

Title: PPT pure state transformations and catalysis

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Abstract: In an effort to better understand the class of operations on a bipartite system which preserve positivity of partial transpose (PPT operations), we have investigated the (non-asymptotic) transformation of pure states to pure states by operations in this class. Under local operations and classical communication (LOCC) Nielsen's majorization criterion provides a necessary and sufficient condition for such a transformation. This can be used to show that under LOCC a phenomenon called catalysis can occur, where an otherwise impossible transformation can be made possible by the provision of an entangled catalyst state, which must be recovered unchanged after the transformation (hence the name). I will present some recent work where we have found a necessary condition for obtaining a given pure state from a maximally entangled state via PPT operations. This condition is conjectured to be sufficient also, and we can prove this for the case where the goal state has Schmidt rank three. We have also shown that catalysis occurs under PPT operations, and have derived a necessary and sufficient condition for PPT pure state transformations where both the initial state and the catalyst are maximally entangled.

PPT Pure State Transformations and Catalysis

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14 May 2008

Outline

- ▶ The partial transpose map, PPT states and PPT operations.
- ▶ Bipartite pure state transformations and LOCC catalysis.
- ▶ Bipartite pure state transformations by PPT operations.
- ▶ Catalysis under PPT operations.

Partial Transpose

- ▶ Let $|ij\rangle = |i\rangle_A \otimes |j\rangle_B$ denote an element of a orthonormal product basis for a bipartite Hilbert space.
- ▶ We define the (linear) partial transpose map by

$$\Gamma(|ij\rangle\langle kl|) = |ij\rangle\langle kl|^\Gamma = |il\rangle\langle kj|.$$

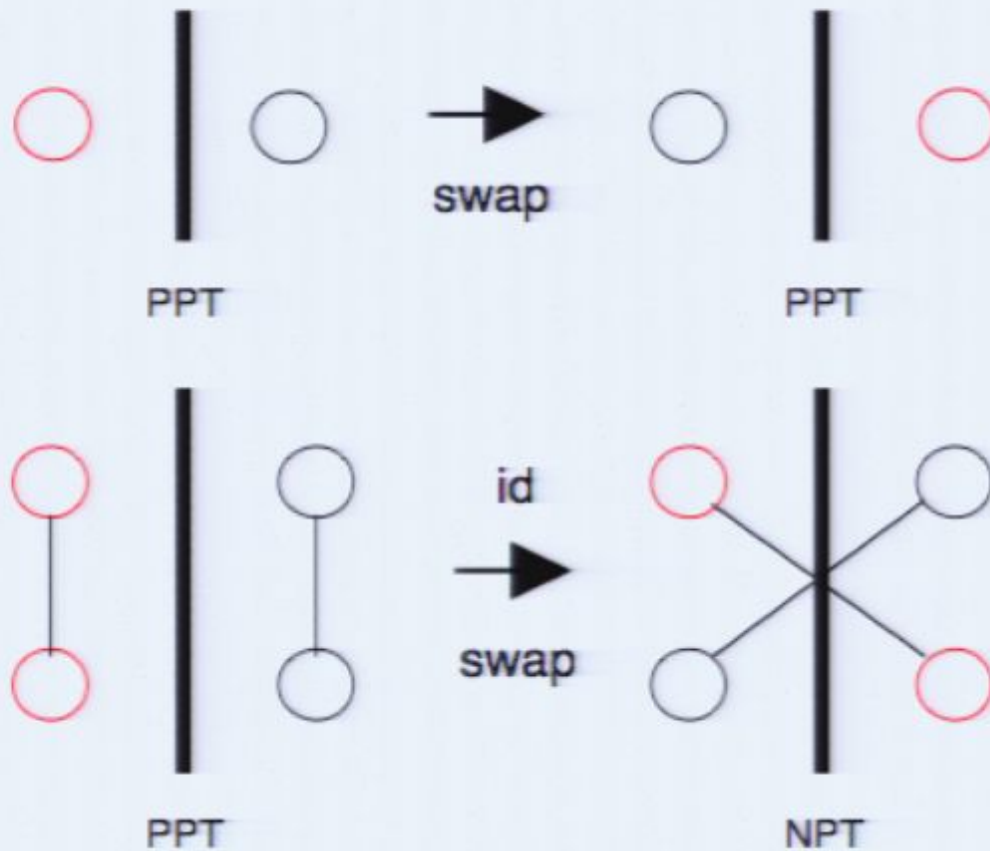
- ▶ Basis dependent, but the eigenvalues (and hence the positivity) of the partial transpose of an operator don't depend on this basis choice.
- ▶ Positive Partial Transpose (PPT) states are simply those density operators satisfying $\rho^\Gamma \geq 0$.
- ▶ Separable states are all PPT. (For 2×2 and 2×3 dimensional systems, PPT states = Separable states.)
- ▶ Converse not true. 3×3 dimensional systems can have entangled PPT states: Bound entanglement.

PPT Operations

- ▶ A general quantum operation Ψ corresponds to a CPTP map.
- ▶ Ψ is PPT¹ iff $\Gamma \circ \Psi \circ \Gamma$ is also completely positive.
- ▶ If ρ is PPT, then $(\Psi \otimes \mathbb{1})\rho$ is also PPT.
- ▶ $LOCC \subset PPT$

PPT Operations \subset PPT preserving operations

Swap operation preserves PPT. Swap \otimes Identity does not.

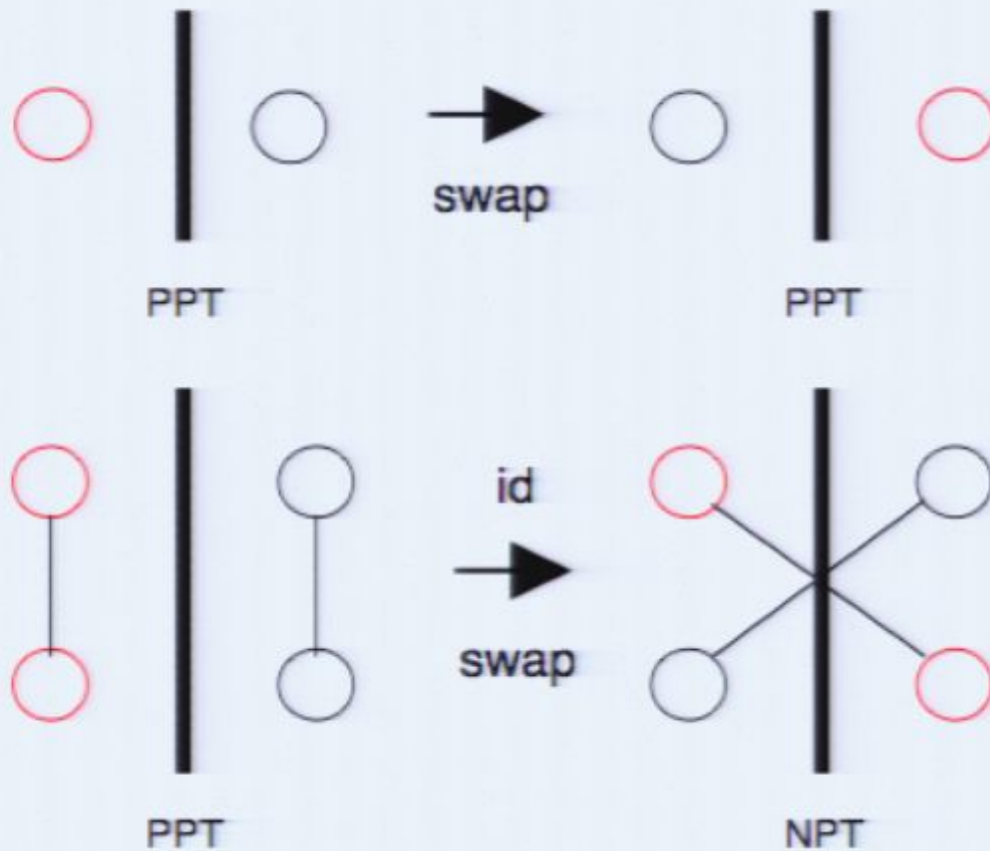


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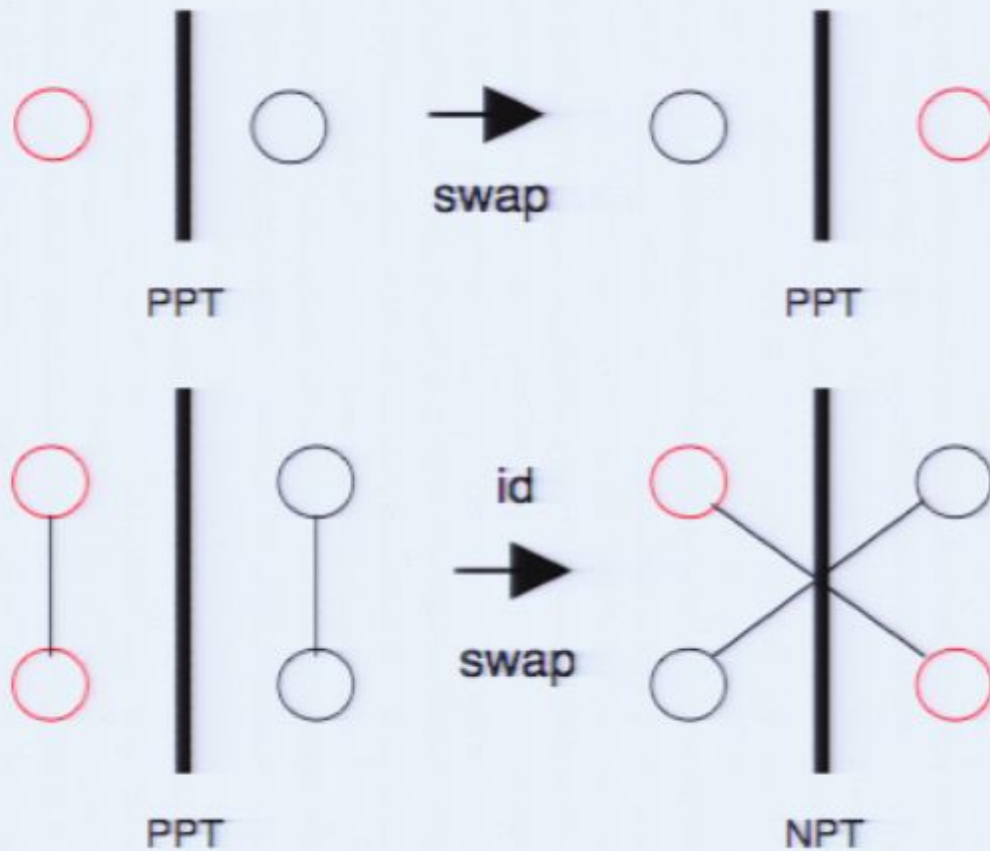
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PPT Operations - Why?

- ▶ Practical use: Determining whether a given CPTP map can be performed by LOCC is generally very hard. It is much simpler to determine whether the map is PPT.
- ▶ Can often use this to rule out an LOCC procedure to do a certain task.
- ▶ If PPT bound entanglement is the only type of bound entanglement then $\text{PPT} = \text{class of operations which cannot create distillable entanglement}$.

PPT Operations

PPT is strictly more powerful than LOCC. E.g.:

- ▶ Can create PPT bound entangled states from separable states.
- ▶ Can increase Schmidt rank of a pure state.

Still, not much is known about the relative power of PPT vs. LOCC.

Bipartite Pure State Transformations

In a given class of operations, can we deterministically transform $|\phi\rangle$ into $|\psi\rangle$?

If the class contains local unitaries, the answer will only depend on the Schmidt coefficients of the initial and final state.

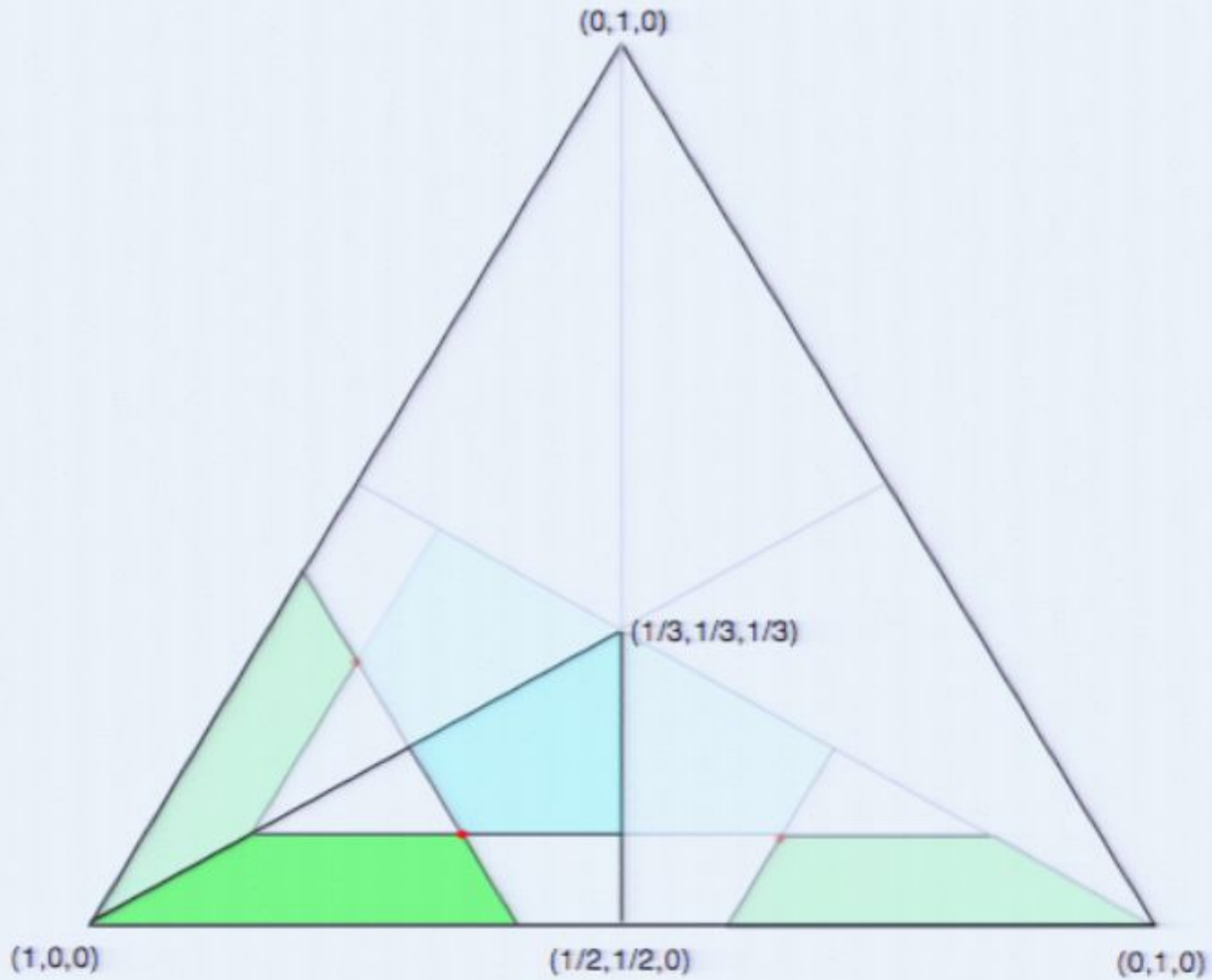
$$\rho_\lambda = \sum_{i=1}^d \sum_{j=1}^d \sqrt{\lambda_i \lambda_j} |ii\rangle\langle jj|, \lambda_i \geq 0, \sum_{i=1}^d \lambda_i = 1$$

For LOCC, a necessary and sufficient condition was given by Nielsen² - the majorization criterion:

$$\rho_\lambda \xrightarrow{LOCC} \rho_\mu \iff \lambda \prec \mu$$

$$\lambda \prec \mu \iff \sum_{i=1}^k \lambda_i^\downarrow \leq \sum_{i=1}^k \mu_i^\downarrow \text{ for } k \in \{1, \dots, d\}$$

LOCC Pure State Transformations



LOCC Catalysis

Using the majorization criterion, it can be shown that a phenomenon analogous to chemical catalysis can occur for pure state transformations³:

$$\lambda_1 = (0.4, 0.4, 0.1, 0.1) \not\prec \lambda_2 = (0.5, 0.25, 0.25) \implies \rho_{\lambda_1} \not\leftrightarrow \rho_{\lambda_2}$$

$$\chi = (0.6, 0.4)$$

$$(\lambda_1 \otimes \chi)^\downarrow = (0.24, 0.24, 0.16, 0.16, 0.06, 0.06, 0.04, 0.04)$$

$$(0.24, 0.48, 0.64, 0.80, 0.86, 0.92, 0.96, 1.00)$$

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$$\lambda_1 \otimes \chi \prec \lambda_2 \otimes \chi, \rho_{\lambda_1} \otimes \rho_\chi \rightarrow \rho_{\lambda_2} \otimes \rho_\chi$$

A maximally entangled state cannot act as a catalyst.

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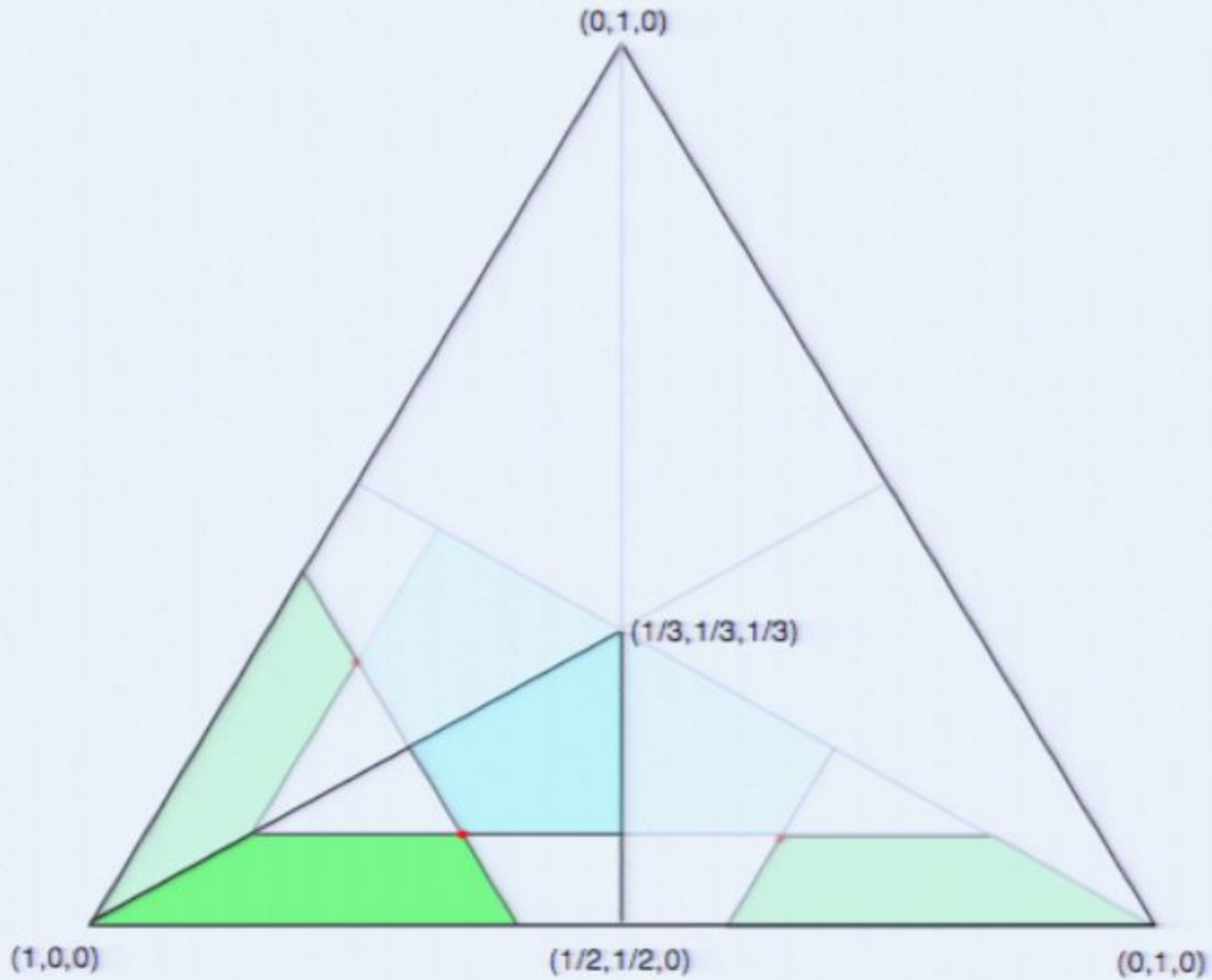
$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \quad (2)$$

$$f_t(\lambda) > f_t(\mu) \text{ for } t \in (-\infty, 0), \quad (3)$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log \left(\sum_{i=1}^d \lambda_i^t \right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, \dots, d\}, \\ -\infty & \text{otherwise.} \end{cases} \quad (4)$$

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Turgut⁴, Klimesh: Necessary and sufficient conditions for a catalyzed LOCC transformation to exist between two pure states, in terms of the Rényi entropies of the Schmidt coefficients (and related functions).

For $t \in [0, \infty]$ the *Rényi entropy at t* is defined by

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PPT Pure State Transformations - Maximally Entangled Initial State

The existence of a PPT map taking $|\phi\rangle$ to $|\psi\rangle$ can be formulated as a semidefinite program.

It is rather complicated but it simplifies greatly if we restrict to the case initial state is maximally entangled state.

For any (pure or mixed) final state ρ , $\Phi_K \xrightarrow{PPT} \rho$ if and only if the solution to the semidefinite program

$$\min\{\text{Tr}(P) \mid P \geq 0, -(K-1)P^\Gamma \leq \rho^\Gamma \leq (K+1)P^\Gamma\}, \quad (5)$$

where P is an hermitian operator on the same Hilbert space as ρ , is less than or equal to one.

For pure ρ we can make further simplifications...

Primal Semidefinite Program

$$\Phi_K \xrightarrow{PPT} \rho_\lambda \iff T(K; \lambda) \leq 1$$

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subject to

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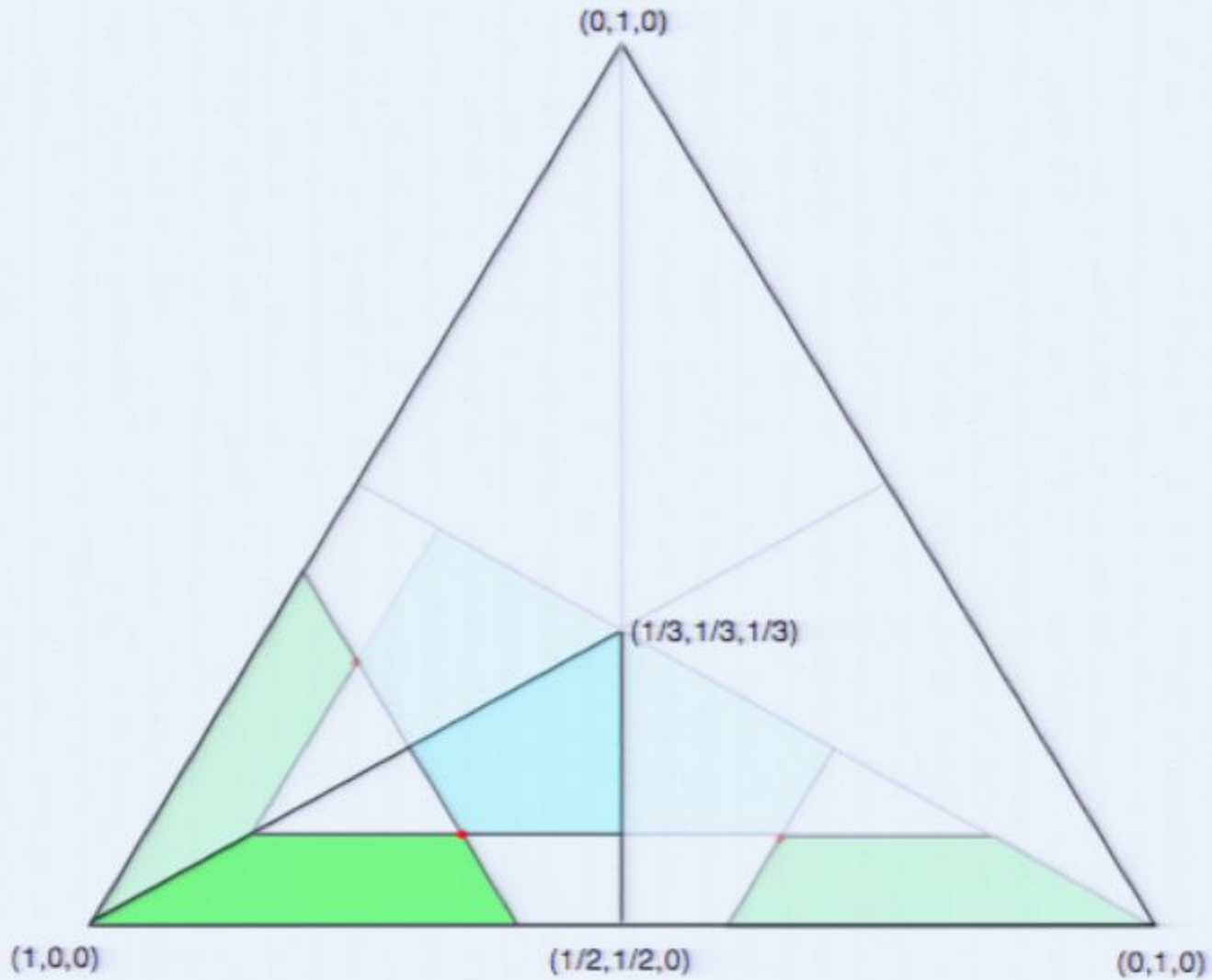
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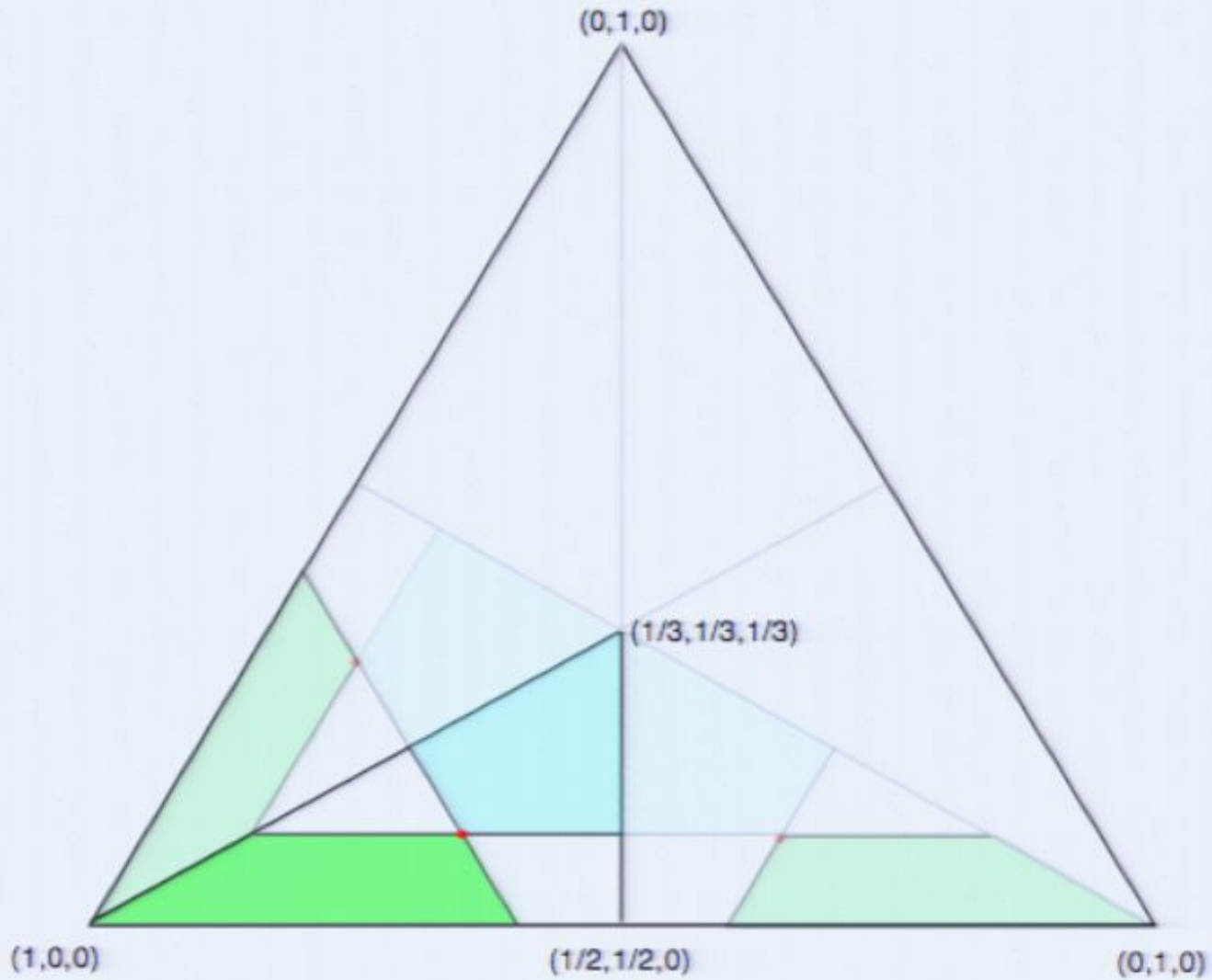
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$$\lambda \prec \mu \iff \sum_{i=1}^k \lambda_i^\downarrow \leq \sum_{i=1}^k \mu_i^\downarrow \text{ for } k \in \{1, \dots, d\}$$

LOCC Catalysis

Given two pure bipartite states ρ_λ and ρ_μ , where $\lambda^\uparrow \neq \mu^\uparrow$ and λ and μ don't both have components equal to zero, there exists a pure state ρ_ξ such that $\rho_\lambda \otimes \rho_\xi \xrightarrow{LOCC} \rho_\mu \otimes \rho_\xi$ if and only if the following conditions are satisfied

$$S_t(\lambda) > S_t(\mu) \text{ for } t \in (0, \infty), \quad (1)$$

$$\sum_i \log \lambda_i > \sum_i \log \mu_i, \quad (2)$$

$$f_t(\lambda) > f_t(\mu) \text{ for } t \in (-\infty, 0), \quad (3)$$

where

$$f_t(\lambda) := \begin{cases} \frac{1}{t-1} \log \left(\sum_{i=1}^d \lambda_i^t \right) & \text{when } \lambda_i \neq 0 \text{ for all } i \in \{1, \dots, d\}, \\ -\infty & \text{otherwise.} \end{cases} \quad (4)$$

PPT Pure State Transformations - General necessary conditions

- ▶ The entanglement cost $E_c^{\text{PPT}}(\rho_\lambda)$ and distillable entanglement $E_d^{\text{PPT}}(\rho_\lambda)$ of ρ_λ are both equal to $S_1(\lambda)$, the entropy of entanglement of the state.
- ▶ The exact distillable $E_{xd}^{\text{PPT}}(\rho_\lambda)$ of ρ_λ is given by $S_\infty(\lambda)$.
- ▶ The exact entanglement cost $E_{xc}^{\text{PPT}}(\rho_\lambda)$ of ρ_λ is $S_{1/2}(\lambda)$.
- ▶ Non-increasing under PPT operations:
- ▶ $\rho_\lambda \xrightarrow{\text{PPT}} \rho_\mu \implies S_t(\lambda) \geq S_t(\mu)$ for $t = 1/2, 1$ and ∞ .

Primal Semidefinite Program

$$\Phi_K \xrightarrow{PPT} \rho_\lambda \iff T(K; \lambda) \leq 1$$

where

$$T(K; \lambda) := \min \sum_{i \geq j}^d s_{ij} + \sum_{i > j}^d a_{ij}$$

subject to

$$s_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K+1}, a_{ij} \geq \frac{\sqrt{\lambda_i \lambda_j}}{K-1}, \sum_{i \geq j} s_{ij} \sigma_{ij}^\Gamma + \sum_{i > j} a_{ij} \alpha_{ij}^\Gamma \geq 0$$

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$$\sigma_{ij} = \begin{cases} (|ij\rangle + |ji\rangle)(\langle ij| + \langle ji|)/2 & \text{when } i \neq j, \\ |ii\rangle\langle ii| & \text{when } i = j, \end{cases}$$

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The dual objective at any dual feasible point is a lower bound on $T(K; \lambda)$.

Additional constraint:

$$\text{rank} \left(\sum_{i \geq j} \mu_{ij} \sigma_{ij}^{\Gamma} + \sum_{i > j} \nu_{ij} \alpha_{ij}^{\Gamma} \right) = 1$$

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PPT Pure State Transformations - Maximally Entangled Initial State

The existence of a PPT map taking $|\phi\rangle$ to $|\psi\rangle$ can be formulated as a semidefinite program.

It is rather complicated but it simplifies greatly if we restrict to the case initial state is maximally entangled state.

For any (pure or mixed) final state ρ , $\Phi_K \xrightarrow{PPT} \rho$ if and only if the solution to the semidefinite program

$$\min\{\text{Tr}(P) \mid P \geq 0, -(K-1)P^\Gamma \leq \rho^\Gamma \leq (K+1)P^\Gamma\}, \quad (5)$$

where P is an hermitian operator on the same Hilbert space as ρ , is less than or equal to one.

For pure ρ we can make further simplifications...

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Rank Constrained Dual Optimum

Let d be the Schmidt rank of the goal state.

$$T_1(K; \lambda) = \frac{K 2^{S_{1/2}(\lambda)} - 1}{K^2 - 1} + \frac{K \left(\left(\sum_{i=1}^{c^*} \sqrt{\lambda_i^\uparrow} \right)^2 - (K + c^* - d) \sum_{i=1}^{c^*} \lambda_i^\uparrow \right)}{(K^2 - 1)(K + c^* - d)},$$

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If none of the integers in the range satisfy this relation then $c^* = d$.

$$T(K; \lambda) \geq T_1(K; \lambda)$$

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Necessary and sufficient condition for $K = 2, d = 3$

In this case we can find a primal solution with primal objective equal to $T_1(K; \lambda)$.

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We conjecture that $T(K; \lambda) = T_1(K; \lambda)$ so that

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PPT Pure State Catalysis

Does catalysis occur under PPT operations?

$$\rho_\lambda \xrightarrow{PPT} \rho_\mu$$

but

$$\rho_\lambda \otimes \rho_\chi \xrightarrow{PPT} \rho_\mu \otimes \rho_\chi?$$

It is always possible to find a primal solution (in general not optimal) giving the upper bound

$$T(K; \lambda) \leq \frac{2^{S_{1/2}(\lambda)} - 1}{K - 1},$$

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Since $S_{1/2}(\lambda \otimes \mu) = S_{1/2}(\lambda) + S_{1/2}(\mu)$ and $S_{1/2}(U_C) = \log C$ we find that

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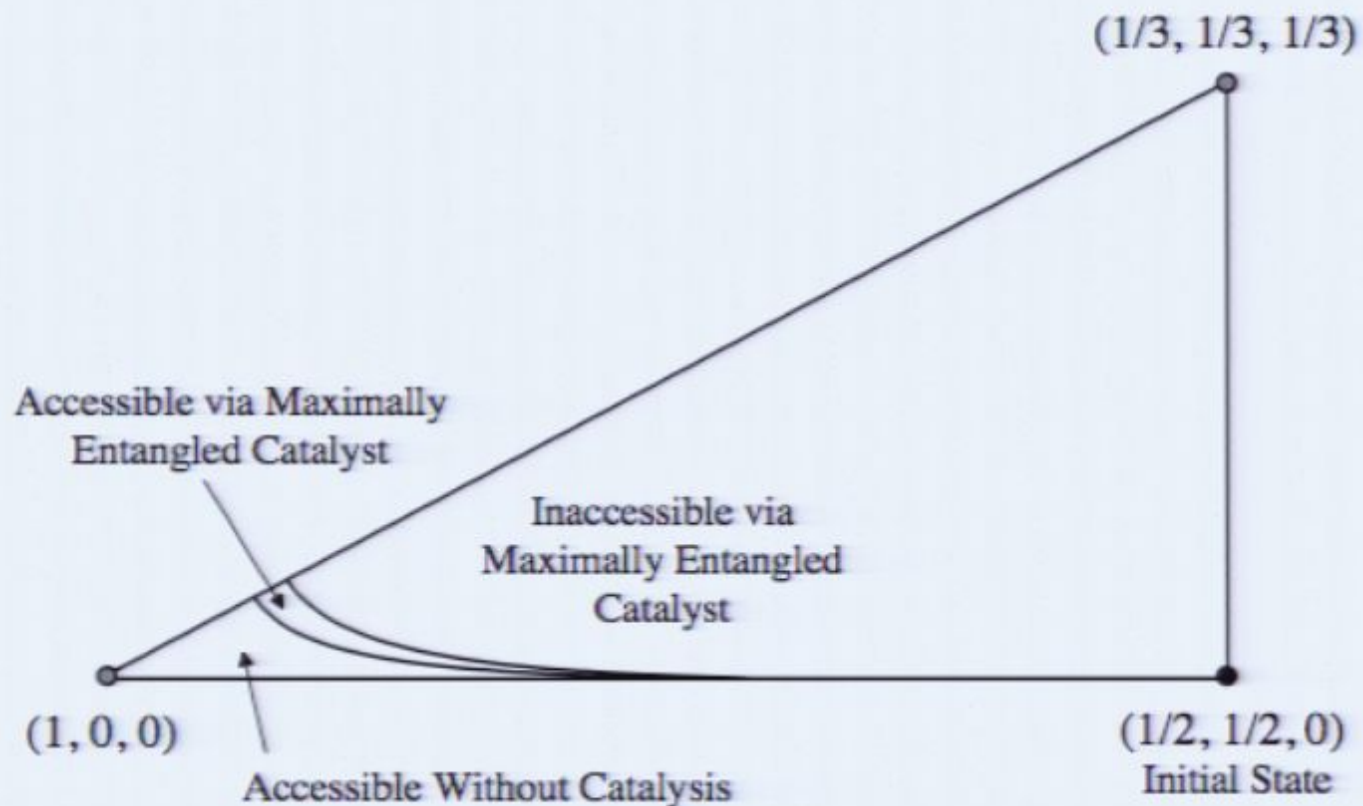
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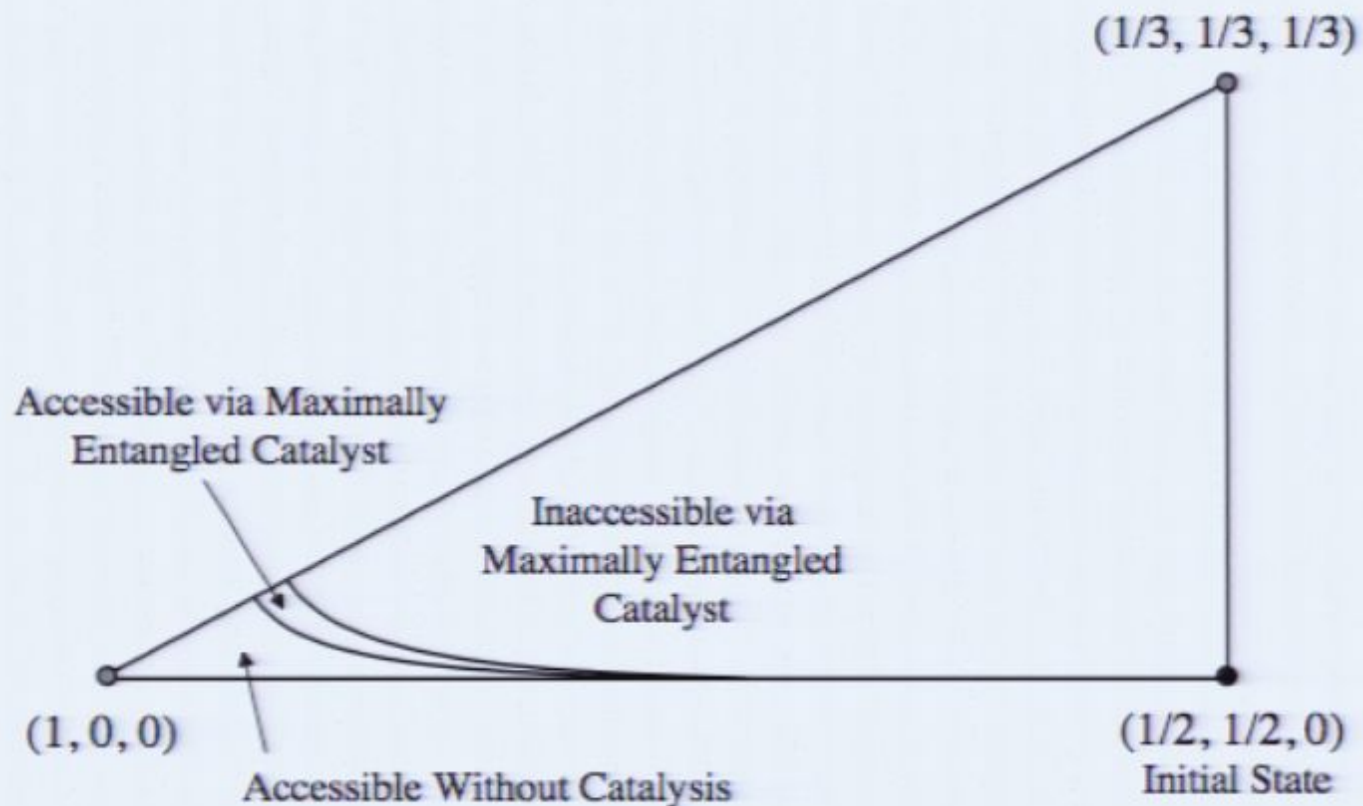
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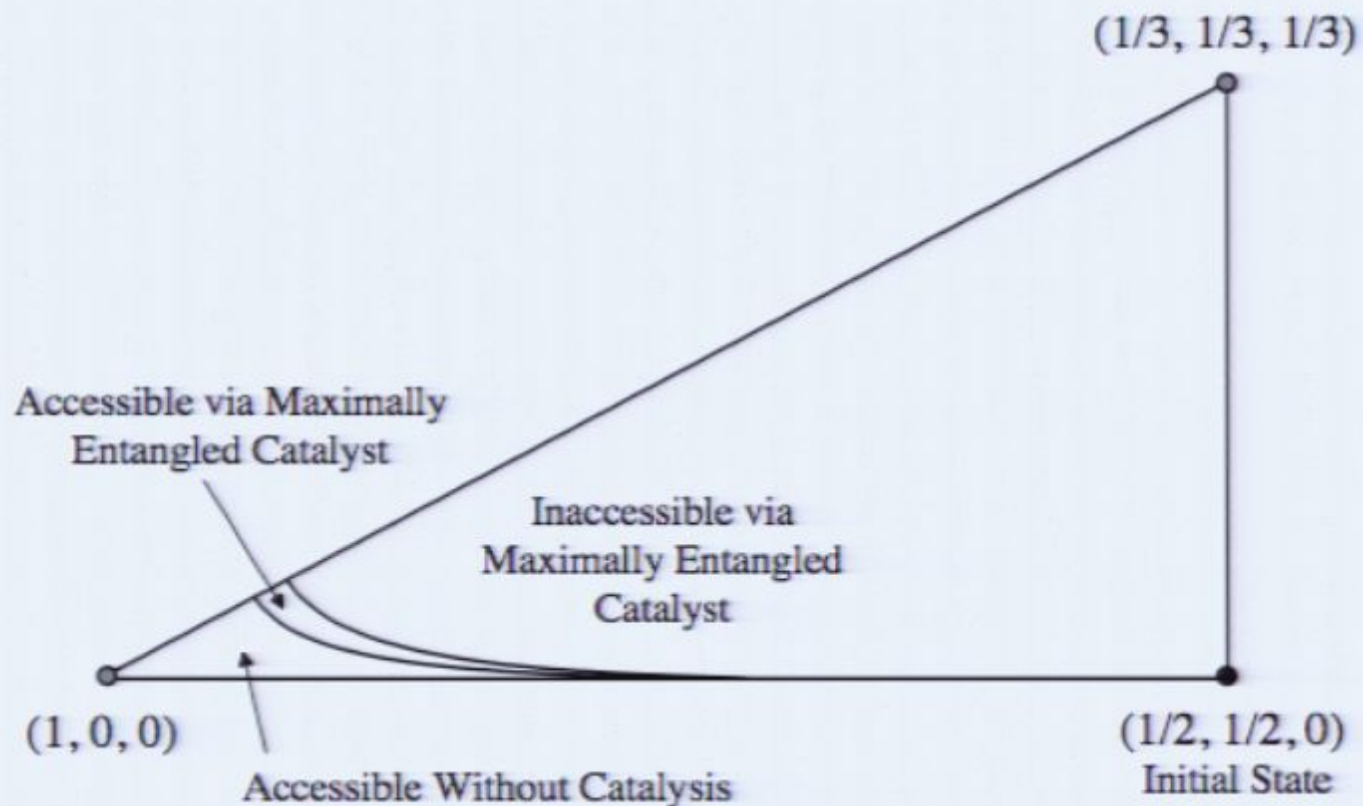
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Summary

- ▶ Necessary and sufficient condition for $\Phi_2 \xrightarrow{PPT} \rho_\lambda$, when ρ_λ has Schmidt rank 3.
- ▶ Necessary condition for $\Phi_K \xrightarrow{PPT} \rho_\lambda$, conjectured to be sufficient.
- ▶ PPT Catalysis occurs. Maximally entangled states can be catalysts.