Title: Topological Quantum Order: A paradigm for the physics of matter

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Abstract: What does a fractional quantum Hall liquid and Kitaev\'s proposals for topological quantum computation have in common? It turns out that they are physical systems that exhibit degenerate ground states with properties seemingly different than ordinary (Landau-type) vacua, such as the ground states of a Heisenberg magnet. For example, those (topologically quantum ordered)states cannot be characterized by (local) order parameters such as magnetization. How does one characterize this new order? I will present a unifying framework which will allow us to engineer physical systems displaying topological quantum order. What are the physical properties of these new orders? How robust are they to temperature effects? What are they useful for? Topologically quantum ordered states of matter seem to be ideal physical systems to store and manipulate quantum information since they are believed to be robust against decoherence with an environment, and thus appropriate for building a quantum computer and quantum memories. I will discuss the role of temperature in the protection of quantum information. Have we finally found a technological application for quantum Hall liquids?

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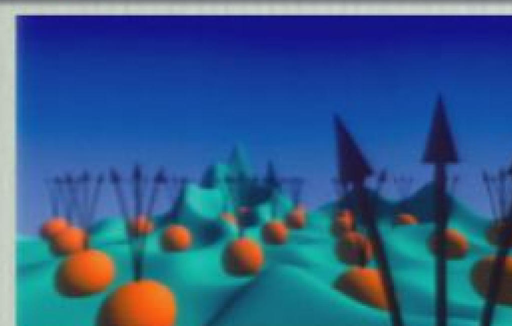
Towards an understanding of a new paradigm in the physics of matter: A new quantum vacuum

Zohar Nussinov
Washington University - St. Louis



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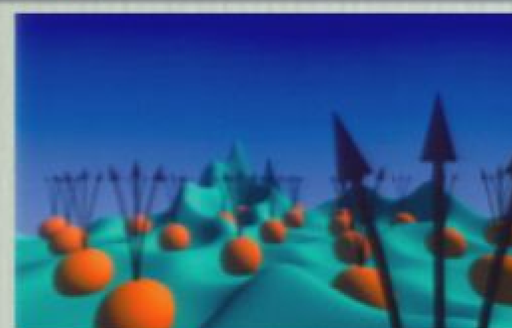

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A Tale of Dimensional Reduction

Gerardo Ortiz
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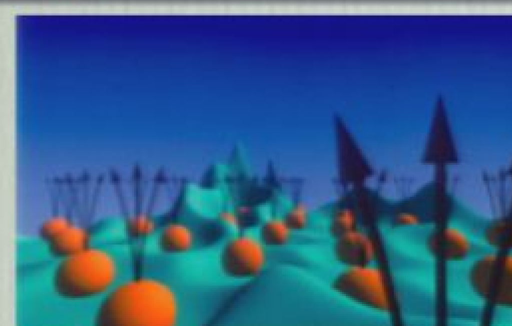

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Who are the main actors?

Condensed Matter Physics: Wen

Towards a classification of quantum orders

Quantum Computation: Kitaev

Computing with anyons

Why TQO?

- New states of matter where the traditional Landau paradigm fails

  A new quantum vacuum

  Can we engineer them?

- Topological Quantum Computation: Hardware Fault-tolerance

  Defeating Decoherence

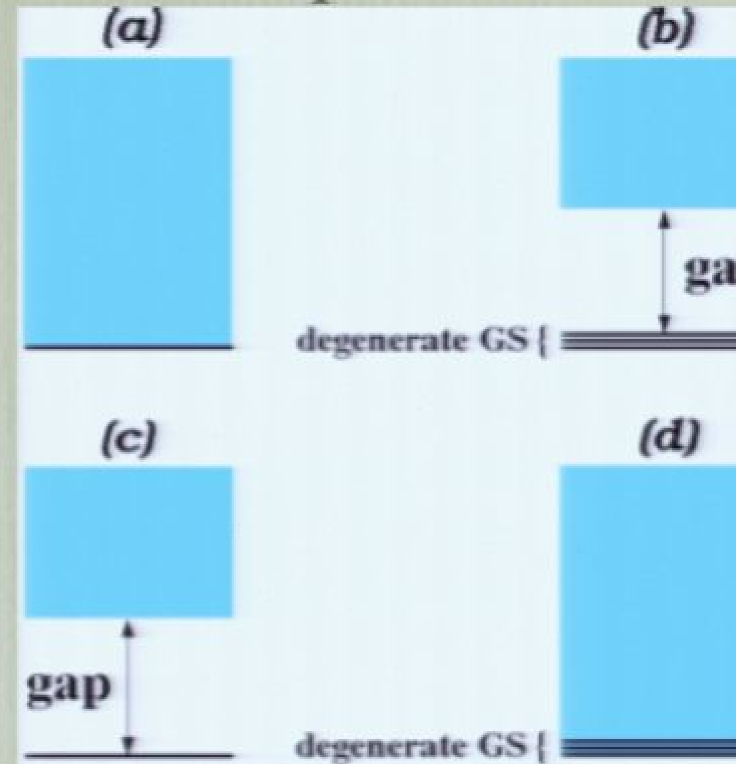
- Functionalities other than computation:

  Quantum Memories

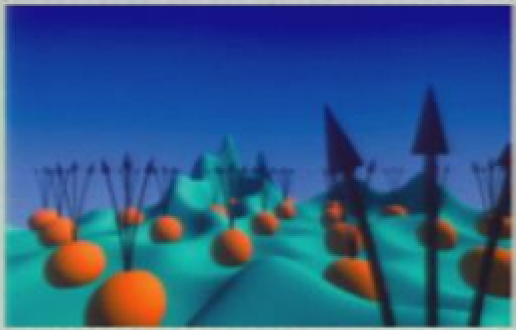
  Precision measurements (quantum metrology)?

  Background independent "emergent" space?

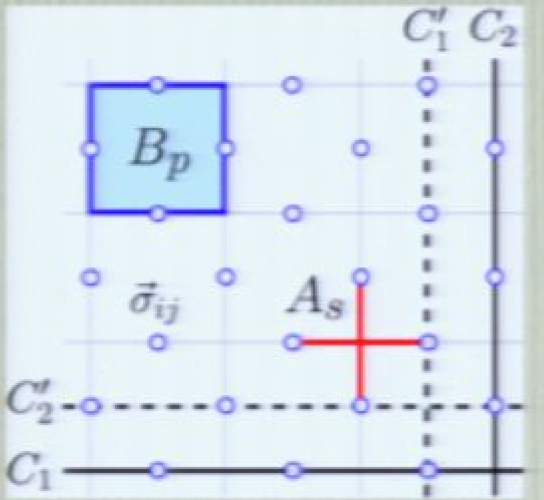
Spectra



Old Examples



- Fractional Quantum Hall Liquids
- Kitaev's Toric code model



$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{j \in \mathrm{star}(s)} \sigma_j^x \qquad B_p = \prod_{j \in \mathrm{boundary}(p)} \sigma_j^z$$

- Some spin liquids

Outline

- What is Topological Quantum Order (TQO) ?
- *d*-dim Gauge-Like Symmetries - Physical Consequences
  Dimensional reduction ($d < D$)
- Linking *d*-GLSs and TQO: A theorem
  Provide a unifying symmetry framework for TQO
  Engineer new model systems with TQO
- Thermal Fragility: Does temperature preclude protection of quantum information ?

References

- *Sufficient Symmetry Conditions for Topological Quantum Order* - Z. Nussinov and G. Ortiz, ***cond-mat/0605316***.
- *A Symmetry Principle for Topological Quantum Order* - Z. Nussinov and G. Ortiz, ***cond-mat/0702377***.
- *On Thermal Fragility of Anyonic Loops in Topologically Quantum Ordered Systems* - Z. Nussinov and G. Ortiz, ***arXiv:0709.2717***.
- *Orbital Order Driven Quantum Criticality* - Z. Nussinov and G. Ortiz, ***arXiv:0801.4391***.

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Orders in Matter: Invariance Principles

Understanding the properties of the Phases of Matter by using Symmetry Principles allows us to characterize them in terms of Universal behaviors

- Global Symmetry Breaking Orders (e.g. Magnets)

  Landau paradigm to matter classification
  in terms of an Order Parameter

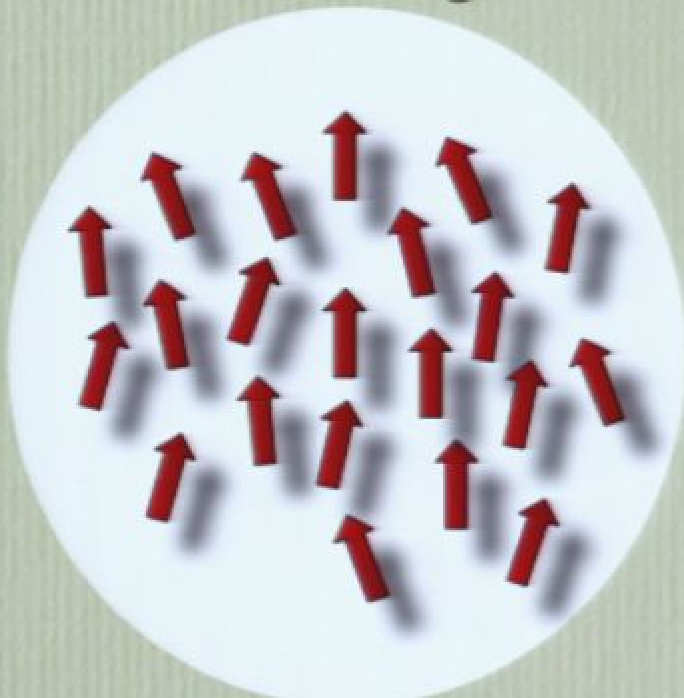
- The new paradigm of Topological Order (e.g. Quantum Hall, Gauge Theories, Spin Liquids, String-Net models) - no obvious broken symmetry

What characterizes these new orders ?

Non-local Order Parameters?????

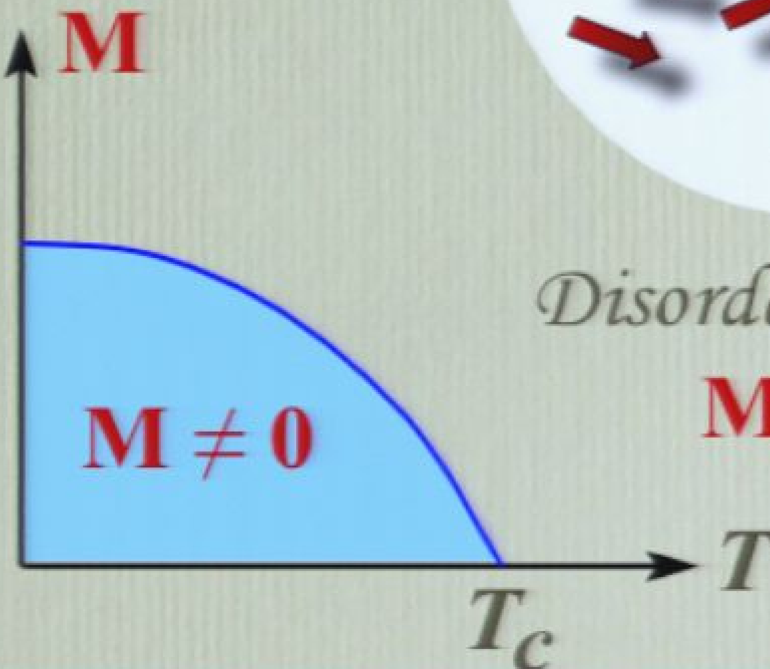
Symmetry and Phase Transitions

$T < T_c$



Broken Symmetry Phase

$\mathbf{M \neq 0}$



$T > T_c$

Disordered Phase

$\mathbf{M = 0}$

Local order parameters

In a ferromagnet, a local expectation value is different for different orthogonal ground states (GSs)

$$\langle g_\alpha | \hat{M} | g_\alpha \rangle \neq \langle g_\beta | \hat{M} | g_\beta \rangle \qquad T = 0$$

Applying different boundary conditions can lead, at sufficiently low temperatures to spontaneous symmetry breaking

$$\langle \hat{M} \rangle_\alpha \neq \langle \hat{M} \rangle_\beta \qquad T \neq 0$$

Local Measurements can distinguish the GSs

Concepts involved in TQO

Symmetry

Degeneracy

Entanglement Entropy

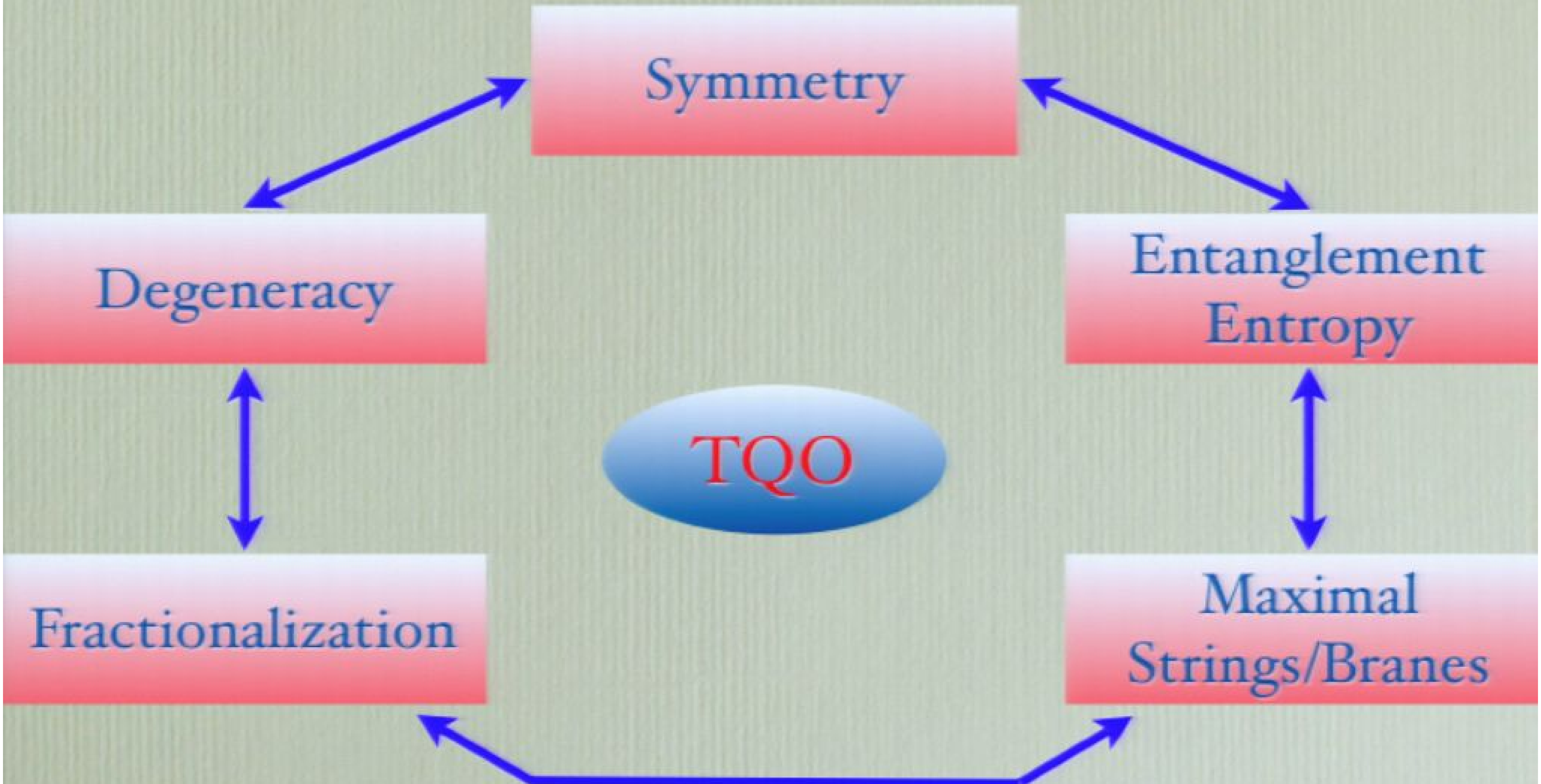
TQO

Fractionalization

Maximal Strings/Branes

It is important to establish
what is needed to display TQO

Concepts involved in TQO



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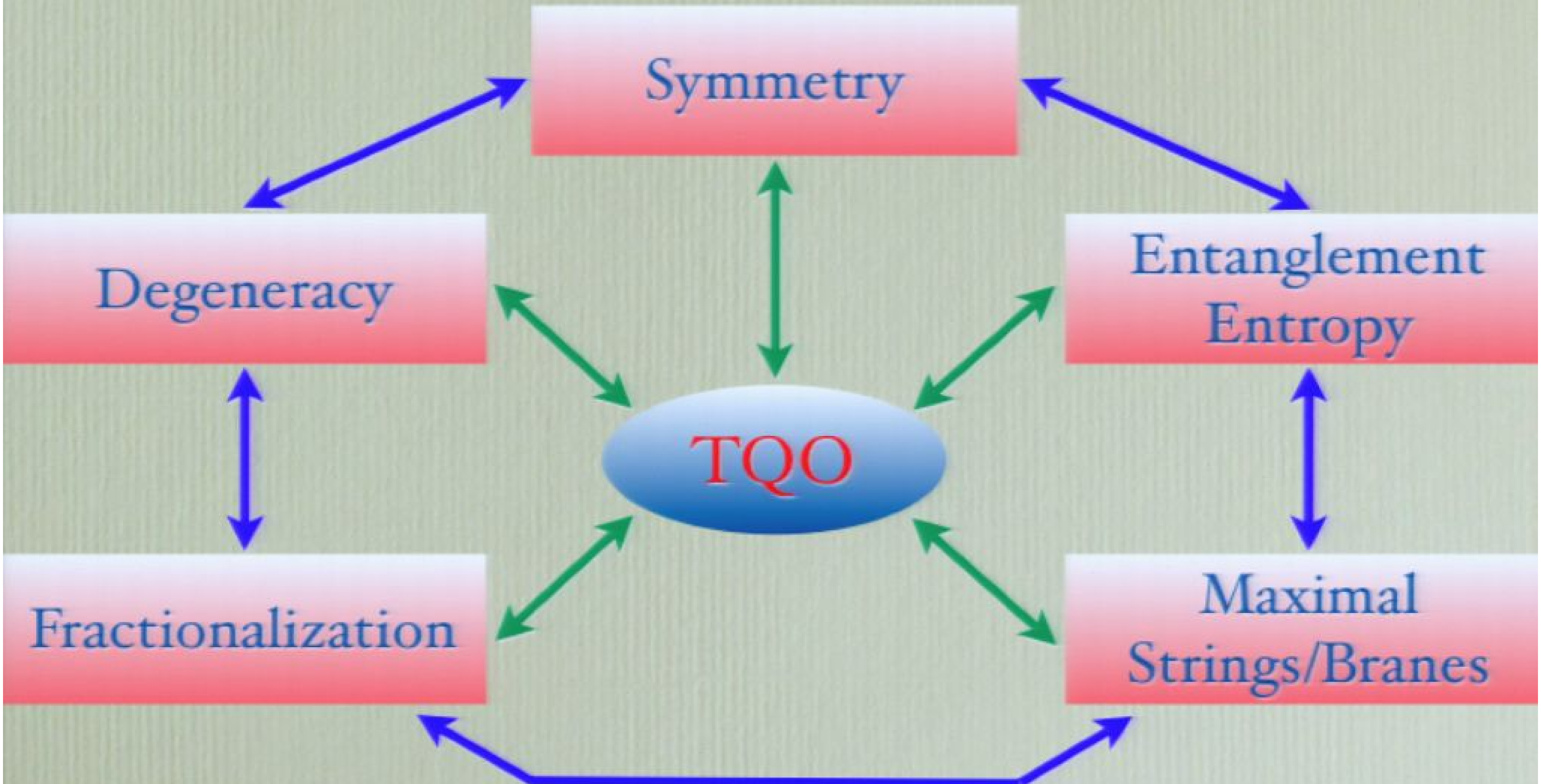
Concepts involved in TQO



It is important to establish
what is needed to display TQO

What is TQO?

Colloquially, TQO is often very loosely referred to as order whose GS degeneracy depends on the surface topology of the manifold on which the physical system is embedded.

Our working definition: Robustness

Non-Distinguishability: Given a quasi-local operator $\hat{V}^m$

$$\langle g_\alpha|\hat{V}^m|g_\beta\rangle = c\,\delta_{\alpha\beta},\ \forall\ \alpha,\beta \in \mathcal{S}_0,$$

Perturbation Theory:

$$\langle g_\alpha|\underbrace{\hat{V}\bar{G}_0\hat{V}\ldots\bar{G}_0\hat{V}}_{m \text{ factors } \hat{V}}|g_\beta\rangle = c\,\delta_{\alpha\beta},\ \forall\ \alpha,\beta \in \mathcal{S}_0$$

$$\bar{G}_0 = (\epsilon_0 - H_0)^{-1}\hat{P}_\perp$$

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What is TQO?

Order is evident only in non-local (topological) quantities

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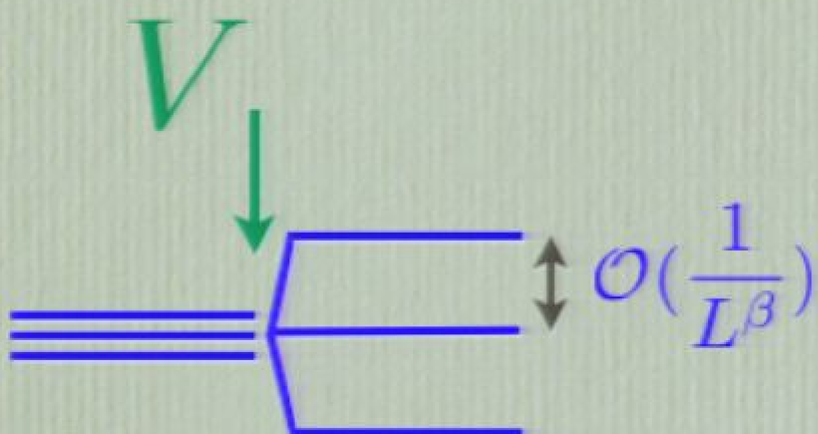
What is TQO? Physical robustness

Let's assume we have a degenerate ground state subspace
D-dim theory

Landau orders

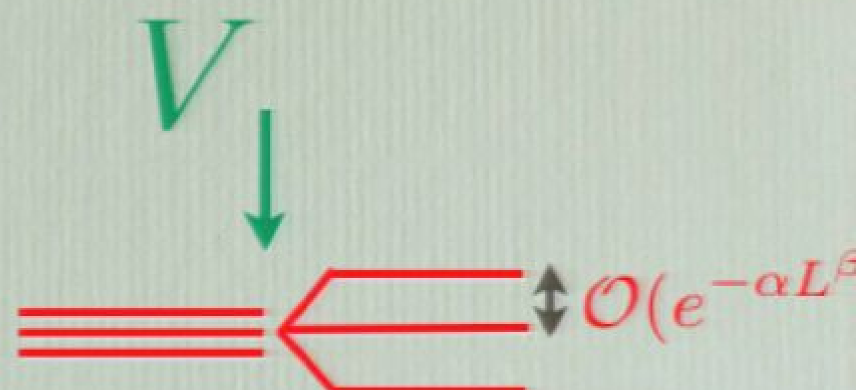


$$\hat{G}_\mu|g_\alpha\rangle = |g_\beta\rangle$$

$$[H,\hat{G}_\mu] = 0$$

Global symmetry

TQO $(d \leq D)$

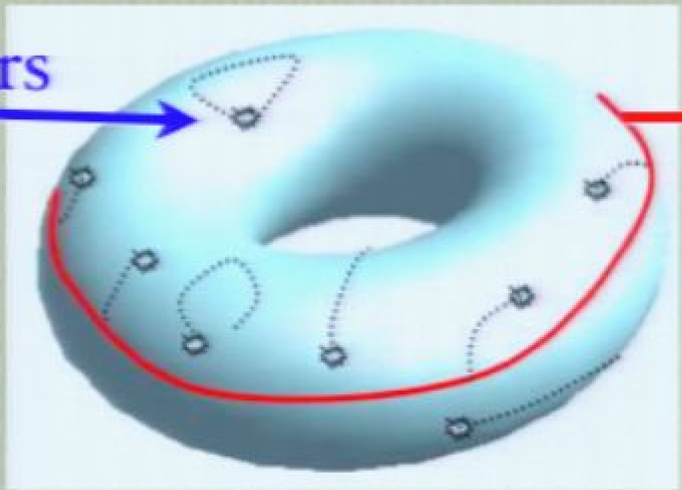


$$\hat{T}_\mu|g_\alpha\rangle = |g_\beta\rangle$$

$$[H,\hat{T}_\mu] = 0$$

$([\hat{T}_\mu,\hat{T}_\nu] \neq 0)$ $\hat{T}_\mu$ is a d-GLS

What is TQO? Conditions error detection



Propagation of errors

V

(quasi-local)

$\hat{T}_\mu$

Logical operators (non-commuting braiding operations

$[H, \hat{T}_\mu] = 0$

Protected subspace: $\hat{P}_0 = \sum_\alpha |g_\alpha\rangle\langle g_\alpha|$

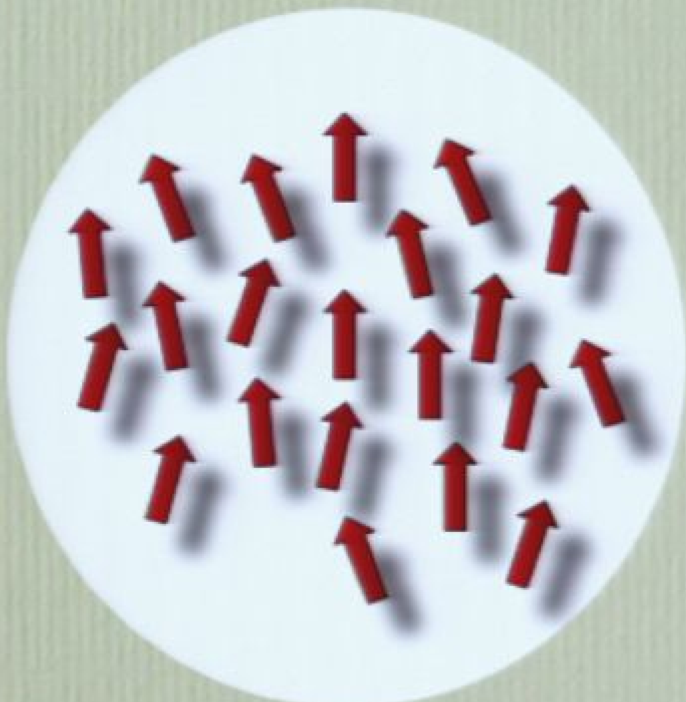
As long as: $[\hat{P}_0 V \hat{P}_0, \hat{T}_\mu] = 0$ Causes no harm to $\hat{T}_\mu$

Kitaev's non-distinguishability condition implies

$$[\hat{P}_0 V \hat{P}_0, \hat{T}_\mu] = 0$$

What is TQO?

Typically we need to correct (local) errors



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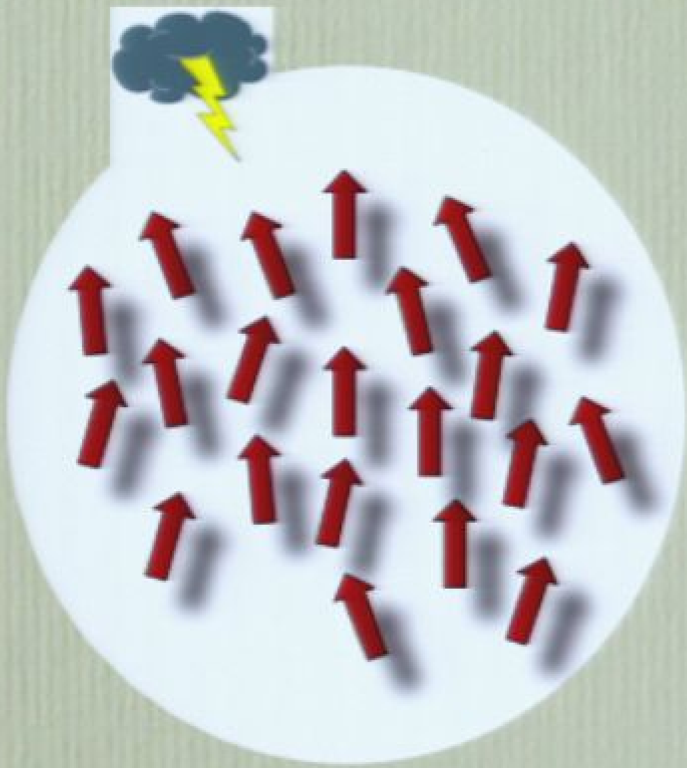

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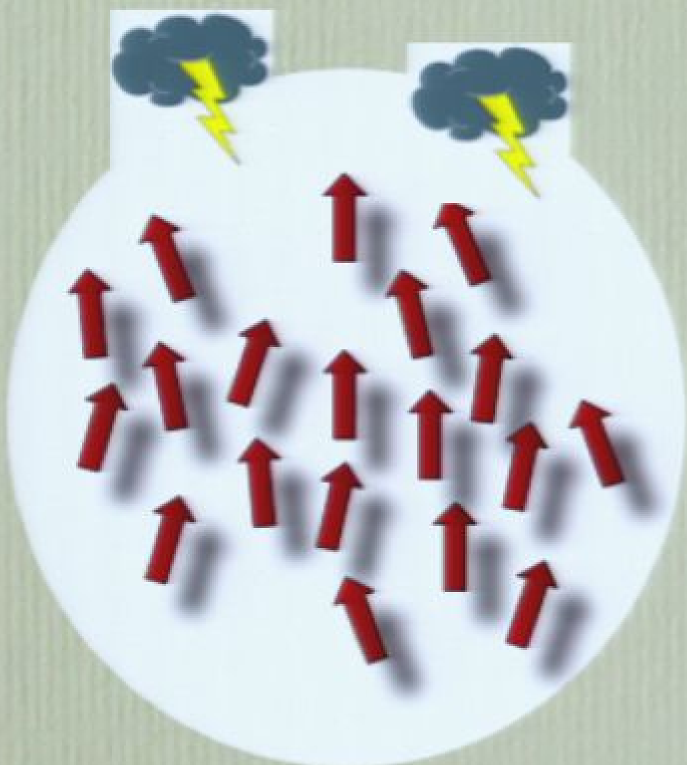

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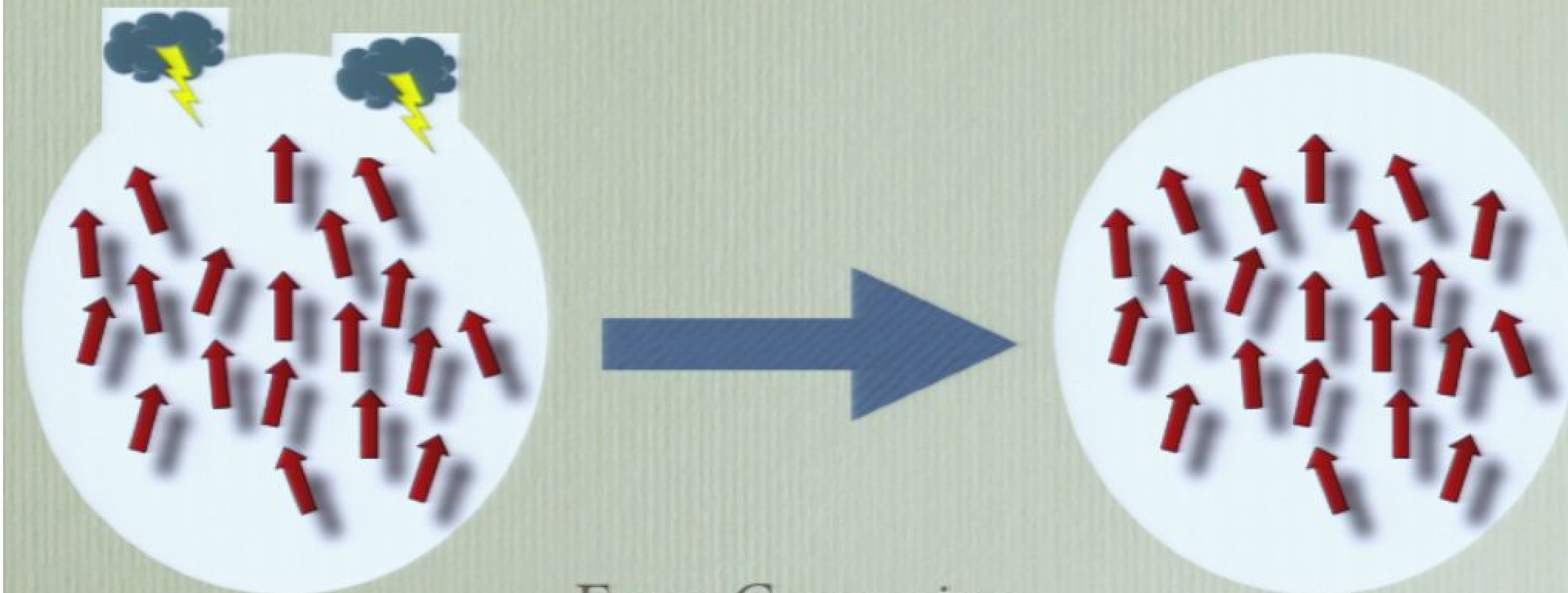
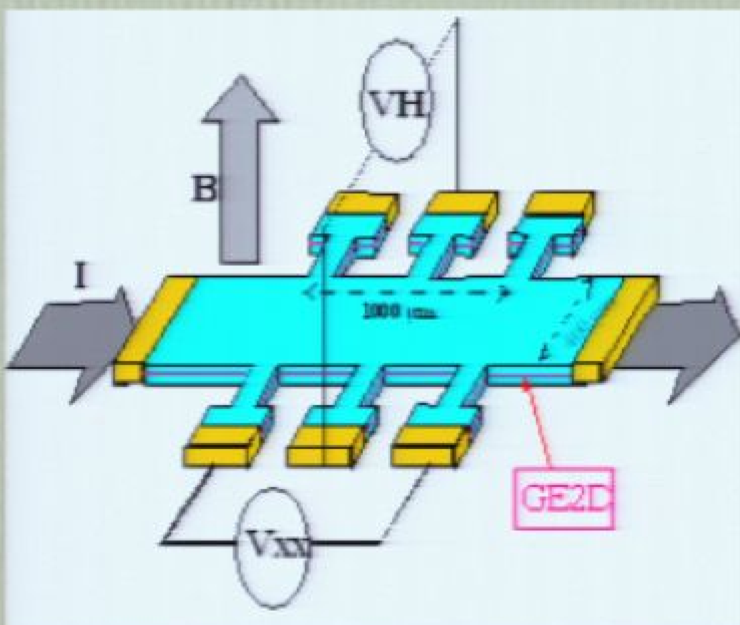


Error Correction

What is TQO? Quantum Protection

The physical idea behind: quantum self-correcting



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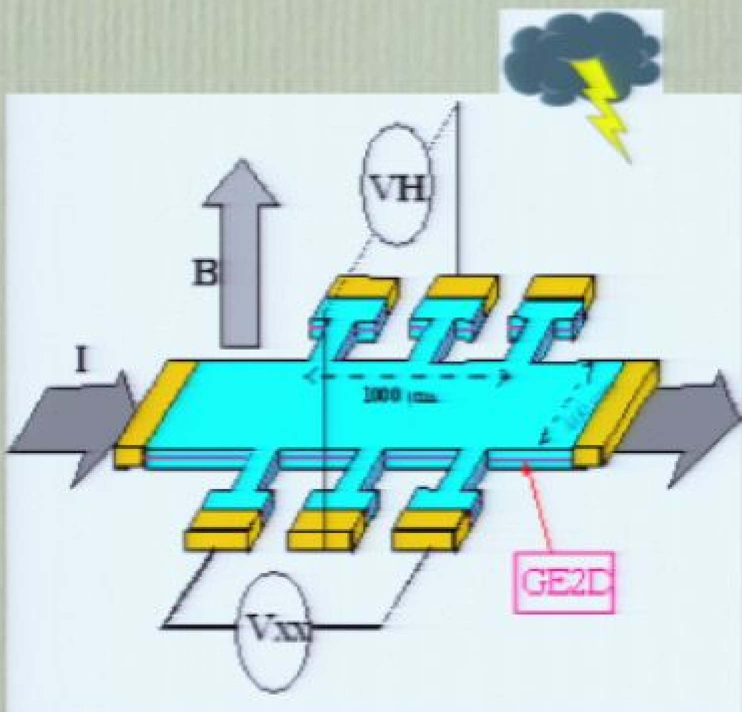

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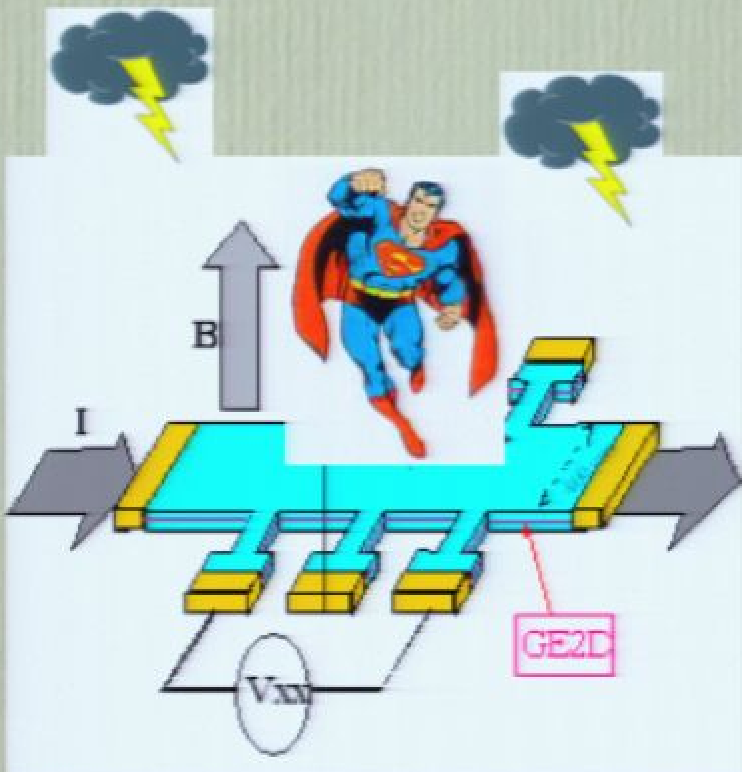

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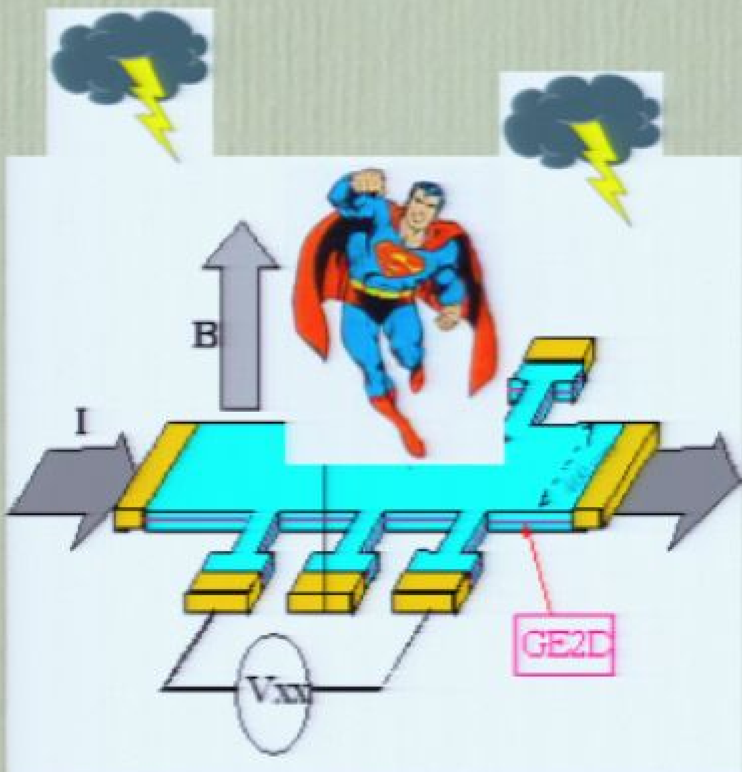
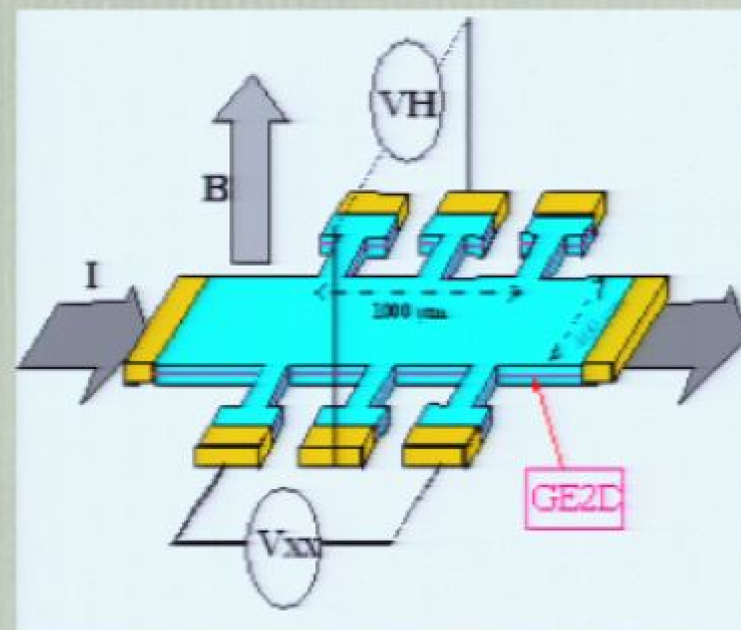


Robust



What is the unifying physical principle behind TQO ?

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Gauge-Like-Symmetries

Given a *D*-dim theory:

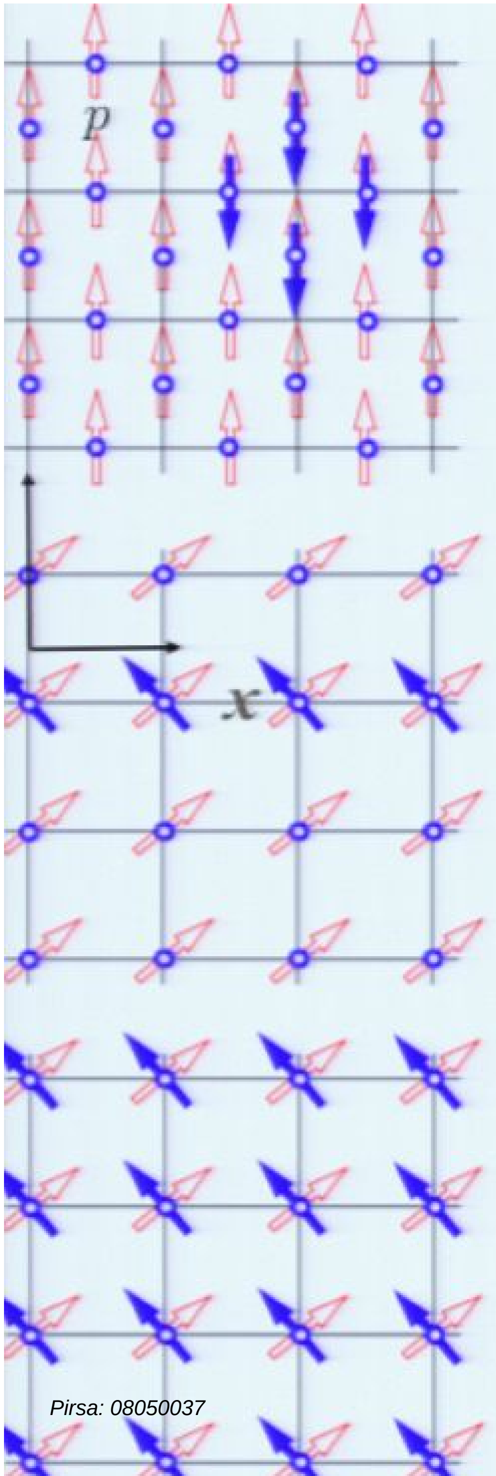
A *d*-dim **GLS** is a group of transformations that leave the theory invariant such that the minimum non empty set of fields that are changed under the symmetry operation occupies a *d*-dim region

$$d \leq D$$

$d=0$ (Gauge)	$d < D$ (Gauge-Like)	$d=D$ (Global)

Group: $\mathcal{G}_d$

Gauge-Like-Symmetries $D = 2$



$d = 0$ (Ising Gauge Theory)

$$H = -K \sum_{p} \sigma^z_{ij}\sigma^z_{jk}\sigma^z_{kl}\sigma^z_{li} \qquad G_i = \prod_{s \in \text{nn}} \sigma^x_{is}$$

$d = 1$ (Orbital Compass Model)

$$H = -\sum_{i} [J_x \sigma^x_i \sigma^x_{i+\hat{e}_x} + J_y \sigma^y_i \sigma^y_{i+\hat{e}_y}]$$

$$O^x = \prod_{j \in C_x} i\sigma^x_j \qquad O^y = \prod_{j \in C_y} i\sigma^y_j$$

$d = D = 2$ (XY model)

$$H = -J \sum_{\langle ij \rangle} [\sigma^x_i \sigma^x_j + \sigma^y_i \sigma^y_j]$$

$$U(\theta) = \prod_j \exp[-(i/2)\theta \sigma^z_j]$$

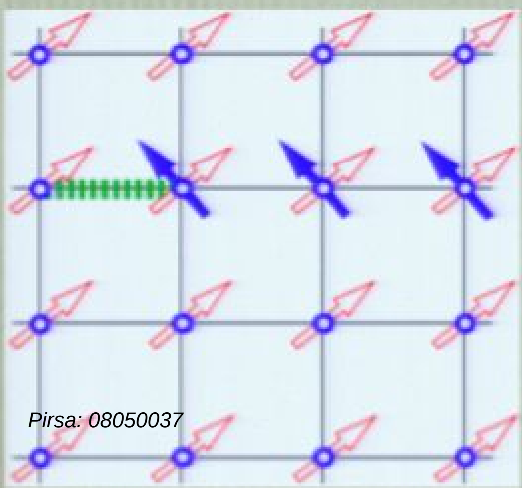
d-GLSs and Topological Phases

There is a connection between Topological Phases and the group generators of *d*-GLSs and its Topological defects

$d=1$ (D=2 Orbital Compass Model) C_x: closed path

$$O^x = e^{i\frac{\pi}{2}\sum_{j\in C_x}\sigma_j^x} = \mathcal{P}e^{i\oint_{C_x}\vec{A}\cdot d\vec{s}}$$

Symmetries are linking operators: $O^\mu|g_\alpha\rangle = |g_\beta\rangle$



Topological defect: C_+: open path

$$D^x = e^{i\frac{\pi}{2}\sum_{j\in C_+}\sigma_j^x} = \mathcal{P}e^{i\int_{C_+}\vec{A}\cdot d\vec{s}}$$

Defect-Antidefect pair creation

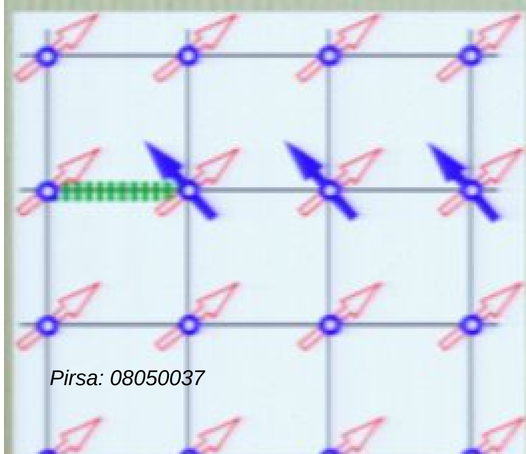
Physical Consequences

For a D-dim system, $d < D$ GLSs lead to
dimensional reduction

- Conservation Laws within d-dim regions:
  Additional conserved currents

- Topological terms that appear in $d+1$ also appear in $D+1$

- Freely propagating d-dim topological defects



$d=1$ soliton in the $D=2$ orbital compass mod
(Finite Energy cost)

To Break or not to Break

Can we spontaneously break a *d*-GLS in a *D*-dim system ?

From the Generalized Elitzur's Theorem: (finite-range int.)

For non-$\mathcal{G}_d$-invariant quantities

- *d=0* SSB is forbidden
- *d=1* SSB is forbidden
- *d=2* (continuous) SSB is forbidden

  d=2 (discrete) SSB may be broken
- *d=2* (continuous with a gap) SSB is forbidden even at *T=0*

Transitions and crossovers are signaled by symmetry-invariant string/brane or Wilson-like loops

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- What is Topological Quantum Order ? [illegible]

- *d*-dim Gauge-Like Symmetries - Physical Consequences Dimensional reduction ($d < D$)

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Fundamental Theorem Linking TQO and GLSs

(Z. Nussinov and G. Ortiz, cond-mat/0605316, 0702377)

Any physical system which displays T=0 TQO, and interactions of finite range and strength, in which all GSs (satisfying the non-distinguishability condition) can be linked by discrete $d < 2$ or continuous $d < 3$ GLSs, has TQO at all temperatures.

(d-GLSs with $d < D$ can mandate the absence of SSB)

Fundamental Theorem
Proof

(Z. Nussinov and G. Ortiz, cond-mat/0605316, 0702377)

For any symmetry $U \in G_d$, we separate

$$V = V_0 + V_\perp$$ Quasi-local operato

where $$[V_0, U] = 0$$

and $$\int dU\, U^\dagger V_\perp U = 0$$

Fundamental Theorem
Proof

Non-symmetry invariant component: $[V_\perp, U] \neq 0$

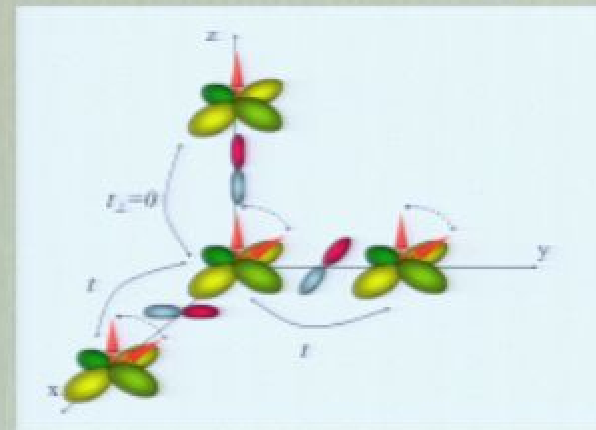
By generalized Elitzur's Theorem: $\langle V_\perp \rangle_\alpha = 0$

(Batista-Nussinov)

Putting all the pieces together: $\langle V \rangle_\alpha = \langle V \rangle_\beta$

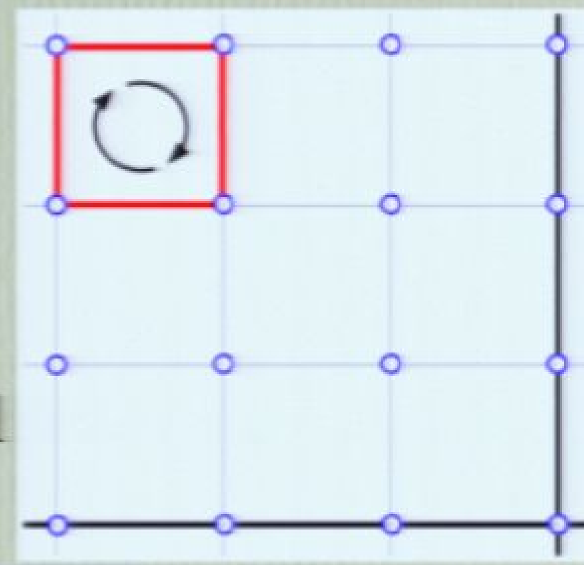
QED

New Examples



- Kugel-Khomskii Hamiltonians (Transition metals)

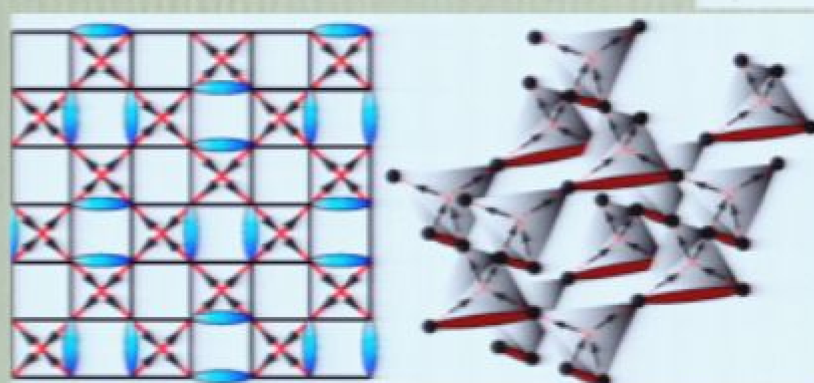
  Orbital compass model



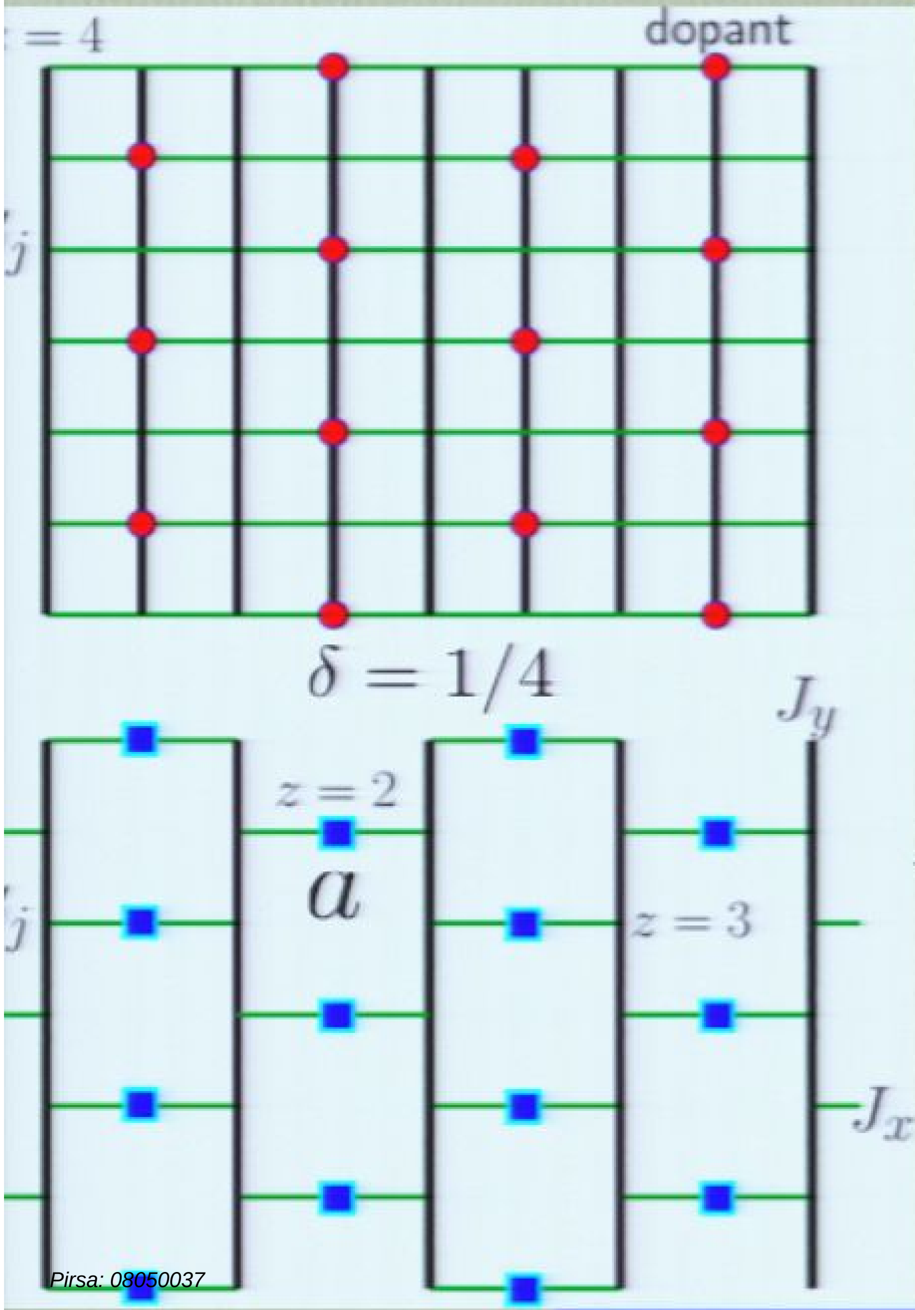
- Superconducting (p+ip) arrays (Sr_2RuO_4)

p+ip) model $\leftrightarrow$ *D*=2 orbital compass model

- Klein spin models



Diluted Orbital Compass Model and Criticality



$$H_{\mathsf{OCM}} = -\sum_j J_\mu \sigma_j^\mu \sigma_{j+\hat{e}_\mu}^\mu$$

After doping: New gauge symmetry

$$\hat{O}_a = \sigma_a^x, \quad [H_{\mathsf{DOCM}}, \hat{O}_a] = 0$$

$$\bar{H}_{\mathsf{DOCM}} \equiv \hat{P}_\ell H_{\mathsf{DOCM}} \hat{P}_\ell$$

$$\hat{P}_\ell = \prod_{a=1}^{N/3} \left(\frac{\mathbb{1} + \eta_a \sigma_a^x}{2}\right) \qquad \eta_a = \pm 1$$

$$\bar{H}_{\mathsf{DOCM}} = -\sum_b \Big(J_x \eta_a \, \sigma_b^x + J_y \, \sigma_b^y \sigma_{b+\hat{e}}^y$$

$$\mathcal{Z} = \mathrm{tr}_{\mathcal{H}} e^{-\beta H_{\mathsf{DOCM}}} = 2^{N/3} \mathcal{Z}_{\mathsf{TFIM}}$$

Quantum critical

$(Ca_3Ru_2O_7)$

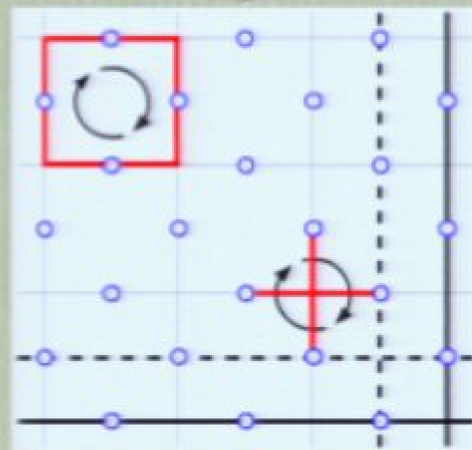
How do we mathematically characterize TQO ?

Insufficient criteria:

- Hamiltonian Spectrum: TQO is a property of states
  (Duality mappings disentangle the non-local order)
- Topological Entanglement Entropy
  (Deviation from an Area law)
- String/Brane Correlations:
  Long-range order of non-local operators

TQO is a property of States not of the Spectrum

Kitaev's toric code model:



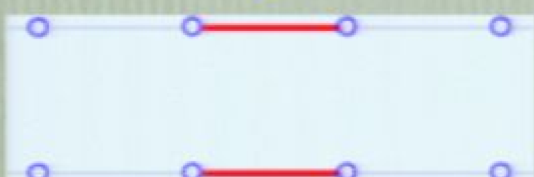
$$H_K = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{ij \in star(s)} \sigma^x_{ij}$$

$$B_p = \prod_{ij \in boundary(p)} \sigma^z_{ij}$$

Duality mappings: Non-local (Identical spectra)

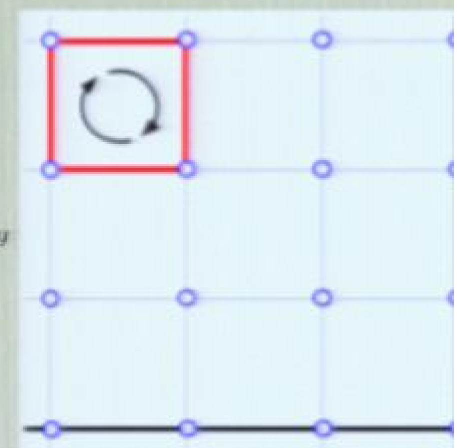
2 Ising chains:



$$H_I = -\sum_s \sigma^z_s \sigma^z_{s+1} - \sum_p \sigma^z_p \sigma^z_{p+1}$$

Wen's plaquette model:

$$H_W = -\sum_i \sigma^x_i \sigma^y_{i+\hat{e}_x} \sigma^x_{i+\hat{e}_x+\hat{e}_y} \sigma^y_{i+\hat{e}_y}$$



(Nussinov-Ortiz 2006)

Outline

- What is Topological Quantum Order (TQO)?
- d-dim Gauge-Like Symmetries - Physical Consequences Dimensional reduction ($d < D$)
- Linking d-GLSs and TQO: A theorem
- **Thermal Fragility**: Does temperature preclude protection of quantum information ?

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- What is Topological Quantum Order (TQO) ?
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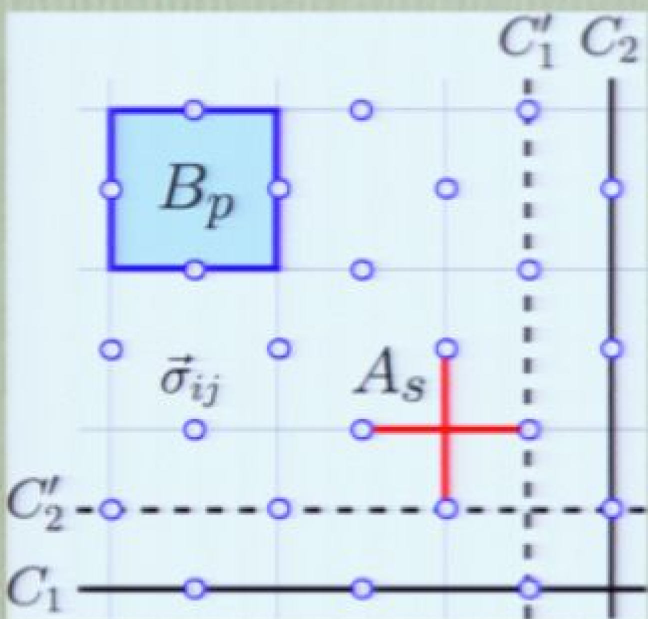
Thermal Fragility

In TQO systems, which have a gap,
does temperature preclude protection of information?



$$H = -\sum_s A_s - \sum_p B_p$$

$$A_s = \prod_{ij\in \mathsf{star}(s)} \sigma_{ij}^x \qquad B_p = \prod_{ij\in \mathsf{plaquette}(p)} \sigma_{ij}^z$$

$$X_{1,2} = \prod_{ij\in C'_{1,2}} \sigma_{ij}^x \qquad Z_{1,2} = \prod_{ij\in C_{1,2}} \sigma_{ij}^z$$

Free-energy is analytic

$$\{X_i, Z_i\} = 0\ ,\ [X_i, Z_j] = 0$$

No thermodynamic phase transition!

Thermal Fragility

For a finite size: By Symmetry

$$\langle Z_1 \rangle = \langle Z_2 \rangle = \langle X_1 \rangle = \langle X_2 \rangle = 0$$

Partition function (2 Ising chains):

$$Z = \mathrm{tr}\left[\exp[-\beta(H - \sum_{i=1,2}(h_{x,i}X_i + h_{z,i}Z_i))]\right]$$

$$= [(2\cosh\beta)^{N_s} + (2\sinh\beta)^{N_s}]^2 \cosh\beta h_1 \cosh\beta h_2$$

$$h_i = \sqrt{h_{x,i}^2 + h_{z,i}^2}$$

$$\langle Z_i \rangle = \lim_{h_{z,i}\to 0^+} \frac{\partial}{\partial(\beta h_{z,i})} \ln Z = \lim_{h_{z,i}\to 0^+} \frac{h_{z,i}}{h_i} \tanh(\beta h_i)$$

$$= 0$$

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Thermal Fragility: Energy-Entropy budget

From a Physics standpoint:

$$\langle Z_1 \rangle = \langle Z_2 \rangle = \langle X_1 \rangle = \langle X_2 \rangle = 0$$

Energy penalty for excitations: Independent of system size

Entropy: Log in the system size → Proliferation of defects

Loss of order: $$N_s \geq \xi = \frac{1}{\ln \coth \beta J} \rightarrow_{\beta \rightarrow \infty} \frac{e^{2\beta J}}{2}$$

$$k_B T \geq \frac{2J}{\ln 2 N_s}$$

Thermal Fragility: Dynamical aspects

Time autocorrelation functions:

$$G_{X_\mu}(t) \equiv \langle X_\mu(0) X_\mu(t) \rangle \sim e^{-(|t|/\tau)^\epsilon}$$

τ is independent of system size

1) Long-times: $|t| \gg \tau = \dfrac{\text{const.}}{1 - \tanh 2\beta J} \to_{\beta\to\infty} e^{\beta\Delta}$

$$\epsilon = 1$$

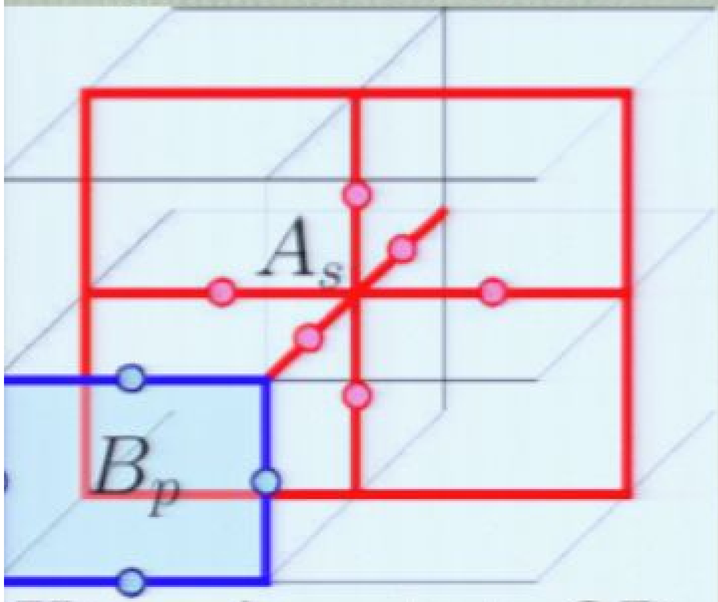
2) Intermediate-times: $\text{const.} \ll |t| \ll \tau = \dfrac{\text{const.}}{1 - \tanh 2\beta J}$

$$\epsilon = 1/2$$

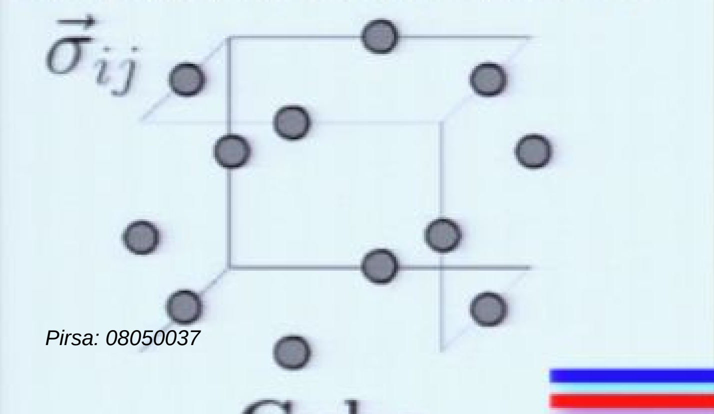
Thermal Fragility and Phase Transitions

What is the relation between the existence of a phase transition and TQO?

(Phase transitions are signaled by non-analyticities in the Free Energy)



Kitaev's toric in $3D$



It has a thermodynamic phase transition:

$$\mathcal{Z}_{3D} = \mathcal{Z}_{3D \text{ Ising gauge}} \times \mathcal{Z}_{1D \text{ Ising}}$$

$$(\beta_c = 0.761423)$$

It displays TQO

However, e.g.

$$\langle Z_{C_\mu} \rangle = \langle \prod_{(ij)\in C_\mu} \sigma^z_{ij} \rangle = 0$$

Loops around Toric cycles

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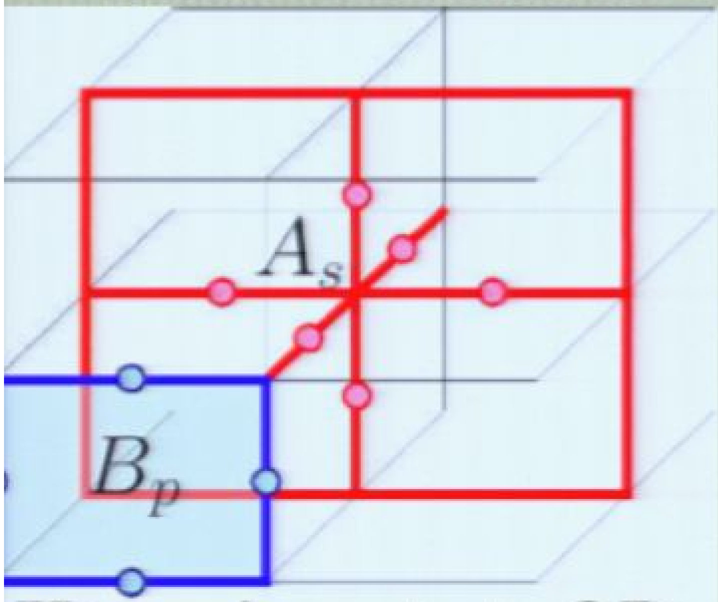

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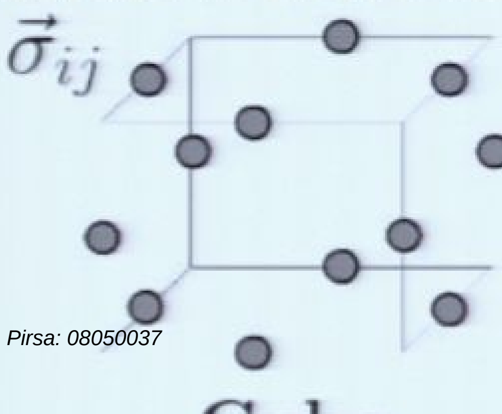

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↓

Loops around Toric cycles

What have we done and proved?

Most significant results:

- Provide a unifying framework
- Fundamental Theorem: A sufficient symmetry condition to have TQO is that the system displays low *d*-dim GLSs
- *d*-dim ($d<D$) GLSs lead to dimensional reduction
- *d*-GLSs can enforce high dimensional fractionalization, unusual topological indices (& related Berry phases)

What have we done and proved?

- The devil is not in the spectrum
- Thermal effects seem to impose severe restrictions on several current suggestions for topological quantum computing (Thermal fragility)
- General entangled systems have string (or higher dimensional "brane") correlators which decay more slowly than the usual two-point correlators
- A goal is to use the symmetry principles to engineer new model Hamiltonians that can be easily realized experimentally.

What remains to be done?

Most significant questions:

- How do we characterize and classify TQO?

  (Entanglement entropy? Generalized entanglement?)

- How do we measure TQO? Experimental probes?

- Most importantly for quantum memories: Conditions under which TQO is protected from thermal effects?

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What remains to be done?

Most significant questions:

- What are TQO states useful for?

  Quantum orders vs Functionalities

- What is the relation between *d*-GLSs and Topological Quantum Field Theories?