Title: Numerical Studies of Graphity Models

Date: Jun 12, 2008 02:00 PM

URL: http://pirsa.org/08060003

Abstract: Graphity models are characterized by configuration spaces in which states correspond to graphs and Hamiltonians that depend on local properties of graphs such as degrees of vertices and numbers of shortcycles. It has been argued that such models can be useful in studying how an extended geometry might emerge from a background independent dynamical system. As statistical systems, graphity models can be studied analytically by estimating their partition functions or numerically by Monte Carlo simulations. In this talk I will present recent results obtained using both of these approaches. In particular, I will describe the transition between the high and low temperature regimes and arguethat matter degrees of freedom must play an important role in order for the graph states dominating in the low temperature regime to resemble interesting extended geometries.

Numerical Studies of Graphity Models

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Universiteit Utrecht

Talk prepared for:
Perimeter Institute
June 12, 2008

Based on: TK, arXiv: 0805.2283

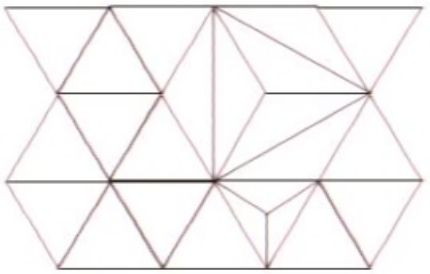
Also see: TK, F. Markopoulou, and S. Severini, arXiv: 0801.0861, PRD (2008).
TK, F. Markopoulou, and L. Smolin, arXiv: hep-th/0611197

Motivation

Why graph(ity) models?

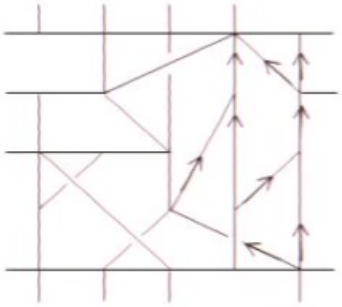
lattice-based approaches:
Lattice QFT, …

e.g. J. Smit, CUP



triangulation-based approaches:
(C)DT, spin foams, …

e.g. J. Ambjorn, J. Jurkiewicz, R. Loll, hep-th/0505154;
J. C. Baez, gr-qc/9905087.



approaches based on non-triangulations:
randomized lattices, matrix models, causal structures, …

e.g. D. Oriti, 0710.3276;
J. Henson, gr-qc/0601121

General graphs are the pre-cursor structure used in many approaches to studying dynamical geometries.

Purpose

The goal of graphity models is to select a background geometry dynamically from the space of all possible graphs.

TK. F. Markopoulou, and L. Smolin, arXiv: hep-th/0611197

TK. F. Markopoulou, and S. Severini, arXiv: 0801.0861

Assumptions	Non-Assumptions
The existence of a microlocal weighting function (e.g. an action or a Hamiltonian)	existence of an underlying manifold
	existence of a dual triangulation

Motivation

Why numerical studies?

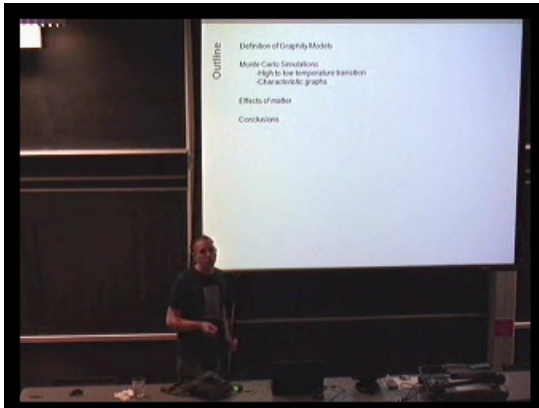
Most models are defined in terms of a path integral or partition function.

$$Z = \int \mathcal{D}[g]\,\mathcal{D}[\phi]\, e^{-S[g,\phi]}$$

$$\langle O_g \rangle = \frac{1}{Z} \int \mathcal{D}[g]\,\mathcal{D}[\phi]\, O_g\, e^{-S[g,\phi]}$$

Except in some very simple circumstances, it is not possible to evaluate Z or $\langle O_g \rangle$ analytically – numerical methods are thus necessary.

e.g. J. D. Christensen, E. R. Livine and S. Speziale, arXiv:0710.0617.



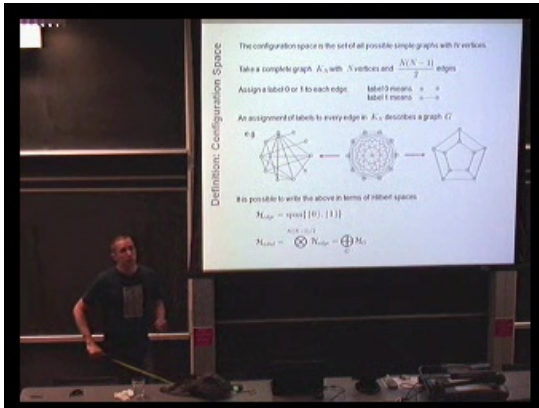
Outline

Definition of Graphity Models

Monte Carlo Simulations
- -High to low temperature transition
- -Characteristic graphs

Effects of matter

Conclusions



Definition: Configuration Space

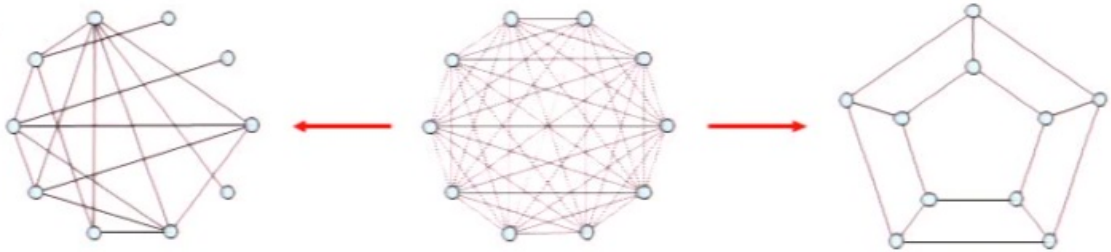
The configuration space is the set of all possible simple graphs with N vertices.

Take a complete graph K_N with N vertices and $\frac{N(N-1)}{2}$ edges

Assign a label 0 or 1 to each edge: label 0 means ○ ○
label 1 means ○—○

An assignment of labels to every edge in K_N describes a graph G

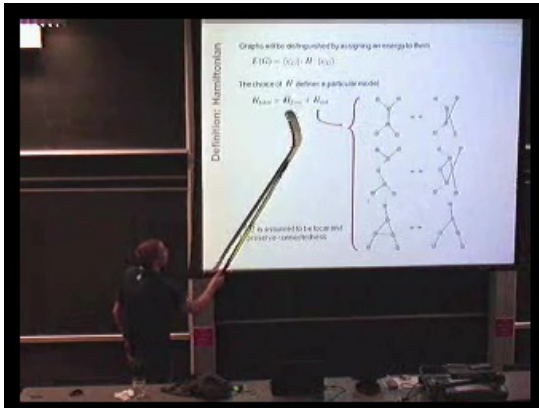
e.g.



It is possible to write the above in terms of Hilbert spaces

$$\mathcal{H}_{edge} = \mathrm{span}\{|0\rangle, |1\rangle\}$$

$$\mathcal{H}_{total} = \bigotimes^{N(N-1)/2} \mathcal{H}_{edge} = \bigoplus_{G} \mathcal{H}_G$$



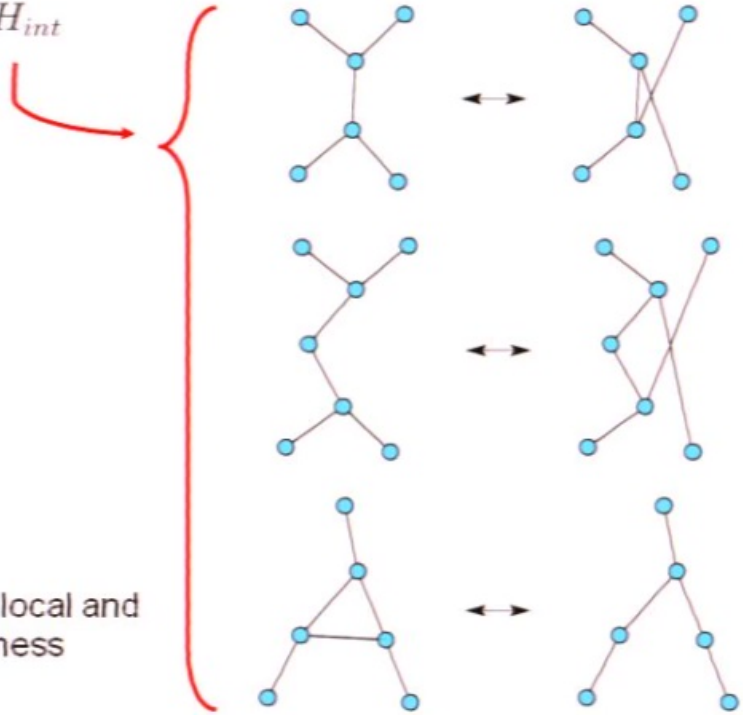
Definition: Hamiltonian

Graphs will be distinguished by assigning an energy to them.

$$E(G) = \langle \upsilon_G | : H : | \upsilon_G \rangle$$

The choice of H defines a particular model.

$$H_{total} = H_{free} + H_{int}$$



H_{int} is assumed to be local and to preserve connectedness

Definition: Hamiltonian

$$H_{free} = H_V + H_B$$

"Valence Term"

general form

$$H_V = g_V \sum_a f_a \left(\sum_b N_{ab}, v_0 \right)$$

a particular choice

$$H_V = g_V \sum_a e^{p\left(v_0 - \sum_b N_{ab}\right)^2}$$

"Cycle Term"

general form

$$H_B = \sum_a H_{B_a}$$

a particular choice

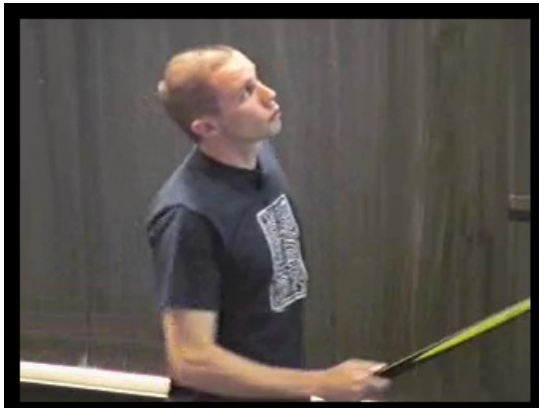
$$H_{B_a} = -g_B \sum_b \delta_{ab}\, e^{r N_{ab}}$$

where $e^{r N_{ab}} = \sum_{L=0}^{\infty} \frac{r^L}{L!} N_{ab}^{(L)}$

Thus the energy is given by

$$E(G) = \left(\sum_a g_V e^{p(v_a - v_0)^2} \right) - \left(\sum_a \sum_L g_B(L) P(a, L) \right)$$

With $P(a, L)$ the number of cycles of length L at vertex a, and $g_B(L) = \frac{r^L}{L!} g_B$



Statistical Interpretation

A graphity model can be treated like any other problem in statistical mechanics.

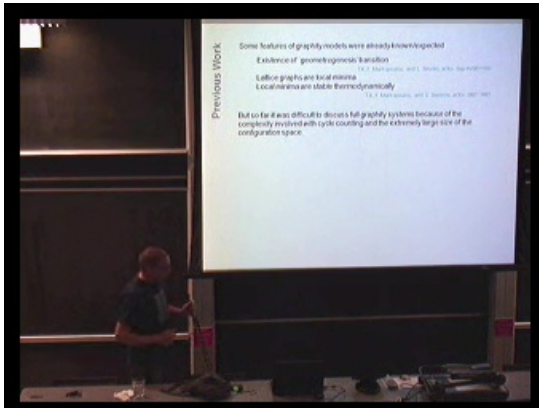
$$Z = \sum_G e^{-\beta E(G)} \qquad \beta = \frac{1}{k_B T} \text{ is the inverse temperature}$$

The basic model can also be supplemented with other (matter) degrees of freedom.

$$\begin{aligned} Z &= \sum_G \sum_M e^{-\beta E(G) - \beta E_M(G,M)} \\ &= \sum_G z(G)\, e^{-\beta E(G)} \\ &= \sum_G e^{-C(G)} \qquad \text{where } C(G) = \beta E(G) - \ln z(G) \end{aligned}$$

Within the statistical interpretation, one can try to compute expectation values for observables, transition temperatures, etc.

One of the primary questions to address is whether a graphity model contains a phase wherein the expected graph state approximates a manifold.



Previous Work

Some features of graphity models were already known/expected

Existence of `geometrogenesis' transition

TK, F. Markopoulou, and L. Smolin, arXiv: hep-th/0611197

Lattice graphs are local minima
Local minima are stable thermodynamically

TK, F. Markopoulou, and S. Severini, arXiv: 0801.0861

But so far it was difficult to discuss full graphity systems because of the complexity involved with cycle counting and the extremely large size of the configuration space.

Monte Carlo Simulations

Monte Carlo simulations are a numerical method for approximating a partition function and expectation values in a complicated model.

Monte Carlo simulations can also be used to optimize (minimize) some function.

For (classical) graphity models this means:

- Specify $N,\ v_0,\ r,\ \beta,$ (and μ)
- Specify an initial graph state
- Run a simulation for long enough
- Collect and analyze the results

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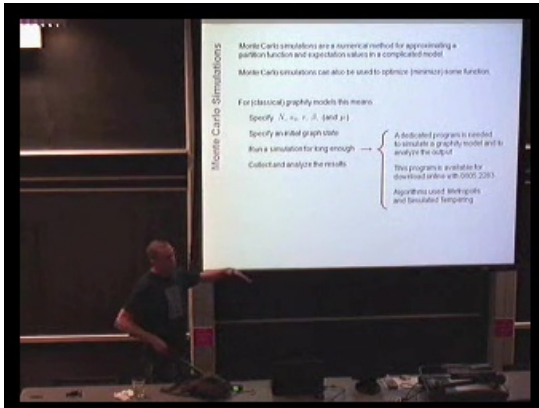

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For (classical) graphity models this means:

- Specify N, v_0, r, β, (and μ)
- Specify an initial graph state
- Run a simulation for long enough → A dedicated program is needed to simulate a graphity model and to analyze the output
- Collect and analyze the results

This program is available for download online with 0805.2283

Algorithms used: Metropolis and Simulated Tempering

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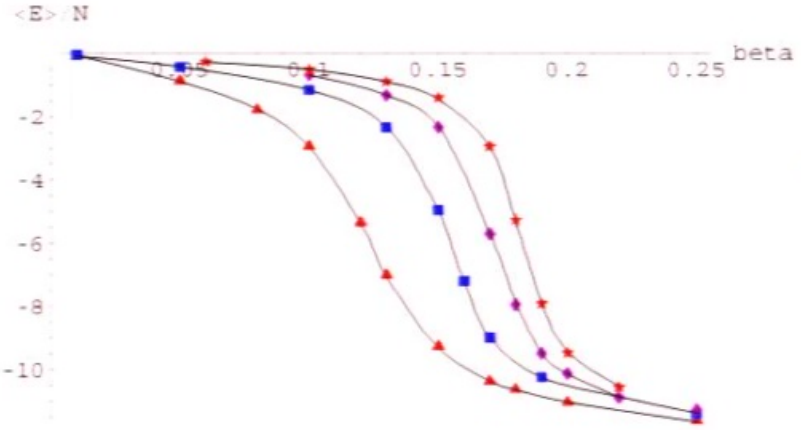
Low and High T Regimes

Results from simulations: $v_0 = 3$ $r = -2.5$ $N = 60.\ 120.\ 180.\ 240$

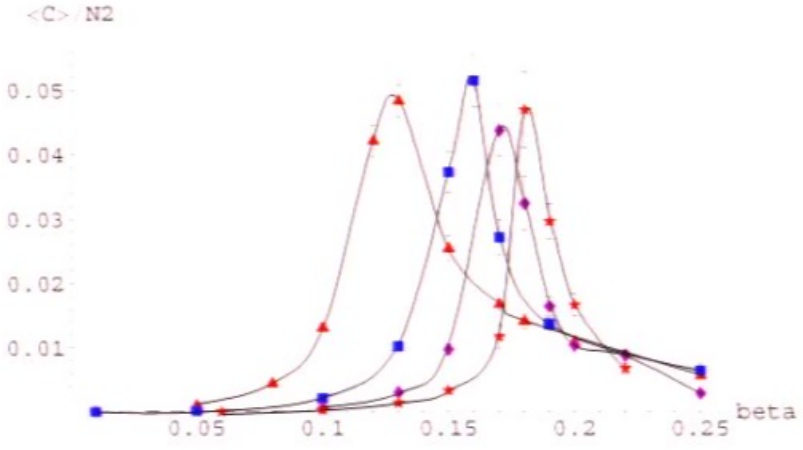


As N increases:

Transition becomes sharper

Transition β increases

Peaks in specific heat indicate possible phase-transition



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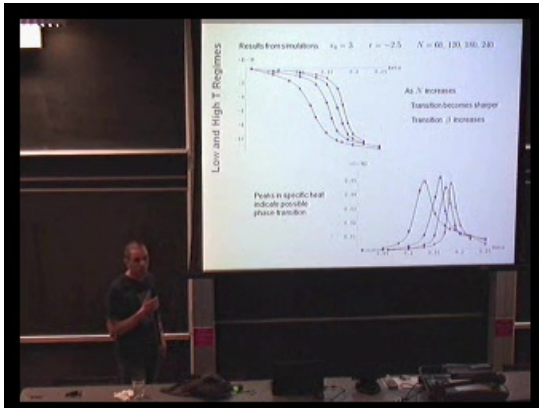

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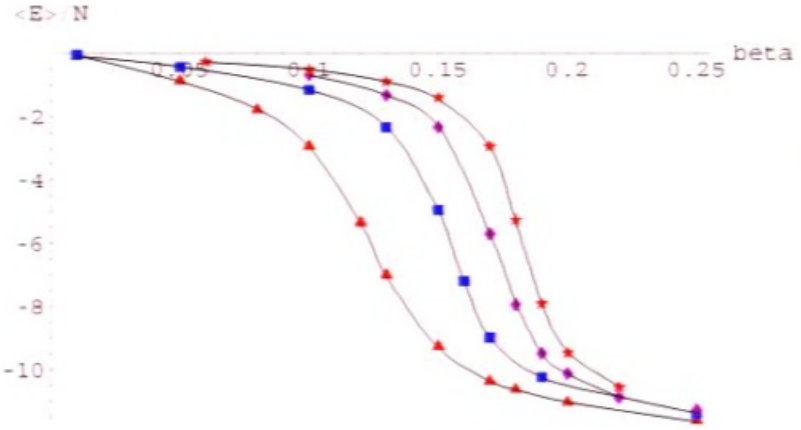

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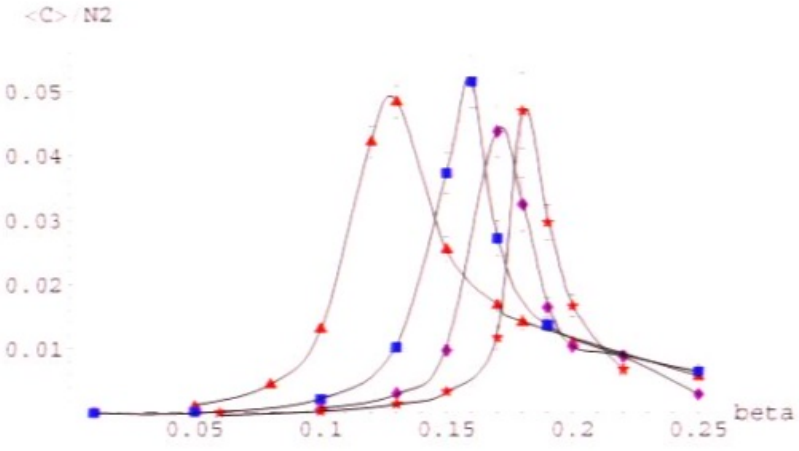


Peaks in specific heat indicate possible phase-transition

Derivation of transition β

Some features found in the simulations can be understood analytically.

The partition function can be written as a sum of terms that describe contributions of various classes of graphs

$$Z = \sum_G e^{-\beta E(G)} = Z_R + Z_L + \cdots$$

The most relevant terms are those describing random graphs and low energy graphs.

$$\ln Z_L \sim N \ln N - \beta N \left(g_V + \epsilon_0\right)$$

$$\ln Z_R \sim \frac{1}{2} N v_0 \ln N - \beta N g_V$$

$\epsilon_0 < 0$ is the average cycle energy per node in the low-energy graph

The transition temperature can be obtained from the above in several ways.

$$\beta_{L.v_0} = -\frac{1}{\epsilon_0}\left(\frac{v_0}{2} - 1\right) \ln N$$

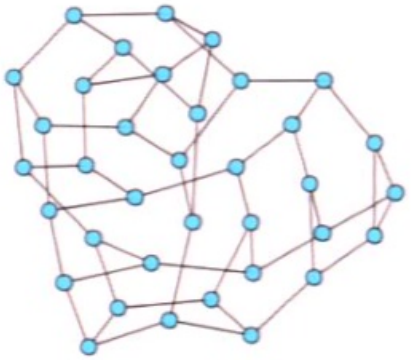
Since β scales with N the transition occurs at zero temperature if $N \to \infty$

This estimated transition temperature agrees with simulations to within 20%

Characteristic Graphs

What are the characteristic graphs at high and low temperature?

$N = 36$
$v_0 = 3$

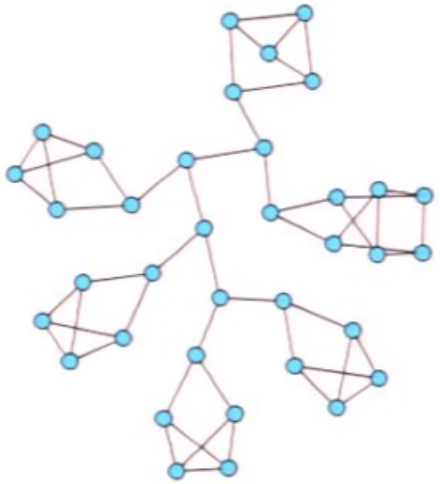


High temperature: regular random graphs

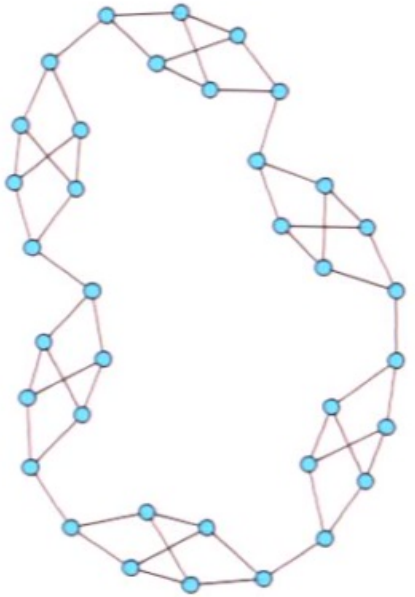
small diameter, few short cycles

Low temperature:
tree and chain graphs

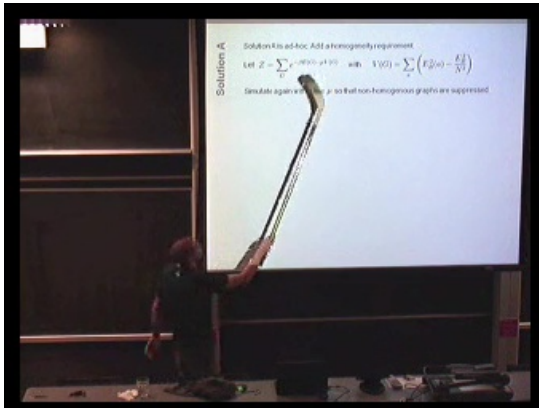
larger diameter,
many short cycles



$r = +2.5$



$r = -2.5$



Solution A

Solution A is ad-hoc: Add a homogeneity requirement.

Let $Z = \sum_G e^{-\beta E(G) - \mu V(G)}$ with $V(G) = \sum_a \left(E_B^2(a) - \frac{E_B^2}{N^2} \right)$

Simulate again with large μ so that non-homogenous graphs are suppressed.

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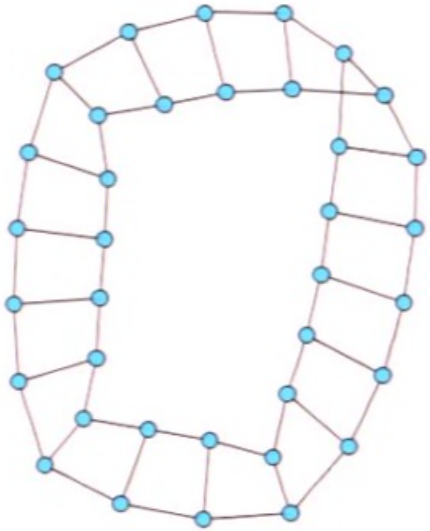
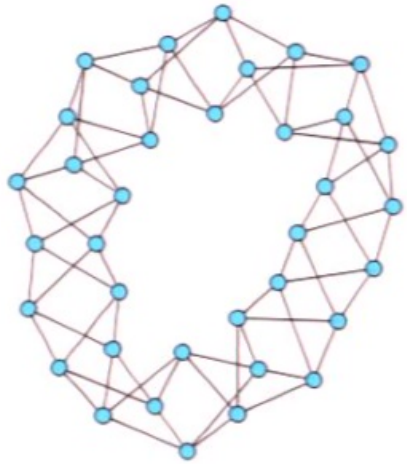
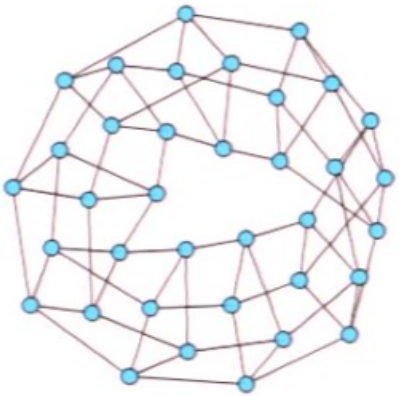


$r = -2.5$
$\beta = 0.2$
$\mu = 4$



$r = -1.5$
$\beta = 0.6$
$\mu = 32$



$r = -2.5$
$\beta = 0.17$
$\mu = 3.8$

Solution B: Warm-up

Solution B involves modeling matter on graphs.

As a warm-up, consider a toy model of a set of particles in a rectangular box with area A and side lengths (L_x, L_y)

$$Z = \sum_{(L_x, L_y) \in \mathcal{C}} z(L_x, L_y) e^{-\beta E_B(L_x, L_y)}$$

For simplicity, take $E_B(L_x, L_y) = 0$

$\mathcal{C}$ is the space of box configurations. Assume:

$$(\sqrt{A}, \sqrt{A}) \in \mathcal{C}$$

$$(\ell, A/\ell) \in \mathcal{C}$$

and

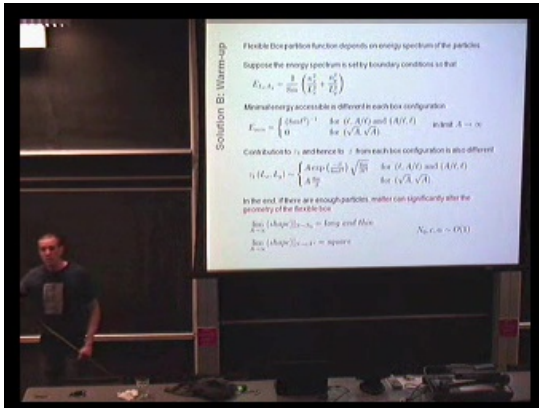
$$|\mathcal{C}| \sim O((A/\ell^2)^\eta)$$

(ℓ is some minimal length)

Take a collection of N particles

$$z(L_x, L_y) = \frac{1}{N!} (z_1(L_x, L_y))^N$$

$$z_1(L_x, L_y) = \sum_{E_{L_x, L_y}} e^{-\beta E_{L_x, L_y}}$$



Solution B: Warm-up

Flexible Box partition function depends on energy spectrum of the particles.

Suppose the energy spectrum is set by boundary conditions so that

$$E_{L_x.L_y} = \frac{1}{8m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)$$

Minimal energy accessible is different in each box configuration

$$E_{min} = \begin{cases} (8m\ell^2)^{-1} & \text{for } (\ell.\, A/\ell) \text{ and } (A/\ell.\, \ell) \\ 0 & \text{for } (\sqrt{A}.\, \sqrt{A}). \end{cases} \qquad \text{in limit } A \to \infty$$

Contribution to z_1 and hence to z from each box configuration is also different

$$z_1\,(L_x.\, L_y) \sim \begin{cases} A \exp\left(\frac{-\beta}{8m\ell^2}\right)\sqrt{\frac{8m}{3\ell^2}} & \text{for } (\ell.\, A/\ell) \text{ and } (A/\ell.\, \ell) \\ A\,\frac{8m}{\beta} & \text{for } (\sqrt{A}.\, \sqrt{A}). \end{cases}$$

In the end, if there are enough particles, matter can significantly alter the geometry of the flexible box

$$\lim_{A\to\infty} \langle shape \rangle|_{N\sim N_0} = long\ and\ thin$$

$$\lim_{A\to\infty} \langle shape \rangle|_{N\sim cA^\alpha} = square$$

$$N_0.\, c.\, \alpha \sim O(1)$$

Solution B: Particles on Graphs

Again write Z for graphs as a sum of parts.

$$Z = \sum_G z(G)\, e^{-\beta E(G)} = Z_R + Z_L + Z_H + \cdots$$

This time, taking matter into account, one has

$$\ln Z^d_{H.C\sim N^{1/d}} \sim N \ln N - \beta N \left(g_V + a_d\, \epsilon_H\right) + \frac{1}{2} N d \ln \frac{8m\ell^2}{\beta}.$$

$$\ln Z^{2d}_{H.C_1 \ll C_2} \sim N \ln N - \beta N \left(g_V + a\epsilon_H + \frac{1}{8m\ell^2}\right) + \frac{1}{2} N \ln \frac{8m\ell^2}{\beta}$$

$$\ln Z_L \sim N \ln N - \beta N \left(g_V + \epsilon_0 + \frac{1}{8m_L\ell^2}\right) \qquad a\epsilon_H < \epsilon_H < 0$$

$$\ln Z_{R.v_0} \sim \frac{1}{2} N v_0 \ln N - \beta N \left(g_V + \frac{1}{8m_R\ell^2}\right) \qquad \epsilon_0 < \epsilon_H < 0$$

Extended manifold-like graph can dominate when

$$\epsilon_H < a\epsilon_H + \frac{1}{8m\ell^2} + \frac{1}{2\beta} \ln \frac{8m\ell^2}{\beta}$$

$$\epsilon_H < \epsilon_0 + \frac{1}{8m_L\ell^2} + \frac{1}{\beta} \ln \frac{8m\ell^2}{\beta}$$

Conclusions

Graphity models are designed to test if the manifold-like structure of the universe is fundamental or if it can be an emergent phenomenon.

Monte Carlo simulations provide concrete information about the high- and low-temperature behavior of the models.

It seems matter must play an important role in these models if they are to describe extended emergent geometries.

Will matter also have a similar effect on the model you are working on?

Thank you.