Title: Stochastic Inflation and Replica Field Theory

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Abstract: We formulate a general replica field-theoretic framework for stochastic inflation in which a manifestation of dimensional reduction is found. The scale above which the latter becomes dominant is explicitly calculated. It is found that for a wide class of stochastic systems considerable changes of the spectral index inevitably occur on super-horizon scales. We work out an explicit relation between the noise correlator and the non-Gaussianity parameter f_{NL}, which thus can effectively be generated entirely by quantum fluctuations.

Stochastic Inflation and Replica Field Theory

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arXiv:0805.1998 [gr-qc]

with Dominik Schwarz

Overview

- Stochastic inflation
  - General idea
  - Statement of the problem
- The methods:
  - Replica trick
  - Gaussian variational method
- Again stochastic inflation
  - ... in the light of replica field theory
  - Results

Stochastic Inflation I

Consider a free, minimally coupled, real scalar test field φ with mass μ, in d-dimensional space-time with metric $g_{\mu\nu}$.

Focus on de Sitter space-time:

$$(g_{\mu\nu}) = \mathrm{diag}\big(-1, a^2(t), \ldots, a^2(t)\big), \quad a(t) = \mathrm{e}^t, \quad H \stackrel{!}{=} 1$$

Split into long and short wavelengths:

$$\boxed{\varphi = \varphi_\mathsf{L} + \varphi_\mathsf{S}}$$

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where

$$\varphi_{\mathsf{S}}(t, \mathbf{x}) = \int \mathrm{d}^{d-1}k \, \mathrm{W}_\kappa\left(\frac{k}{a(t)} - \epsilon\right) \left[\hat{\mathrm{a}}(k)\, u(t, k)\, \mathrm{e}^{-\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} + \mathrm{h.c.}\right]$$

with $\mathrm{W}_\kappa \xrightarrow{\kappa \to 0} \Theta$, where $0 < \epsilon \ll 1$.

Stochastic Inflation II

Think of φ_S as generating a (quantum) bath in which φ_L evolves.

[Starobinsky '85]

The field equation $(\Box + \mu^2)\,\varphi = 0$ implies

$$\boxed{(\Box + \mu^2)\varphi_L = \mathrm{h}}$$

where h includes φ_S and is a Gaussian stochastic variable
$\rightsquigarrow$ Gaussian random force $\quad\leftrightarrow\quad$ random potential $\mathrm{V}_D(\varphi_L)$

This captures the leading-log$[a(t)]$ contribution to $\epsilon, p, \ldots$.

[Woodard, et al. '05]

So far so good...

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Arbitrary interactions / non-linear noise potentials and gradient terms?

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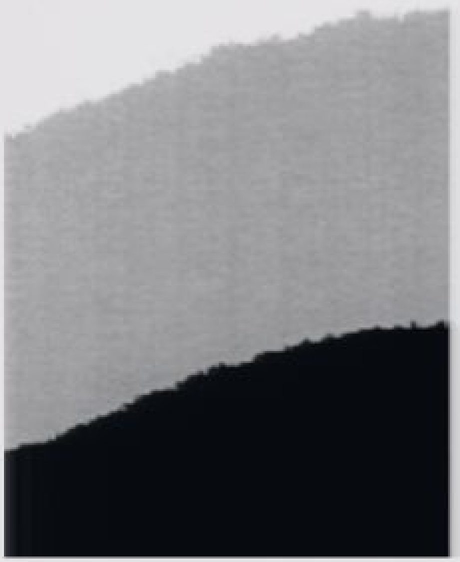
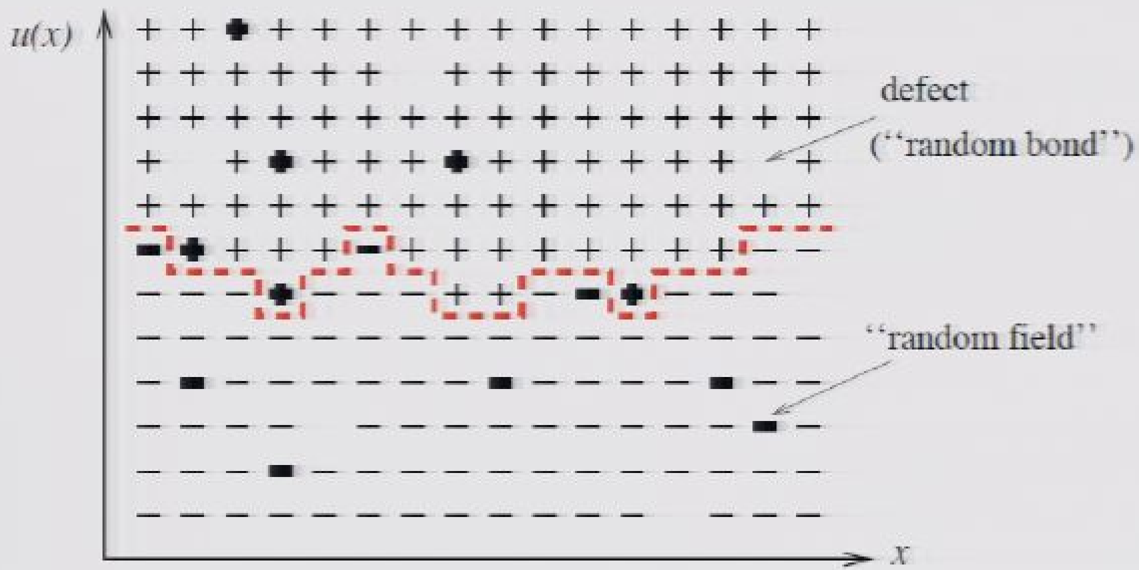
Arbitrary interactions / non-linear noise potentials and gradient terms?

$\rightsquigarrow$ Use replica trick and Gaussian variational method! [F.K., Schwarz '08]

New Methods ... from Statistical Physics





(from Kay Wiese)

Replica Trick

Observables are averages.

Average $\overline{\mathcal{F}} = -T\,\overline{\ln\{\mathcal{Z}\}}$ is complicated.

$\rightsquigarrow$ Use replica trick

$$\overline{\mathcal{F}} = -T \lim_{n\to 0} \frac{1}{n} \ln\left\{\overline{\mathcal{Z}^n}\right\}$$

That means: compute

$$\overline{\mathcal{Z}^n} = \int \prod_{a=1}^{n} \mathcal{D}[\vec{u}_a]\, \overline{\exp\left\{-\beta \sum_{b=1}^{n} \mathcal{H}[\vec{u}_b]\right\}}$$

for $n \in \mathbb{N}$, and perform limit $n \to 0$ at the end.

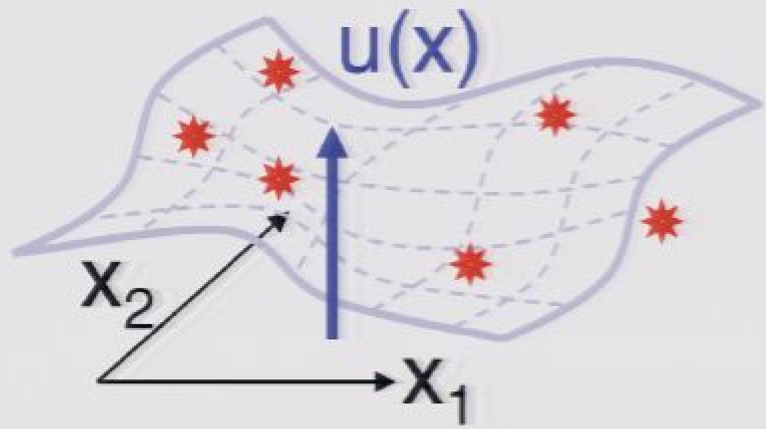
Cumulants

Suppose $\mathrm{V}_D(x, \vec{u}_a(x))$ is Gaussian distributed with

$$\overline{\mathrm{V}_D(x, \vec{u}_a(x))} = 0$$

$$\overline{\mathrm{V}_D(x, \vec{u}_a(x))\,\mathrm{V}_D(y, \vec{u}_b(y))} = \phi(x-y)\,\mathrm{R}(\vec{u}_a(x)\cdot\vec{u}_b(y))$$

Special case I: $\phi(x-y) = \delta^{(d-1)}(x-y) \quad \rightsquigarrow \quad$ point-like disorder



Special case II: $\mathrm{V}_D(\vec{u}_a) \sim \vec{u}_a \quad \rightsquigarrow \quad \mathrm{R}(\vec{u}_a\cdot\vec{u}_b) \sim \vec{u}_a\cdot\vec{u}_b$

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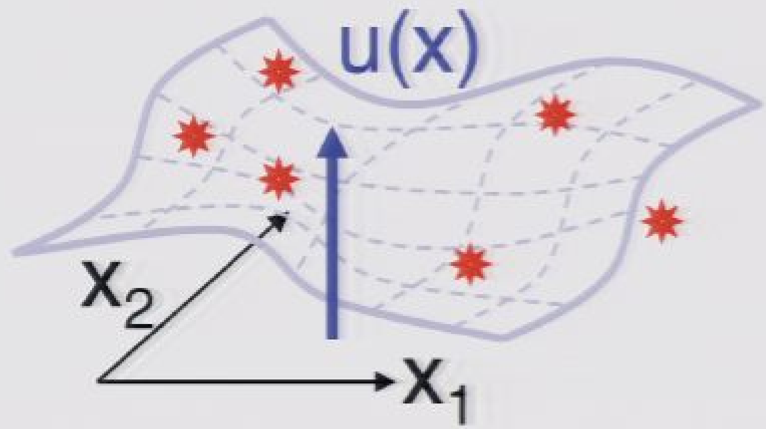

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Gaussian Variational Method

Any trial Hamiltonian $\mathcal{H}_0 := \frac{1}{2}\sum_{ab}\int_k \mathrm{G}^{-1}{}_{ab}(k)\, \vec{u}_a(k) \cdot \vec{u}_b(-k)$ implies

$$\mathcal{F} \leq \mathcal{F}_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_{\mathcal{H}_0} =: \mathcal{F}_{var} \quad \text{Feynman-Jensen}$$

with $\mathcal{F}_{(0)} := -T \ln \mathrm{Tr}\Big\{ \exp\{-\beta\, \mathcal{H}_{(0)}\}\Big\}$

Make the ansatz

$$\mathrm{G}^{-1}{}_{ab}(k) := \mathrm{G}_0^{-1}(k)\, \delta_{ab} - \sigma_{ab}$$

Use self-energy-matrix (σ_{ab}) as a variational parameter.

One finds

$$\boxed{\sigma_{ab}(p) = \int_x \phi(x)\, \mathrm{e}^{\mathrm{i}px}\, \widehat{\mathrm{R}}'\Big(\int_k \mathrm{e}^{\mathrm{i}kx} \mathrm{G}_{ab}(k)\Big)}$$

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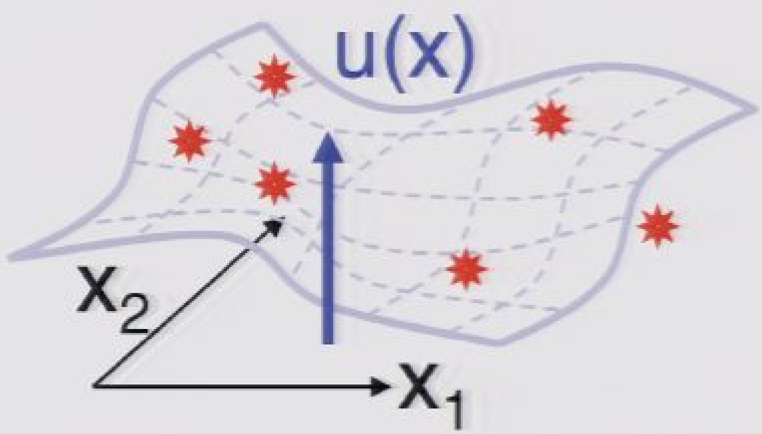

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Application to Stochastic Inflation: Results I

Important cosmological observable: Power spectrum $\mathrm{P}(k)$

Relation to propagator of φ_{L}: $\mathrm{P}(k) \sim k^{d-1}\,\mathrm{G}(k) \sim k^{n-1}$

Presented methods imply

$$\boxed{\mathrm{G}(t,k) = \mathrm{G}_0(t,k) + \sigma(t,k)\mathrm{G}_0^2(t,k)}$$

$$d = 4 \Rightarrow n: \qquad 1 + \frac{2}{3}\mu^2 \qquad -3 + 2\,\mu^2 \qquad + \quad \mathcal{O}(\mu^4)$$

$\rightsquigarrow$ new dimensional reduction part $\leftrightarrow$ $\begin{cases} \text{modeling non-gaussianity!} \\ \text{violation of scale invariance!} \end{cases}$

Determine scale of equal balance $k_*(t)$:

$$\mathrm{G}_0(t,k_*) \stackrel{!}{=} \sigma(t,k_*)\mathrm{G}_0^2(t,k_*)$$

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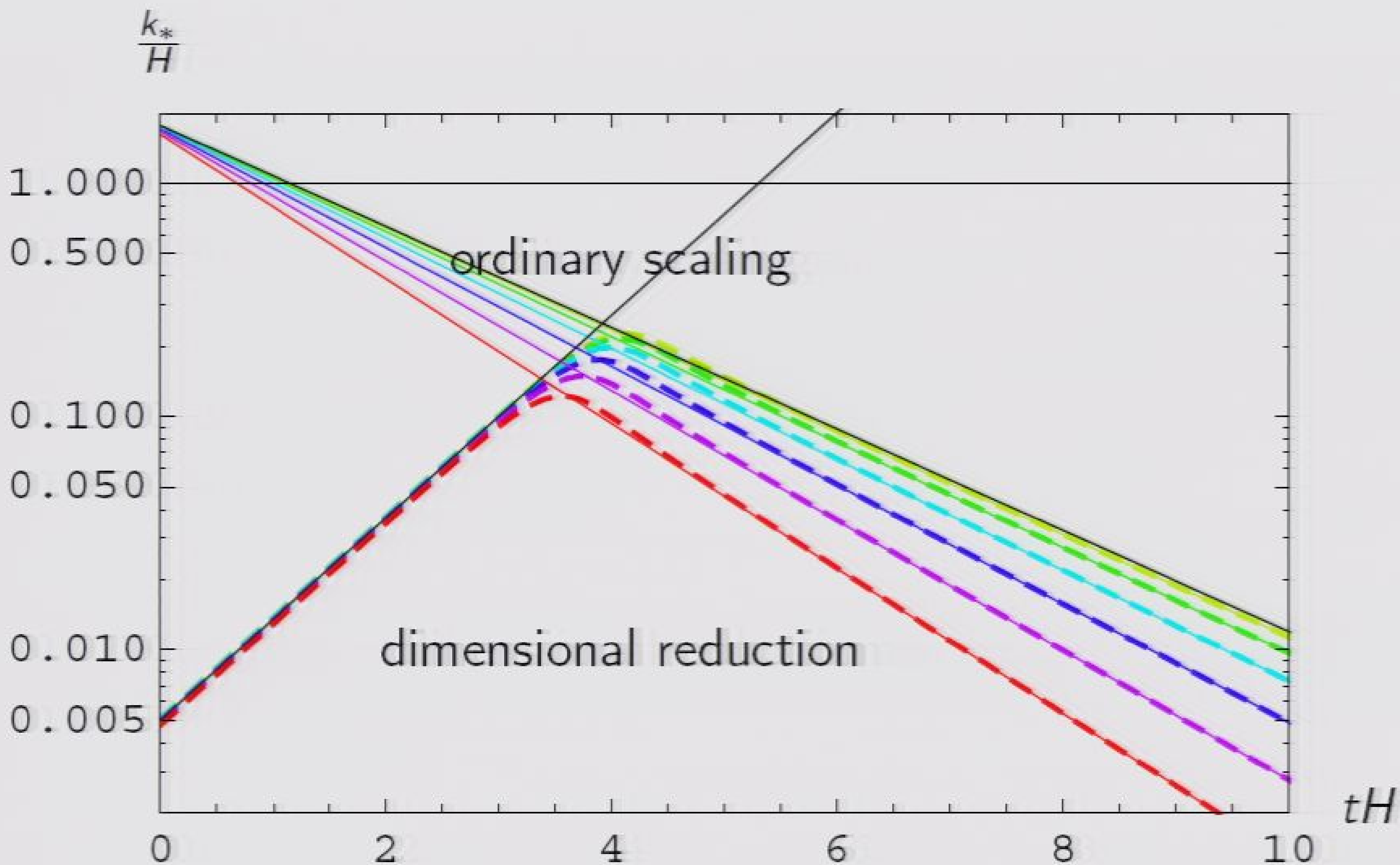
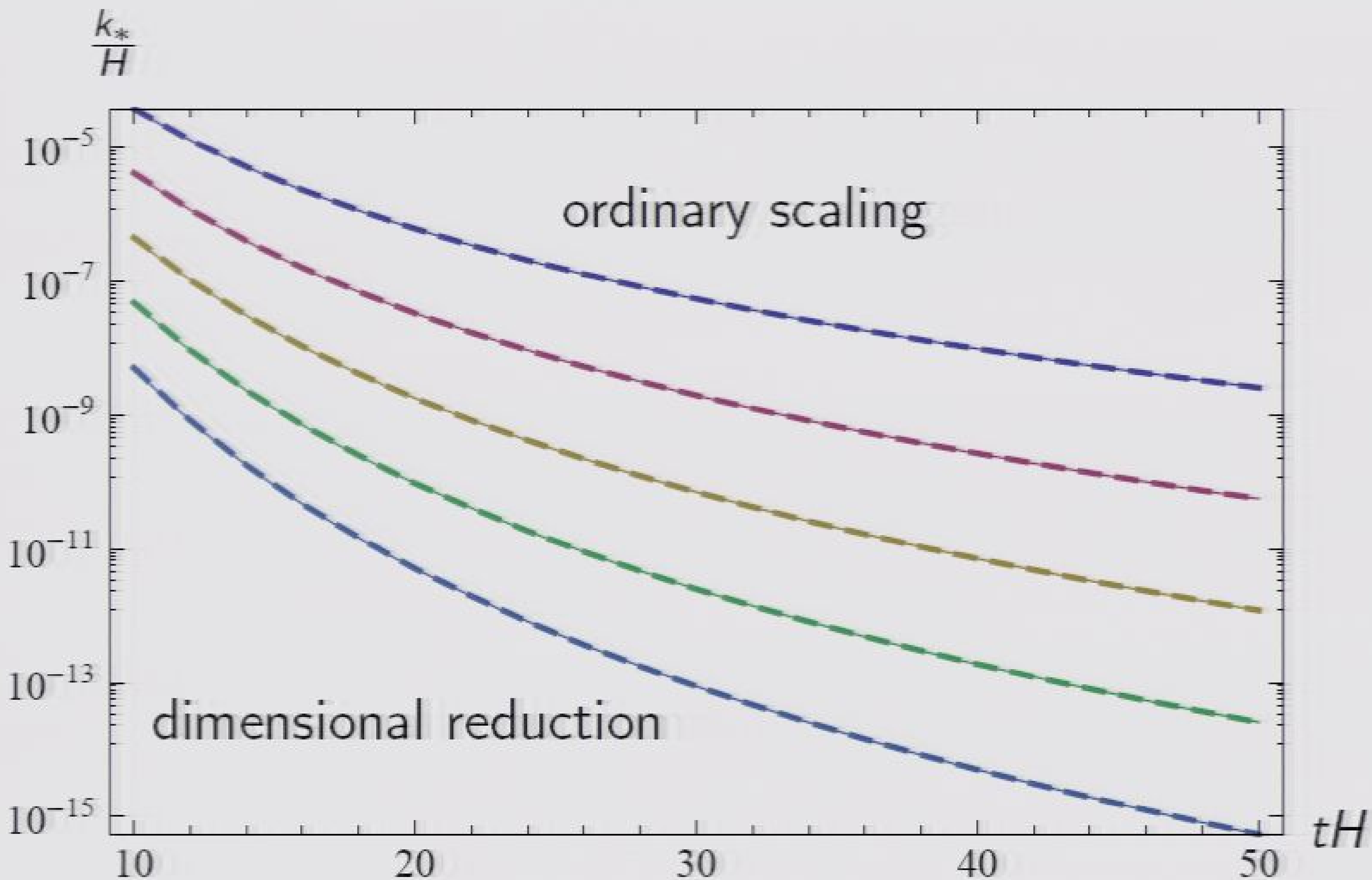
Results II (de Sitter)



Results III (power-law)



Summary and Outlook

- Application of replica field theory to stochastic inflation
- Cosmological dimensional reduction
  $\rightarrow$ deviation from scale invariance
- Modelling non-Gaussianities $(\rightarrow f_{NL}(t,k))$
- Time behaviour of scale $k_*(t)$

- Currently working on
  - Other geometries (e.g. power-law inflation)
  - Dependency on filter functions W_κ and parameters (ϵ, κ)
  - Arbitrary potentials $\leftrightarrow$ Replica Symmetry Breaking

- Open problem: stochastic geometry

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