|A(t_1)|\psi\rangle = \langle\psi|A(t_2)|\psi\rangle = 1$ and $\langle\psi|A(t_1)^2|\psi\rangle = \langle\psi|A(t_2)^2|\psi\rangle = 2$.

However, we have $\langle\psi|A(t_1)^3|\psi\rangle = 4$ but $\langle\psi|A(t_2)^3|\psi\rangle = 3$.

Thus, the third moment of A is changed, so that $A(t_1) \equiv_\psi A(t_2)$ does not hold.

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4 Quantum equality — Diffculty 2

Problem: Does the relation

$$\frac{dA(t)}{dt}\psi = 0 \tag{1}$$

for all $t \in (0,1)$ **ensures that** $A(0) \equiv_\psi A(1)$**?**

Answer: NO.

Example:

$$A(t) = Q + tP \tag{2}$$

$$P\psi = 0, \tag{3}$$

where Q, P **are the position and momentum operators of a one-dimensional unit mass, satisfy Eq. (1). However,** $A(0) \equiv_\psi A(1)$ **does not hold, since** $A(0) = Q$ **and** $A(1) = Q + P$ **has no joint probability in any states.**

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5 Quantum equality — Difficulty 3

There is a prevailing opinion that quantum mechanics only speaks of probability distribution,

so that a measurement of an observable A in a state ψ is correct if and only if the measurement reproduces the correct probability distribution, or

$A \otimes I$ and $U^{\dagger}(I \otimes M)U$ has the same probability distribution in $\psi \otimes \xi$.

However, $A \equiv_{\psi} B \Leftrightarrow$ "A and B has the same probability distribution" does not hold.

Special Case 2

In the case where $[A, B] = 0$, $A \equiv_\psi B$ if and only if the joint probability distribution of A and B in ψ concentrates on the diagonal set.

Problem. Does the condition that A and B should have the same probability distribution in the state ψ ensure the relation $A \equiv_\psi B$?

Answer. NO.

Suppose

$\psi = \phi \otimes \phi$ for some ϕ, and

$A = B \otimes I$ and $B = I \otimes C$ for some C.

In this case, A and B have the same probability distribution, and commuting, but they are statistically independent in the case where ϕ is not an eigenstate of B.

6 Quantum axioms for finite dimensional systems

Axiom 1 (Representations of observables and staes)

finite dimensional quantum system S$\rightarrow$ finite dimensional Hilbert space $\mathcal{H}_{\mathrm{S}}$

observables of S $\rightarrow$ self-adjoint operators on $\mathcal{H}_{\mathrm{S}}$

states of S $\rightarrow$ density operators on $\mathcal{H}_{\mathrm{S}}$

Axiom 2 (Time evolution)

$H_{\mathbf{S}}$: **Hamiltonian**

$\rho(t)$: **state at the time** $t \in (t_1, t_2)$

S is isolated in (t_1, t_2)

$\rightarrow$

$$i\hbar \frac{d}{dt}\rho(t) = [H_{\mathbf{S}}, \rho(t)] \quad (4)$$

Axiom SF (Statistical Formula)

The probability distribution of mutually commuting observable $A_1, \ldots, A_n$ in ρ:

$$\Pr\{\mathbf{x}_1 = a_1, \ldots, \mathbf{x}_n = a_n | \rho\} = \mathrm{Tr}[E^{A_1}(a_1) \cdots E^{A_n}(a_n)\rho], \qquad (5)$$

where $E^A(a)$ is the projection on $\{\psi \in \mathcal{H}_{\mathbf{S}} \mid A\psi = a\psi\}$.

- Problem: What is the meaning of the values of observables.
- In the following, we set postulates for "values of observables", and reconstruct the standard formulation and an interpretation of the "values of observables."

7 Quantum observational propositions

- To clarify the logical principle behind the Statistical Formula, we define observational propositions.

Axiom 3-1 (Definition of atomic observational propositions) For any observable A of **A** and any real number a, we have an atomic observational proposition $A = a$. Any atomic observational propositions is an observational proposition.

Axiom 3-2 (Introduction of logical symbols) If ϕ is an observational proposition, then so is $\neg\phi$. If ϕ_1 and ϕ_2 are observational propositions, then so are $\phi_1 \wedge \phi_2$ and $\phi_1 \vee \phi_2$.

Axiom 3-3 (Definition of observational propositions) Every observational proposition is constructed from atomic ones by only finite time introductions of logical symbols.

- **The observational proposition $A = a$ means that "The value of the observable A is equal to the real number a." $\neg$, $\wedge$, $\vee$ means negation, conjunction, and disjunction, respectively.**

8 State-dependent truth of observational propositions

- By the following rules, we introduce the relation $\psi \Vdash \phi$ meaning that the observational proposition ψ is true in the vector state ψ.

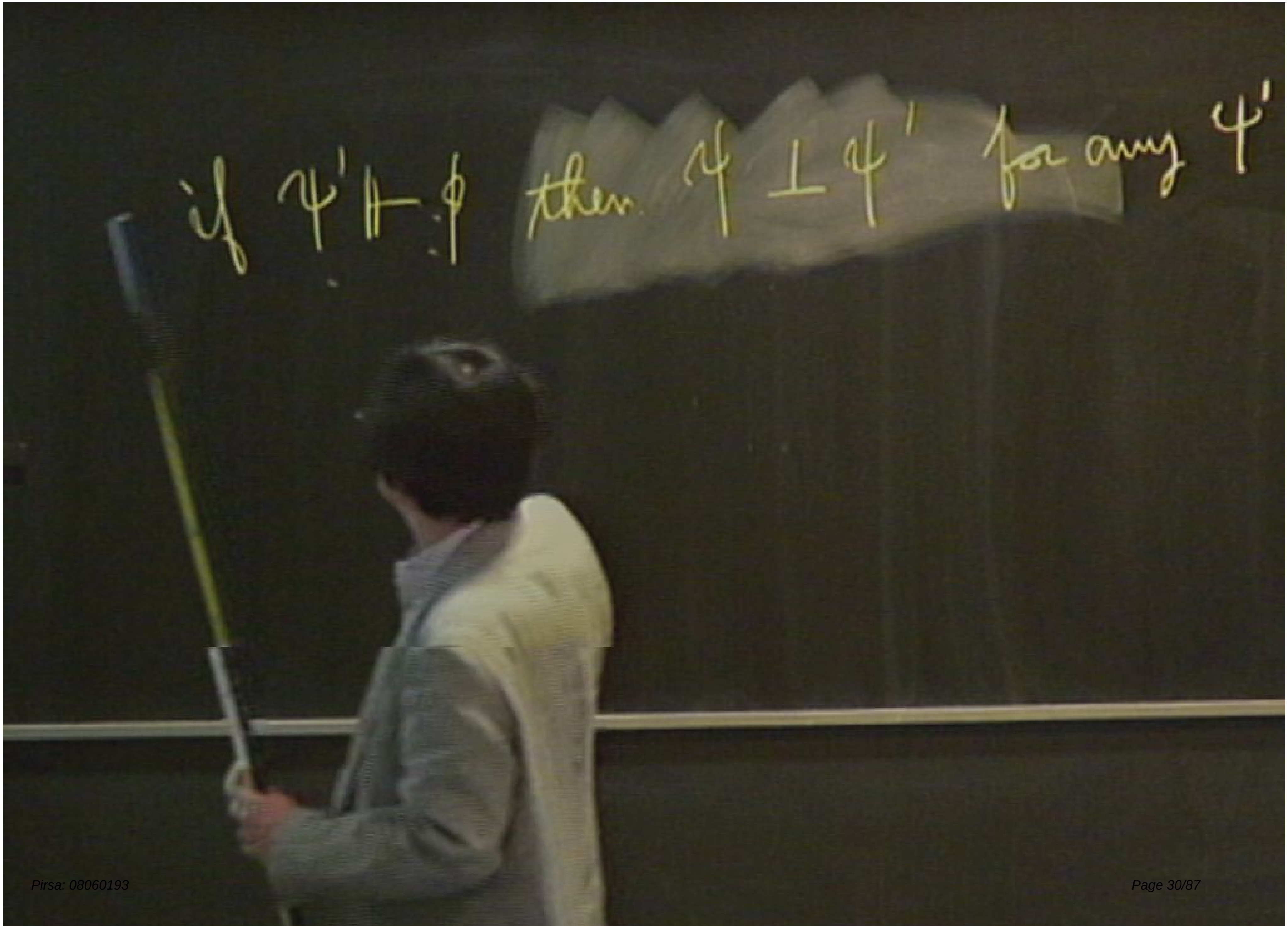
Axiom 4-1 $\psi \Vdash A = a \Leftrightarrow A\psi = a\psi$.

Axiom 4-2 $\psi \Vdash \neg\phi \Leftrightarrow$ **if** $\psi' \Vdash \phi$ **for any** ψ' **then** $\psi \perp \psi'$.

Axiom 4-3 $\psi \Vdash \phi_1 \wedge \phi_2 \Leftrightarrow \psi \Vdash \phi_1$ **and** $\psi \Vdash \phi_2$.

Axiom 4-4 $\psi \Vdash \phi_1 \vee \phi_2 \Leftrightarrow$ **there are** $\psi_1, \psi_2 \neq 0$ **such that** $\frac{\psi_1}{\|\psi_1\|} \Vdash \phi_1$ **and** $\frac{\psi_2}{\|\psi_2\|} \Vdash \phi_2$ **and** $\psi = \psi_1 + \psi_2$.

$\psi' H \cdot \phi$ for any ψ' such that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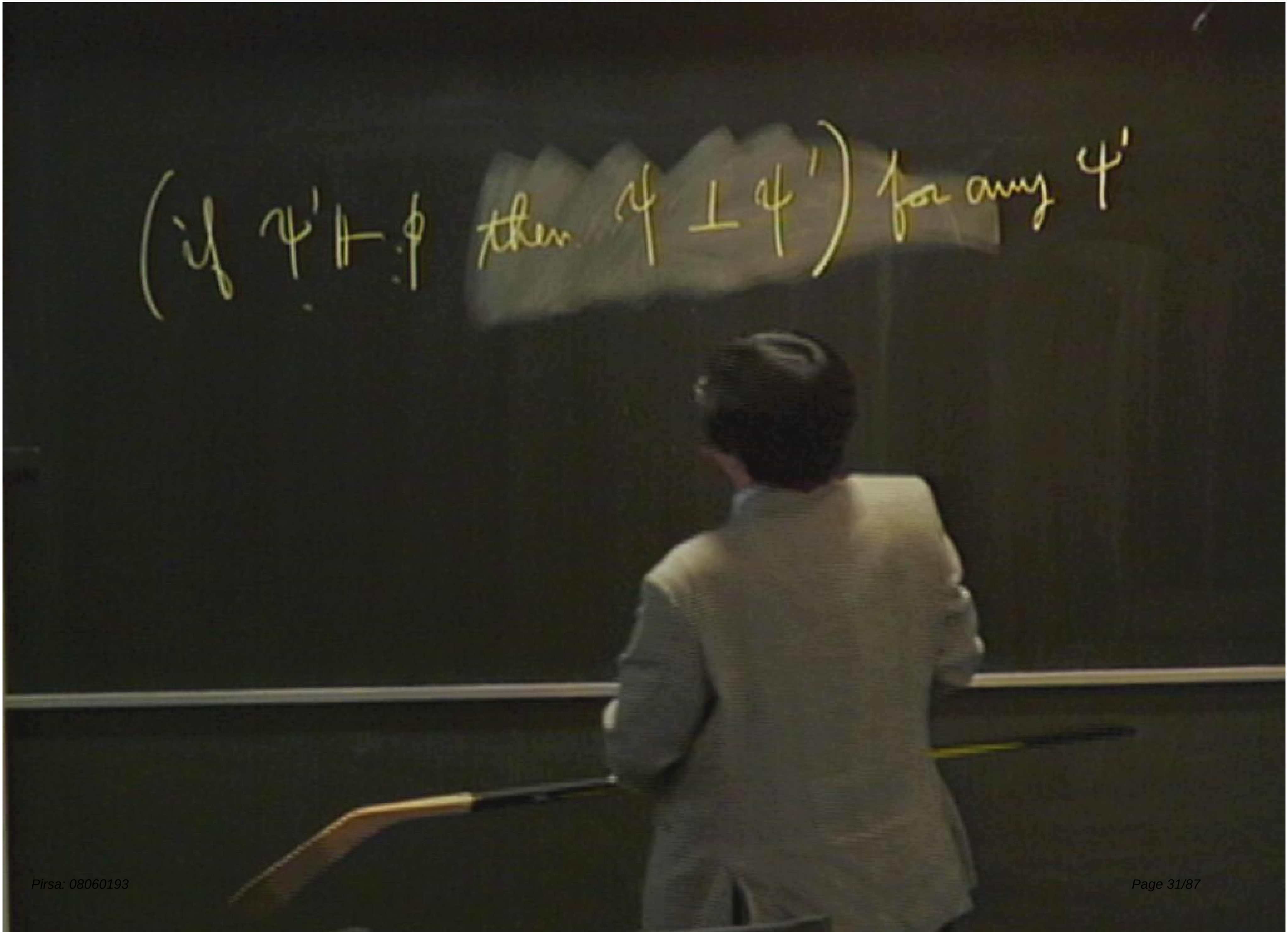
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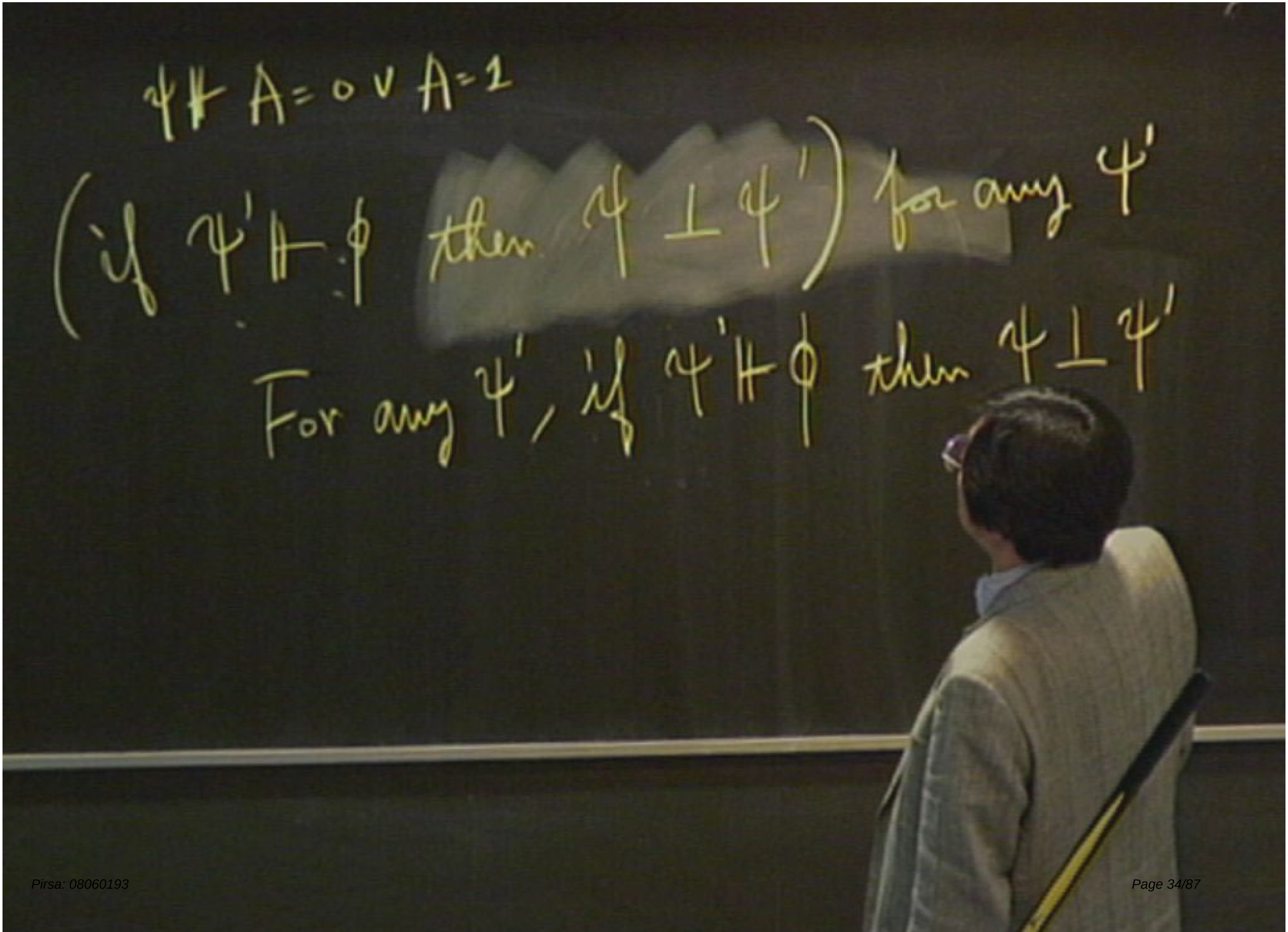
ψ ⊮ A=0 ∨ A=1
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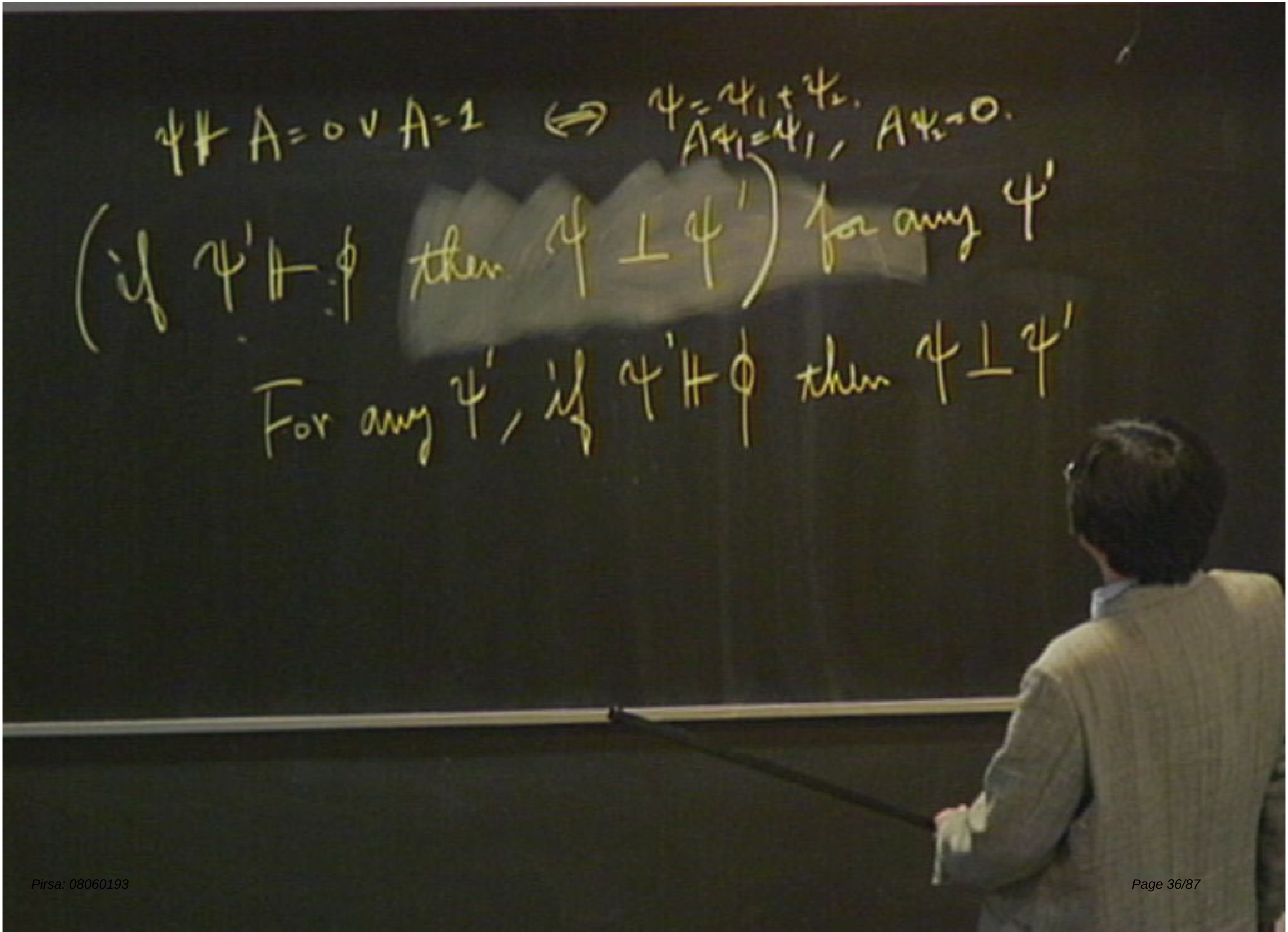
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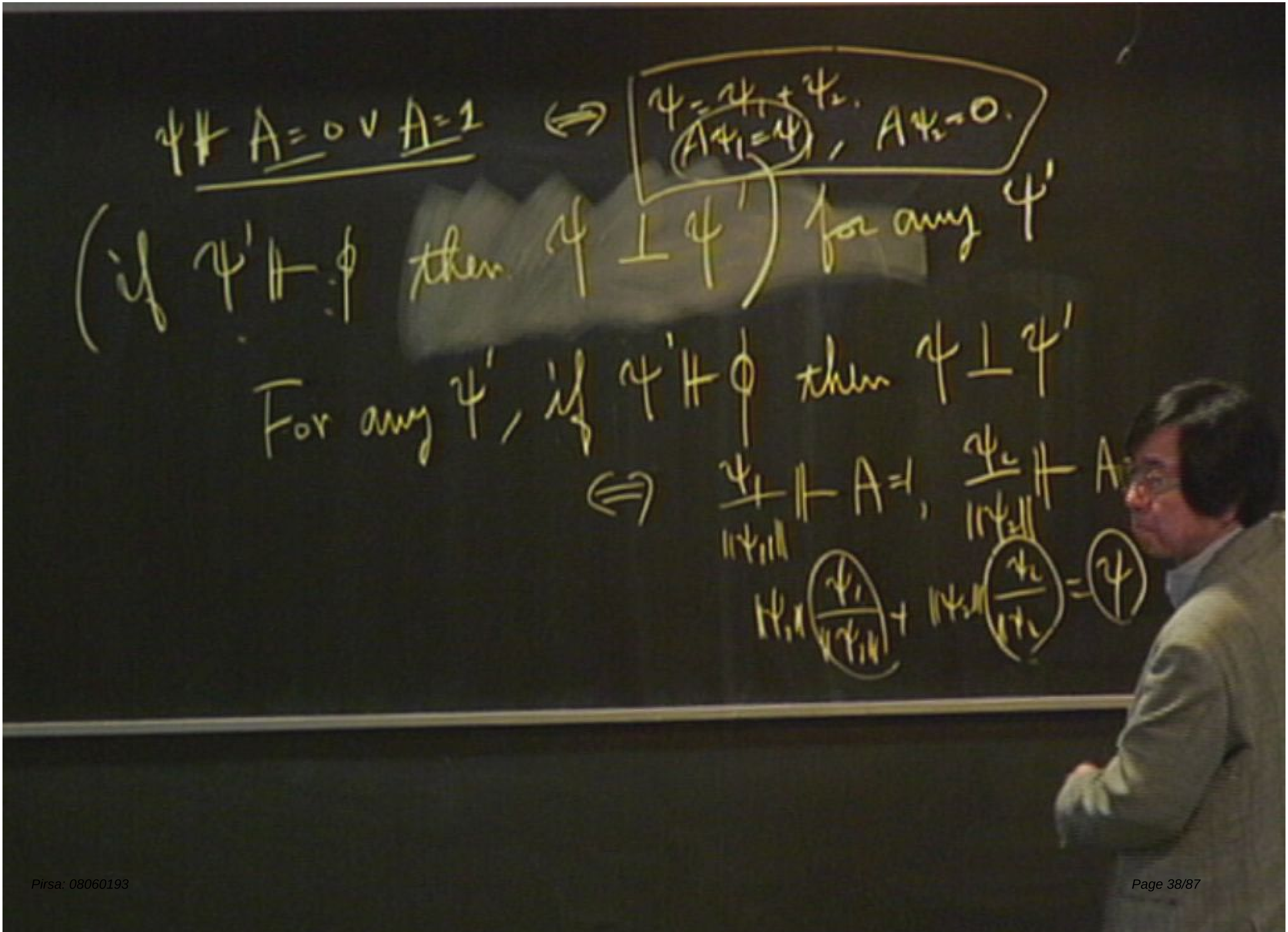
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9 Quantum Logic

- The set of closed subspaces of a Hilbert space is a partially ordered set with inclusion, and a complete complemented lattice with the orthogonal complementation $M \to M^{\perp}$.
- The lattice operations satisfy

$$M_1 \wedge M_2 = M_1 \cap M_2 \qquad M_1 \vee M_2 = M_1 + M_2$$

- An operator P is called a projection if $P = P^{\dagger} = P^2$. The projection operators and the closed subspaces are in one-to-one correspondence.
- We call the lattice of projections as a quantum logic and denote it by $\mathcal{Q}(\mathcal{H})$.
- The projection with range M is denoted by $\mathcal{P}(M)$.

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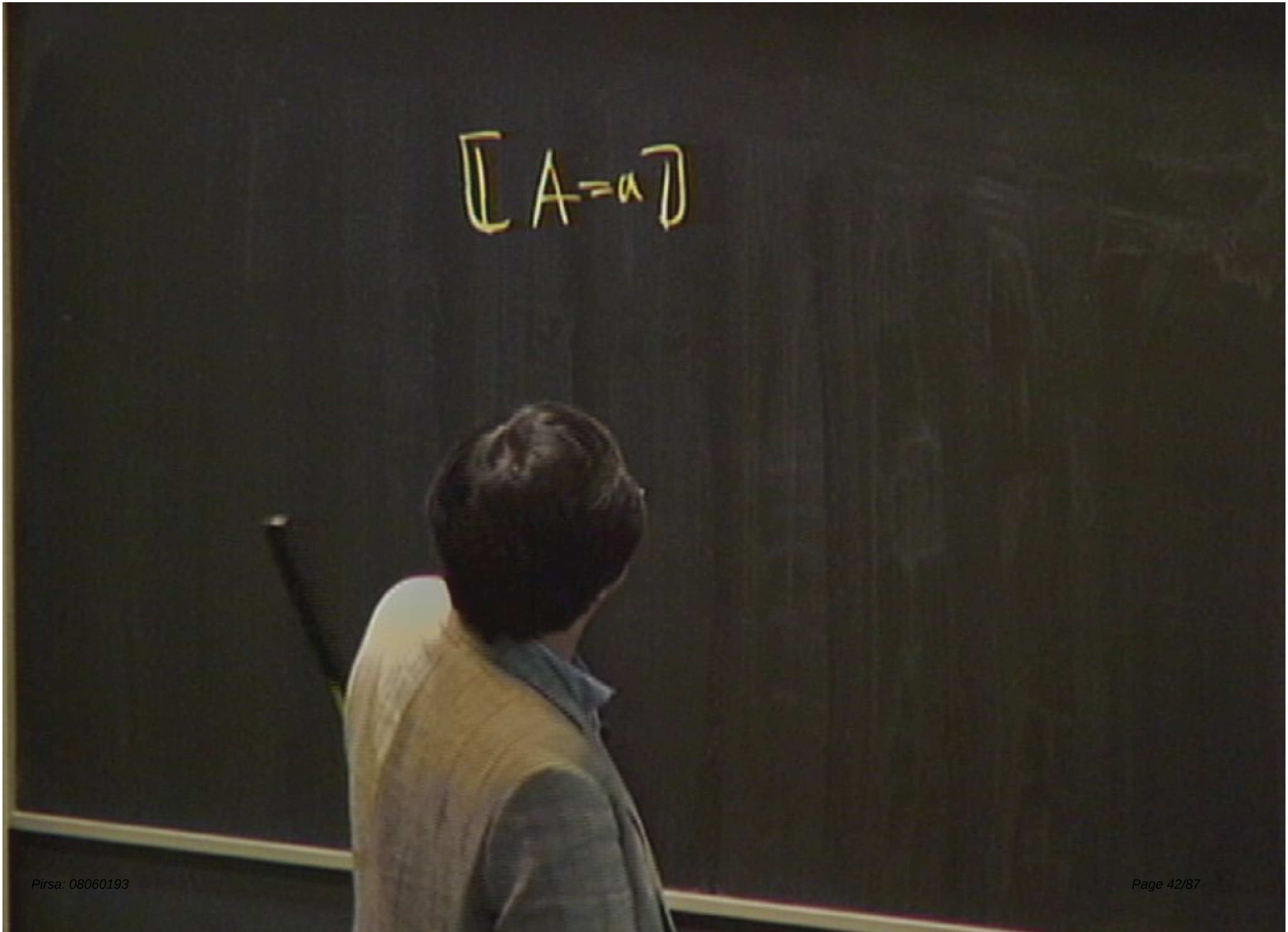
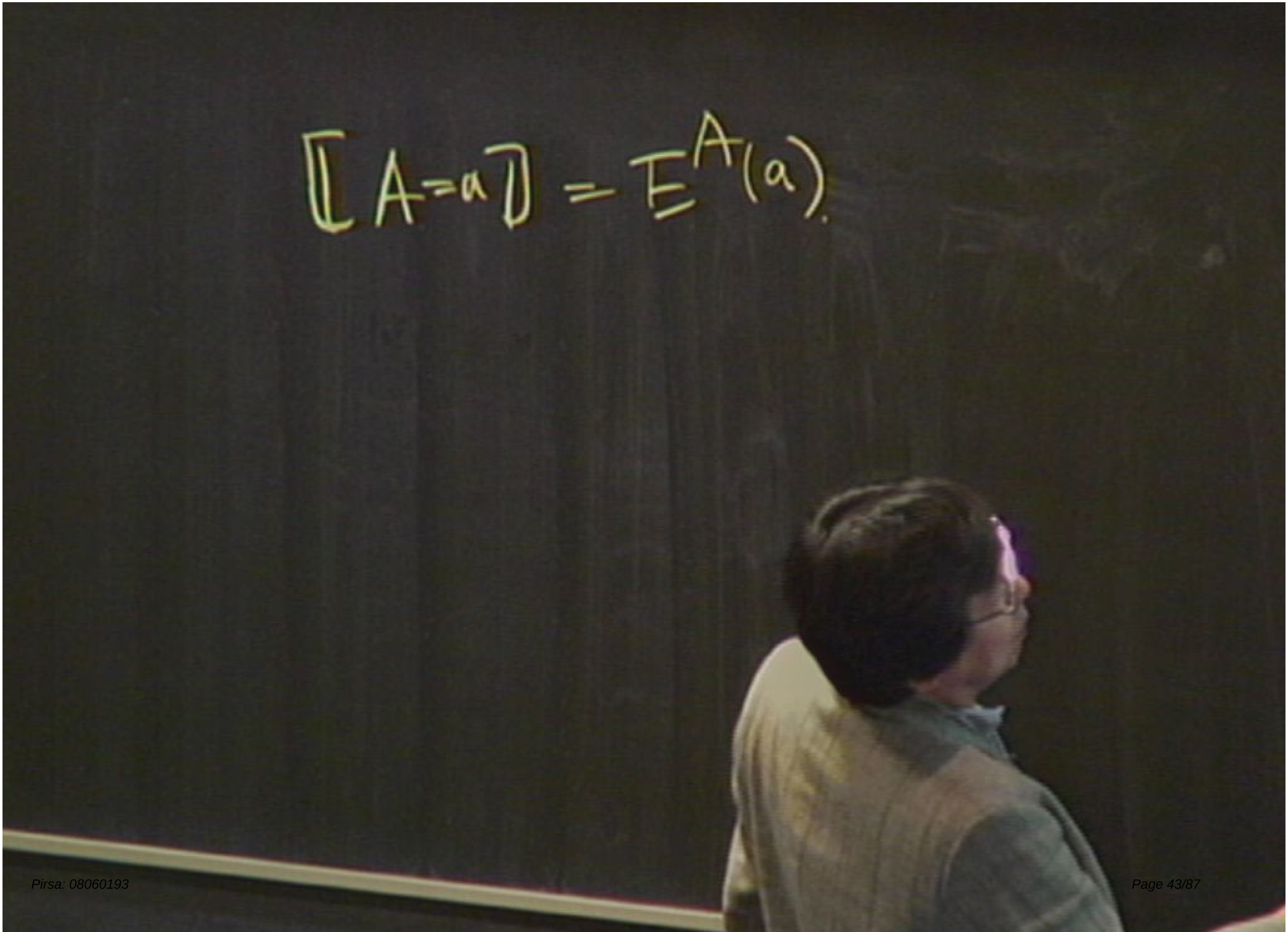

⟦A=a⟧


$[\![A=a]\!] = E^{A}(a)$

10 Truth value and probability of observational propositions

- **The truth value of an observational proposition ϕ is defined by**

$$[\![\phi]\!] = \mathcal{P}\{\psi \in \mathcal{H} \mid \psi \neq 0 \Rightarrow \frac{\psi}{\|\psi\|} \Vdash \phi\}.$$

- **Theorem: Truth values of observational propositions ϕ satisfies the following.**

(i) $[\![A \in I]\!] = E^A(I)$.

(ii) $[\![\neg\phi]\!] = [\![\phi]\!]^\perp$.

(iii) $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \wedge [\![\phi_2]\!]$.

(iv) $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \vee [\![\phi_2]\!]$.

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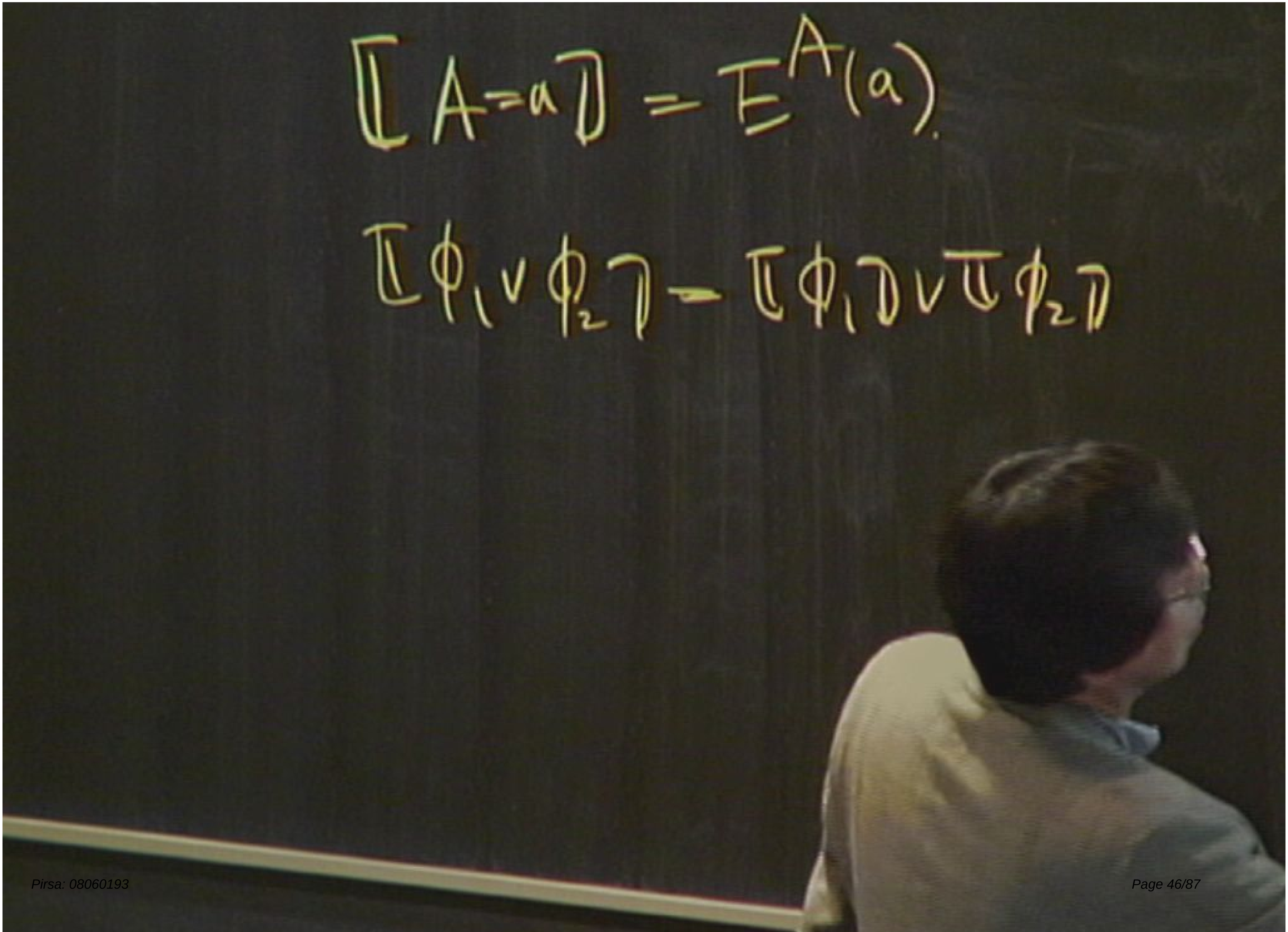
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11 Statistical formula

Axiom 5 **(Statistical formula) The probability of an observational proposition ϕ being true in a state ρ is given by**

$$\Pr\{\phi|\rho\} = \mathrm{Tr}[[\![\phi]\!]\rho].$$

- **Definition: We say that observational proposition ϕ is true in state ρ if $\Pr\{\phi|\rho\} = 1$.**
- **We obtain the following.**

$$\Pr\{A_1 = a_1 \wedge \cdots \wedge A_n = a_n|\rho\} = \mathrm{Tr}[E^{A_1}(a_1) \wedge \cdots \wedge E^{A_n}(a_n)\rho] \tag{6}$$

- **If $A_1, \ldots, A_n$ are commuting, we have**

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12 Spin 1/2 particles

- **The x, y, z components of a particle with spin 1/2 are given by**

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$(\hbar = 2)$.

- **For any $a, b = \pm 1$ we have**

$$[\![\sigma_z = a \wedge \sigma_x = b]\!] = 0.$$

For any state ρ, we have

$$\Pr\{\sigma_z = a \wedge \sigma_x = b | \rho\} = 0.$$

13 Counter example of distributive law

- **From $[\![\sigma_x = 1 \vee \sigma_x = -1]\!] = I$, we have**

$$[\![\sigma_z = 1 \wedge (\sigma_x = 1 \vee \sigma_x = -1)]\!] = [\![\sigma_z = 1]\!] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand, we have

$$\begin{aligned} &[\![(\sigma_z = 1 \wedge \sigma_x = 1) \vee (\sigma_z = 1 \wedge \sigma_x = -1)]\!] \\ &\quad = [\![\sigma_z = 1 \wedge \sigma_x = 1]\!] \vee [\![\sigma_z = 1 \wedge \sigma_x = -1]\!] \\ &\quad = 0, \end{aligned}$$

so that the distributive law does not hold.

14 Contextuality of values of observables

- The range of values of observables depends on the context as to what observables are considered simultaneously.

- Example:

$$
\begin{aligned}
\text{the range of } \sigma_z &= \{-1, +1\} \\
\text{the range of } \sigma_x &= \{-1, +1\} \\
\text{the range of } (\sigma_z, I) &= \{(-1, 1), (+1, 1)\} \\
\text{the range of } (\sigma_z, \sigma_x) &= \emptyset \text{ (empty set)}
\end{aligned}
$$

- In general, if $[A, B] = 0$ the range of (A, B) is $\mathrm{Sp}(A) \times \mathrm{Sp}(B)$. However, if A and B are non-commutative, it is not the case. If the range of (A, B) is empty, they can be considered as complementary (totally noncommutative).

15 Simultaneous determinateness of observables

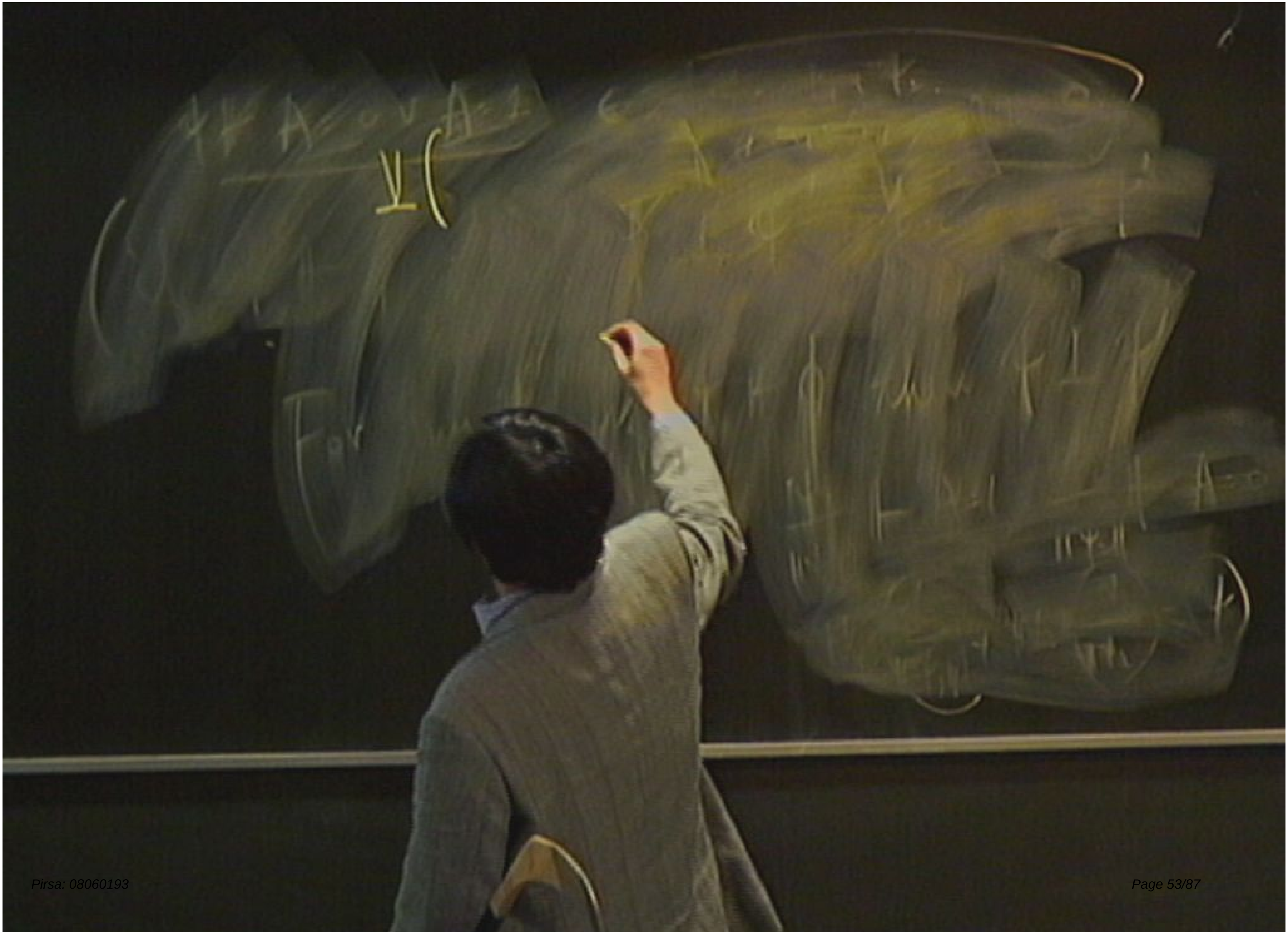
- **Definition: The symbol** $(A_1, \ldots, A_n)$ **meaning the** n**-tuple quantum ordered pair of observables** $A_1, \ldots, A_n$ **is introduced by**

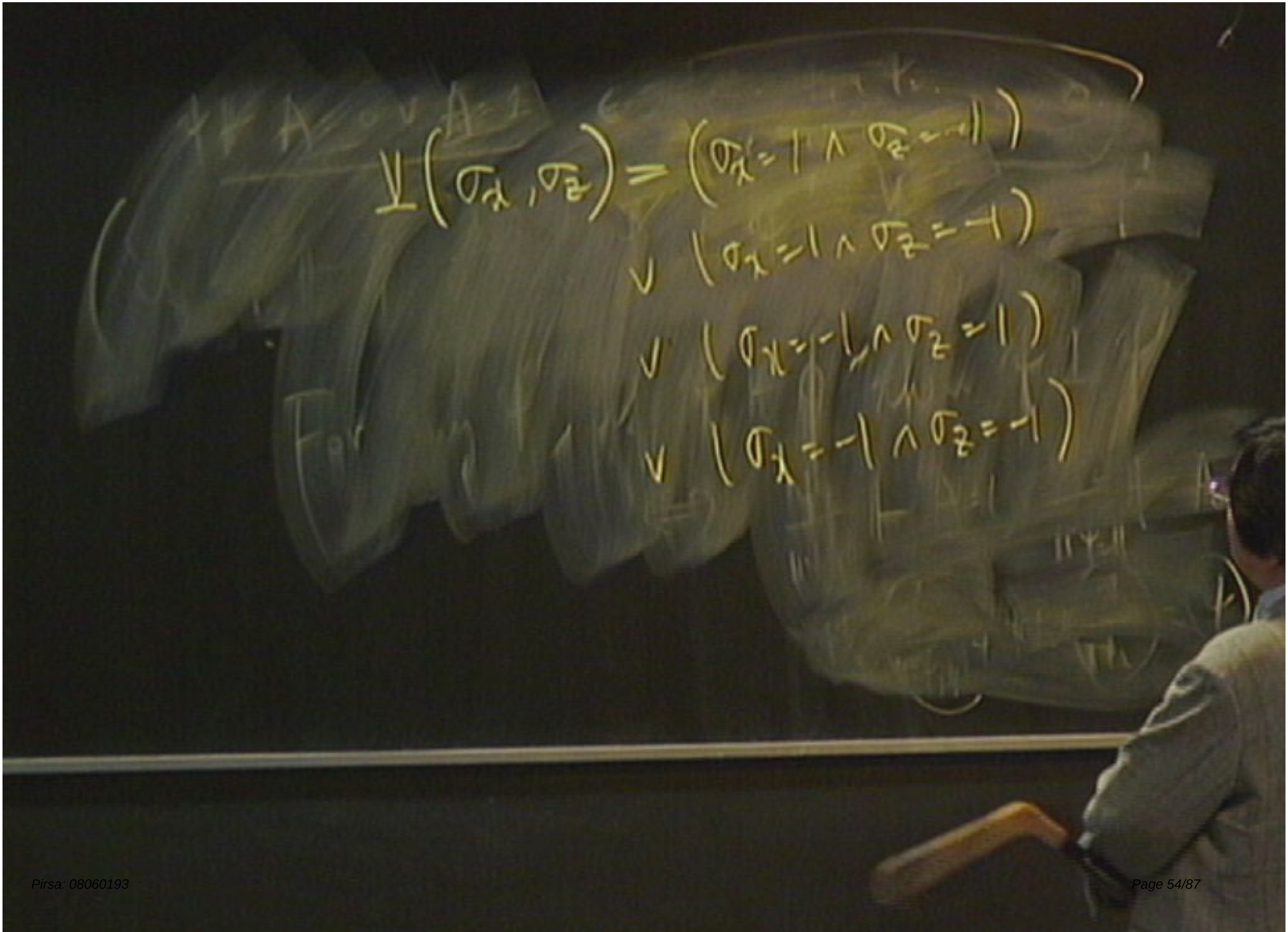
$$\text{“}(A_1, \ldots, A_n) = (a_1, \ldots, a_n)\text{”} = \text{“}A_1 = a_1 \wedge \cdots \wedge A_n = a_n\text{”}$$

- **Definition: For any observables** $A_1, \ldots, A_n$ **we define the observational proposition** $\underline{\vee}(A_1, \ldots, A_n)$ **meaning that** $A_1, \ldots, A_n$ **are simultaneously determined by**

$$\text{“}\underline{\vee}(A_1, \ldots, A_n)\text{”} = \text{“} \bigvee_{a_1 \in \mathrm{Sp}(A_1), \ldots, a_n \in \mathrm{Sp}(A_n)} (A_1, \ldots, A_n) = (a_1, \ldots, a_n)\text{”}$$

- **If** $\rho \Vdash \underline{\vee}(A_1, \ldots, A_n)$**, we say that observables** $A_1, \ldots, A_n$ **are simultaneously determined in state** ρ**.**




V(σx, σz) = (σx = 1 ∧ σz = 1)
∨ (σx = 1 ∧ σz = −1)
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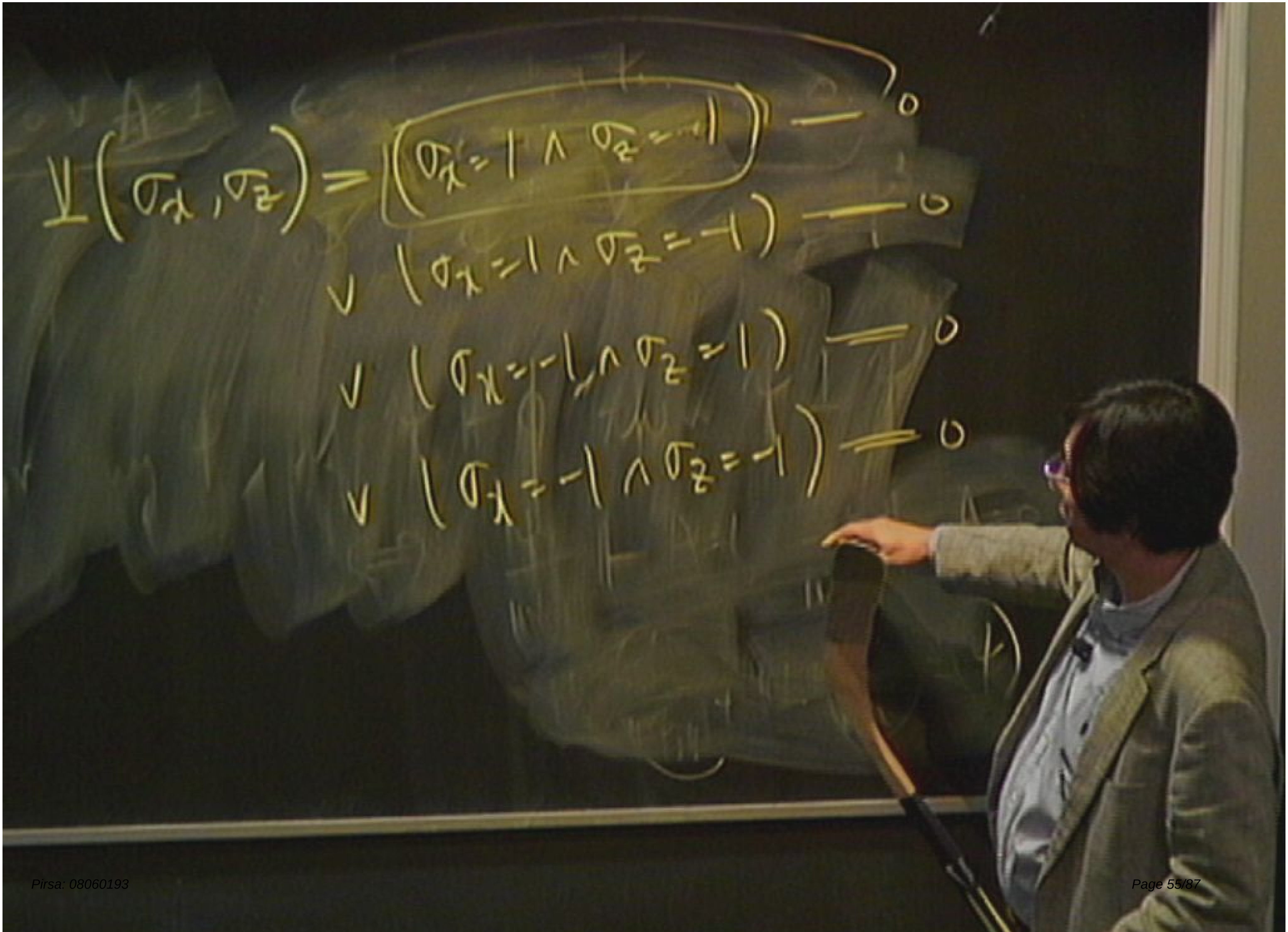
Ψ(σx, σz) ⇒ (σx = 1 ∧ σz = 1) — 0
∨ (σx = 1 ∧ σz = −1) — 0
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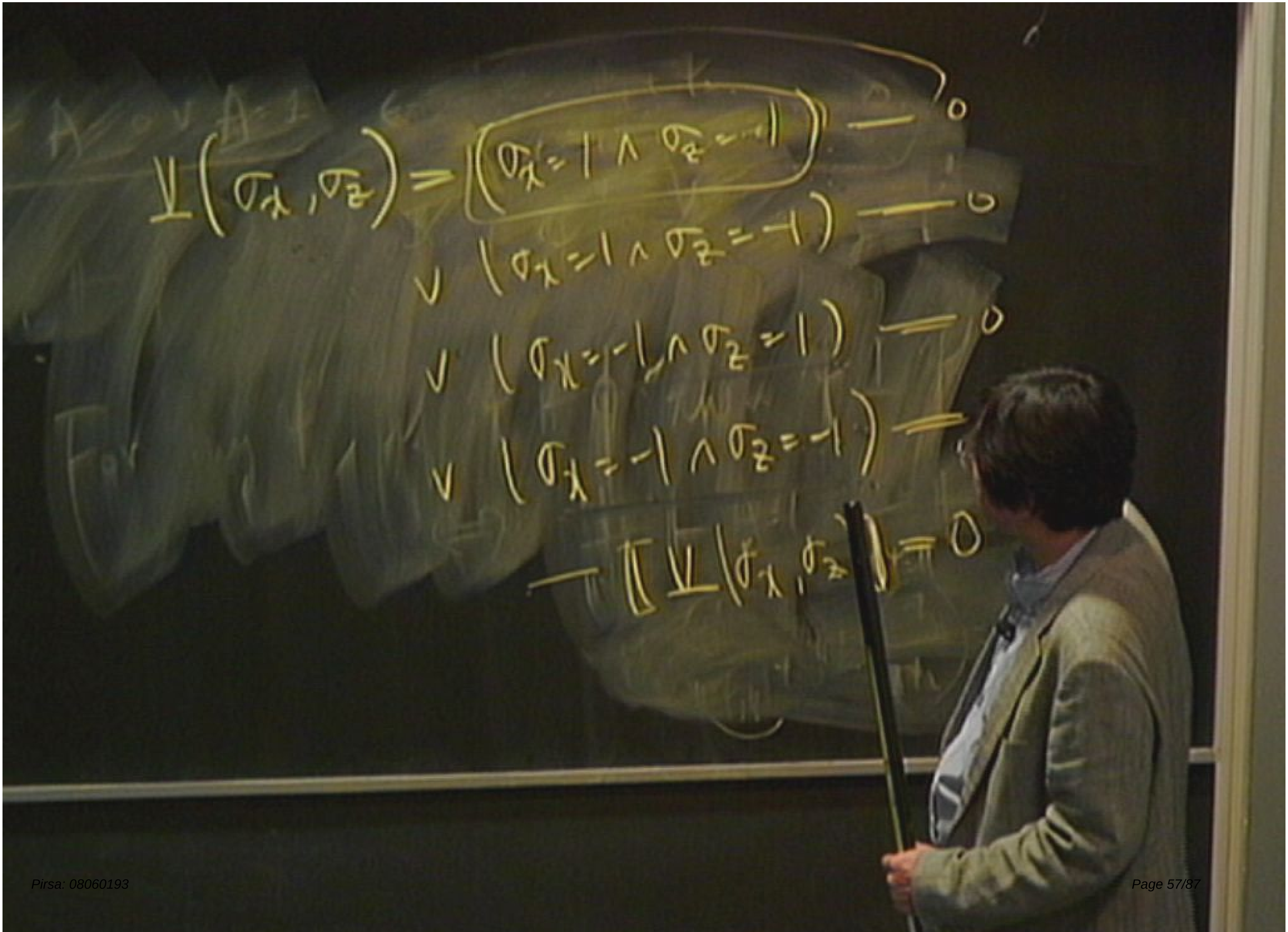
∨ (σx = −1 ∧ σz = −1) — 0

- **Proposition:** Observables $A_1, \ldots, A_n$ are simultaneously determined in a vector state ψ if and only if the state ψ is a superposition of simultaneous eigenvectors of $A_1, \ldots, A_n$.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16 Classical joint probability distribution of the values of observables

- **Definition:** n-ary function $P(x_1, \ldots, x_n)$ is called a classical joint probability distribution of observables $A_1, \ldots, A_n$ in a state ρ is

  (i) $P(x_1, \ldots, x_n) \geq 0$

  (ii) $\sum_{x_1,\ldots,x_n} P(x_1, \ldots, x_n) = 1$

  (iii) $P(a_1, \ldots, a_n) = \Pr\{(A_1, \ldots, A_n) = (a_1, \ldots, a_n)|\rho\}$

- **Theorem:** There exists the classical joint probability distribution of observables $A_1, \ldots, A_n$ in a state if and only if $\rho \Vdash \underline{\vee}(A_1, \ldots, A_n)$. In this case, for any polynomial $f(A_1, \ldots, A_n)$ of observables $A_1, \ldots, A_n$, we have

$$\mathrm{Tr}[f(A_1, \ldots, A_n)\rho] = \sum_{x_1,\ldots,x_n} f(x_1, \ldots, x_n)P(x_1, \ldots, x_n)$$

17 Equality of the values of observables

- **Definition: For any observables A, B, we define the observational proposition $A = B$ meaning that the value of A and the value of B are equal by**

$$\text{“}A = B\text{”} = \text{“}\bigvee_{a \in \mathrm{Sp}(A)} (A, B) = (a, a)\text{”}. \tag{8}$$

- **If $\rho \vdash A = B$, we say that the value of A and the value of B are equal in state ρ.**

- **Theorem: The relation $\rho \Vdash A = B$ is an equivalence relation between A and B. In particular, transitive, i.e., $\rho \Vdash A = B$ and $\rho \Vdash B = C$ imply $\rho \Vdash A = C$.**

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- **Theorem:** $[\![A_1 = A_2 \wedge \cdots \wedge A_{n-1} = A_n]\!] \leq [\![\underline{\vee}(A_1, \ldots, A_n, B_1, \ldots, B_n)]\!]$.

- **Theorem:** $\rho \Vdash A = B \Leftrightarrow$ **there exists the classical joint probability distribution** $P(x_1, x_2)$ **of** A, B **in** ρ **such that**

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19 Measurement of the value of an observable

- **Definition: We say that a measuring process $(\mathcal{K}, \xi, U, M)$ is a measurement of the value of an observable A in a state ρ if the value of the meter after measurement and the value of the observable A in the state $\rho \otimes |\xi\rangle\langle\xi|$ are equal, i.e.,**

$$\rho \otimes |\xi\rangle\langle\xi| \Vdash A \otimes I = U^\dagger (I \otimes M) U.$$

- **Theorem: A measuring process is a measurement of an observable A in a state ρ if and only if the POVM Π of $(\mathcal{K}, \xi, U, M)$ satisfies**

$$\mathrm{Tr}[\Pi(y) E^A(x) \rho] = \delta_{x,y} \mathrm{Tr}[E^A(x) \rho].$$

- **Theorem: A measuring process is a measurement of an observable A in any state if and only if $\Pi = E^A$.**

18 Measuring process

- **Definition:** A measuring process for $\mathcal{H}$ is a 4-tuple $(\mathcal{K}, \xi, U, M)$ consisting of a finite dimensional Hilbert space $\mathcal{K}$ a unit vector ξ in $\mathcal{K}$, a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, and a self-adjoint operator M on $\mathcal{K}$.

- **Definition:** The POVM Π operation T, and instrument $\mathcal{I}$ of any measuring process $(\mathcal{K}, \xi, U, M)$ is defined by

$$\Pi(x) = \mathrm{Tr}_{\mathcal{K}}[U^\dagger(I \otimes E^M(x))U(I \otimes |\xi\rangle\langle\xi|)], \tag{9}$$

$$T\rho = \mathrm{Tr}_{\mathcal{K}}[U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger], \tag{10}$$

$$\mathcal{I}(x)\rho = \mathrm{Tr}_{\mathcal{K}}[(I \otimes E^M(x))U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger]. \tag{11}$$

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20 Disturbance in measurement

- **Definition:** We say that a measuring process does not disturb the value of an observable B in a state ρ if the values of B before the measurement and after the measurement are equal, i.e.,

$$\rho \otimes |\xi\rangle\langle\xi| \Vdash B \otimes I = U^{\dagger}(B \otimes I)U.$$

- **Theorem:** A measuring process $(\mathcal{K}, \xi, U, M)$ does not disturb the value of an observable B in a state ρ if and only if the operation T of the measuring process $(\mathcal{K}, \xi, U, M)$ satisfies

$$\mathrm{Tr}[T^{\dagger}(E^{B}(y))E^{B}(x)\rho] = \delta_{x,y}\mathrm{Tr}[E^{B}(x)\rho].$$

21 Simultaneous measurement

- **Definition: We say that a measuring process** $(\mathcal{K}, \xi, U, M)$ **and polynomials** $f_1, \ldots, f_n$ **is a simultaneous measurement of values of observables** $A_1, \ldots, A_n$ **in a state** ρ **if**

$$\rho \otimes |\xi\rangle\langle\xi| \Vdash A_j \otimes I = U^\dagger (I \otimes f(M_j)) U,$$

where $j = 1, \ldots, n$**.**

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$$\mathrm{Tr}[\Pi(f_j^{-1}(y)) E^{A_j}(x)\rho] = \delta_{x,y} \mathrm{Tr}[E^{A_j}(x)\rho],$$

where $j = 1, \ldots, n$**.**

22 Successive measurement and simultaneous measurement

- **Definition:** We say that a successive measurement carried out by a sequence of measuring processes $\mathbf{M}_j = (\mathcal{K}_j, \xi_j, U_j, M_j)$ $(j = 1, \ldots, n)$ is a simultaneous measurement of observables $A_1, \ldots, A_n$ in a state ρ, if for any $j = 1, \ldots, n$ we have

$$\rho \otimes |\xi_1\rangle\langle\xi_1| \otimes \cdots \otimes |\xi_j\rangle\langle\xi_j| \vdash I \otimes \cdots I \otimes A_j = \tilde{U}_1^\dagger \cdots \tilde{U}_j^\dagger (I \otimes \cdots \otimes M_j) \tilde{U}_j \cdots \tilde{U}_1$$

where $\tilde{U}_j$ is the extension of U_j to $\otimes_{j=1}^n \mathcal{K}_j$.

- **Theorem:** In the sequence of measuring processes $\mathbf{M}_j = (\mathcal{K}_j, \xi_j, U_j, M_j)$ $(j = 1, \ldots, n)$, if $\mathbf{M}_1, \ldots, \mathbf{M}_j$ does not disturb the values of observables $A_j, \ldots, A_n$ in a state ρ, then the successive measurement carried out by that sequence is a simultaneous measurement of $A_1, \ldots, A_n$ in ρ.

23 EPR measurement

- The observables $\sigma_z \otimes I$ and $\sigma_x \otimes I$ in the composite system of two spin 1/2 particles are simultaneously determinate in no states.

- However, they are simultaneously measurable in the following state ψ:

$$\psi = \frac{1}{2}(|\sigma_x = +1\rangle|\sigma_x = +1\rangle + |\sigma_x = -1\rangle|\sigma_x = -1\rangle)$$

- To show this let $A = \sigma_z \otimes I$, $B = \sigma_x \otimes I$, $C = I \otimes \sigma_x$. Since A and C are commuting operators, there is an measuring process $(\mathcal{K}, \xi, U, M)$ and functions f, g. Then, we have

$$\psi \otimes \xi \Vdash A(0) = f(M(\Delta t)), \quad \psi \otimes \xi \Vdash C(0) = g(M(\Delta t)).$$

On the other hand, by a property of the state ψ we have

$$\psi \otimes \xi \vdash C(0) = B(0).$$

By the transitivity of quantum equality, we have

$$\psi \otimes \xi \Vdash B(0) = g(M(\Delta t)).$$

Thus, A, B are measured simultaneously.

24 Difference between the simultaneous determinateness and simultaneous measurability

- **Definition (Projection onto the cyclic subspace):**

$$
\begin{aligned}
C(A,\psi) &= \mathcal{P}\{f(A)\psi \mid f(x)\text{: \textbf{polynomial}}\ \} \\
C(A,B,\psi) &= \mathcal{P}\{f(A,B)\psi \mid f(x,y)\text{: \textbf{polynomial}}\}
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- **Theorem: Two observables A, B are simultaneously determinate in a state $\psi \Leftrightarrow$**
  There exists a POVM $\Pi(x,y)$ on $\mathbf{R}^2$ such that

$$\begin{aligned} \sum_y \Pi(x,y) C(A,B,\psi) &= E^A(x) C(A,B,\psi), \\ \sum_x \Pi(x,y) C(A,B,\psi) &= E^B(y) C(A,B,\psi). \end{aligned}$$

- **Theorem: Two observables A, B are simultaneously measurable in a state $\psi \Leftrightarrow$**
  There exists a POVM $\Pi(x,y)$ on $\mathbf{R}^2$ such that

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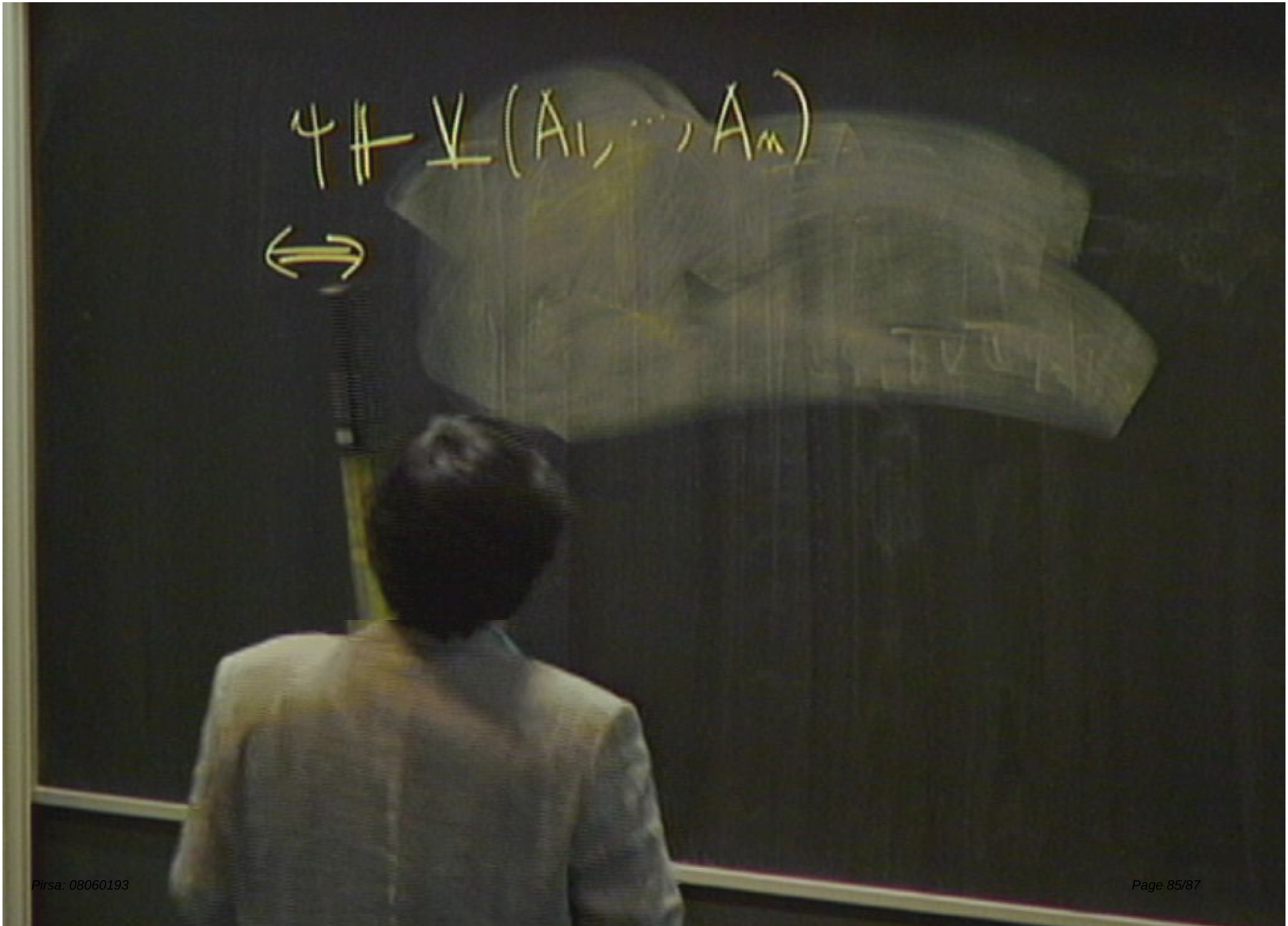
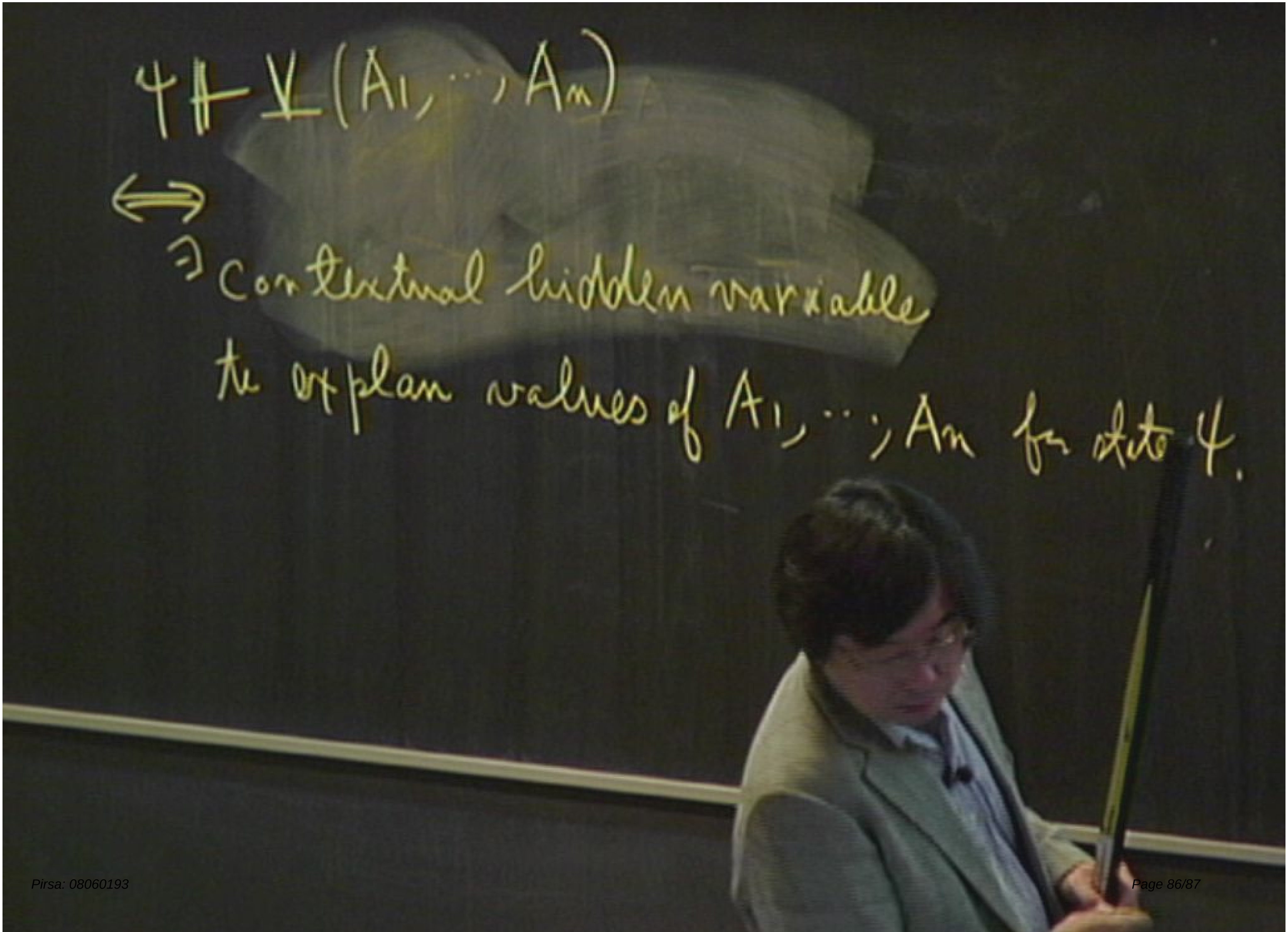

ψ ⊮ ⋁(A1, …, Am)
⟺


ψ ⊮ V(A1, ..., An)
⟺
∃ contextual hidden variable
to explain values of A1, ..., An for state ψ.

10 Truth value and probability of observational propositions

- **The truth value of an observational proposition ϕ is defined by**

$$[\![\phi]\!] = \mathcal{P}\{\psi \in \mathcal{H} \mid \psi \neq 0 \Rightarrow \frac{\psi}{\|\psi\|} \Vdash \phi\}.$$

- **Theorem: Truth values of observational propositions ϕ satisfies the following.**

(i) $[\![A \in I]\!] = E^A(I)$.

(ii) $[\![\neg\phi]\!] = [\![\phi]\!]^{\perp}$.

(iii) $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \wedge [\![\phi_2]\!]$.

(iv) $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \vee [\![\phi_2]\!]$.