Title: The geometry of the AdS/CFT correspondence

Date: May 08, 2009 04:30 PM

URL: http://pirsa.org/08050063

Abstract: I will describe how the geometry of supersymmetric AdS solutions of type IIB string theory may be rephrased in terms of the geometry of generalized (in the sense of Hitchin) Calabi-Yau cones. Calabi-Yau cones, and hence Sasaki-Einstein manifolds, are a special case, and thus the geometrical structure described may be considered a form of generalized Sasaki-Einstein geometry. Generalized complex geometry naturally describes many features of the AdS/CFT correspondence. For example, a certain type changing locus is identified naturally with the moduli space of the dual CFT. There is also a generalized Reeb vector field, which defines a foliation with a transverse generalized Hermitian structure. For solutions with non-zero D3-brane charge, the generalized Calabi-Yau cone is also equipped with a canonical symplectic structure, and this captures many quantities of physical interest, such as the central charge and conformal dimensions of certain operators, in the form of Duistermaat-Heckman type integrals.

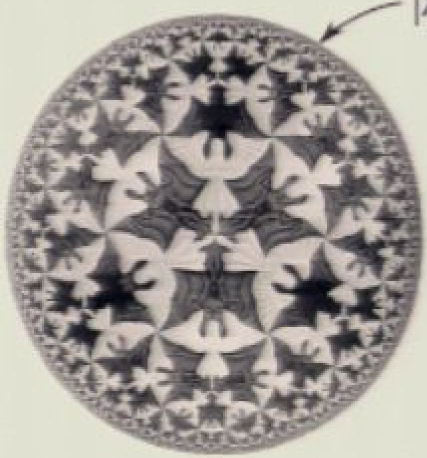
The geometry of the AdS/CFT correspondence

James Sparks (Oxford)

Based on work with M. Gabella, J. P. Gauntlett, E. Palti, D. Waldram

Connections in Geometry and Physics
May 8-10, 2009, Perimeter Institute

Anti de Sitter spacetime in $\mathbf{d+1}$ dimensions is the maximally symmetric (under $\mathbf{SO(d,2)}$) solution to the Einstein equation $\mathrm{Ric}[\mathbf{g}] = -\mathbf{d}\mathbf{g}$. This is the Lorentz signature version of hyperbolic space $(\mathbf{H}, \mathbf{g_H})$.



$$\mathbf{H} = \{\mathbf{x} \in \mathbb{E}^{d+1} \mid |\mathbf{x}| < 1\} ,$$

$$\mathbf{g_H} = \frac{4\sum_{i=1}^{d+1} \mathrm{d}x_i \otimes \mathrm{d}x_i}{(1-|\mathbf{x}|^2)^2} .$$

May compactify $\mathbf{H}$ to $\bar{\mathbf{H}} = \{|\mathbf{x}| \leq 1\}$ with metric $\mathbf{g}_{\bar{\mathbf{H}}} = \mathbf{f}^2\mathbf{g_H}$, where $\mathbf{f}$ is a smooth positive function on $\mathbf{H}$ with a simple zero on $\partial\bar{\mathbf{H}} = \mathbf{S}^d$.

Induces the standard conformal structure $[\mathbf{g}_{\mathbf{S}^d}]$ on $\mathbf{S}^d$.

AdS/CFT conjecture: Quantum gravity on $\mathbf{H}$ = conformal field theory on $\mathbf{S}^d$

This is best understood in string theory, or rather its supergravity limits. Here I shall focus on type IIB supergravity. This is a form of General Relativity on a ten-manifold $(\mathbf{M}, \mathbf{g})$, with a very special matter content.

In addition to the (Lorentz signature) metric $\mathbf{g}$ on $\mathbf{M}$, there are also form fields $\phi \in \Omega^0(\mathbf{M}, \mathbb{R})$, $\mathsf{H} \in \Omega^3(\mathbf{M}, \mathbb{R})$, $\mathsf{F}_\alpha \in \Omega^\alpha(\mathbf{M}, \mathbb{R})$, $\alpha = 1, 3, 5$, with $*\mathsf{F}_5 = \mathsf{F}_5$. These must satisfy the Bianchi identities $\mathrm{d}\mathsf{H} = 0$, $(\mathrm{d} - \mathsf{H}\wedge)\mathsf{F} = 0$ $(\mathsf{F} = \mathsf{F}_1 + \mathsf{F}_3 + \mathsf{F}_5)$, and Einstein equations

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$$
\begin{aligned}
\mathrm{Ric}[g]_{ij} &= \frac{1}{2}\partial_i\phi\partial_j\phi + \frac{1}{2}e^{2\phi}F_iF_j - \frac{1}{8}g_{ij}\left(e^{-\phi}|H|^2 + e^{\phi}|F_3|^2\right) \\
&\quad + \frac{1}{4}\left(e^{-\phi}H_{imn}H_j{}^{mn} + e^{\phi}F_{imn}F_j{}^{mn}\right) + \frac{1}{96}F_{imnpq}F_j{}^{mnpq}\,, \\
\nabla^2\phi &= e^{2\phi}|F_1|^2 - \frac{1}{2}e^{-\phi}|H|^2 + \frac{1}{2}e^{\phi}|F_3|^2\,, \\
\mathrm{d}(e^{-\phi} * H) &= -F_5 \wedge F_3 + e^{\phi}F_1 \wedge *F_3\,, \\
\mathrm{d}^\dagger(e^{2\phi}F_1) &= e^{\phi}\langle H, F_3\rangle, \qquad \mathrm{d}(e^{\phi} * F_3) = F_5 \wedge H\,.
\end{aligned}
$$

This theory is very special. In particular, the above system of second order equations are integrability conditions for a first order system of equations for a spinor $\epsilon \in \Gamma(\mathsf{S}^+\mathsf{M})$:

$$
\begin{aligned}
0 &= \left(\nabla_i + \frac{i}{4}e^{\phi}\,F_i + \frac{i}{192}F_{imnpq}\Gamma^{mnpq}\right)\epsilon \\
&\quad -\frac{1}{96}\left(G_{mnp}\Gamma_i{}^{mnp} - 9G_{imn}\Gamma^{mn}\right)\epsilon^c\,, \\
0 &= \frac{i}{24}G_{ijk}\Gamma^{ijk}\epsilon + \frac{i}{2}\left(\partial_i\phi + ie^{\phi}\,F_i\right)\Gamma^i\epsilon^c\,.
\end{aligned}
$$

Here I have defined $G \equiv -ie^{\phi/2}F_3 - e^{-\phi/2}H \in \Omega^3(M,\mathbb{C})$.

Γ_i, $i = 1,\ldots,10$, generate the Clifford algebra for g: $\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2g_{ij}$ (and $\Gamma_{i_1\cdots i_n} \equiv \Gamma_{[i_1}\cdots\Gamma_{i_n]}$).

Note: when $\phi = H = F_\alpha = 0$, this reduces to a parallel spinor $\nabla\epsilon = 0$, which is well-known to imply Ricci-flatness as an integrability condition.

Definition: Any solution to the first order system is called a supersymmetric (SUSY) supergravity solution.

For applications to AdS/CFT, we are interested in product solutions $\mathbf{M} = \mathbf{AdS}_5 \times \mathbf{Y}$, where $(\mathbf{Y}, g_{\mathbf{Y}})$ is a compact Riemannian 5-manifold, and

$$g = e^{2\Delta}(g_{AdS} + g_{\mathbf{Y}}) \ .$$

where $\Delta \in \Omega^0(\mathbf{Y}, \mathbb{R})$. $\mathbf{SO(4,2)}$-invariance $\Rightarrow$ all form fields are pull-backs of forms on $\mathbf{Y}$, except $\mathbf{F}_5$:

$$\mathbf{F}_5 = \mathbf{f}(\mathrm{vol}_{AdS} + \mathrm{vol}_{\mathbf{Y}}) \ .$$

where $\mathbf{f} \in \mathbb{R}$ is a constant by the Bianchi identity.

Each SUSY supergravity solution of this form gives rise to a (super-)conformal field theory on $\mathbf{S}^4$, via the AdS/CFT conjecture.

A special class of SUSY solutions is given by taking $\phi = \mathbf{H} = \mathbf{F}_1 = \mathbf{F}_3 = 0$, but $\mathbf{F}_5$ (hence $\mathbf{f}$) non-zero.

Then the first order supersymmetry equations reduce to

$$\left(\nabla_\zeta + \frac{i}{2}\zeta\cdot\right)\psi = 0 \, ,$$

$\forall\zeta \in \Gamma(\mathbf{TY})$, where $\cdot$ denotes Clifford multiplication for $(\mathbf{Y}, \mathbf{g_Y})$, and $\psi \in \Gamma(\mathbf{SY})$ is said to be a Killing spinor on $\mathbf{Y}$.

It is well-known that this is equivalent to $(\mathbf{Y}, \mathbf{g_Y})$ being a Sasaki-Einstein 5-manifold. In particular, $\mathbf{g_Y}$ is Einstein with positive Ricci curvature: $\mathrm{Ric}[\mathbf{g_Y}] = 4\mathbf{g_Y}$. An essentially equivalent definition is

Definition: $(\mathbf{Y}, \mathbf{g_Y})$ is Sasaki-Einstein iff the metric cone $(\mathbb{R}_{>0} \times \mathbf{Y}, \mathrm{d}\mathbf{r}^2 + \mathbf{r}^2\mathbf{g_Y})$ is both Kähler and Ricci-flat.

This cone metric also appears naturally in ten dimensions, for general solutions. We write the AdS_5 metric in a Poincaré patch as

$$g_{\mathrm{AdS}} = \frac{dr^2}{r^2} + r^2 g_{\mathbb{E}^{3,1}} \ .$$

Then

$$g = e^{-\frac{\phi}{2}} \left(e^{2A} g_{\mathbb{E}^{3,1}} + g_X \right) \ .$$

where

$$e^{2A} \equiv e^{2\Delta + \frac{\phi}{2}} r^2 \ , \qquad g_X = \frac{e^{2A}}{r^4}(dr^2 + r^2 g_Y) \ .$$

Hence we may equivalently think of a SUSY AdS_5 solution as a SUSY $\mathbb{E}^{3,1} \times \mathbf{X}$ solution, where the "internal manifold" $\mathbf{X} \cong \mathbb{R}_{>0} \times \mathbf{Y}$ is (conformal to) a cone.

(Graña-Minasian-Petrini-Tomasiello) showed all SUSY $\mathbb{E}^{3,1} \times \mathbf{X}$ solutions require that $\mathbf{X}$ is generalized Calabi-Yau, in the sense of (Hitchin).

Generalized geometry studies geometry of $\mathbf{TX} \oplus \mathbf{T^*X}$, rather than $\mathbf{TX}$.

(More generally generalized tangent bundle, an extension of $\mathbf{TX}$ by $\mathbf{T^*X}$ given by a connective structure on a gerbe with curvature the 3-form $\mathbf{H}$.)

Natural $\mathbf{O(d,d)}$-invariant metric on this bundle ($\dim \mathbf{X} = \mathbf{d}$):
$\eta(\mathbf{V},\mathbf{V}) = \mathbf{i}_\zeta \alpha$. Here $\mathbf{V} = \zeta + \alpha$, $\zeta =$ vector field, $\alpha =$ one-form.

Lie bracket replaced by the ($\mathbf{H}$-twisted) Courant bracket:

$$[\zeta + \alpha, \chi + \beta]_{\mathbf{H}} \equiv [\zeta, \chi]_{\mathrm{Lie}} + \mathcal{L}_\zeta \beta - \mathcal{L}_\chi \alpha - \frac{1}{2} \mathrm{d}(\mathbf{i}_\zeta \beta - \mathbf{i}_\chi \alpha) + \mathbf{i}_\chi \mathbf{i}_\zeta \mathbf{H} \, .$$

Note $\mathrm{so}(\mathsf{T} \oplus \mathsf{T}^*) = \Lambda^2(\mathsf{T} \oplus \mathsf{T}^*) = \mathrm{End}(\mathsf{T}) \oplus \Lambda^2\mathsf{T}^* \oplus \Lambda^2\mathsf{T}$.

In particular, there is the orthogonal B-transform

$$\zeta + \alpha \longrightarrow \zeta + (\alpha - \mathbf{i}_\zeta \omega) ,$$

where ω is a two-form (think of as a skew map $\omega : \mathsf{T} \longrightarrow \mathsf{T}^*$).

Closed B-transforms are symmetries of the Courant bracket $\Rightarrow$ automorphism group in generalized geometry is $\mathrm{Diff}(\mathsf{X}) \ltimes \Omega^2_{\mathrm{closed}}(\mathsf{X})$ (more precisely, should replace last factor with curvatures of unitary line bundles on X, gerbey).

Infinitesimally, generated by vector field χ and two-form $\omega = \mathrm{d}\beta$, then the generalized Lie derivative of $\mathsf{U} = \zeta + \alpha$ along $\mathsf{V} = \chi + \beta$ is

$$\mathbb{L}_\mathsf{V}\mathsf{U} = [\chi, \zeta]_{\mathrm{Lie}} + (\mathcal{L}_\chi \alpha - \mathbf{i}_\zeta \mathrm{d}\beta) .$$

A generalized almost complex structure $\mathcal{J}$ is a section of $\mathbf{End}(\mathsf{T} \oplus \mathsf{T}^*)$, orthogonal wrt η, $\mathcal{J}^2 = -\mathbf{1}$.

$\pm\mathbf{i}$ eigenspaces L, $\bar{\mathsf{L}} \subset (\mathsf{T} \oplus \mathsf{T}^*) \otimes_{\mathbb{R}} \mathbb{C}$ are maximal isotropic.

Example: almost complex structure I, almost symplectic form ω:

$$\mathcal{J}_1 = \begin{pmatrix} \mathsf{I} & 0 \\ 0 & -\mathsf{I}^* \end{pmatrix}, \mathcal{J}_2 = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}.$$

By definition, $\mathcal{J}$ is integrable iff L is closed under the (twisted) Courant bracket.

For $\mathsf{H} = 0$, $\mathcal{J}_1$, $\mathcal{J}_2$ integrable iff I is integrable, $\mathrm{d}\omega = 0$, respectively.

The standard spin representation of **Spin(d, d)** is simply the (complexified) bundle of forms $\mathcal{S}X = \Lambda(T^*X)$ on X. Reducible: $\mathcal{S}_\pm X = \Lambda^{\mathrm{even/odd}} T^*X$.

Clifford action of V on spinor $\Omega \in \Omega^*(X, \mathbb{C})$ is $V \cdot \Omega = i_\varsigma \Omega + \alpha \wedge \Omega$.

generalized almost complex structures $\xleftrightarrow{1-1}$ pure spinor lines $\subset \Lambda(T^*X \otimes \mathbb{C})$.

Pure spinor $\Omega =$ spinor with maximal isotropic annihilator space $L_\Omega \subset (T \oplus T^*) \otimes \mathbb{C}$. Then $\mathcal{J}$ is defined by saying $L_\Omega = +i$ eigenspace of $\mathcal{J}$.

We say two generalized almost complex structures $\mathcal{J}_1, \mathcal{J}_2$ are compatible if $[\mathcal{J}_1, \mathcal{J}_1] = 0$ and $-\mathcal{J}_1\mathcal{J}_2 \equiv G$ is a generalized metric: $\frac{1}{2}(1 \pm G)$ projects onto $C_\pm$, where η is $\pm$-ve definite on $C_\pm$. May write

$$G = \begin{pmatrix} g_X^{-1}B & g_X^{-1} \\ g_X - Bg_X^{-1}B & -Bg_X^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} 0 & g_X^{-1} \\ g_X & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

where $g_X =$ metric on X, $B =$ two-form. In fact, we may identify $dB = H$.

The conditions for a SUSY $\mathbb{E}^{3,1} \times \mathbf{X}$ solution may be recast in terms of two compatible generalized almost complex structures $\mathcal{J}_-$, $\mathcal{J}_+$ (Graña *et al*), or more precisely sections Ω_-, Ω_+ of the corresponding pure spinor lines.

For an $\mathbf{AdS_5}$ solution, after some work these may be written

$$\begin{aligned} \mathrm{d}\Omega_- &= 0 , \\ \mathrm{d}\Omega_+ &= \mathrm{d}\mathbf{A} \wedge \bar{\Omega}_+ + \frac{\mathrm{i}}{8} \mathrm{e}^{3\mathrm{A}} \mathrm{e}^{-\mathrm{B}} \star (\mathrm{F}_1 - \mathrm{F}_3 + \mathrm{F}_5) , \end{aligned}$$

where $\star =$ Hodge star for g_{X}. Here $|\Omega_\pm|^2 = \frac{1}{8} \mathrm{e}^{6\mathrm{A}-2\phi}$, where the norm is given by the Mukai pairing.

In particular, $\mathrm{d}\Omega_- = 0 \Rightarrow \mathcal{J}_-$ is integrable, and $\mathbf{X}$ is generalized Calabi-Yau.

In general, $\mathcal{J}_+$ is not integrable $\Rightarrow$ generalized Hermitian structure.

We may say $\mathbf{X}$, equipped with Ω_-, is a generalized Calabi-Yau cone.

Recall $\exists$ vector field $r\partial_r$ on $\mathbf{X} = \mathbb{R}_{>0} \times \mathbf{Y}$. May show

$$\mathcal{L}_{r\partial_r}\Omega_\pm = 3\Omega_\pm \ ,$$

and hence

$$\mathbb{L}_{r\partial_r}\mathcal{J}_\pm = 0 \ .$$

Thus $r\partial_r$ is a generalized holomorphic vector field (for $\mathcal{J}_-$). Since $\mathbf{G} = -\mathcal{J}_-\mathcal{J}_+$, also $\mathbb{L}_{r\partial r}\mathbf{G} = 0$ and $r\partial_r$ is generalized Killing.

In general, $\mathbb{L}_V\mathbf{G} = 0 \Rightarrow \mathcal{L}_\zeta g_X = 0$ and $\mathcal{L}_\zeta B = d\alpha$.

In the Sasaki-Einstein/Calabi-Yau case, we note that

$$\Omega_- = \frac{1}{8}\Omega_{3,0} ,$$
$$\Omega_+ = -\frac{ir^3}{8}\exp\left(\frac{i}{r^2}\omega_{1,1}\right) .$$

where $\Omega_{3,0}$ is the holomorphic $(3,0)$-form and $\omega_{1,1}$ is the Kähler two-form on $\mathbf{X}$.

It is well-known that $r\partial_r$ is a holomorphic vector field. Moreover the Reeb vector field

$$\xi \equiv I(r\partial_r) .$$

where $I =$ complex structure on $\mathbf{X}$, is holomorphic, Killing, and unit length on $\mathbf{Y} = \{r = 1\} \subset \mathbf{X}$.

By analogy with the Calabi-Yau case, we define

$$\xi = \mathcal{J}_{-}(\mathrm{r}\partial_{\mathrm{r}}) \ .$$

Note this is now, in general, a generalized vector field. It is immediate that

$$\mathbb{L}_{\xi}\Omega_{-} = \mathrm{d}(\xi \cdot \Omega_{-}) = -\mathrm{i}\mathrm{d}(\mathrm{r}\partial_{\mathrm{r}} \cdot \Omega_{-}) = -\mathrm{i}\mathcal{L}_{\mathrm{r}\partial_{\mathrm{r}}}\Omega_{-} = -3\mathrm{i}\Omega_{-} \ ,$$

and hence $\mathbb{L}_{\xi}\mathcal{J}_{-} = 0$. Thus ξ is also generalized holomorphic.

After considerable effort, one can use the other SUSY equation to prove

$$\mathbb{L}_{\xi}\Omega_{+} = 0 \ .$$

implying ξ is also generalized Killing. May also show that the generalized Lie derivative of $\mathrm{e}^{-\mathrm{B}}\mathrm{F}$ vanishes $\Rightarrow$ ξ generates full symmetry of the solution.

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By analogy with the Calabi-Yau case, we define

$$\xi = \mathcal{J}_-(r\partial_r) \ .$$

Note this is now, in general, a generalized vector field. It is immediate that

$$\mathbb{L}_\xi \Omega_- = d(\xi \cdot \Omega_-) = -i d(r\partial_r \cdot \Omega_-) = -i\mathcal{L}_{r\partial_r}\Omega_- = -3i\Omega_- \ ,$$

and hence $\mathbb{L}_\xi \mathcal{J}_- = 0$. Thus ξ is also generalized holomorphic.

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A general pure spinor Ω may be written

$$\Omega = \alpha\theta_1 \wedge \cdots \wedge \theta_k \wedge e^{-b+i\omega}$$

where $\alpha =$ complex function, θ_i are complex one-forms, and $\mathbf{b}$, ω are real two-forms. Here $k \in \mathbb{N}$ is called the type.

Assuming $\mathbf{X}$ is not Calabi-Yau (everywhere type 3), on a dense open set Ω_- is type 1:

$$\Omega_- = \theta \wedge e^{-b_- + i\omega_-} ,$$

where $\theta =$ complex one-form.

The locus where Ω_- becomes type 3 is precisely where a pointlike (space-filling) probe D3-brane on $\mathbf{X}$ is supersymmetric. May show this locus is Kähler in the induced structure. In AdS/CFT, this is naturally interpreted as the moduli space of the conformal field theory.

Let's switch to the other pure spinor, Ω_+. May rewrite the single complex equation as

$$\begin{aligned} d(e^{-A}\mathrm{Re}\Omega_+) &= 0 \, , \\ d(e^{A}\mathrm{Im}\Omega_+) &= \frac{1}{8}e^{4A}e^{-B} \star (F_1 - F_3 + F_5) \, . \end{aligned}$$

Assuming $F_5 \neq 0$ ($\Leftrightarrow f \neq 0$), we see Ω_+ is type 0. Physically, $F_5 \neq 0$ says the solution has non-zero D3-brane charge.

After some calculation, may show

$$\Omega_+ = -\frac{i}{32} f r^4 e^{-A} e^{-b_+ + i e^{2A} r^{-4} \omega_+} \, .$$

This implies

$$\frac{f^2}{16} e^{-2A+2\phi} r^{-4} \frac{1}{3!} \omega_+^3 = \mathrm{vol}_X$$

is the Riemannian volume form of (X, g_X), implying the two-form ω_+ is non-degenerate, and also

$$d\omega_+ = 0 \ .$$

Thus for solutions with non-zero D3-brane charge, X is also equipped with a canonical symplectic structure! This was surprising, as Ω_+ is not integrable.

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Note: In the Calabi-Yau case, $\omega_+ = \omega_{1,1}$ is simply the Kähler form.

May check $\mathcal{L}_{r\partial_r}\omega_+ = 2\omega_+$ and hence

$$\omega_+ = \frac{1}{2}\mathrm{d}(r^2\sigma) \ .$$

where $\sigma =$ (pull-back of) a contact one-form on $\mathbf{Y} = \{r = 1\} \subset \mathbf{X}$.

Moreover,

$$\xi_v \lrcorner \sigma = 1, \qquad \xi_v \lrcorner \mathrm{d}\sigma = 0 \ ,$$

where $\xi_v =$ vector component of $\xi \Rightarrow$ may call ξ a generalized Reeb vector field.

This generalizes the Sasaki-Einstein case, where all of these formulae also hold. In general, $|\xi_v|\ |_{\mathbf{Y}}$ is not constant, but it is nowhere zero, and hence defines a one-dimensional foliation of $\mathbf{Y}$.

There are some nice physical applications of these formulae. The central charge $\mathrm{a} \in \mathbb{R}$ of a solution is defined by

$$\mathrm{a} \equiv \frac{\int_Y e^{8\Delta} \mathrm{vol}_Y}{2(2\pi)^5} .$$

This is an important quantity in the dual conformal field theory, essentially a count of massless degrees of freedom. In particular, if ξ_v generates a $U(1)$ action – that is, all its orbits are closed – the field theory implies $\mathrm{a} \in \mathbb{Q}$.

The central charge a of the dual field theory is given by the contact volume

$$\mathrm{a} = \frac{(2\pi)^3 N^2}{4 \int_Y \sigma \wedge d\sigma \wedge d\sigma} .$$

Here we have imposed flux quantization:

$$\mathbb{Z} \ni N = \frac{1}{(2\pi)^4} \int_Y (F_5 + H \wedge C_2) .$$

Suppose ξ_v generates a $\mathbf{U(1)}$ action. This is locally free (since ξ_v is nowhere zero) $\Rightarrow \mathbf{V} = \mathbf{Y}/\mathbf{U(1)}$ = orbifold, $\mathcal{L} = \mathbf{Y} = \mathbf{U(1)}$ orbibundle over $\mathbf{V}$. Then the contact volume is (essentially) just the Chern number $\int_{\mathbf{V}} c_1(\mathcal{L})^2 \in \mathbb{Q}$. This proves $\mathbf{a} \in \mathbb{Q}$ is essentially just a Chern number.

One may also write

$$\mathbf{a} = \frac{(2\pi)^3 N^2}{4 \int_{\mathbf{X}} e^{-r^2/2} \frac{\omega_+^3}{3!}} .$$

This is a Duistermaat-Heckman integral, where $\mathcal{H} = r^2/2$ = Hamiltonian function for ξ_v: $d\mathcal{H} = -\xi_v \lrcorner \omega_+$.

This formerly localizes where $\xi_v = 0$, which is at the tip of the cone. One can obtain a general localization formula, in terms of Chern classes and weights, given an equivariant symplectic resolution of $(\mathbf{X}, \omega_+)$ (Martelli-JFS-Yau). Particularly simple when $(\mathbf{X}, \omega_+)$ is symplectic toric.

Also of interest are certain three-submanifolds Σ_3 of $\mathbf{Y}$. The cones over these are generalized complex submanifolds of $\mathbf{X}$, in the sense of (Gualtieri). This may be phrased as a generalized calibration condition.

These submanifolds may be identified with certain operators $\mathcal{O}_{\Sigma_3}$ in the dual conformal field theory. Skipping the details, and the physics, these have conformal dimension (eigenvalue under rescalings)

$$\Delta(\mathcal{O}_{\Sigma_3}) = \frac{2\pi \mathrm{N} \int_{\Sigma_3} \sigma \wedge \mathrm{d}\sigma}{\int_{\mathrm{Y}} \sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma} \,.$$

The field theory again predicts these are rational numbers for ξ_{v} generating a $\mathbf{U(1)}$ isometry. Indeed, the above is essentially the Chern number $\int_{\Sigma_2} c_1(\mathcal{L})$, where $\Sigma_2 = \Sigma_3/\mathbf{U(1)}$.

Example: There are not many known non-Sasaki-Einstein examples. In fact it is only in recent years that Sasaki-Einstein geometry (in particular constructions, examples, obstructions) has flourished.

The (Pilch-Warner) solution is defined on $\mathbf{M} = \mathbf{AdS_5} \times \mathbf{S^5}$. In fact, it is in some sense the (end-point of a) deformation of the round Einstein metric on $\mathbf{S^5}$:

$$
\begin{aligned}
g_Y \;=\; & \frac{1}{9}\Bigg[6\mathrm{d}\vartheta^2 + \frac{6\cos^2\vartheta}{3-\cos 2\vartheta}(\sigma_1^2+\sigma_2^2) + \frac{6\sin^2 2\vartheta}{(3-\cos 2\vartheta)^2}\sigma_3^2 \\
& +4\left(\mathrm{d}\varphi + \frac{2\cos^2\vartheta}{3-\cos 2\vartheta}\sigma_3\right)^2\Bigg] .
\end{aligned}
$$

where $0 \leq \vartheta \leq \frac{\pi}{2}$, $0 \leq \varphi \leq 2\pi$, and σ_i, $i = 1, 2, 3$, are left-invariant one-forms on $SU(2)$.

We also have $e^{4\Delta} = \frac{f}{8}(3 - \cos 2\vartheta)$, and non-zero F_3 and H.

One can show this solution is toric ($\mathbf{U(1)}^3$ invariant), and $(\mathbf{X}, \omega_+) =$ standard toric symplectic structure on $\mathbb{R}^6 \setminus \{0\}$.

Reeb vector field is computed to be $\xi_{\mathsf{v}} = \frac{3}{2}\frac{\partial}{\partial\phi_1} + \frac{3}{4}\frac{\partial}{\partial\phi_2} + \frac{3}{4}\frac{\partial}{\partial\phi_3}$. The central charge is then easily computed by localization:

$$\frac{\mathsf{N}^2}{4\mathsf{a}} = \sum_{\text{fixed pts}} \prod_{\mathrm{i}=1}^{3} \frac{1}{\langle \xi, \mathbf{u}_{\mathrm{i}} \rangle} = \frac{1}{\frac{3}{2} \cdot \frac{3}{4} \cdot \frac{3}{4}} = \frac{32}{27} \, .$$

where $\mathbf{u}_{\mathrm{i}} =$ tangent space weights, agreeing with a conformal field theory calculation.

The type changing locus of Ω_- is a copy of $\mathbb{C}^2 \subset \mathbb{R}^6$ given by $\vartheta = \mathbf{0}$. The dual conformal field theory is known explicitly (it is the IR fixed point of a mass deformation of $\mathcal{N} = 4$ SYM), and its moduli space is indeed $\mathbb{C}^2$.

Some open problems/questions:

- One can perform a generalized reduction (in the sense of (Bursztyn-Cavalcanti-Gualtieri)) of $\mathbf{X}$ along ξ, to obtain a transverse generalized Hermitian structure to the corresponding foliation of $\mathbf{Y}$. This structure is a generalization of Fano Kähler-Einstein geometry (in progress).

- In the Sasaki-Einstein case, one can also write $\frac{N^2}{4a} = \lim_{t \searrow 0} t^3 \sum_{i=0}^{\infty} \exp(-t\lambda_i)$, where $\{\lambda_i\}_{i=1}^{\infty}$ denotes the holomorphic spectrum on $\mathbf{X}$. That is, the λ_i are weights of holomorphic functions on $\mathbf{X}$ under $r\partial_r$. Does this generalize to a sum over generalized holomorphic objects, or more precisely generalized cohomology of $\bar{\partial}_{\mathcal{J}_-}$?

- For the Sasaki-Einstein case, (Gauntlett-Martelli-JFS-Yau) showed that small eigenvalues λ_i can obstruct the existence of Sasaki-Einstein metrics (via the Lichnerowicz bound). Is there an analogue in generalized geometry?

- In (Martelli-JFS-Yau) we showed that the Einstein-Hilbert action on metrics on **Y**, restricted to Sasakian metrics, is a function of the Reeb vector field ξ_{v}. This is strictly convex, provided $\Omega_- = \frac{1}{8}\Omega_{3,0}$ has weight 3, and if a Sasaki-Einstein metric exists on **Y** with this fixed complex structure on **X**, it is the unique critical point. Moreover, this action was shown to be a rational function of ξ_{v}, with rational coefficients. Hence the critical point is algebraic. The field theory predicts this. I expect all of this to generalize.
- Can one reduce the transverse generalized Hermitian structure to some kind of Monge-Ampère equation, analogous to Fano Kähler-Einstein? Can one prove an existence theorem for solutions in the toric case?
- In the Sasaki-Einstein case, if the Calabi-Yau cone admits a crepant resolution $\tilde{\mathbf{X}}$ the dual CFT is a quiver gauge theory, where the category $\mathrm{Rep}(\Gamma, \mathbf{W})$ of the quiver Γ with superpotential **W** is derived equivalent to the category $\mathrm{Coh}(\tilde{\mathbf{X}})$. What is the generalized analogue of this?
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