Title: PPT pure state transformations and catalysis

Date: May 14, 2008 04:00 PM

URL: http://pirsa.org/08050040

Abstract: In an effort to better understand the class of operations on a bipartite system which preserve positivity of partial transpose (PPT operations), we have investigated the (non-asymptotic) transformation of pure states to pure states by operations in this class. Under local operations and classical communication (LOCC) Nielsen\'s majorization criterion provides a necessary and sufficient condition for such a transformation. This can be used to show that under LOCC a phenomenon called catalysis can occur, where an otherwise impossible transformation can be made possible by the provision of an entangled catalyst state, which must be recovered unchanged after the transformation (hence the name). I will present some recent work where we have found a necessary condition for obtaining a given pure state from a maximally entangled state via PPT operations. This condition is conjectured to be sufficient also, and we can prove this for the case where the goal state has Schmidt rank three. We have also shown that catalysis occurs under PPT operations, and have derived a necessary and sufficient condition for PPT pure state transformations where both the initial state and the catalyst are maximally entangled.

PPT Pure State Transformations and Catalysis

William Matthews and Andreas Winter

University of Bristol

14 May 2008

Outline

- The partial transpose map, PPT states and PPT operations.
- Bipartite pure state transformations and LOCC catalysis.
- Bipartite pure state transformations by PPT operations.
- Catalysis under PPT operations.

Partial Transpose

- Let $|ij\rangle = |i\rangle_A \otimes |j\rangle_B$ denote an element of a orthonormal product basis for a bipartite Hilbert space.
- We define the (linear) partial transpose map by

$$\Gamma(|ij\rangle\langle kl|) = |ij\rangle\langle kl|^{\Gamma} = |il\rangle\langle kj|.$$

- Basis dependent, but the eigenvalues (and hence the positivity) of the partial transpose of an operator don't depend on this basis choice.
- Positive Partial Transpose (PPT) states are simply those density operators satisfying $\rho^{\Gamma} \geq 0$.
- Separable states are all PPT. (For 2×2 and 2×3 dimensional systems, PPT states = Separable states.)
- Converse not true. 3×3 dimensional systems can have entangled PPT states: Bound entanglement.

PPT Operations

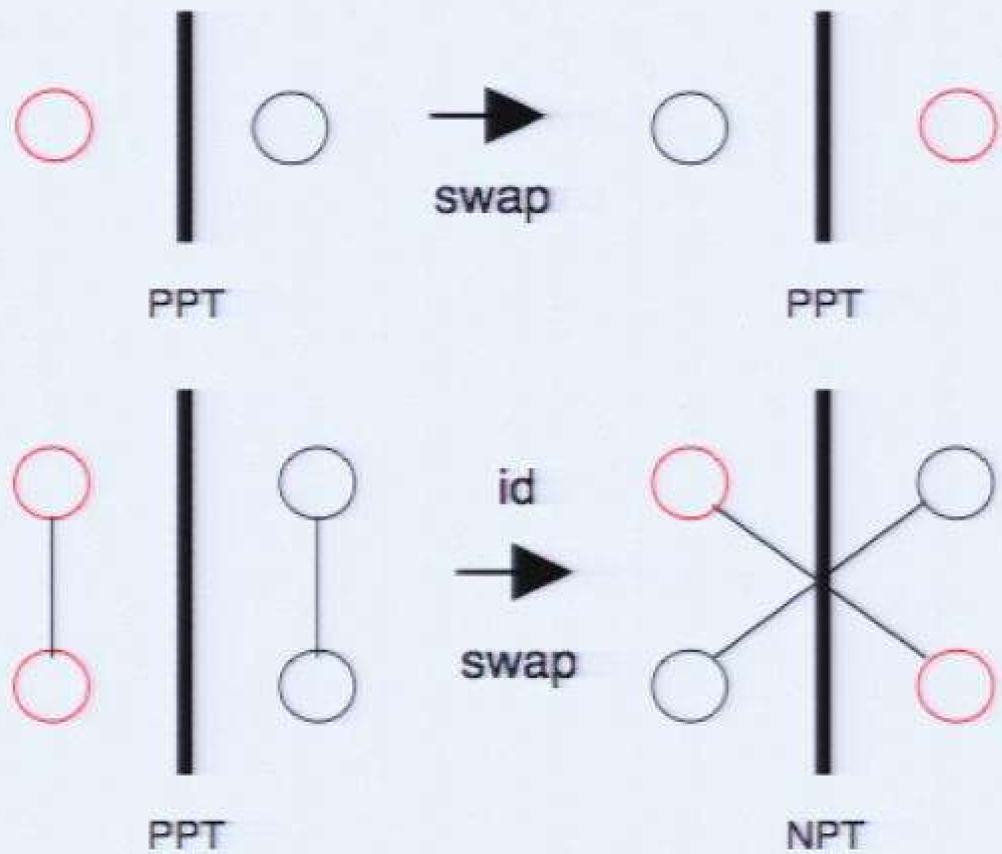
- A general quantum operation Ψ corresponds to a CPTP map.
- Ψ is PPT [1] iff $\Gamma \circ \Psi \circ \Gamma$ is also completely positive.
- If ρ is PPT, then $(\Psi \otimes \mathbb{1})\rho$ is also PPT.
- $LOCC \subset PPT$

---

[1] E. M. Rains, "A semidefinite program for distillable entanglement", IEEE Trans. Inf. Theory, 47(7):2921-2933 (2001).

PPT Operations ⊂ PPT preserving operations

Swap operation preserves PPT. Swap ⊗ Identity does not.



PPT Operations

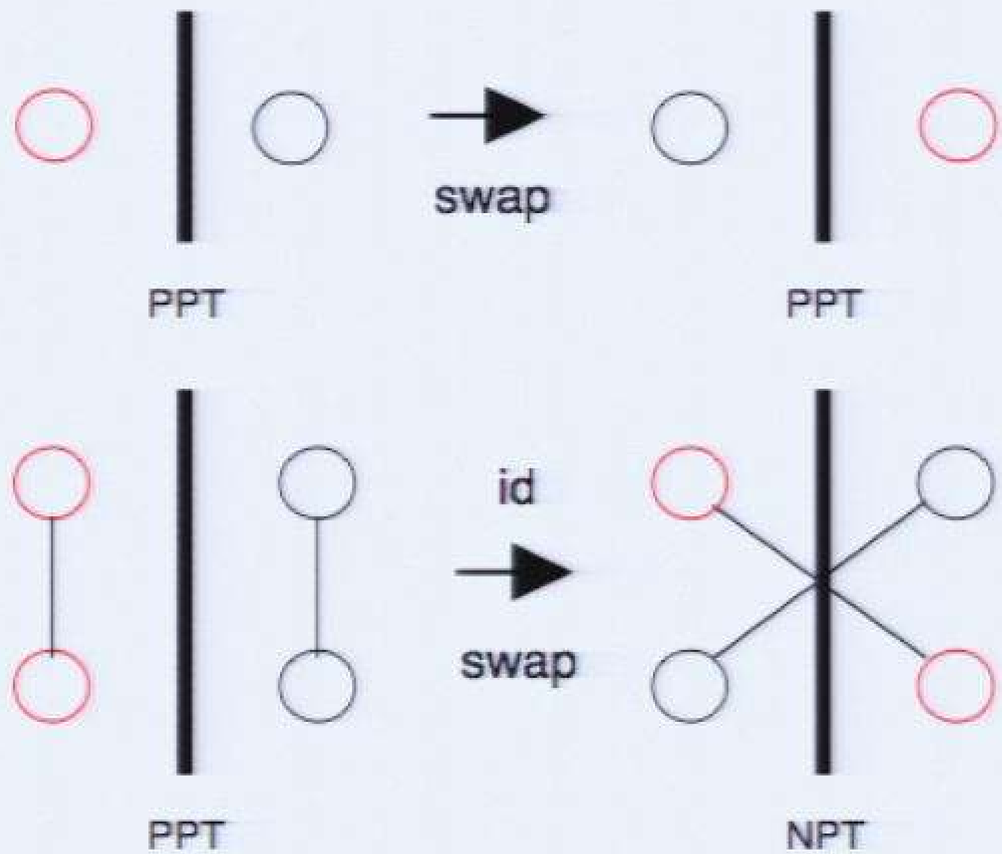
- A general quantum operation Ψ corresponds to a CPTP map.
- Ψ is PPT [1] iff $\Gamma \circ \Psi \circ \Gamma$ is also completely positive.
- If ρ is PPT, then $(\Psi \otimes \mathbb{1})\rho$ is also PPT.
- $LOCC \subset PPT$

---

[1] E. M. Rains, "A semidefinite program for distillable entanglement", IEEE Trans. Inf. Theory 47(7):2921-2933 (2001).

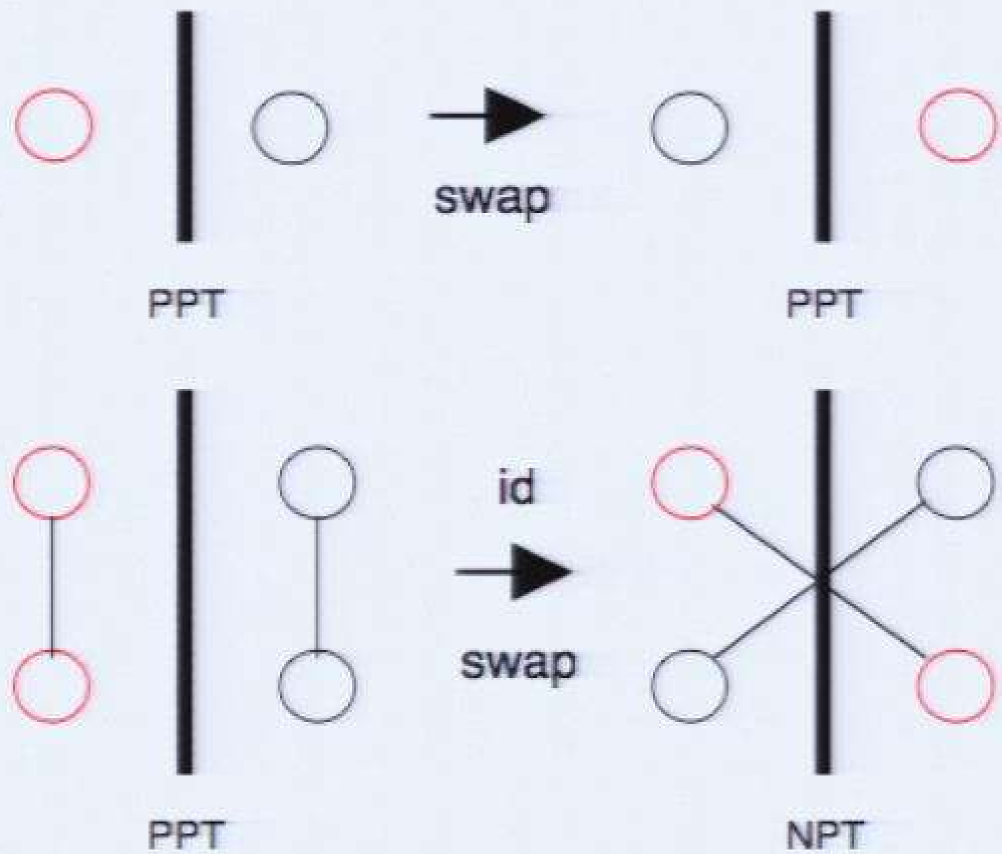
PPT Operations $\subset$ PPT preserving operations

Swap operation preserves PPT. Swap $\otimes$ Identity does not.



PPT Operations ⊂ PPT preserving operations

Swap operation preserves PPT. Swap $\otimes$ Identity does not.



PPT Operations

- A general quantum operation Ψ corresponds to a CPTP map.
- Ψ is PPT [1] iff $\Gamma \circ \Psi \circ \Gamma$ is also completely positive.
- If ρ is PPT, then $(\Psi \otimes \mathbb{1})\rho$ is also PPT.
- $LOCC \subset PPT$

---

[1] E. M. Rains, "A semidefinite program for distillable entanglement", IEEE Trans. Inf. Theory, 47(7):2921-2933 (2001).

PPT Operations - Why?

- Practical use: Determining whether a given CPTP map can be performed by LOCC is generally very hard. It is much simpler to determine whether the map is PPT.
- Can often use this to rule out an LOCC procedure to do a certain task.
- If PPT bound entanglement is the only type of bound entanglement then PPT = class of operations which cannot create distillable entanglement.

PPT Operations

PPT is strictly more powerful than LOCC. E.g.:

- Can create PPT bound entangled states from separable states.
- Can increase Schmidt rank of a pure state.

Still, not much is known about the relative power of PPT vs. LOCC.

Bipartite Pure State Transformations

In a given class of operations, can we deterministically transform $|\phi\rangle$ into $|\psi\rangle$?
If the class contains local unitaries, the answer will only depend on the Schmidt coefficients of the initial and final state.

$$\rho_\lambda = \sum_{i=1}^{d}\sum_{j=1}^{d}\sqrt{\lambda_i\lambda_j}|ii\rangle\langle jj|, \lambda_i \geq 0, \sum_{i=1}^{d}\lambda_i = 1$$

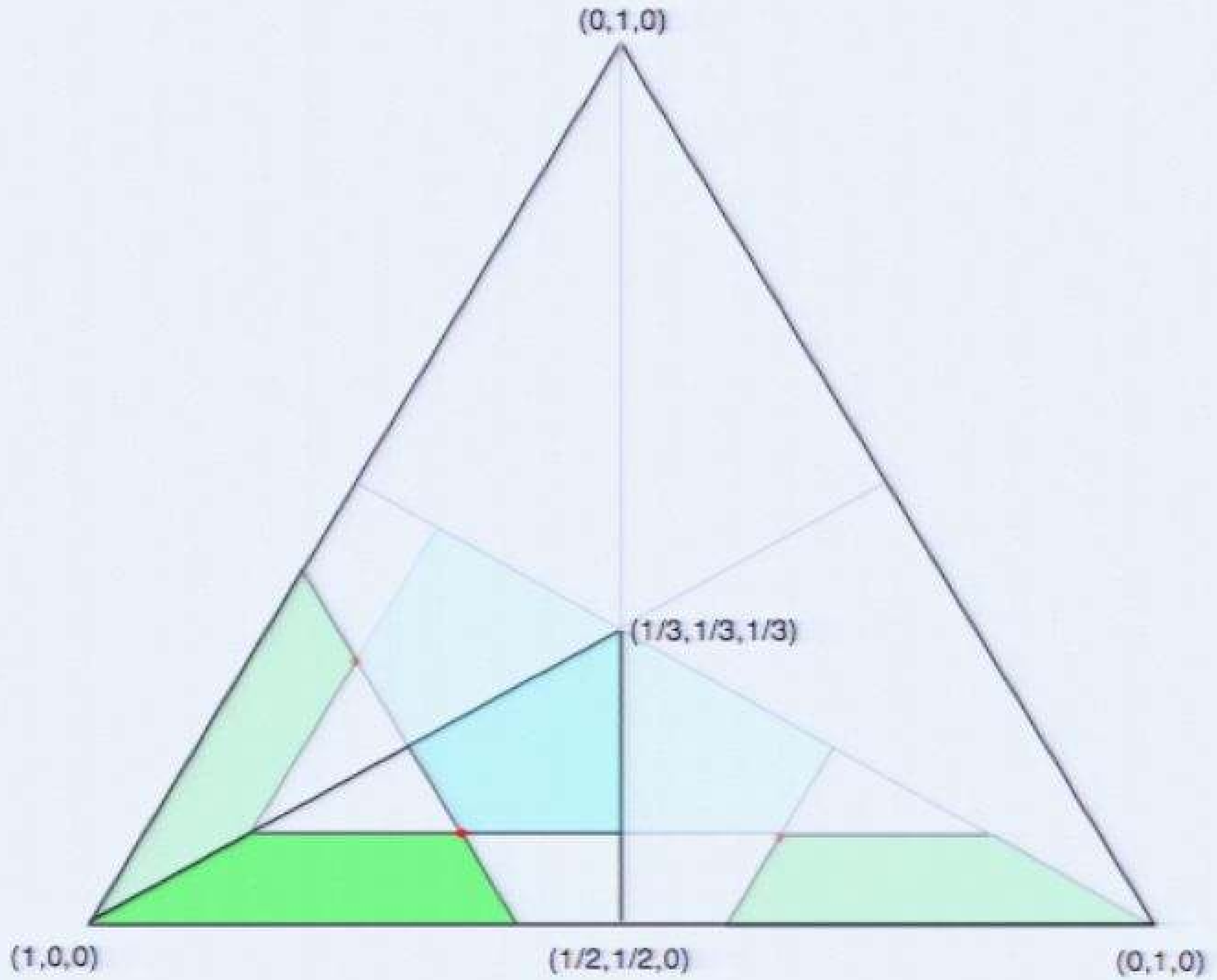
For LOCC, a necessary and sufficient condition was given by Nielsen [2] - the majorization criterion:

$$\rho_\lambda \overset{LOCC}{\rightarrow} \rho_\mu \iff \lambda \prec \mu$$

$$\lambda \prec \mu \iff \sum_{i=1}^{k}\lambda_i^{\downarrow} \leq \sum_{i=1}^{k}\mu_i^{\downarrow} \text{ for } k \in \{1, ..., d\}$$

[2] M. A. Nielsen, "Conditions for a class of entanglement manipulations", Phys. Rev. Lett. 83, 436 – 439 (1999)

LOCC Pure State Transformations



LOCC Catalysis

Using the majorization criterion, it can be shown that a phenomenon analogous to chemical catalysis can occur for pure state transformations[3]:

$$\lambda_1 = (0.4, 0.4, 0.1, 0.1) \nprec \lambda_2 = (0.5, 0.25, 0.25) \implies \rho_{\lambda_1} \not\leftrightarrow \rho_{\lambda_2}$$

$$\chi = (0.6, 0.4)$$

$$\begin{aligned}
(\lambda_1 \otimes \chi)^{\downarrow} &= (0.24, \quad 0.24, \quad 0.16, \quad 0.16, \quad 0.06, \quad 0.06, \quad 0.04, \quad 0.04) \\
& (0.24, \quad 0.48, \quad 0.64, \quad 0.80, \quad 0.86, \quad 0.92, \quad 0.96, \quad 1.00) \\
(\lambda_2 \otimes \chi)^{\downarrow} &= (0.30, \quad 0.20, \quad 0.15, \quad 0.15, \quad 0.10, \quad 0.10, \quad 0.00, \quad 0.00) \\
& (0.30, \quad 0.50, \quad 0.65, \quad 0.80, \quad 0.90, \quad 1.00, \quad 1.00, \quad 1.00)
\end{aligned}$$

$$\lambda_1 \otimes \chi \prec \lambda_2 \otimes \chi, \rho_{\lambda_1} \otimes \rho_\chi \rightarrow \rho_{\lambda_2} \otimes \rho_\chi$$

A maximally entangled state cannot act as a catalyst.

[3] D. Jonathan, M. B. Plenio, "Entanglement-Assisted Local Manipulation of Pure Quantum States", Phys. Rev. Lett. 83, 3566 – 3569 (1999).

LOCC Catalysis

Given two pure bipartite states ρ_λ and ρ_μ, where $\lambda^\uparrow \neq \mu^\uparrow$ and λ and μ don't both have components equal to zero, there exists a pure state ρ_ξ such that $\rho_\lambda \otimes \rho_\xi \stackrel{LOCC}{\rightarrow} \rho_\mu \otimes \rho_\xi$ if and only if the following conditions are satisfied

$$S_t(\lambda) > S_t(\mu) \ \text{ for } t \in (0, \infty), \tag{1}$$

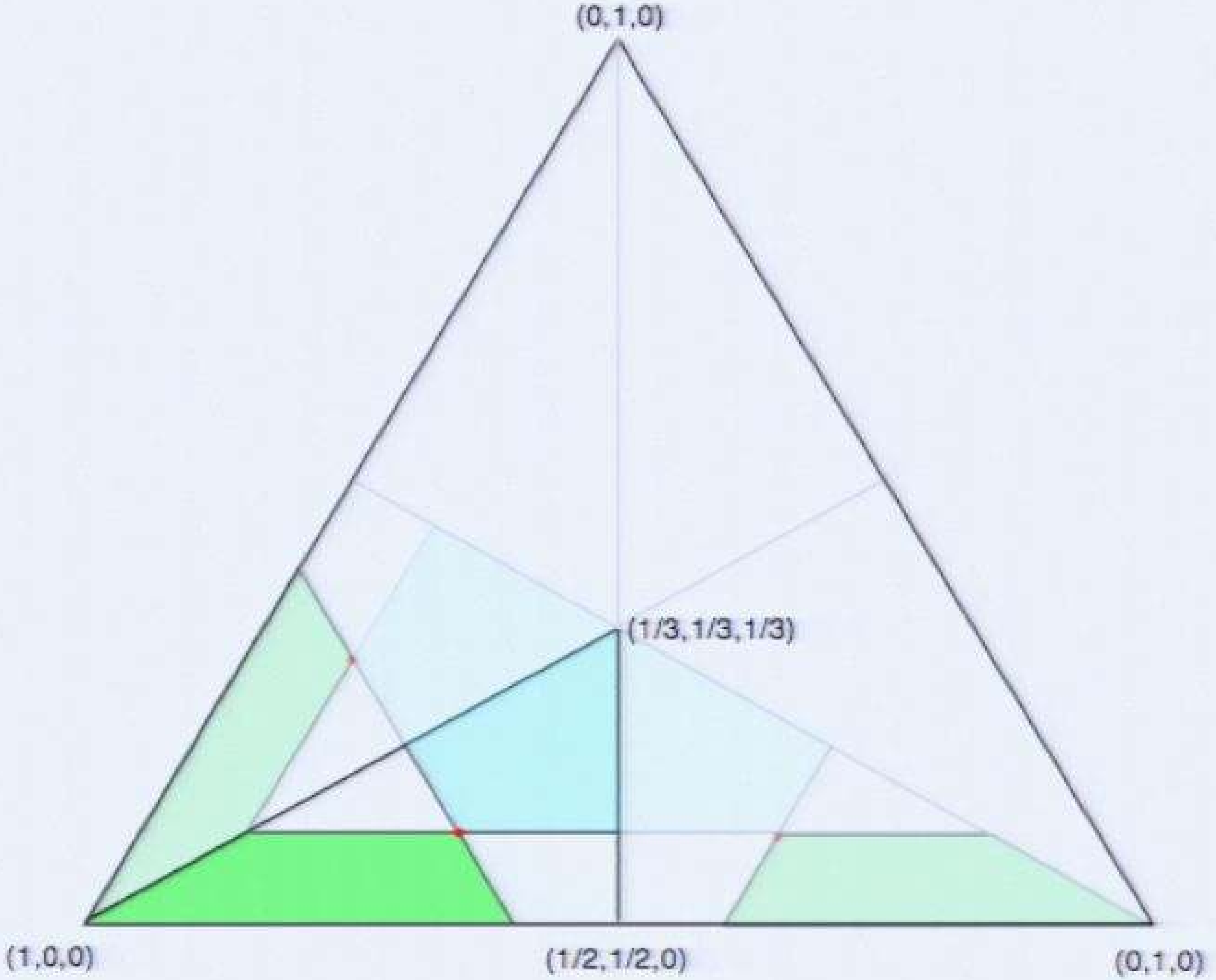
$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \tag{2}$$

$$f_t(\lambda) > f_t(\mu) \ \text{ for } t \in (-\infty, 0), \tag{3}$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log\left(\sum_{i=1}^d \lambda_i^t\right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, ..., d\}, \\ -\infty & \text{otherwise.} \end{cases} \tag{4}$$

LOCC Pure State Transformations



LOCC Catalysis

Using the majorization criterion, it can be shown that a phenomenon analogous to chemical catalysis can occur for pure state transformations[3]:

$$\lambda_1 = (0.4, 0.4, 0.1, 0.1) \nprec \lambda_2 = (0.5, 0.25, 0.25) \implies \rho_{\lambda_1} \nleftrightarrow \rho_{\lambda_2}$$

$$\chi = (0.6, 0.4)$$

$$\begin{aligned}
(\lambda_1 \otimes \chi)^{\downarrow} &= (0.24,\quad 0.24,\quad 0.16,\quad 0.16,\quad 0.06,\quad 0.06,\quad 0.04,\quad 0.04) \\
&\ (0.24,\quad 0.48,\quad 0.64,\quad 0.80,\quad 0.86,\quad 0.92,\quad 0.96,\quad 1.00) \\
(\lambda_2 \otimes \chi)^{\downarrow} &= (0.30,\quad 0.20,\quad 0.15,\quad 0.15,\quad 0.10,\quad 0.10,\quad 0.00,\quad 0.00) \\
&\ (0.30,\quad 0.50,\quad 0.65,\quad 0.80,\quad 0.90,\quad 1.00,\quad 1.00,\quad 1.00)
\end{aligned}$$

$$\lambda_1 \otimes \chi \prec \lambda_2 \otimes \chi, \rho_{\lambda_1} \otimes \rho_\chi \to \rho_{\lambda_2} \otimes \rho_\chi$$

A maximally entangled state cannot act as a catalyst.

[3] D. Jonathan, M. B. Plenio, "Entanglement-Assisted Local Manipulation of Pure Quantum States", Phys. Rev. Lett. 83, 3566 – 3569 (1999)

LOCC Catalysis

Turgut[4], Klimesh: Necessary and sufficient conditions for a catalyzed LOCC transformation to exist between two pure states, in terms of the Rényi entropies of the Schmidt coefficients (and related functions).
For $t \in [0, \infty]$ the *Rényi entropy at t* is defined by

$$S_t(\lambda) = \begin{cases} \frac{1}{1-t} \log \left(\sum_{i=1}^{d} \lambda_i^t \right) & \text{for } t \in (0,1) \cup (1, \infty), \\ \log |\lambda| & \text{for } t = 0, \\ H(\lambda) & \text{for } t = 1, \\ -\log \lambda_1^{\downarrow} & \text{for } t = \infty, \end{cases}$$

where $|\lambda|$ denotes the number of non-zero components of λ (i.e. the Schmidt rank) and $H(\lambda) = -\sum_i \lambda_i \log \lambda_i$ is the Shannon entropy of λ.

---

[4] S. Turgut, "Necessary and sufficient conditions for the trumping relation", arXiv:0707.0444

LOCC Catalysis

Given two pure bipartite states ρ_λ and ρ_μ, where $\lambda^\uparrow \neq \mu^\uparrow$ and λ and μ don't both have components equal to zero, there exists a pure state ρ_ξ such that $\rho_\lambda \otimes \rho_\xi \overset{LOCC}{\rightarrow} \rho_\mu \otimes \rho_\xi$ if and only if the following conditions are satisfied

$$S_t(\lambda) > S_t(\mu) \text{ for } t \in (0, \infty), \tag{1}$$

$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \tag{2}$$

$$f_t(\lambda) > f_t(\mu) \text{ for } t \in (-\infty, 0), \tag{3}$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log \left(\sum_{i=1}^{d} \lambda_i^t \right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, ..., d\}, \\ -\infty & \text{otherwise.} \end{cases} \tag{4}$$

LOCC Catalysis

Turgut[4], Klimesh: Necessary and sufficient conditions for a catalyzed LOCC transformation to exist between two pure states, in terms of the Rényi entropies of the Schmidt coefficients (and related functions).
For $t \in [0, \infty]$ the *Rényi entropy at t* is defined by

$$S_t(\lambda) = \begin{cases} \frac{1}{1-t} \log\left(\sum_{i=1}^{d} \lambda_i^t\right) & \text{for } t \in (0,1) \cup (1,\infty), \\ \log|\lambda| & \text{for } t = 0, \\ H(\lambda) & \text{for } t = 1, \\ -\log \lambda_1^{\downarrow} & \text{for } t = \infty, \end{cases}$$

where $|\lambda|$ denotes the number of non-zero components of λ (i.e. the Schmidt rank) and $H(\lambda) = -\sum_i \lambda_i \log \lambda_i$ is the Shannon entropy of λ.

---

[4] S. Turgut, "Necessary and sufficient conditions for the trumping relation", arXiv:0707.0444

LOCC Catalysis

Given two pure bipartite states ρ_λ and ρ_μ, where $\lambda^\uparrow \neq \mu^\uparrow$ and λ and μ don't both have components equal to zero, there exists a pure state ρ_ξ such that $\rho_\lambda \otimes \rho_\xi \stackrel{LOCC}{\to} \rho_\mu \otimes \rho_\xi$ if and only if the following conditions are satisfied

$$S_t(\lambda) > S_t(\mu) \;\; \text{for } t \in (0, \infty)\,, \tag{1}$$

$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \tag{2}$$

$$f_t(\lambda) > f_t(\mu) \;\; \text{for } t \in (-\infty, 0)\,, \tag{3}$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log\left(\sum_{i=1}^{d} \lambda_i^t\right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, ..., d\}, \\ -\infty & \text{otherwise.} \end{cases} \tag{4}$$

LOCC Catalysis

Turgut[4], Klimesh: Necessary and sufficient conditions for a catalyzed LOCC transformation to exist between two pure states, in terms of the Rényi entropies of the Schmidt coefficients (and related functions).
For $t \in [0, \infty]$ the *Rényi entropy at t* is defined by

$$S_t(\lambda) = \begin{cases} \frac{1}{1-t} \log \left(\sum_{i=1}^{d} \lambda_i^t \right) & \text{for } t \in (0,1) \cup (1, \infty), \\ \log |\lambda| & \text{for } t = 0, \\ H(\lambda) & \text{for } t = 1, \\ -\log \lambda_1^{\downarrow} & \text{for } t = \infty, \end{cases}$$

where $|\lambda|$ denotes the number of non-zero components of λ (i.e. the Schmidt rank) and $H(\lambda) = -\sum_i \lambda_i \log \lambda_i$ is the Shannon entropy of λ.

---

[4] S. Turgut, "Necessary and sufficient conditions for the trumping relation", arXiv:0707.0444

LOCC Catalysis

Given two pure bipartite states ρ_λ and ρ_μ, where $\lambda^\uparrow \neq \mu^\uparrow$ and λ and μ don't both have components equal to zero, there exists a pure state ρ_ξ such that $\rho_\lambda \otimes \rho_\xi \overset{LOCC}{\rightarrow} \rho_\mu \otimes \rho_\xi$ if and only if the following conditions are satisfied

$$S_t(\lambda) > S_t(\mu) \ \text{ for } t \in (0, \infty), \tag{1}$$

$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \tag{2}$$

$$f_t(\lambda) > f_t(\mu) \ \text{ for } t \in (-\infty, 0), \tag{3}$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log \left(\sum_{i=1}^{d} \lambda_i^t \right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, ..., d\}, \\ -\infty & \text{otherwise.} \end{cases} \tag{4}$$

PPT Pure State Transformations - General necessary conditions

- The entanglement cost $E_c^{\mathrm{PPT}}(\rho_\lambda)$ and distillable entanglement $E_d^{\mathrm{PPT}}(\rho_\lambda)$ of ρ_λ are both equal to $S_1(\lambda)$, the entropy of entanglement of the state.
- The exact distillable $E_{xd}^{\mathrm{PPT}}(\rho_\lambda)$ of ρ_λ is given by $S_\infty(\lambda)$.
- The exact entanglement cost $E_{xc}^{\mathrm{PPT}}(\rho_\lambda)$ of ρ_λ is $S_{1/2}(\lambda)$.
- Non-increasing under PPT operations:
- $\rho_\lambda \stackrel{PPT}{\to} \rho_\mu \implies S_t(\lambda) \geq S_t(\mu)$ for t =1/2, 1 and ∞.

LOCC Catalysis

Turgut[4], Klimesh: Necessary and sufficient conditions for a catalyzed LOCC transformation to exist between two pure states, in terms of the Rényi entropies of the Schmidt coefficients (and related functions).

For $t \in [0, \infty]$ the *Rényi entropy at t* is defined by

$$S_t(\lambda) = \begin{cases} \frac{1}{1-t} \log \left(\sum_{i=1}^{d} \lambda_i^t \right) & \text{for } t \in (0,1) \cup (1, \infty), \\ \log |\lambda| & \text{for } t = 0, \\ H(\lambda) & \text{for } t = 1, \\ -\log \lambda_1^{\downarrow} & \text{for } t = \infty, \end{cases}$$

where $|\lambda|$ denotes the number of non-zero components of λ (i.e. the Schmidt rank) and $H(\lambda) = -\sum_i \lambda_i \log \lambda_i$ is the Shannon entropy of λ.

---

[4] S. Turgut, “Necessary and sufficient conditions for the trumping relation”, arXiv:0707.0444

LOCC Catalysis

Using the majorization criterion, it can be shown that a phenomenon analogous to chemical catalysis can occur for pure state transformations[3]:

$$\lambda_1 = (0.4, 0.4, 0.1, 0.1) \not\prec \lambda_2 = (0.5, 0.25, 0.25) \implies \rho_{\lambda_1} \not\leftrightarrow \rho_{\lambda_2}$$

$$\chi = (0.6, 0.4)$$

$$\begin{array}{rl} (\lambda_1 \otimes \chi)^{\downarrow} = & (0.24, \quad 0.24, \quad 0.16, \quad 0.16, \quad 0.06, \quad 0.06, \quad 0.04, \quad 0.04) \\ & (0.24, \quad 0.48, \quad 0.64, \quad 0.80, \quad 0.86, \quad 0.92, \quad 0.96, \quad 1.00) \\ (\lambda_2 \otimes \chi)^{\downarrow} = & (0.30, \quad 0.20, \quad 0.15, \quad 0.15, \quad 0.10, \quad 0.10, \quad 0.00, \quad 0.00) \\ & (0.30, \quad 0.50, \quad 0.65, \quad 0.80, \quad 0.90, \quad 1.00, \quad 1.00, \quad 1.00) \end{array}$$

$$\lambda_1 \otimes \chi \prec \lambda_2 \otimes \chi, \rho_{\lambda_1} \otimes \rho_\chi \to \rho_{\lambda_2} \otimes \rho_\chi$$

A maximally entangled state cannot act as a catalyst.

[3] D. Jonathan, M. B. Plenio, "Entanglement-Assisted Local Manipulation of Pure Quantum States", Phys. Rev. Lett. 83, 3566 – 3569 (1999).

LOCC Catalysis

Turgut[4], Klimesh: Necessary and sufficient conditions for a catalyzed LOCC transformation to exist between two pure states, in terms of the Rényi entropies of the Schmidt coefficients (and related functions).
For $t \in [0, \infty]$ the *Rényi entropy at t* is defined by

$$S_t(\lambda) = \begin{cases} \frac{1}{1-t} \log\left(\sum_{i=1}^{d} \lambda_i^t\right) & \text{for } t \in (0,1) \cup (1,\infty), \\ \log|\lambda| & \text{for } t = 0, \\ H(\lambda) & \text{for } t = 1, \\ -\log \lambda_1^{\downarrow} & \text{for } t = \infty, \end{cases}$$

where $|\lambda|$ denotes the number of non-zero components of λ (i.e. the Schmidt rank) and $H(\lambda) = -\sum_i \lambda_i \log \lambda_i$ is the Shannon entropy of λ.

[4] S. Turgut, "Necessary and sufficient conditions for the trumping relation", arXiv:0707.0444

LOCC Catalysis

Turgut[4], Klimesh: Necessary and sufficient conditions for a catalyzed LOCC transformation to exist between two pure states, in terms of the Rényi entropies of the Schmidt coefficients (and related functions).
For $t \in [0, \infty]$ the *Rényi entropy at t* is defined by

$$S_t(\lambda) = \begin{cases} \frac{1}{1-t} \log \left(\sum_{i=1}^{d} \lambda_i^t \right) & \text{for } t \in (0,1) \cup (1, \infty), \\ \log |\lambda| & \text{for } t = 0, \\ H(\lambda) & \text{for } t = 1, \\ -\log \lambda_1^{\downarrow} & \text{for } t = \infty, \end{cases}$$

where $|\lambda|$ denotes the number of non-zero components of λ (i.e. the Schmidt rank) and $H(\lambda) = -\sum_i \lambda_i \log \lambda_i$ is the Shannon entropy of λ.

---

[4] S. Turgut, "Necessary and sufficient conditions for the trumping relation", arXiv:0707.0444

LOCC Catalysis

Given two pure bipartite states ρ_λ and ρ_μ, where $\lambda^\uparrow \neq \mu^\uparrow$ and λ and μ don't both have components equal to zero, there exists a pure state ρ_ξ such that $\rho_\lambda \otimes \rho_\xi \stackrel{LOCC}{\rightarrow} \rho_\mu \otimes \rho_\xi$ if and only if the following conditions are satisfied

$$S_t(\lambda) > S_t(\mu) \;\text{ for } t \in (0, \infty)\,, \tag{1}$$

$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \tag{2}$$

$$f_t(\lambda) > f_t(\mu) \;\text{ for } t \in (-\infty, 0)\,, \tag{3}$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log \left(\sum_{i=1}^{d} \lambda_i^t \right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, ..., d\}, \\ -\infty & \text{otherwise.} \end{cases} \tag{4}$$

PPT Pure State Transformations - General necessary conditions

- The entanglement cost $E_c^{\mathrm{PPT}}(\rho_\lambda)$ and distillable entanglement $E_d^{\mathrm{PPT}}(\rho_\lambda)$ of ρ_λ are both equal to $S_1(\lambda)$, the entropy of entanglement of the state.
- The exact distillable $E_{xd}^{\mathrm{PPT}}(\rho_\lambda)$ of ρ_λ is given by $S_\infty(\lambda)$.
- The exact entanglement cost $E_{xc}^{\mathrm{PPT}}(\rho_\lambda)$ of ρ_λ is $S_{1/2}(\lambda)$.
- Non-increasing under PPT operations:
- $\rho_\lambda \overset{PPT}{\rightarrow} \rho_\mu \implies S_t(\lambda) \geq S_t(\mu)$ for t =1/2, 1 and ∞.

PPT Pure State Transformations - Maximally Entangled Initial State

The existence of a PPT map taking $|\phi\rangle$ to $|\psi\rangle$ can be formulated as a semidefinite program.
It is rather complicated but it simplifies greatly if we restrict to the case initial state is maximally entangled state.
For any (pure or mixed) final state ρ, $\Phi_K \overset{PPT}{\rightarrow} \rho$ if and only if the solution to the semidefinite program

$$\min\{\mathrm{Tr}\,(P)\,|P \geq 0, -(K-1)\,P^{\Gamma} \leq \rho^{\Gamma} \leq (K+1)\,P^{\Gamma}\}, \qquad (5)$$

where P is an hermitian operator on the same Hilbert space as ρ, is less than or equal to one.
For pure ρ we can make further simplifications...

Primal Semidefinite Program

$$\Phi_K \stackrel{PPT}{\to} \rho_\lambda \iff T(K;\lambda) \le 1$$

where

$$T(K;\lambda) := \min \sum_{i \ge j}^{d} s_{ij} + \sum_{i>j}^{d} a_{ij}$$

subject to

$$s_{ij} \ge \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \ge \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \ge j} s_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} a_{ij}\alpha_{ij}^{\Gamma} \ge 0$$

with

$$\sigma_{ij} = \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \ne j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases}$$

$$\alpha_{ij} = (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2.$$

PPT Pure State Transformations - Maximally Entangled Initial State

The existence of a PPT map taking $|\phi\rangle$ to $|\psi\rangle$ can be formulated as a semidefinite program.
It is rather complicated but it simplifies greatly if we restrict to the case initial state is maximally entangled state.
For any (pure or mixed) final state ρ, $\Phi_K \overset{PPT}{\rightarrow} \rho$ if and only if the solution to the semidefinite program

$$\min\{\mathrm{Tr}\,(P)\,|P \geq 0, -(K-1)\,P^{\Gamma} \leq \rho^{\Gamma} \leq (K+1)\,P^{\Gamma}\}, \quad (5)$$

where P is an hermitian operator on the same Hilbert space as ρ, is less than or equal to one.
For pure ρ we can make further simplifications...

Primal Semidefinite Program

$$\Phi_K \overset{PPT}{\rightarrow} \rho_\lambda \iff T(K;\lambda) \leq 1$$

where

$$T(K;\lambda) := \min \sum_{i \geq j}^{d} s_{ij} + \sum_{i > j}^{d} a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij} \sigma_{ij}^{\Gamma} + \sum_{i > j} a_{ij} \alpha_{ij}^{\Gamma} \geq 0$$

with

$$\begin{aligned} \sigma_{ij} &= \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases} \\ \alpha_{ij} &= (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2. \end{aligned}$$

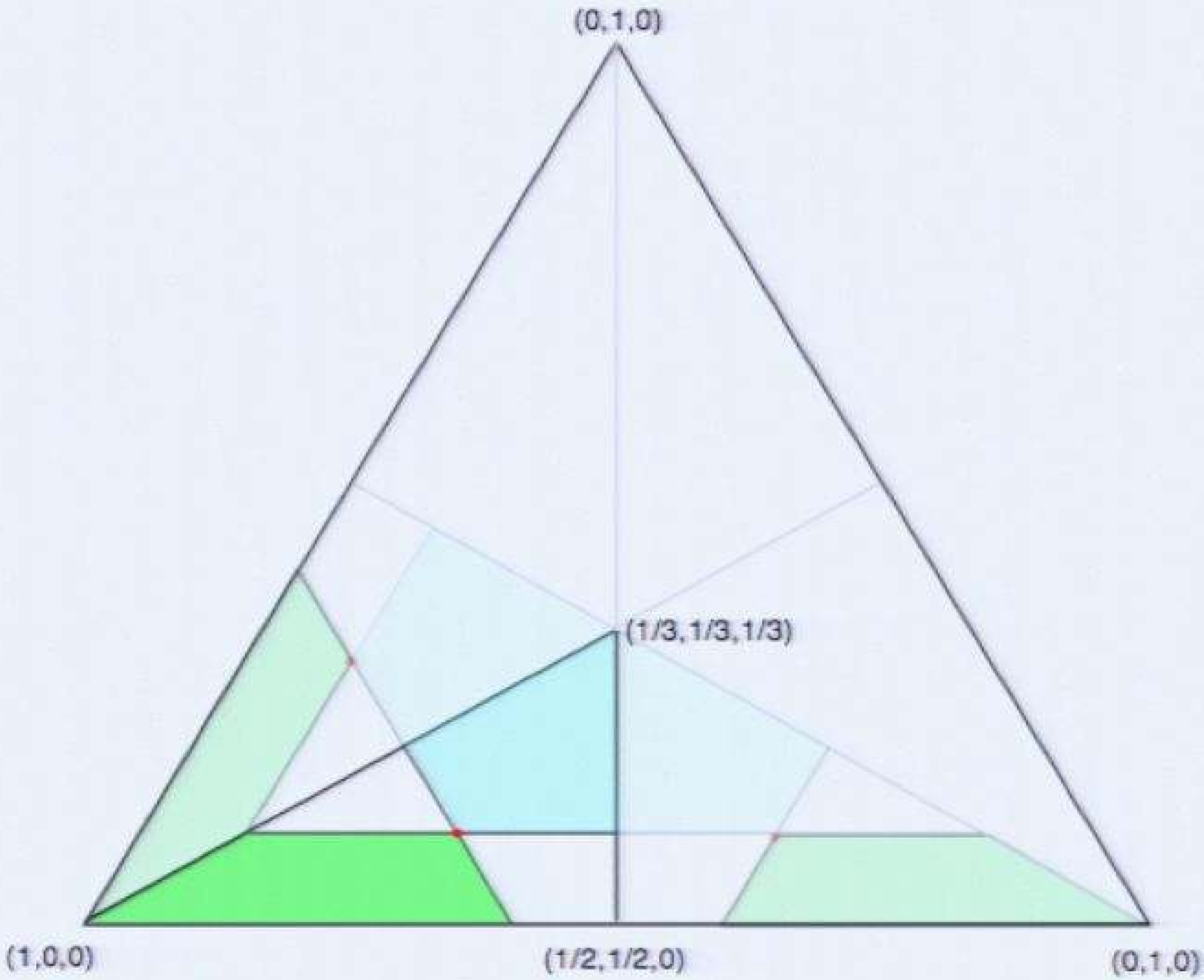
PPT Pure State Transformations - Maximally Entangled Initial State

The existence of a PPT map taking $|\phi\rangle$ to $|\psi\rangle$ can be formulated as a semidefinite program.
It is rather complicated but it simplifies greatly if we restrict to the case initial state is maximally entangled state.
For any (pure or mixed) final state ρ, $\Phi_K \stackrel{PPT}{\rightarrow} \rho$ if and only if the solution to the semidefinite program

$$\min\{\mathrm{Tr}\,(P)\,|P \geq 0, -(K-1)\,P^{\Gamma} \leq \rho^{\Gamma} \leq (K+1)\,P^{\Gamma}\}, \quad (5)$$

where P is an hermitian operator on the same Hilbert space as ρ, is less than or equal to one.
For pure ρ we can make further simplifications...

LOCC Pure State Transformations



LOCC Catalysis

Using the majorization criterion, it can be shown that a phenomenon analogous to chemical catalysis can occur for pure state transformations[3]:

$$\lambda_1 = (0.4, 0.4, 0.1, 0.1) \nprec \lambda_2 = (0.5, 0.25, 0.25) \implies \rho_{\lambda_1} \not\leftrightarrow \rho_{\lambda_2}$$

$$\chi = (0.6, 0.4)$$

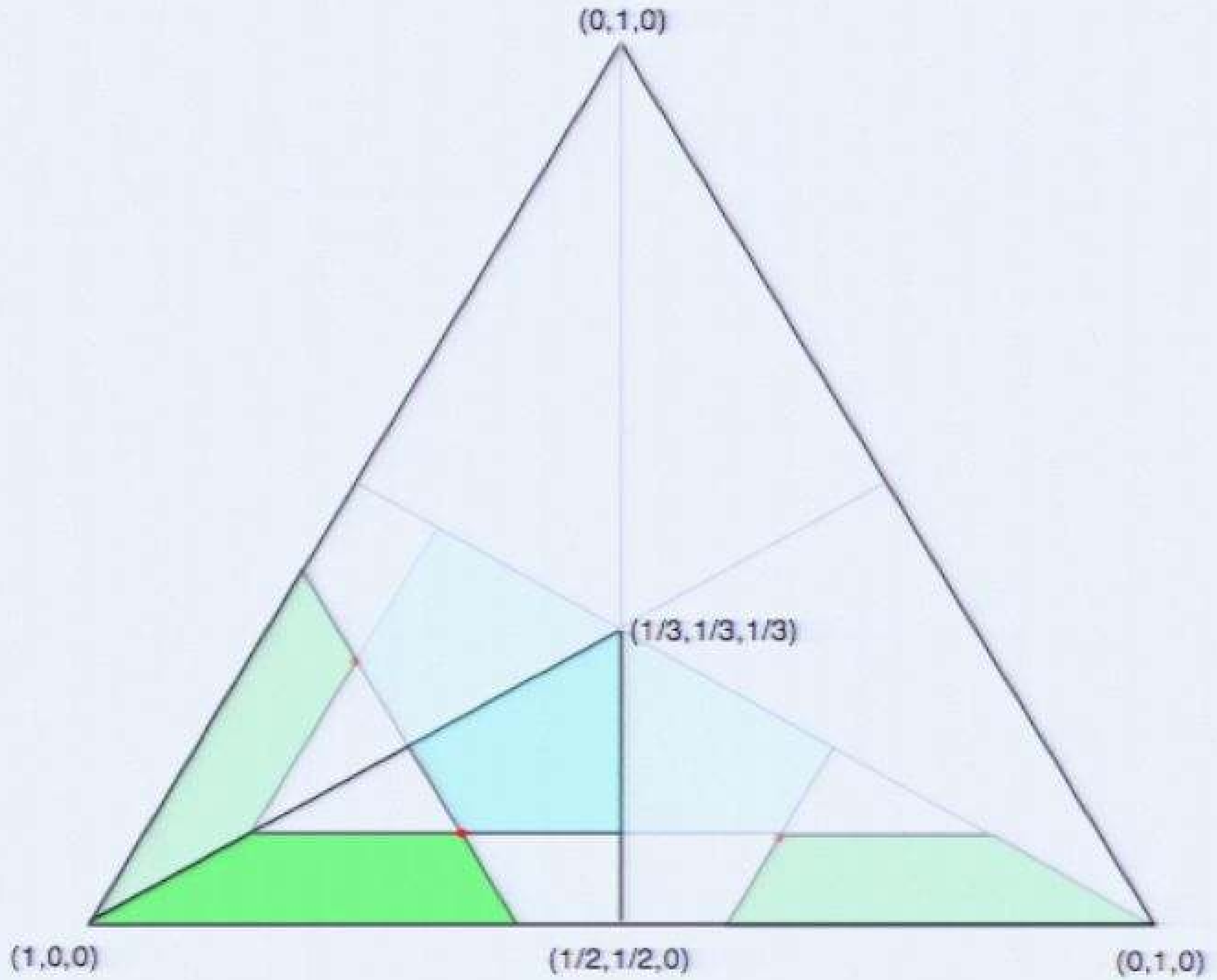
$$\begin{aligned}
(\lambda_1 \otimes \chi)^{\downarrow} &= (0.24, \quad 0.24, \quad 0.16, \quad 0.16, \quad 0.06, \quad 0.06, \quad 0.04, \quad 0.04) \\
& (0.24, \quad 0.48, \quad 0.64, \quad 0.80, \quad 0.86, \quad 0.92, \quad 0.96, \quad 1.00) \\
(\lambda_2 \otimes \chi)^{\downarrow} &= (0.30, \quad 0.20, \quad 0.15, \quad 0.15, \quad 0.10, \quad 0.10, \quad 0.00, \quad 0.00) \\
& (0.30, \quad 0.50, \quad 0.65, \quad 0.80, \quad 0.90, \quad 1.00, \quad 1.00, \quad 1.00)
\end{aligned}$$

$$\lambda_1 \otimes \chi \prec \lambda_2 \otimes \chi, \rho_{\lambda_1} \otimes \rho_\chi \rightarrow \rho_{\lambda_2} \otimes \rho_\chi$$

A maximally entangled state cannot act as a catalyst.

[3] D. Jonathan, M. B. Plenio, "Entanglement-Assisted Local Manipulation of Pure Quantum States", Phys. Rev. Lett. 83, 3566 - 3569 (1999).

LOCC Pure State Transformations



Bipartite Pure State Transformations

In a given class of operations, can we deterministically transform $|\phi\rangle$ into $|\psi\rangle$?
If the class contains local unitaries, the answer will only depend on the Schmidt coefficients of the initial and final state.

$$\rho_\lambda = \sum_{i=1}^{d}\sum_{j=1}^{d}\sqrt{\lambda_i\lambda_j}|ii\rangle\langle jj|, \lambda_i \geq 0, \sum_{i=1}^{d}\lambda_i = 1$$

For LOCC, a necessary and sufficient condition was given by Nielsen [2] - the majorization criterion:

$$\rho_\lambda \overset{LOCC}{\rightarrow} \rho_\mu \iff \lambda \prec \mu$$

$$\lambda \prec \mu \iff \sum_{i=1}^{k}\lambda_i^{\downarrow} \leq \sum_{i=1}^{k}\mu_i^{\downarrow} \text{ for } k \in \{1,...,d\}$$

[2] M. A. Nielsen, "Conditions for a class of entanglement manipulations", Phys. Rev. Lett. **83**, 436 - 439 (1999)

LOCC Catalysis

Given two pure bipartite states ρ_λ and ρ_μ, where $\lambda^\uparrow \neq \mu^\uparrow$ and λ and μ don't both have components equal to zero, there exists a pure state ρ_ξ such that $\rho_\lambda \otimes \rho_\xi \stackrel{LOCC}{\rightarrow} \rho_\mu \otimes \rho_\xi$ if and only if the following conditions are satisfied

$$S_t(\lambda) > S_t(\mu) \ \text{ for } t \in (0, \infty), \tag{1}$$

$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \tag{2}$$

$$f_t(\lambda) > f_t(\mu) \ \text{ for } t \in (-\infty, 0), \tag{3}$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log\left(\sum_{i=1}^d \lambda_i^t\right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, \ldots, d\}, \\ -\infty & \text{otherwise.} \end{cases} \tag{4}$$

PPT Pure State Transformations - General necessary conditions

- The entanglement cost $E_c^{\text{PPT}}(\rho_\lambda)$ and distillable entanglement $E_d^{\text{PPT}}(\rho_\lambda)$ of ρ_λ are both equal to $S_1(\lambda)$, the entropy of entanglement of the state.
- The exact distillable $E_{xd}^{\text{PPT}}(\rho_\lambda)$ of ρ_λ is given by $S_\infty(\lambda)$.
- The exact entanglement cost $E_{xc}^{\text{PPT}}(\rho_\lambda)$ of ρ_λ is $S_{1/2}(\lambda)$.
- Non-increasing under PPT operations:
- $\rho_\lambda \overset{PPT}{\rightarrow} \rho_\mu \implies S_t(\lambda) \geq S_t(\mu)$ for t =1/2, 1 and ∞.

Primal Semidefinite Program

$$\Phi_K \stackrel{PPT}{\rightarrow} \rho_\lambda \iff T(K;\lambda) \leq 1$$

where

$$T(K;\lambda) := \min \sum_{i \geq j}^{d} s_{ij} + \sum_{i>j}^{d} a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} a_{ij}\alpha_{ij}^{\Gamma} \geq 0$$

with

$$\sigma_{ij} = \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases}$$

$$\alpha_{ij} = (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2.$$

Dual Semidefinite Program

The semidefinite program dual to the above is

$$T(K;\lambda) = \max \frac{K2^{S_{1/2}(\lambda)}-1}{K^2-1} - \left(\sum_{i \geq j} \frac{\mu_{ij}\sqrt{\lambda_i \lambda_j}}{K+1} + \sum_{i>j} \frac{\nu_{ij}\sqrt{\lambda_i \lambda_j}}{K-1} \right)$$

subject to

$$\mu_{ij} \leq 1, \nu_{ij} \leq 1, \sum_{i \geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij}\alpha_{ij}^{\Gamma} \geq 0.$$

The dual objective at any dual feasible point is a lower bound on $T(K;\lambda)$.
Additional constraint:

$$\text{rank}\left(\sum_{i \geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij}\alpha_{ij}^{\Gamma} \right) = 1$$

Primal Semidefinite Program

$$\Phi_K \overset{PPT}{\to} \rho_\lambda \iff T(K;\lambda) \leq 1$$

where

$$T(K;\lambda) := \min \sum_{i \geq j}^{d} s_{ij} + \sum_{i>j}^{d} a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij} \sigma_{ij}^{\Gamma} + \sum_{i>j} a_{ij} \alpha_{ij}^{\Gamma} \geq 0$$

with

$$\begin{aligned} \sigma_{ij} &= \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases} \\ \alpha_{ij} &= (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2. \end{aligned}$$

Dual Semidefinite Program

The semidefinite program dual to the above is

$$T(K;\lambda) = \max \frac{K2^{S_{1/2}(\lambda)} - 1}{K^2 - 1} - \left(\sum_{i \geq j} \frac{\mu_{ij}\sqrt{\lambda_i \lambda_j}}{K+1} + \sum_{i>j} \frac{\nu_{ij}\sqrt{\lambda_i \lambda_j}}{K-1} \right)$$

subject to

$$\mu_{ij} \leq 1, \nu_{ij} \leq 1, \sum_{i \geq j} \mu_{ij} \sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij} \alpha_{ij}^{\Gamma} \geq 0.$$

The dual objective at any dual feasible point is a lower bound on $T(K;\lambda)$.
Additional constraint:

$$\text{rank}\left(\sum_{i \geq j} \mu_{ij} \sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij} \alpha_{ij}^{\Gamma} \right) = 1$$

Primal Semidefinite Program

$$\Phi_K \stackrel{PPT}{\rightarrow} \rho_\lambda \iff T(K;\lambda) \leq 1$$

where

$$T(K;\lambda) := \min \sum_{i \geq j}^{d} s_{ij} + \sum_{i>j}^{d} a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij} \sigma_{ij}^{\Gamma} + \sum_{i>j} a_{ij} \alpha_{ij}^{\Gamma} \geq 0$$

with

$$\begin{aligned} \sigma_{ij} &= \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases} \\ \alpha_{ij} &= (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2. \end{aligned}$$

Dual Semidefinite Program

The semidefinite program dual to the above is

$$T(K;\lambda) = \max \frac{K2^{S_{1/2}(\lambda)} - 1}{K^2 - 1} - \left(\sum_{i \geq j} \frac{\mu_{ij}\sqrt{\lambda_i \lambda_j}}{K+1} + \sum_{i > j} \frac{\nu_{ij}\sqrt{\lambda_i \lambda_j}}{K-1} \right)$$

subject to

$$\mu_{ij} \leq 1, \nu_{ij} \leq 1, \sum_{i \geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i > j} t_{ij}\alpha_{ij}^{\Gamma} \geq 0.$$

The dual objective at any dual feasible point is a lower bound on $T(K;\lambda)$.
Additional constraint:

$$\text{rank}\left(\sum_{i \geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i > j} t_{ij}\alpha_{ij}^{\Gamma} \right) = 1$$

Primal Semidefinite Program

$$\Phi_K \stackrel{PPT}{\to} \rho_\lambda \iff T(K;\lambda) \leq 1$$

where

$$T(K;\lambda) := \min \sum_{i \geq j}^{d} s_{ij} + \sum_{i>j}^{d} a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} a_{ij}\alpha_{ij}^{\Gamma} \geq 0$$

with

$$\begin{aligned} \sigma_{ij} &= \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases} \\ \alpha_{ij} &= (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2. \end{aligned}$$

Dual Semidefinite Program

The semidefinite program dual to the above is

$$T(K;\lambda) = \max \frac{K2^{S_{1/2}(\lambda)} - 1}{K^2 - 1} - \left(\sum_{i \geq j} \frac{\mu_{ij}\sqrt{\lambda_i \lambda_j}}{K+1} + \sum_{i>j} \frac{\nu_{ij}\sqrt{\lambda_i \lambda_j}}{K-1} \right)$$

subject to

$$\mu_{ij} \leq 1, \nu_{ij} \leq 1, \sum_{i \geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij}\alpha_{ij}^{\Gamma} \geq 0.$$

The dual objective at any dual feasible point is a lower bound on $T(K;\lambda)$.
Additional constraint:

$$\text{rank}\left(\sum_{i \geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij}\alpha_{ij}^{\Gamma} \right) = 1$$

Primal Semidefinite Program

$$\Phi_K \overset{PPT}{\rightarrow} \rho_\lambda \iff T(K;\lambda) \leq 1$$

where

$$T(K;\lambda) := \min \sum_{i \geq j}^{d} s_{ij} + \sum_{i>j}^{d} a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij} \sigma_{ij}^{\Gamma} + \sum_{i>j} a_{ij} \alpha_{ij}^{\Gamma} \geq 0$$

with

$$\begin{aligned} \sigma_{ij} &= \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases} \\ \alpha_{ij} &= (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2. \end{aligned}$$

PPT Pure State Transformations - Maximally Entangled Initial State

The existence of a PPT map taking $|\phi\rangle$ to $|\psi\rangle$ can be formulated as a semidefinite program.
It is rather complicated but it simplifies greatly if we restrict to the case initial state is maximally entangled state.
For any (pure or mixed) final state ρ, $\Phi_K \stackrel{PPT}{\rightarrow} \rho$ if and only if the solution to the semidefinite program

$$\min\{\mathrm{Tr}\,(P)\,|P \geq 0, -(K-1)\,P^{\Gamma} \leq \rho^{\Gamma} \leq (K+1)\,P^{\Gamma}\}, \qquad (5)$$

where P is an hermitian operator on the same Hilbert space as ρ, is less than or equal to one.
For pure ρ we can make further simplifications...

Primal Semidefinite Program

$$\Phi_K \overset{PPT}{\rightarrow} \rho_\lambda \iff T(K;\lambda) \leq 1$$

where

$$T(K;\lambda) := \min \sum_{i \geq j}^{d} s_{ij} + \sum_{i>j}^{d} a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} a_{ij}\alpha_{ij}^{\Gamma} \geq 0$$

with

$$\begin{aligned} \sigma_{ij} &= \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases} \\ \alpha_{ij} &= (|ij\rangle - |ji\rangle)(\langle ij| - \langle ji|)/2. \end{aligned}$$

Dual Semidefinite Program

The semidefinite program dual to the above is

$$T(K;\lambda) = \max \frac{K2^{S_{1/2}(\lambda)}-1}{K^2-1} - \left(\sum_{i\geq j} \frac{\mu_{ij}\sqrt{\lambda_i\lambda_j}}{K+1} + \sum_{i>j} \frac{\nu_{ij}\sqrt{\lambda_i\lambda_j}}{K-1} \right)$$

subject to

$$\mu_{ij} \leq 1, \nu_{ij} \leq 1, \sum_{i\geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij}\alpha_{ij}^{\Gamma} \geq 0.$$

The dual objective at any dual feasible point is a lower bound on $T(K;\lambda)$.
Additional constraint:

$$\text{rank}\left(\sum_{i\geq j} \mu_{ij}\sigma_{ij}^{\Gamma} + \sum_{i>j} t_{ij}\alpha_{ij}^{\Gamma} \right) = 1$$

Rank Constrained Dual Optimum

Let d be the Schmidt rank of the goal state.

$$T_1(K;\lambda) = \frac{K2^{S_{1/2}(\lambda)} - 1}{K^2 - 1} + \frac{K\left(\left(\sum_{i=1}^{c^*}\sqrt{\lambda_i^\uparrow}\right)^2 - (K + c^* - d)\sum_{i=1}^{c^*}\lambda_i^\uparrow\right)}{(K^2-1)(K+c^*-d)},$$

where c^* is the smallest number $c \in \{1 + d - K, ..., d - 1\}$ satisfying

$$\frac{\sum_{i=1}^{c}\sqrt{\lambda_i^\uparrow}}{K + c - d} \leq \sqrt{\lambda_{c+1}^\uparrow}.$$

If none of the integers in the range satisfy this relation then $c^* = d$.

$$T(K;\lambda) \geq T_1(K;\lambda)$$

$$\Phi_K \stackrel{PPT}{\rightarrow} \rho_\lambda \implies T_1(K;\lambda) \leq 1$$

Necessary and sufficient condition for $K = 2$, $d = 3$

In this case we can find a primal solution with primal objective equal to $T_1(K;\lambda)$.
$T = T_1$ for $K = 2$, $d = 3$.
The necessary and sufficient condition for the feasibility of $\Phi_2 \overset{PPT}{\rightarrow} \rho_\lambda$ can be summarized by

$$2\left(\sqrt{\lambda_3^{\uparrow}\lambda_2^{\uparrow}} + \sqrt{\lambda_3^{\uparrow}\lambda_1^{\uparrow}}\right) + 4\sqrt{\lambda_1^{\uparrow}\lambda_2^{\uparrow}} \leq 1.$$

We conjecture that $T(K;\lambda) = T_1(K;\lambda)$ so that

$$\Phi_K \overset{PPT}{\rightarrow} \rho_\lambda \iff T_1(K;\lambda) \leq 1.$$

PPT Pure State Catalysis

Does catalysis occur under PPT operations?

$$\rho_\lambda \overset{PPT}{\nrightarrow} \rho_\mu$$

but

$$\rho_\lambda \otimes \rho_\chi \overset{PPT}{\rightarrow} \rho_\mu \otimes \rho_\chi ?$$

It is always possible to find a primal solution (in general not optimal) giving the upper bound

$$T(K;\lambda) \le \frac{2^{S_{1/2}(\lambda)} - 1}{K-1},$$

and to find a dual solution which gives the lower bound

$$T(K;\lambda) \ge \frac{K2^{S_{1/2}(\lambda)} - 1}{K^2-1}.$$

PPT Pure State Catalysis

Since $S_{1/2}(\lambda \otimes \mu) = S_{1/2}(\lambda) + S_{1/2}(\mu)$ and $S_{1/2}(U_C) = \log C$ we find that

$$\frac{KC^2 2^{S_{1/2}(\lambda)} - 1}{K^2C^2 - 1} \le T(KC; \lambda \otimes U_C) \le \frac{C2^{S_{1/2}(\lambda)} - 1}{KC - 1}.$$

Taking the limit $C \to \infty$, the upper and lower bound both tend to

$$2^{S_{1/2}(\lambda)}/K$$

so a necessary and sufficient condition for the transformation $\Phi_K \otimes \Phi_C \overset{PPT}{\rightarrow} \rho_\lambda \otimes \Phi_C$ to be possible for some suffiently large C is

$$2^{S_{1/2}(\lambda)}/K \le 1$$

or equivalently

$$S_{1/2}(\lambda) \le \log K.$$

PPT Pure State Catalysis

Since $S_{1/2}(\lambda \otimes \mu) = S_{1/2}(\lambda) + S_{1/2}(\mu)$ and $S_{1/2}(U_C) = \log C$ we find that

$$\frac{KC^2 2^{S_{1/2}(\lambda)} - 1}{K^2C^2 - 1} \leq T(KC; \lambda \otimes U_C) \leq \frac{C2^{S_{1/2}(\lambda)} - 1}{KC - 1}.$$

Taking the limit $C \to \infty$, the upper and lower bound both tend to

$$2^{S_{1/2}(\lambda)}/K$$

so a necessary and sufficient condition for the transformation $\Phi_K \otimes \Phi_C \stackrel{PPT}{\to} \rho_\lambda \otimes \Phi_C$ to be possible for some suffiently large C is

$$2^{S_{1/2}(\lambda)}/K \leq 1$$

or equivalently

$$S_{1/2}(\lambda) \leq \log K.$$

PPT Pure State Catalysis

Since $S_{1/2}(\lambda \otimes \mu) = S_{1/2}(\lambda) + S_{1/2}(\mu)$ and $S_{1/2}(U_C) = \log C$ we find that

$$\frac{KC^2 2^{S_{1/2}(\lambda)} - 1}{K^2C^2 - 1} \leq T(KC; \lambda \otimes U_C) \leq \frac{C2^{S_{1/2}(\lambda)} - 1}{KC - 1}.$$

Taking the limit $C \to \infty$, the upper and lower bound both tend to

$$2^{S_{1/2}(\lambda)}/K$$

so a necessary and sufficient condition for the transformation $\Phi_K \otimes \Phi_C \overset{PPT}{\rightarrow} \rho_\lambda \otimes \Phi_C$ to be possible for some suffiently large C is

$$2^{S_{1/2}(\lambda)}/K \leq 1$$

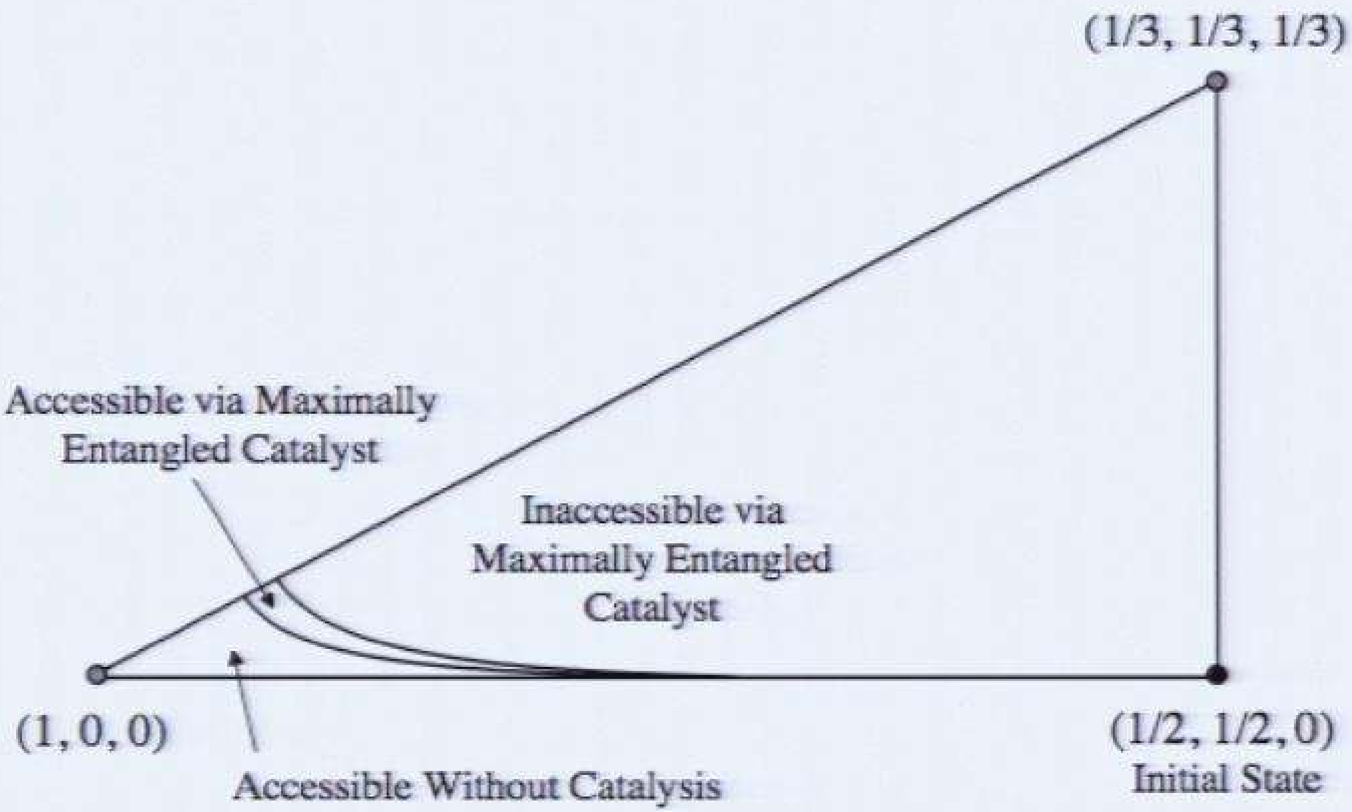
or equivalently

$$S_{1/2}(\lambda) \leq \log K.$$

PPT Pure State Catalysis: $K = 2$, $d = 3$

$2\sqrt{\lambda_3^\uparrow \lambda_2^\uparrow} + 2\sqrt{\lambda_3^\uparrow \lambda_1^\uparrow} + 4\sqrt{\lambda_1^\uparrow \lambda_2^\uparrow} \leq 1$ without a catalyst.

$2\sqrt{\lambda_3^\uparrow \lambda_2^\uparrow} + 2\sqrt{\lambda_3^\uparrow \lambda_1^\uparrow} + 2\sqrt{\lambda_1^\uparrow \lambda_2^\uparrow} \leq 1$ when arbitrarily large maximally entangled catalysts are available.



Maximally entangled states can be catalysts under PPT.

PPT Pure State Catalysis

Since $S_{1/2}(\lambda \otimes \mu) = S_{1/2}(\lambda) + S_{1/2}(\mu)$ and $S_{1/2}(U_C) = \log C$ we find that

$$\frac{KC^2 2^{S_{1/2}(\lambda)} - 1}{K^2C^2 - 1} \leq T(KC; \lambda \otimes U_C) \leq \frac{C2^{S_{1/2}(\lambda)} - 1}{KC - 1}.$$

Taking the limit $C \to \infty$, the upper and lower bound both tend to

$$2^{S_{1/2}(\lambda)}/K$$

so a necessary and sufficient condition for the transformation $\Phi_K \otimes \Phi_C \stackrel{PPT}{\rightarrow} \rho_\lambda \otimes \Phi_C$ to be possible for some suffiently large C is

$$2^{S_{1/2}(\lambda)}/K \leq 1$$

or equivalently

$$S_{1/2}(\lambda) \leq \log K.$$

PPT Pure State Catalysis

Does catalysis occur under PPT operations?

$$\rho_\lambda \overset{PPT}{\nrightarrow} \rho_\mu$$

but

$$\rho_\lambda \otimes \rho_\chi \overset{PPT}{\rightarrow} \rho_\mu \otimes \rho_\chi?$$

It is always possible to find a primal solution (in general not optimal) giving the upper bound

$$T\left(K;\lambda\right) \leq \frac{2^{S_{1/2}(\lambda)} - 1}{K-1},$$

and to find a dual solution which gives the lower bound

$$T\left(K;\lambda\right) \geq \frac{K2^{S_{1/2}(\lambda)} - 1}{K^2-1}.$$

PPT Pure State Catalysis

Since $S_{1/2}(\lambda \otimes \mu) = S_{1/2}(\lambda) + S_{1/2}(\mu)$ and $S_{1/2}(U_C) = \log C$ we find that

$$\frac{KC^2 2^{S_{1/2}(\lambda)} - 1}{K^2 C^2 - 1} \leq T(KC; \lambda \otimes U_C) \leq \frac{C 2^{S_{1/2}(\lambda)} - 1}{KC - 1}.$$

Taking the limit $C \to \infty$, the upper and lower bound both tend to

$$2^{S_{1/2}(\lambda)}/K$$

so a necessary and sufficient condition for the transformation $\Phi_K \otimes \Phi_C \overset{PPT}{\rightarrow} \rho_\lambda \otimes \Phi_C$ to be possible for some suffiently large C is

$$2^{S_{1/2}(\lambda)}/K \leq 1$$

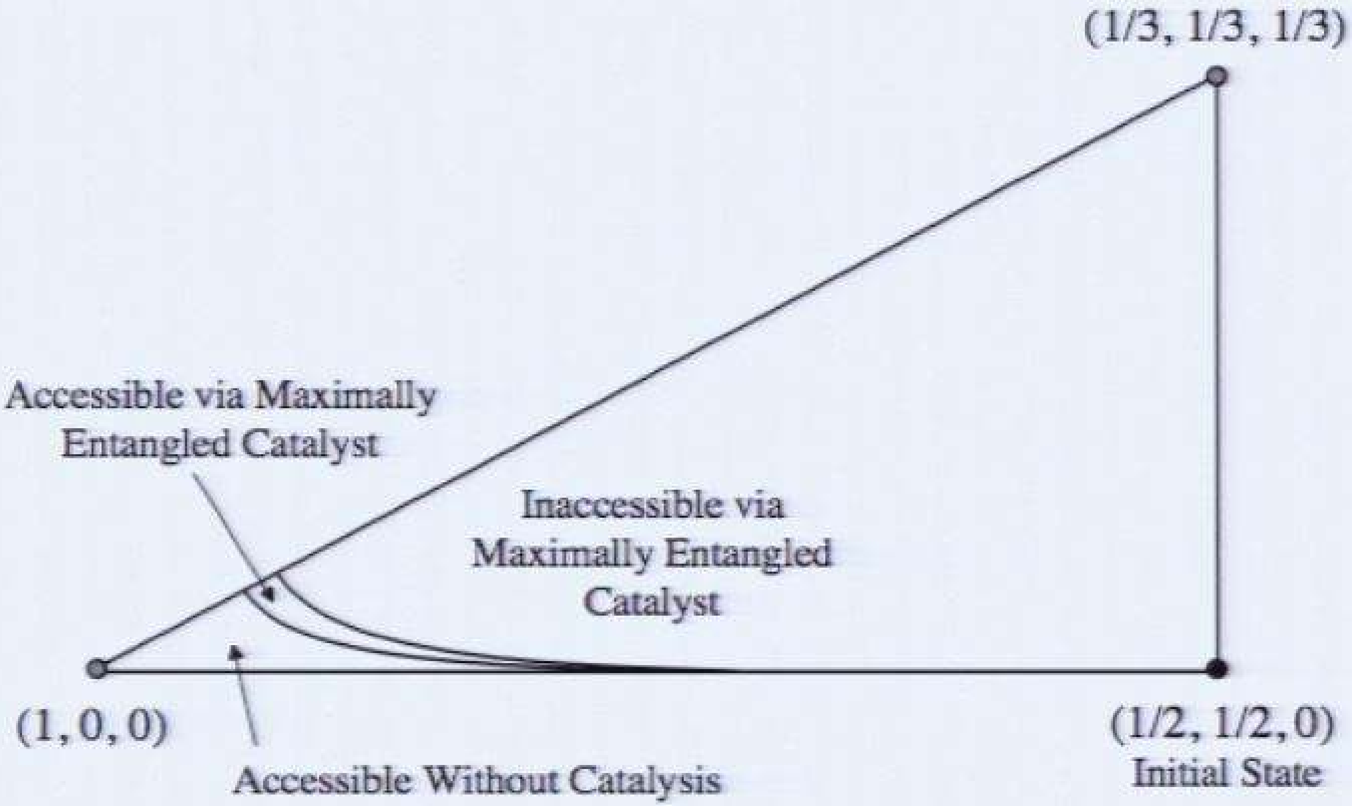
or equivalently

$$S_{1/2}(\lambda) \leq \log K.$$

PPT Pure State Catalysis: $K=2,\, d=3$

$2\sqrt{\lambda_3^\uparrow \lambda_2^\uparrow} + 2\sqrt{\lambda_3^\uparrow \lambda_1^\uparrow} + 4\sqrt{\lambda_1^\uparrow \lambda_2^\uparrow} \leq 1$ without a catalyst.

$2\sqrt{\lambda_3^\uparrow \lambda_2^\uparrow} + 2\sqrt{\lambda_3^\uparrow \lambda_1^\uparrow} + 2\sqrt{\lambda_1^\uparrow \lambda_2^\uparrow} \leq 1$ when arbitrarily large maximally entangled catalysts are available.



Maximally entangled states can be catalysts under PPT.

Necessary and sufficient condition for $K = 2$, $d = 3$

In this case we can find a primal solution with primal objective equal to $T_1(K;\lambda)$.
$T = T_1$ for $K = 2$, $d = 3$.
The necessary and sufficient condition for the feasibility of $\Phi_2 \overset{PPT}{\rightarrow} \rho_\lambda$ can be summarized by

$$2\left(\sqrt{\lambda_3^{\uparrow}\lambda_2^{\uparrow}} + \sqrt{\lambda_3^{\uparrow}\lambda_1^{\uparrow}}\right) + 4\sqrt{\lambda_1^{\uparrow}\lambda_2^{\uparrow}} \leq 1.$$

We conjecture that $T(K;\lambda) = T_1(K;\lambda)$ so that

$$\Phi_K \overset{PPT}{\rightarrow} \rho_\lambda \iff T_1(K;\lambda) \leq 1.$$

Rank Constrained Dual Optimum

Let d be the Schmidt rank of the goal state.

$$T_1(K;\lambda) = \frac{K2^{S_{1/2}(\lambda)} - 1}{K^2 - 1} + \frac{K\left(\left(\sum_{i=1}^{c^*}\sqrt{\lambda_i^\uparrow}\right)^2 - (K + c^* - d)\sum_{i=1}^{c^*}\lambda_i^\uparrow\right)}{(K^2-1)(K + c^* - d)},$$

where c^* is the smallest number $c \in \{1 + d - K, ..., d-1\}$ satisfying

$$\frac{\sum_{i=1}^{c}\sqrt{\lambda_i^\uparrow}}{K + c - d} \le \sqrt{\lambda_{c+1}^\uparrow}.$$

If none of the integers in the range satisfy this relation then $c^* = d$.

$$T(K;\lambda) \ge T_1(K;\lambda)$$

$$\Phi_K \overset{PPT}{\rightarrow} \rho_\lambda \implies T_1(K;\lambda) \le 1$$

PPT Pure State Catalysis

Since $S_{1/2}(\lambda \otimes \mu) = S_{1/2}(\lambda) + S_{1/2}(\mu)$ and $S_{1/2}(U_C) = \log C$ we find that

$$\frac{KC^2 2^{S_{1/2}(\lambda)} - 1}{K^2C^2 - 1} \leq T(KC; \lambda \otimes U_C) \leq \frac{C2^{S_{1/2}(\lambda)} - 1}{KC - 1}.$$

Taking the limit $C \to \infty$, the upper and lower bound both tend to

$$2^{S_{1/2}(\lambda)}/K$$

so a necessary and sufficient condition for the transformation $\Phi_K \otimes \Phi_C \stackrel{PPT}{\to} \rho_\lambda \otimes \Phi_C$ to be possible for some suffiently large C is

$$2^{S_{1/2}(\lambda)}/K \leq 1$$

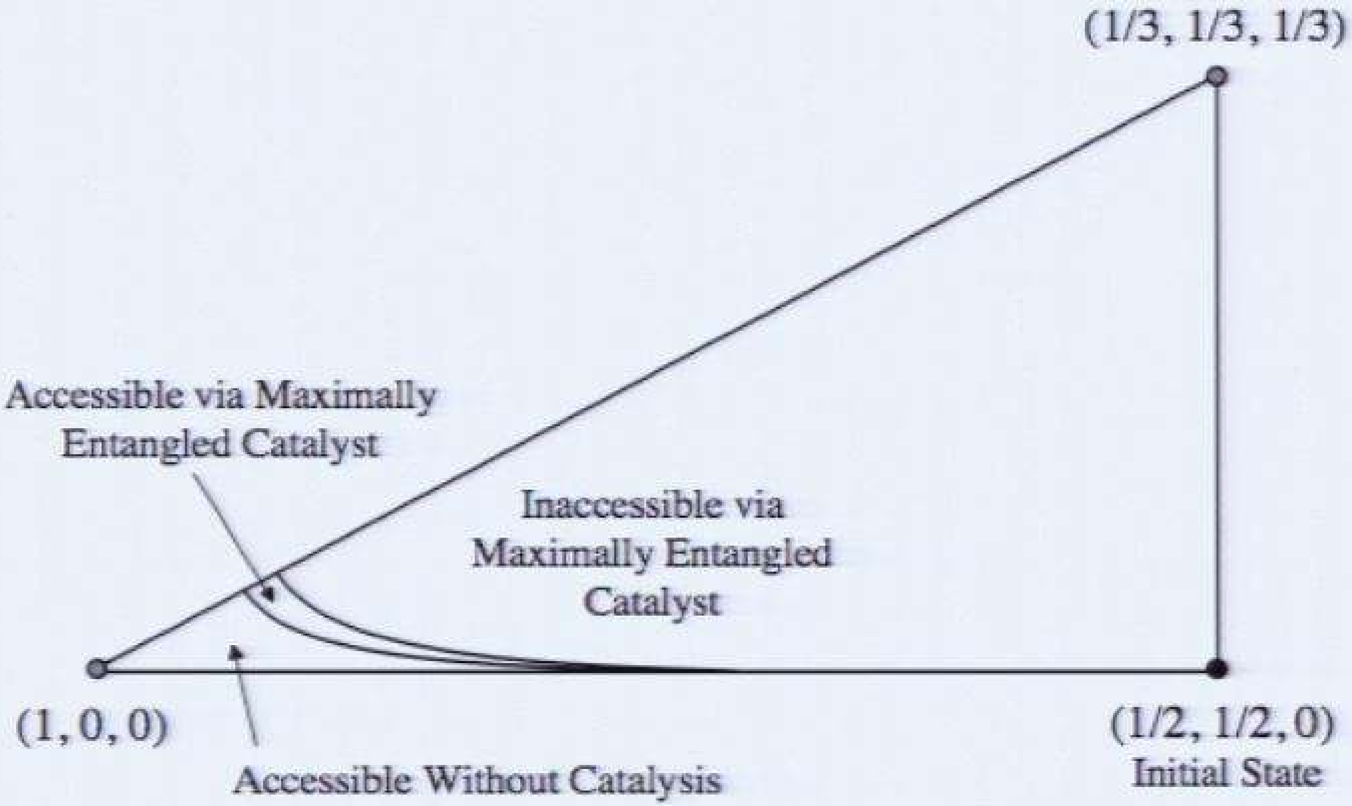
or equivalently

$$S_{1/2}(\lambda) \leq \log K.$$

PPT Pure State Catalysis: $K=2,\, d=3$

$2\sqrt{\lambda_3^\uparrow \lambda_2^\uparrow} + 2\sqrt{\lambda_3^\uparrow \lambda_1^\uparrow} + 4\sqrt{\lambda_1^\uparrow \lambda_2^\uparrow} \leq 1$ without a catalyst.

$2\sqrt{\lambda_3^\uparrow \lambda_2^\uparrow} + 2\sqrt{\lambda_3^\uparrow \lambda_1^\uparrow} + 2\sqrt{\lambda_1^\uparrow \lambda_2^\uparrow} \leq 1$ when arbitrarily large maximally entangled catalysts are available.



Maximally entangled states can be catalysts under PPT.

Summary

- Necessary and sufficient condition for $\Phi_2 \overset{PPT}{\rightarrow} \rho_\lambda$, when ρ_λ has Schmidt rank 3.
- Necessary condition for $\Phi_K \overset{PPT}{\rightarrow} \rho_\lambda$, conjectured to be sufficient.
- PPT Catalysis occurs. Maximally entangled states can be catalysts.