

Title: Tales from graphland

Date: Apr 30, 2008 09:50 AM

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Abstract: This talk will report recent work on two themes that relate concepts in graph theory to problems in quantum information theory. We will discuss the quantum analogue of expander graphs which prove to be of key importance when de-randomizing algorithms in classical computer science. Using powerful ideas of discrete phase space methods, efficiently implementable quantum expanders can be constructed based on an argument that barely fills three lines. We also briefly report news on novel measurement-based models of quantum computing, based on quantum systems distributed on a graph, beyond one-way computing. Work done in collaboration with D. Gross D. Gross, J. Eisert, '\Quantum Margulis expanders\'', Quant. Inf. Comp. (2008), arXiv:0710.0651. D. Gross, J. Eisert, '\Quantum computational wires\'', in preparation (2008). D. Gross, J. Eisert, N. Schuch, D. Perez-Garcia, '\Measurement-based quantum computation beyond the one-way model\'', Phys. Rev. A 76, 052315 (2007), arXiv:0706.3401. D. Gross, J. Eisert, '\Novel schemes for measurement-based quantum computation\'', Phys. Rev. Lett. 98, 220503 (2007).

# Tales from graphland

Jens Eisert, Imperial College London

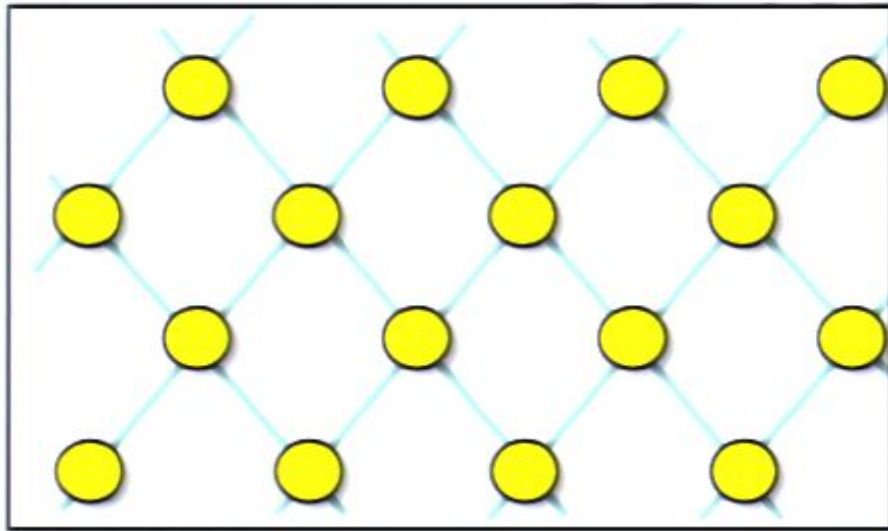
Joint work with D Gross

*Phys Rev Lett* **98**, 220503 (2007)

*Phys Rev A* **76**, 052315 (2007)

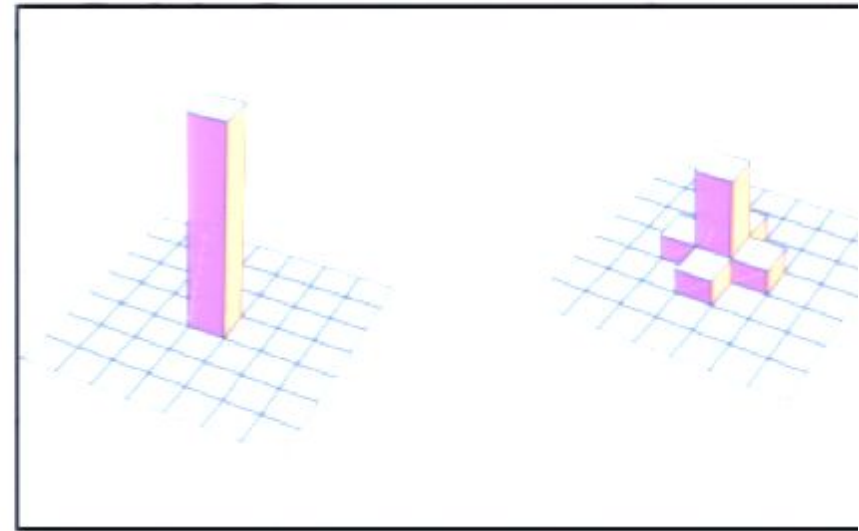
*Quant Inf Comp* (2008)

*In preparation* (2008)

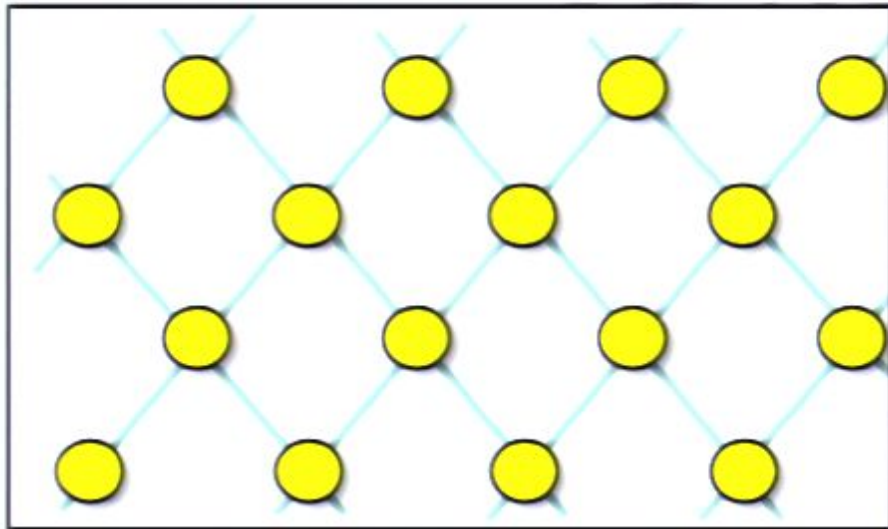


- **“Generalized graph states”**

The computational power of quantum wires

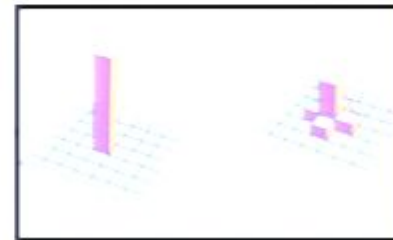


- **Quantum expanders and expander graphs**

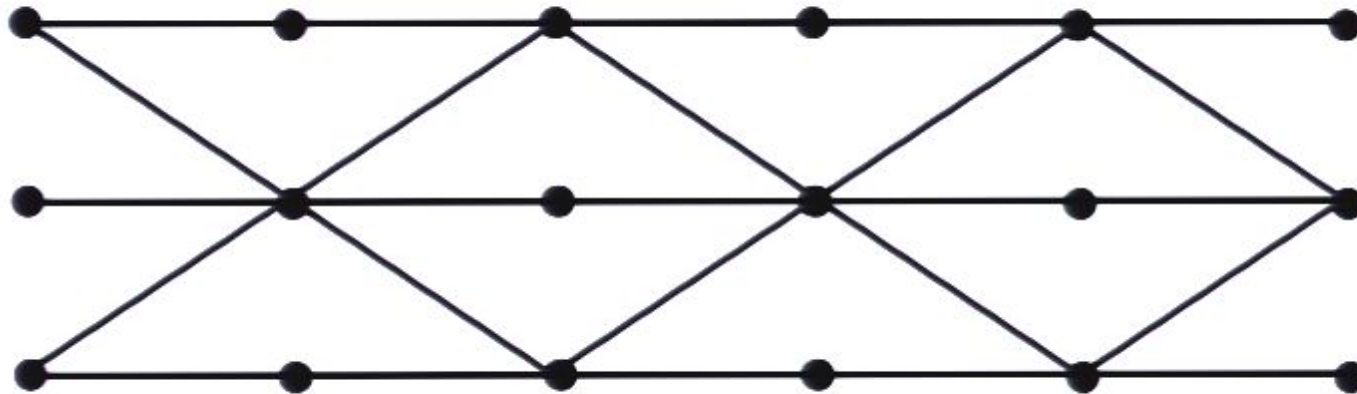


- **“Generalized graph states”**

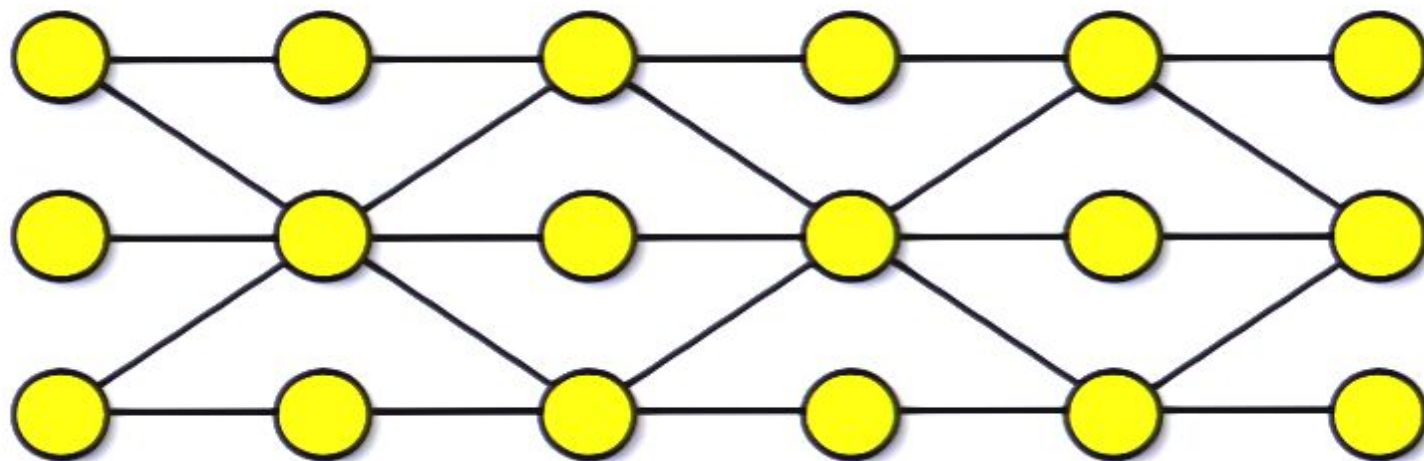
The computational power of quantum wires



- **Quantum expanders**  
and expander graphs



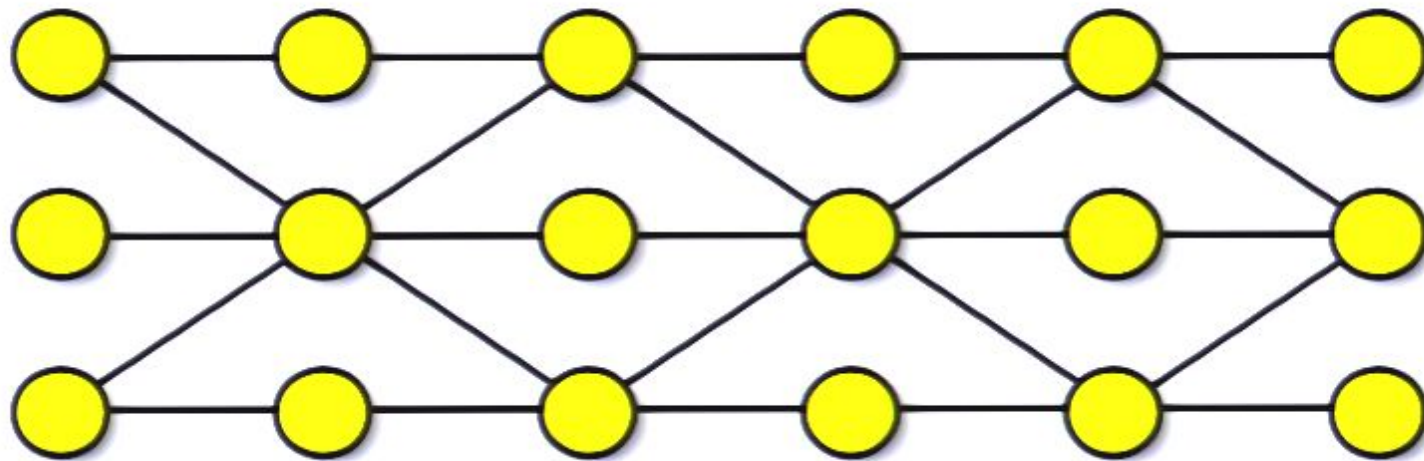
- Given a simple undirected graph  $G = (L, E)$
- There will be a **graph state** associated with it



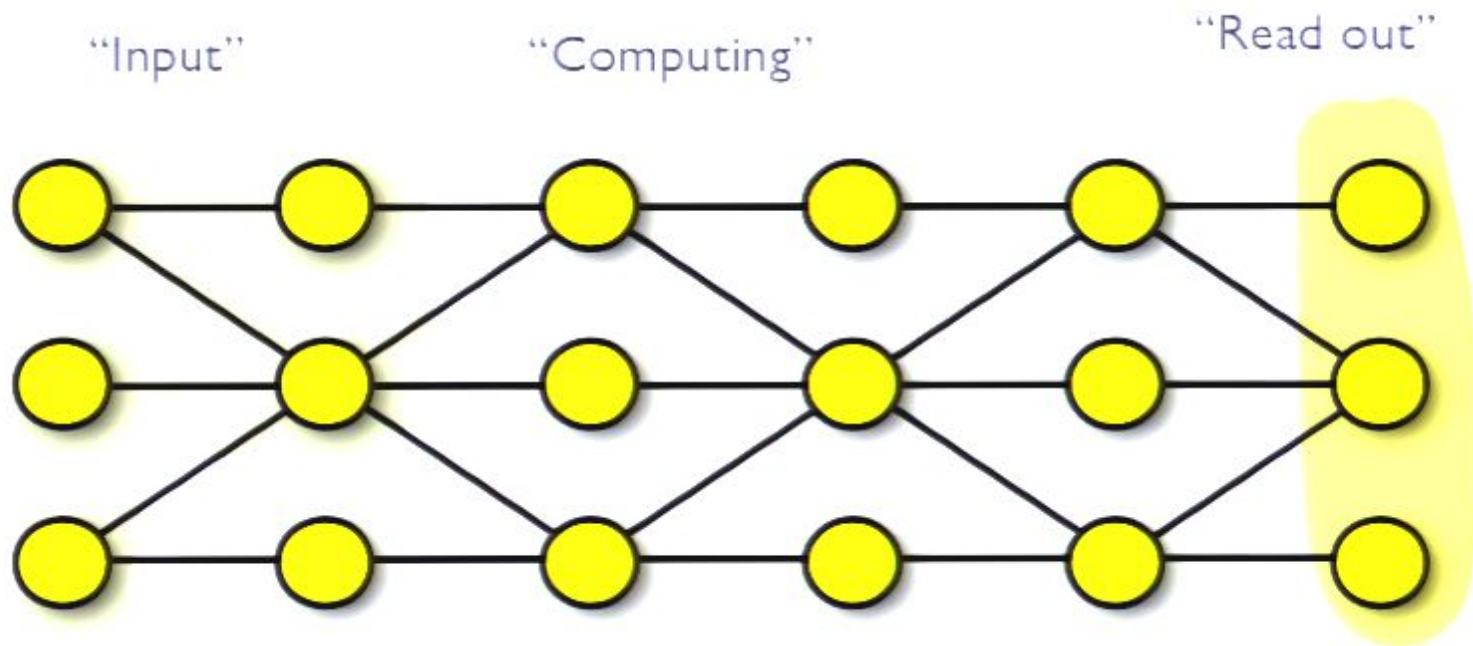
- Given a simple undirected graph  $G = (L, E)$
- Associate vertices with  $\mathbb{C}^2$ , prepare each in  $|+\rangle = |0\rangle + |1\rangle$ , and apply phase gate

$$P = \text{diag}(1, 1, 1, -1)$$

to each edge  $E$  of graph

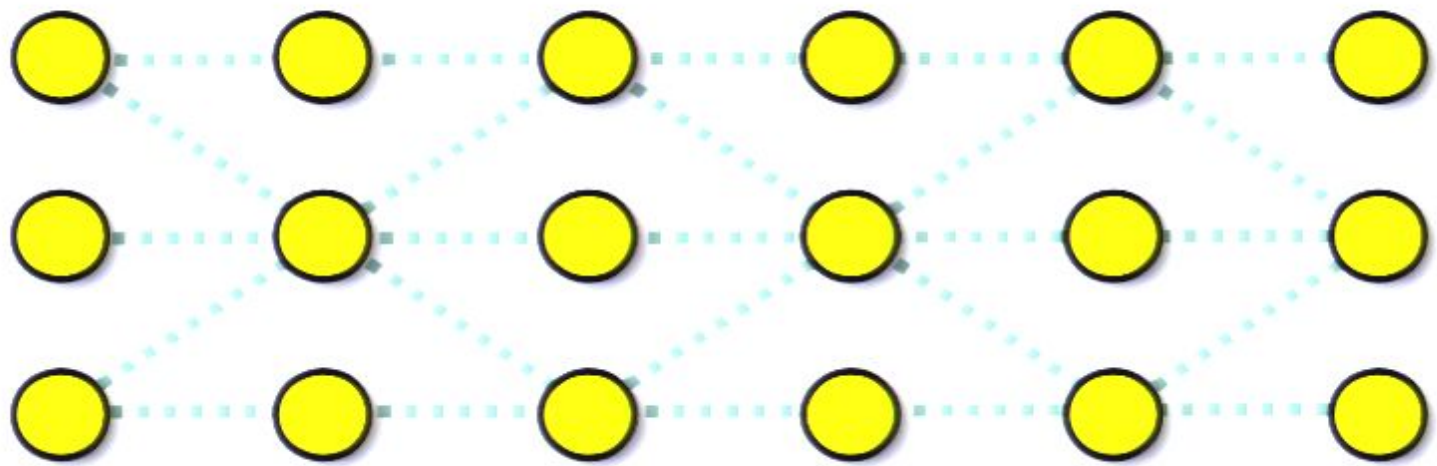


- Then, local measurements are just as powerful as the circuit model for QC
- “Graph states can be universal resources”



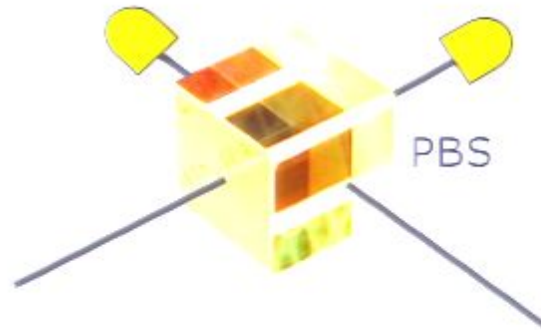
- Then, local measurements are just as powerful as the circuit model for QC
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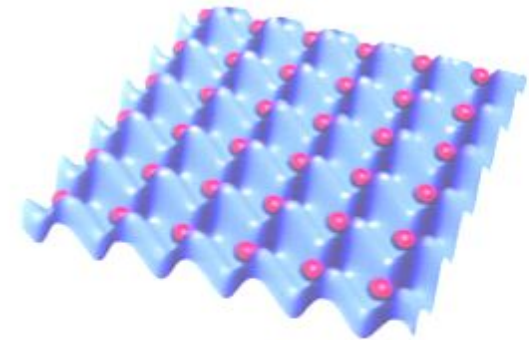


- **Now, very nice, but what if we do not have a graph state?**
- “Generalized graph states with computational power”?

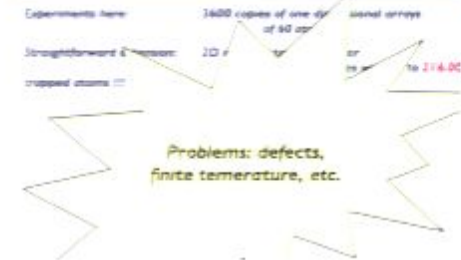
- Optical preparation?



- Cold atoms in optical lattices?

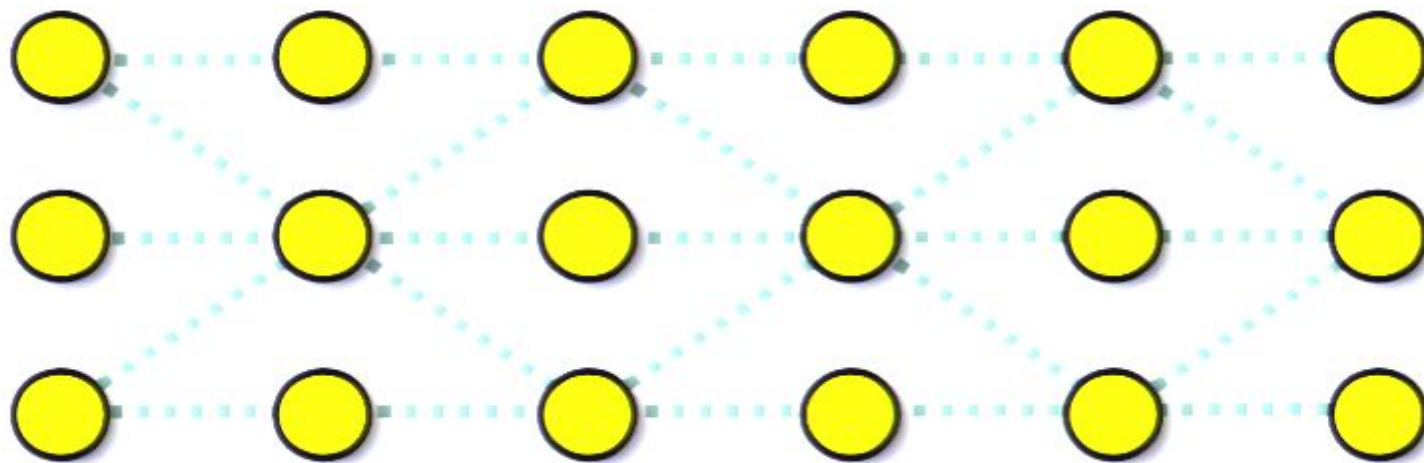


#### 1D/2D Controlled Collisions



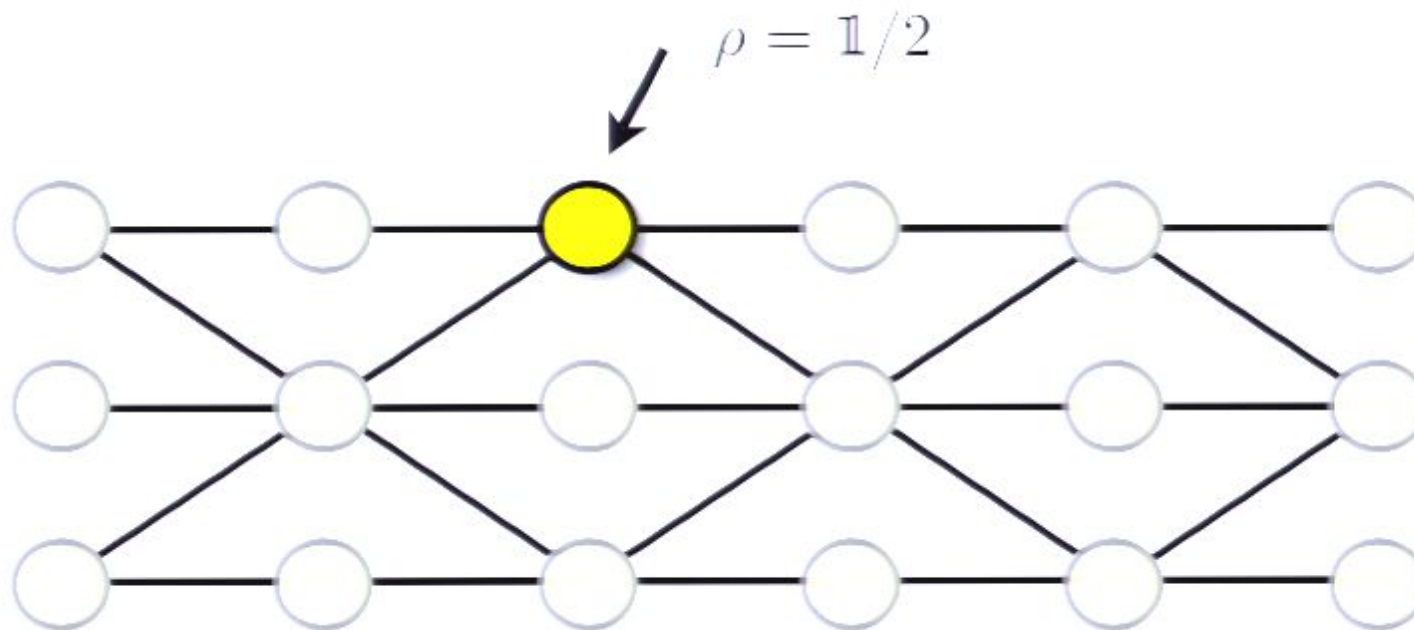
- **Physics motivation:**

- Different meaningful preparation mechanisms, states tailored to architecture
- Protected via ground states



- **Computer science motivation:**

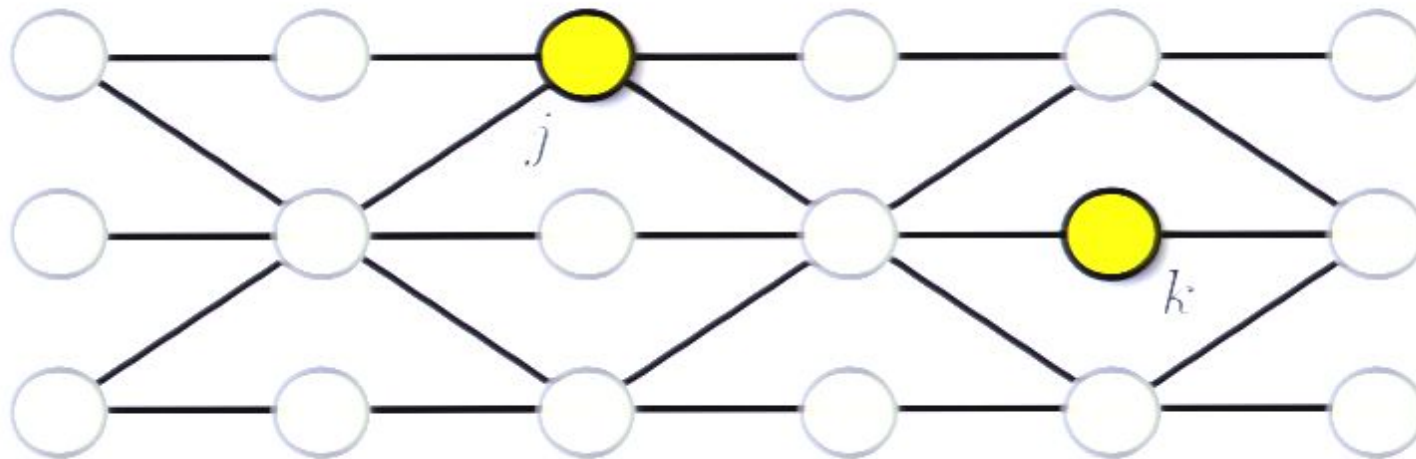
- Explore properties of “universal states and phases”
- Efficient classical simulability (see Maarten) vs. universality for QC
- Computational complexity of tensor contractions



- **Properties of the cluster:**

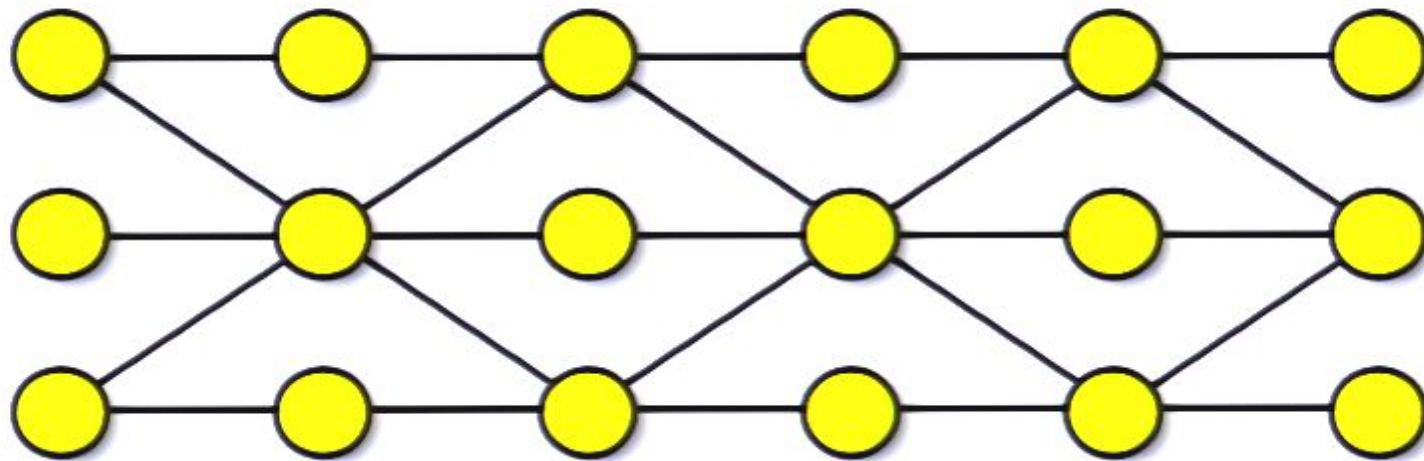
- Graph states have a number of interesting, but also very extremal properties
  - Maximal entanglement!

$$\langle O_j O_k \rangle - \langle O_j \rangle \langle O_k \rangle = 0$$



- **Properties of the cluster:**

- Graph states have a number of interesting, but also very extremal properties
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  - No long-range correlations!



- **Properties of the cluster:**

- Graph states have a number of interesting, but also very extremal properties
  - Maximal entanglement!
  - No long-range correlations!
  - Maximal localizable entanglement!
  - Preparable via QCAs

## • **What states have universal computational power?**

For years, essentially only graph states known

Briegel, Raussendorf, *Phys Rev Lett* **86** (2001)

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Compare also Childs, Leung, Nielsen, *Phys Rev A* **71** (2005)

## • **Recent progress:**

Classes of alternative models

Gross, Eisert, *Phys Rev Lett* **98** (2007)

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Compare also results presented by Katerina and Akimasa

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## • Agenda today:

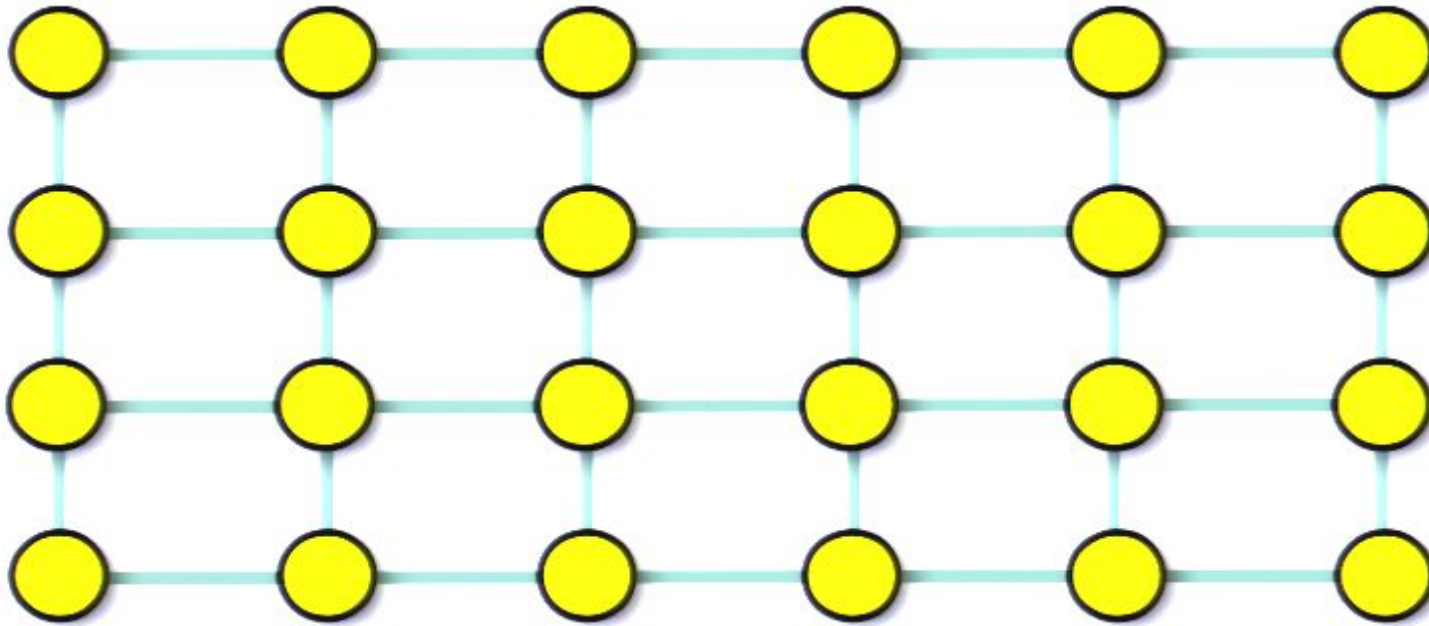
Steps towards a *systematic* toolbox for constructing new models

## • If there is time: teaser on some completely different connection

between graphs and quantum information: **Quantum margulis expanders**

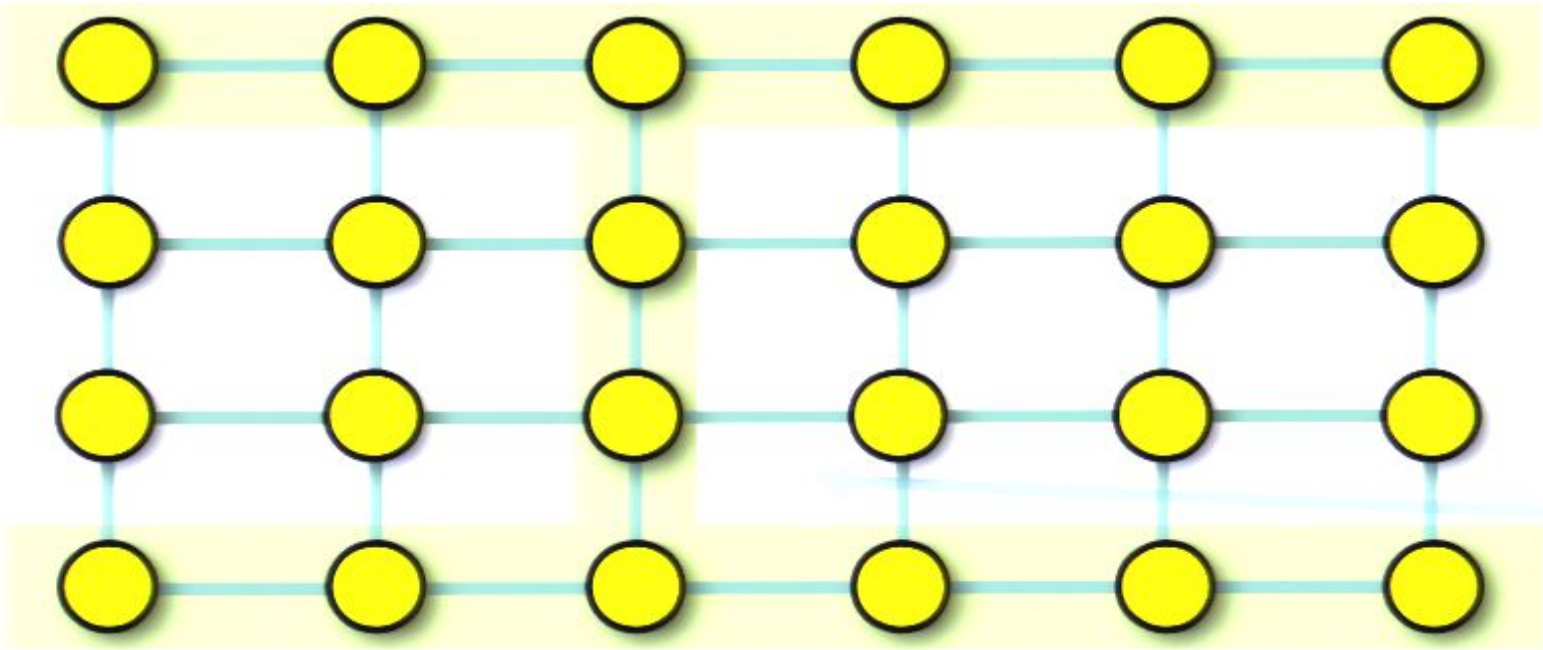


- 
- New measurement-based models:  
Framework of a toolbox





Time →

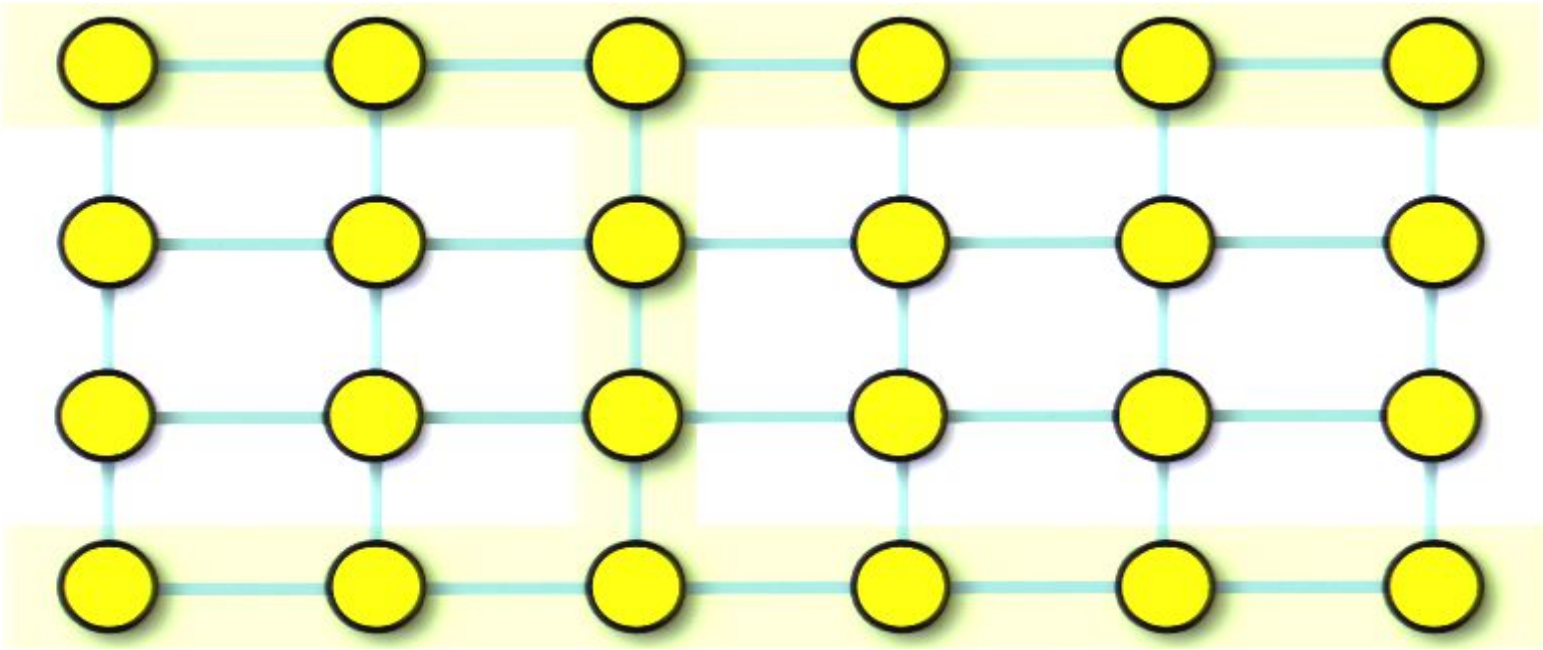


“Gates”

“Wires”



Time →

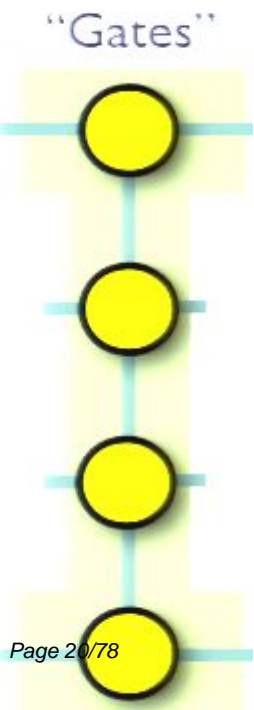


=



“Wires”

+



“Gates”



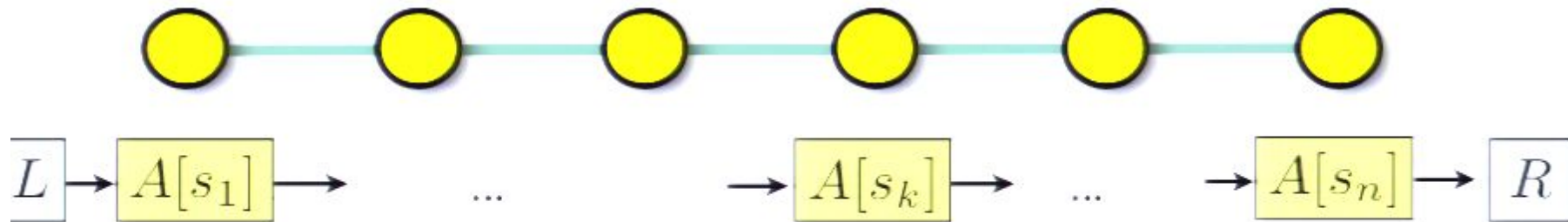
- Much can be gained by looking at states in right way:
  - **Matrix-product states** (as being generated in DMRG),
  - **higher dimensional** “tensor product states”, “PEPS” on graphs

Tools from many-body physics



- **Formal ingredients:**

- Substructure  $\mathbb{C}^d$  (“**correlation space**”) for some **dimension**  $d$
- Two complex  $d \times d$  **matrices**  $A[0], A[1]$
- Two vectors  $|L\rangle, |R\rangle \in \mathbb{C}^d$  representing **boundary conditions**

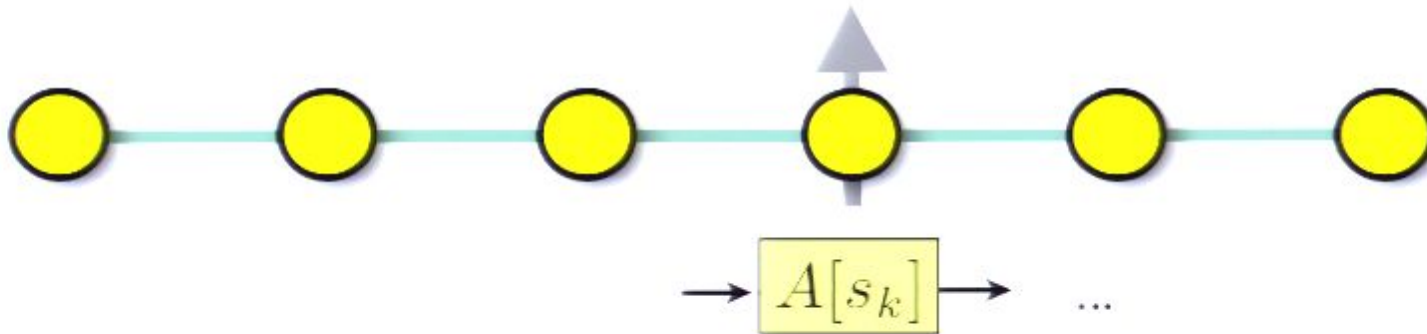


- **Matrix-product state:**

$$\langle s_1 \dots s_n | \psi \rangle = \langle R | A[s_n] \dots A[s_1] | L \rangle$$

“Physical state”

“Matrices in abstract correlation space”



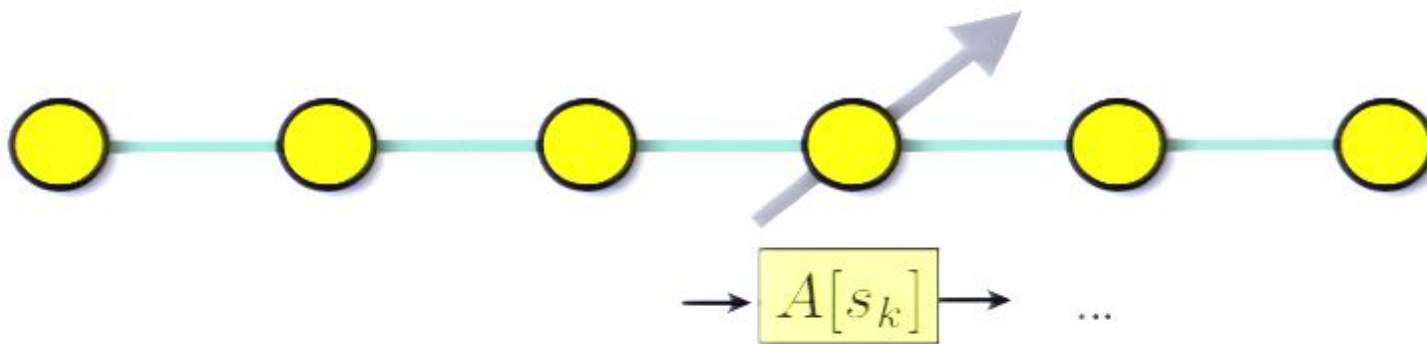
- **Matrix-product state:**

$$\langle s_1, \dots, s_n | \psi \rangle = \langle R | A[s_n] \dots A[s_1] | L \rangle$$

- **Intuition:**

$A[0], A[1]$  operators acting on “correlation space”  $\mathbb{C}^d$





- Now **measurements**:

- Refine for **general local projections** onto  $|\phi\rangle$

$$A[\phi] = \langle 0|\phi\rangle A[0] + \langle 1|\phi\rangle A[1]$$

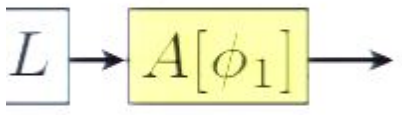
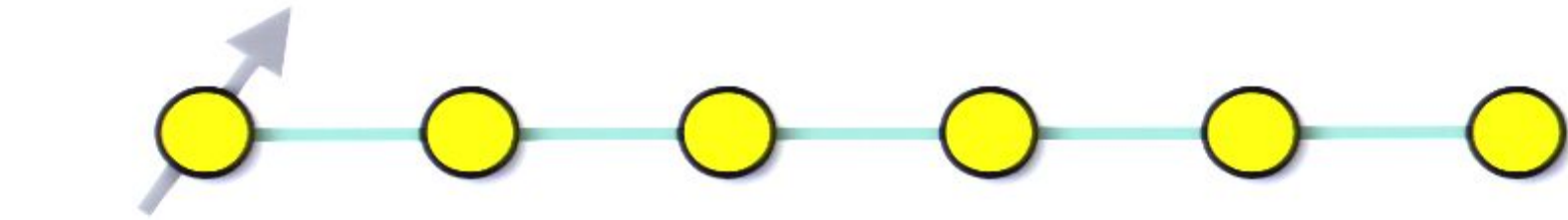
- Then:  $\langle o_1 \dots o_n | \psi \rangle = \langle R | A[o_n] \dots A[o_1] | L \rangle$

- This can be reread as **computation in correlation space!**

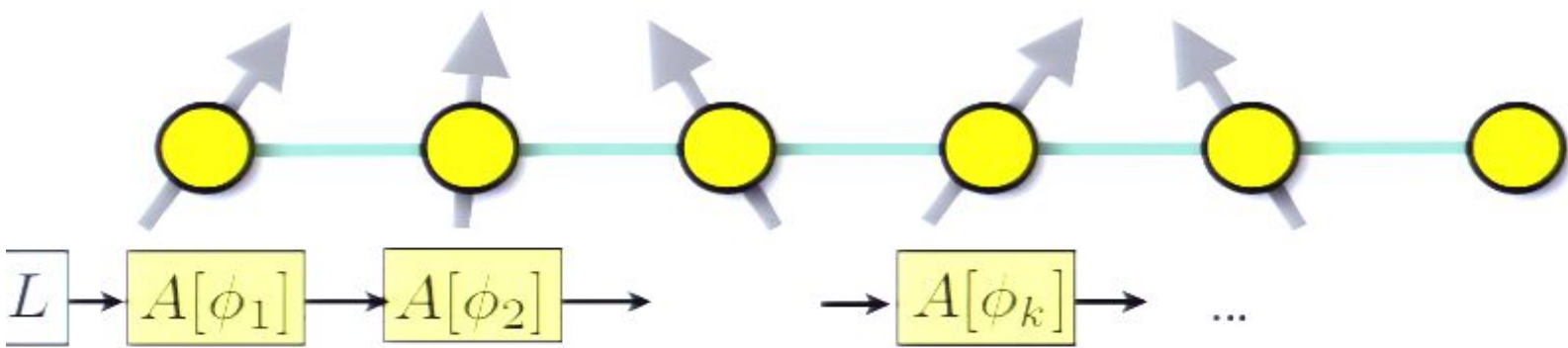


$L$  →

“Preparation”

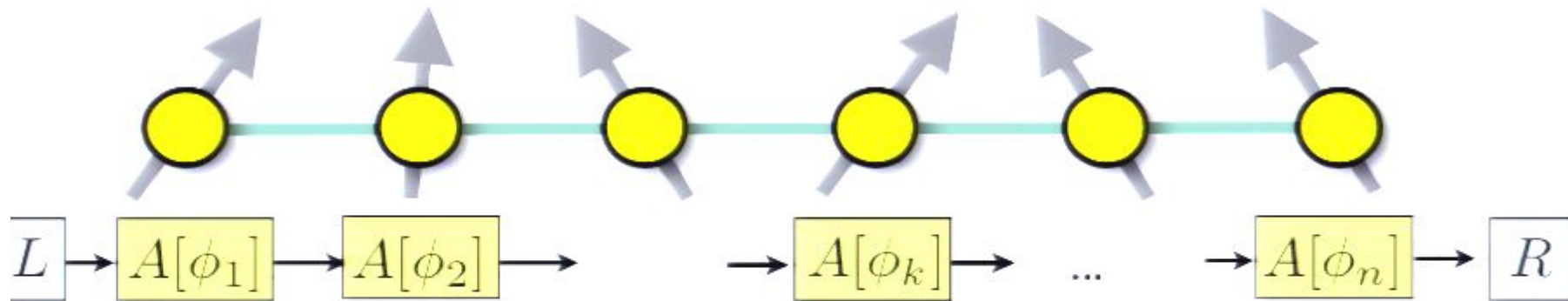


“Preparation”      “Processing”



“Preparation”

“Processing”



“Preparation”

“Processing”

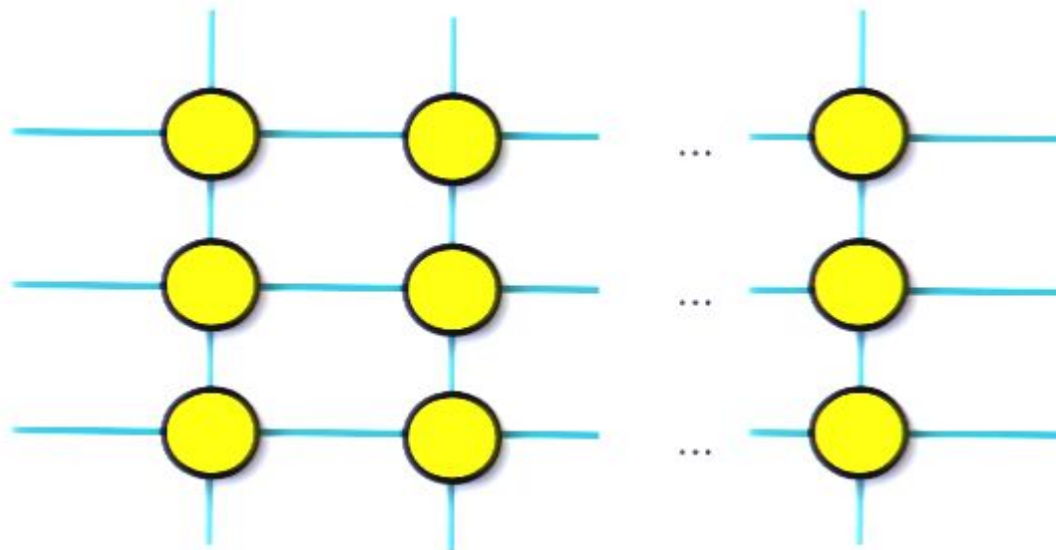
“Prob of occuring”

- “Measurement is driving the computation forward in correlation space”

- “It is all in the correlations”, here in the MPS correlation space

- **General framework:**

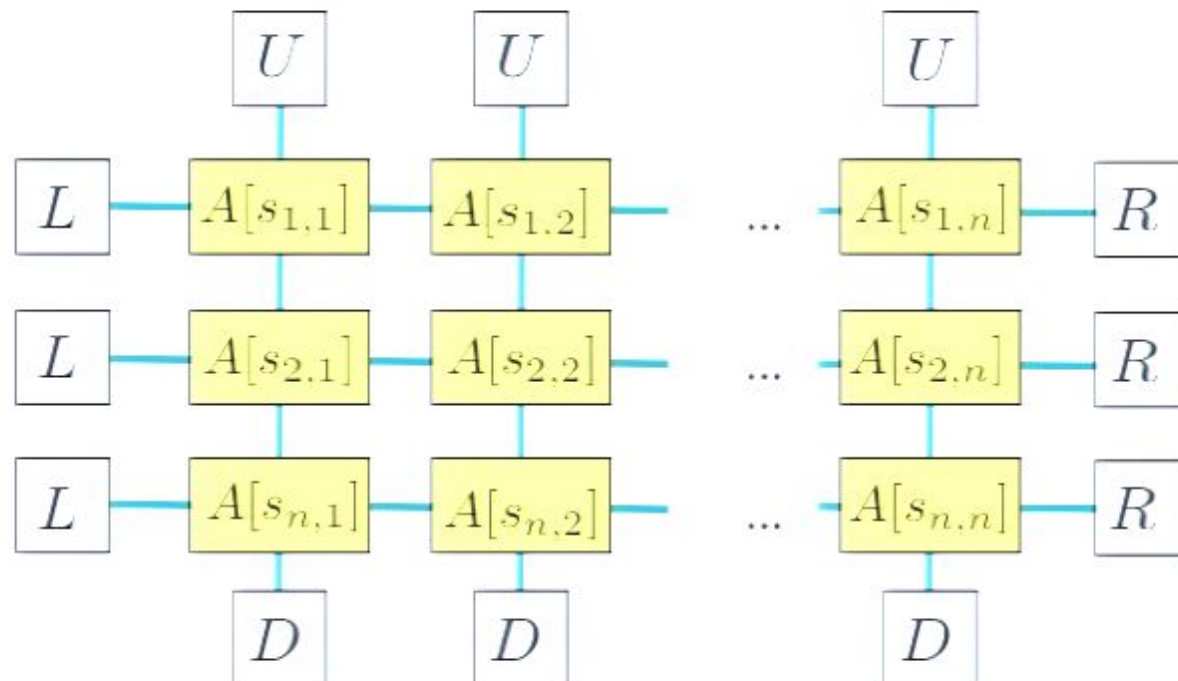
- **General lattices defined by graphs**  $G = (L, E)$ , tensor networks with tensors of degree specified by vertex degree



- **General framework:**

- **General lattices defined by graphs**  $G = (L, E)$ , tensor networks with tensors of degree specified by vertex degree

$A[s]$  now map  $\mathbb{C}_{\text{left}}^d \otimes \mathbb{C}_{\text{down}}^d \rightarrow \mathbb{C}_{\text{up}}^d \otimes \mathbb{C}_{\text{right}}^d$



- Classification of quantum computational wires





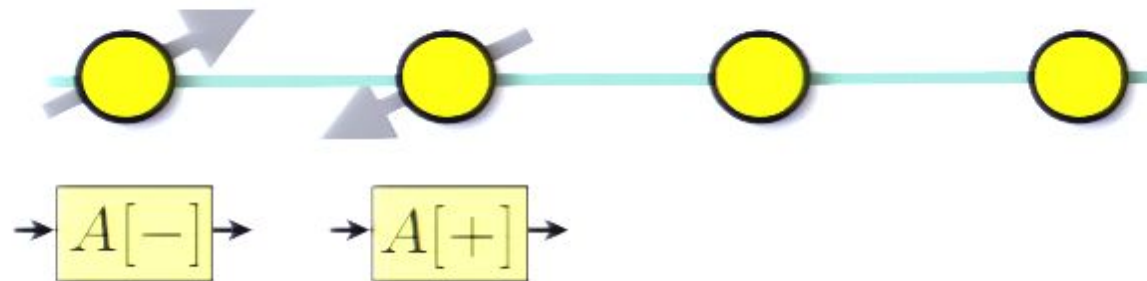
- As a primitive to systematically build up new models:
  - states on a 1D **chain** of qubits
  - allowing for **transport** of a logical qubit
  - **preparable** by nearest-neighbor gates
  - translationally **invariant**
  - can be **coupled** to 2D resource



- Simplest example: 1D cluster

$$\rightarrow \boxed{A[0]} \rightarrow = |+\rangle\langle 0| \quad \rightarrow \boxed{A[1]} \rightarrow = |-\rangle\langle 1|$$

- Measure in  $|+\rangle, |-\rangle$ -basis

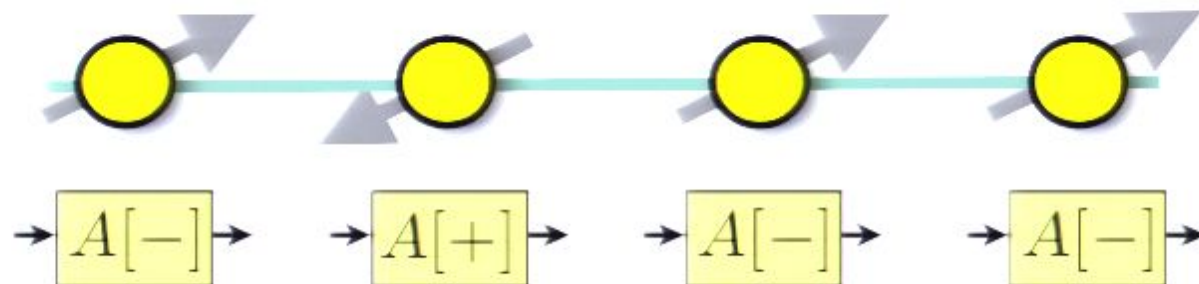


$$= HZ \quad H$$

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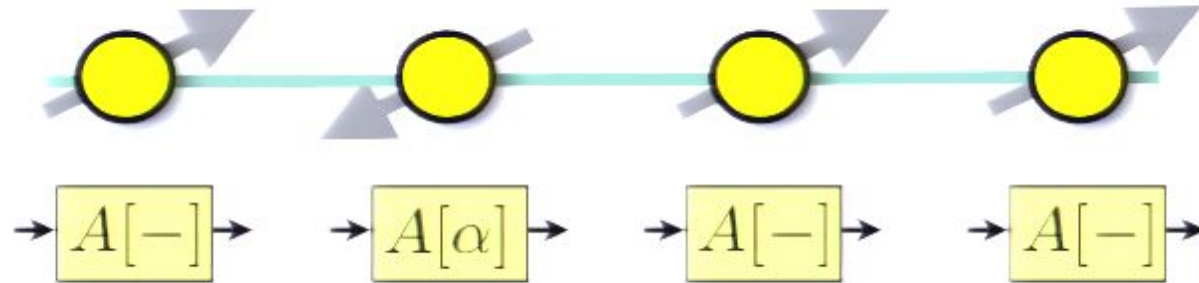
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**• Characteristics:**

1. "Always on interaction"  $H$
2. "Byproduct"  $Z$
3.  $p(+)=p(-)=1/2$



$= HZ \quad HS(\alpha) \quad HZ \quad HZ$

- **4.** characteristic: one-parameter “freedom of choice”

$$A[\sin \alpha |+\rangle + i \cos \alpha |-\rangle] \sim H \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} =: HS(\alpha)$$

## • Interpretation:

1. “always on operation”  $H$

allows for universality

2. “by-product operator”  $Z$

necessary evil of entanglement

3.  $p(+)=p(-)=1/2$

signature of maximal entanglement of cluster

4. Freedom of choice

allows to adapt to different algorithms



- **Axioms for computational wires:**

- a) *states on 1D chain of qubits*

- b) *preparable by nearest-neighbor unitary gates*

- c) *translationally invariant*

- d) *allowing for transport of a logical qubit*

- e) *can be coupled to 2D resource*



- **Axioms for computational wires:**

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- a matrix-product state with  $d = 2$

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$D = 2$ ,  $A[0], A[1]$  are  $2 \times 2$  -matrices

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for some basis  $|\psi_1\rangle, |\psi_2\rangle$  the matrices  $A[\psi_1], A[\psi_2]$  are up to factor unitary

**e)** *can be coupled to 2D resource*

## • Structure of quantum computational wires:

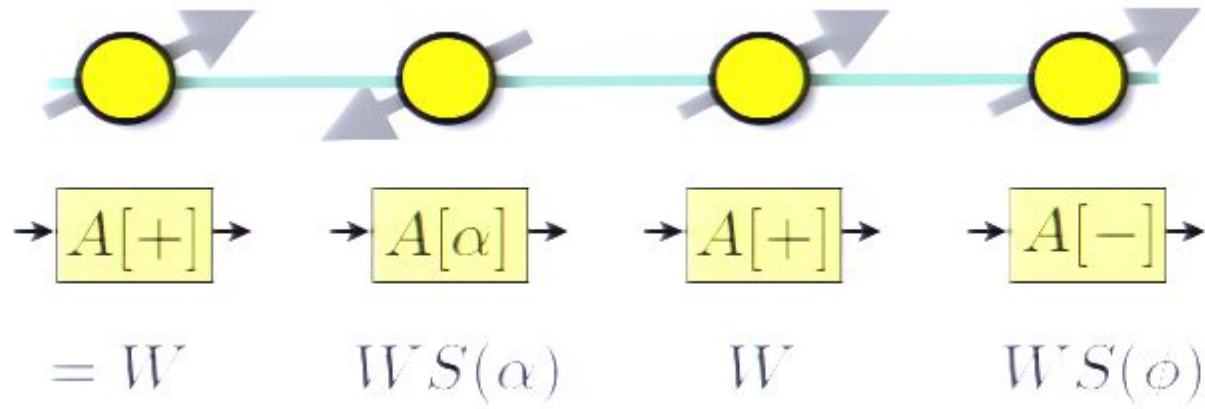
• A quantum wire is described by

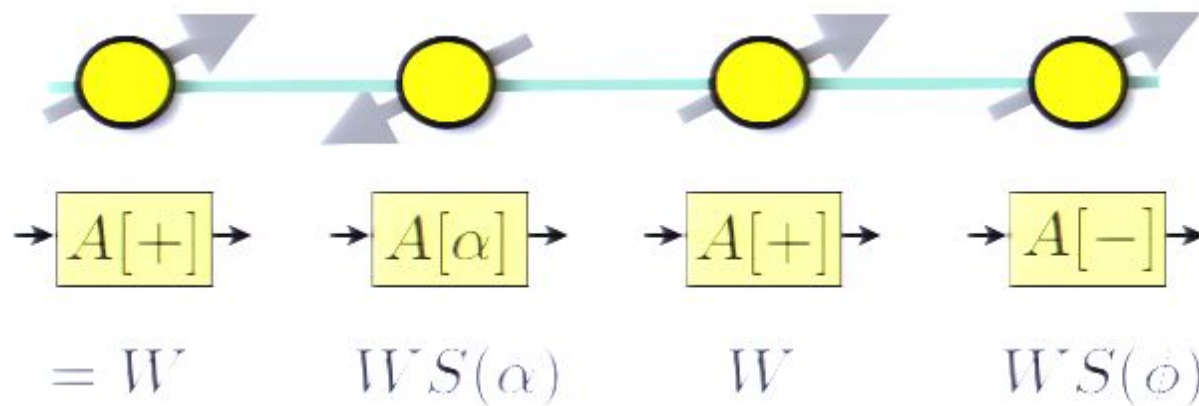
1. an **always on interaction**  $W \in U(2)$
2. a **by-product angle**  $\phi \in [0, 2\pi)$
3. a **bias parameter**  $\theta \in [0, 2\pi)$

$$A[+] = \sin \theta W$$

$$A[-] = \cos \theta W S(\phi)$$

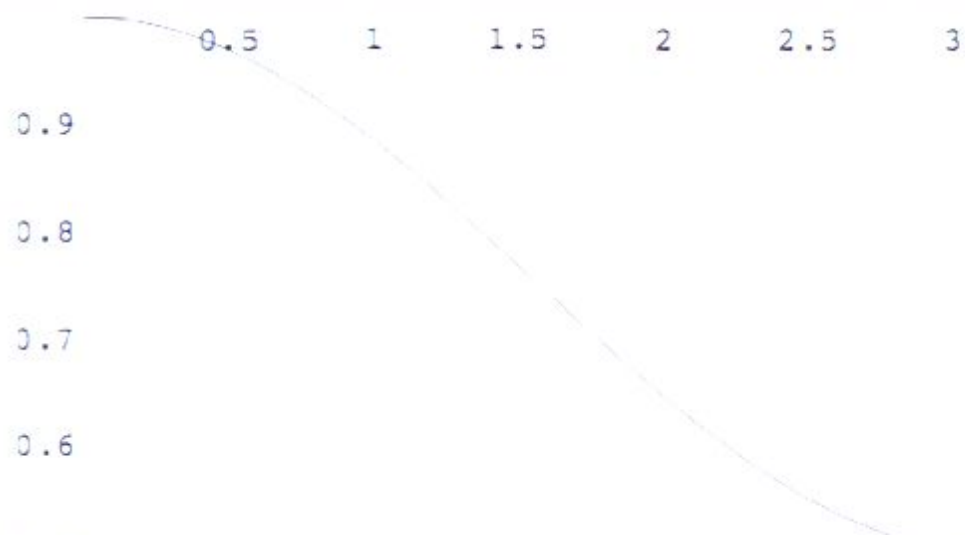
4. lucky mathematical coincidence: is always one-parameter degree of freedom!



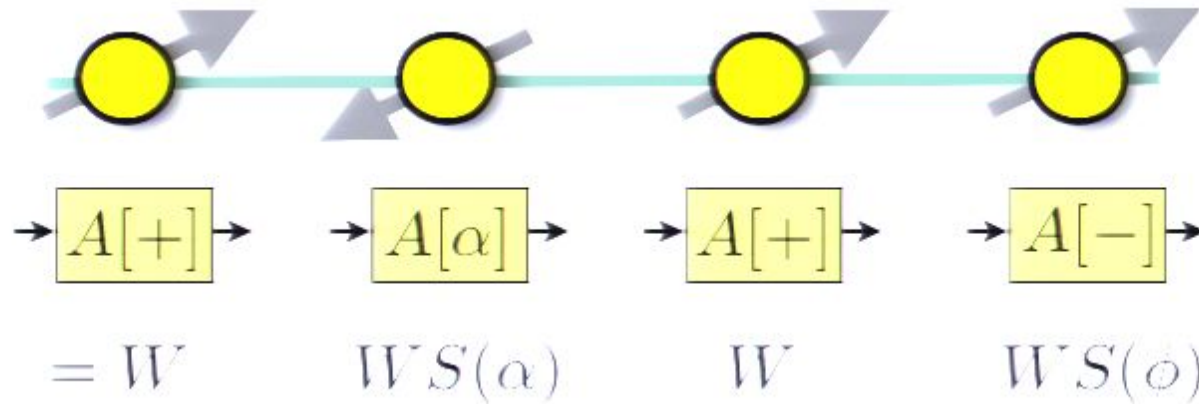


**Observation:** No maximal entanglement, yet still “universal”!

$$p(+)=\sin^2\theta, \quad p(-)=\cos^2(\theta)$$



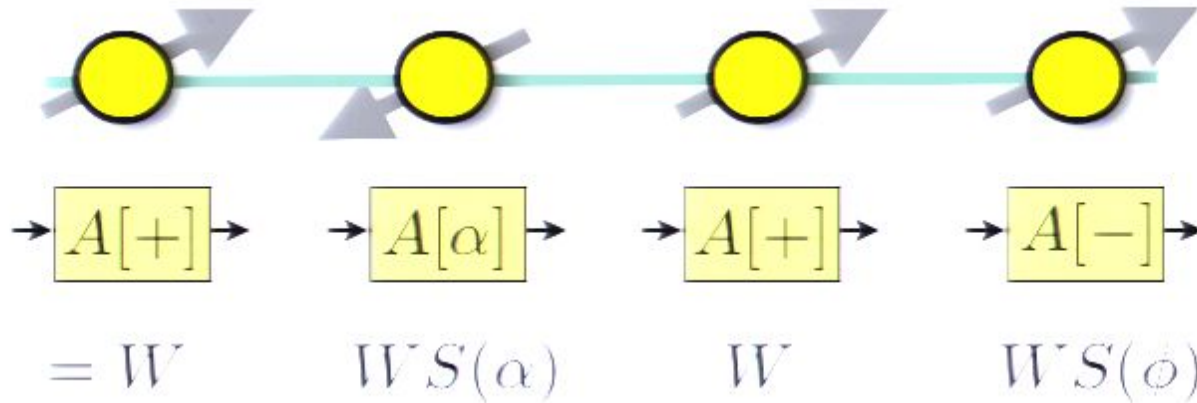
$$\phi = \pi$$



• Or,  $W = \exp(i\pi/kX)$ ,  $k > 2$

$$\theta = \pi/4, \phi = \pi$$

$$\langle Z_i Z_{i+n} \rangle \sim \xi^n$$



- Or,  $W = \exp(i\pi/kX)$ ,  $k > 2$

$$\theta = \pi/4, \phi = \pi$$

**Observation:** Long-range correlations are no obstacle to universality!



- **Physical examples:**

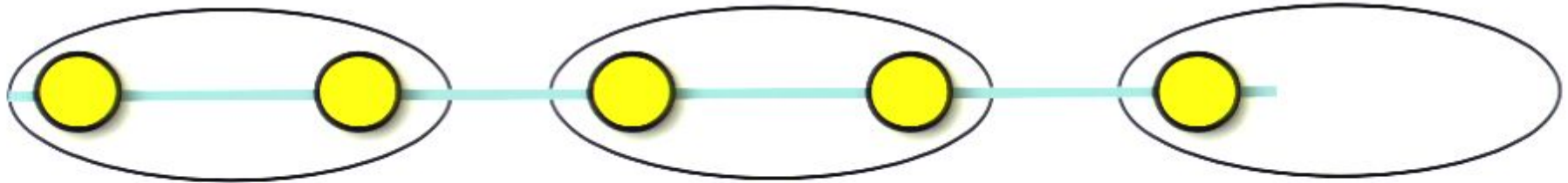


- **Gapped ground state** of (slightly modified) **AKLT Hamiltonian**

$$H = \sum_j h_{j,j-1}$$

**Observation:** Ground states of NN-Hamiltonians can be quantum wires

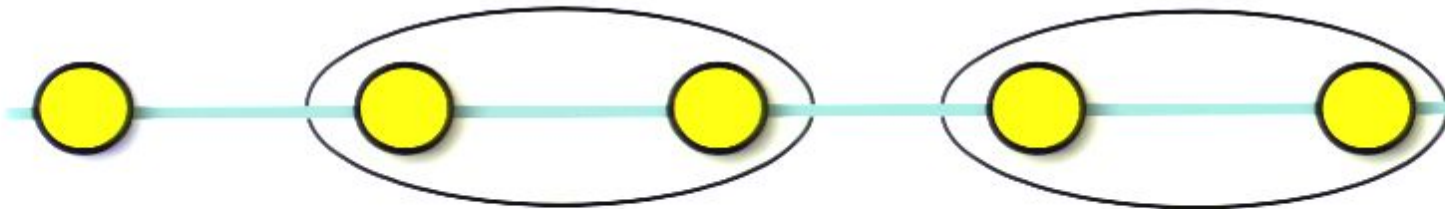
- **Physical examples:**



- **Bose-Hubbard wires:** Use tunneling to neighbors

$$H = \sum_j (a_L^{(j)})^\dagger a_R^{(j+1)}$$

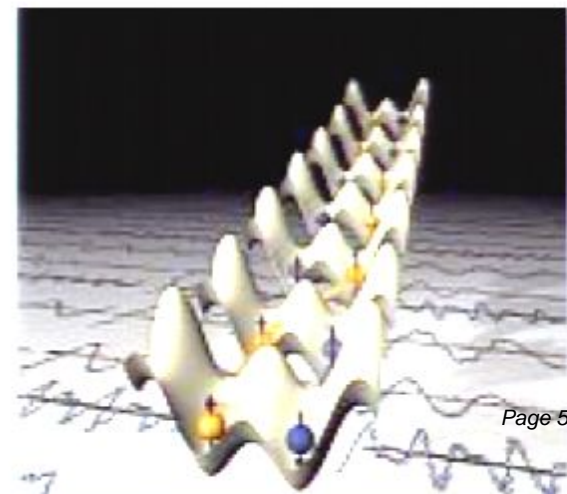
- **Physical examples:**



- **Bose-Hubbard wires:** Use tunneling to neighbors

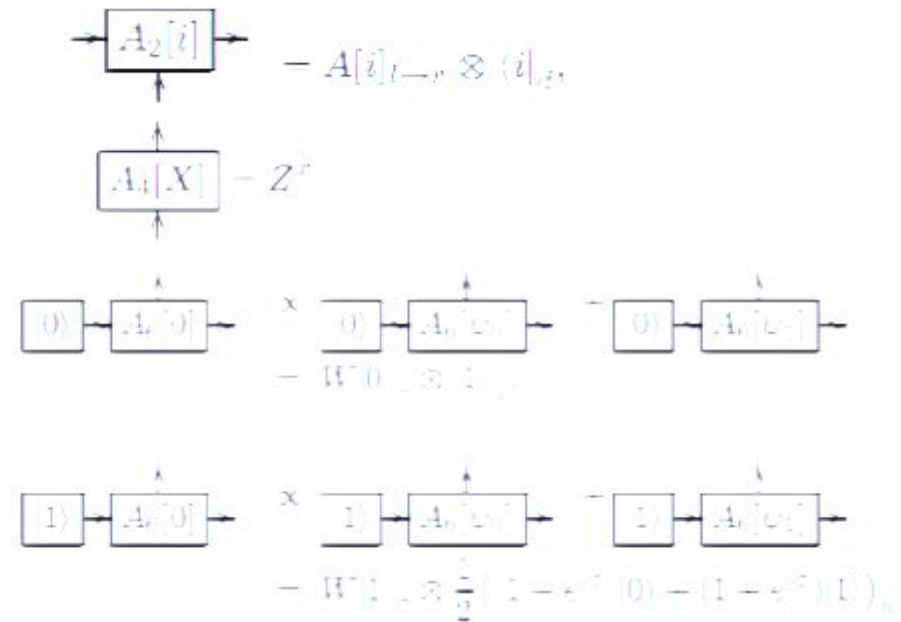
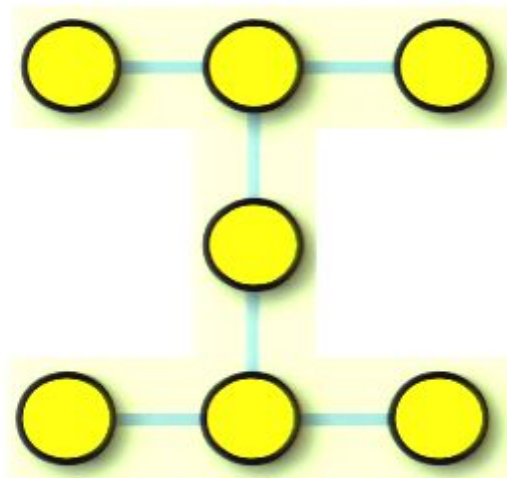
$$H = \sum_j (a_L^{(j)})^\dagger a_R^{(j+1)}$$

- Wire with maximal localizable entanglement + arbitrary rotation in a plane
- Ideas towards realization: Use atoms in **optical superlattices**

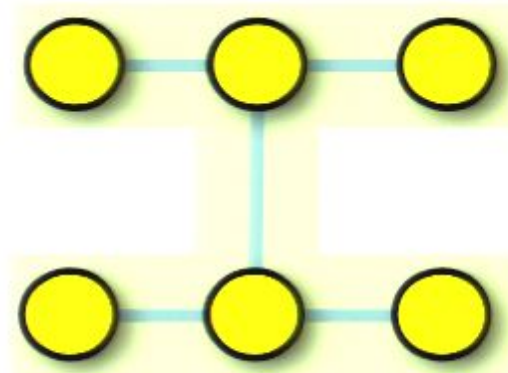


- **Steps towards classification of 2D switches** (is technical, unfortunately)

- Switch based on *phase gates*:

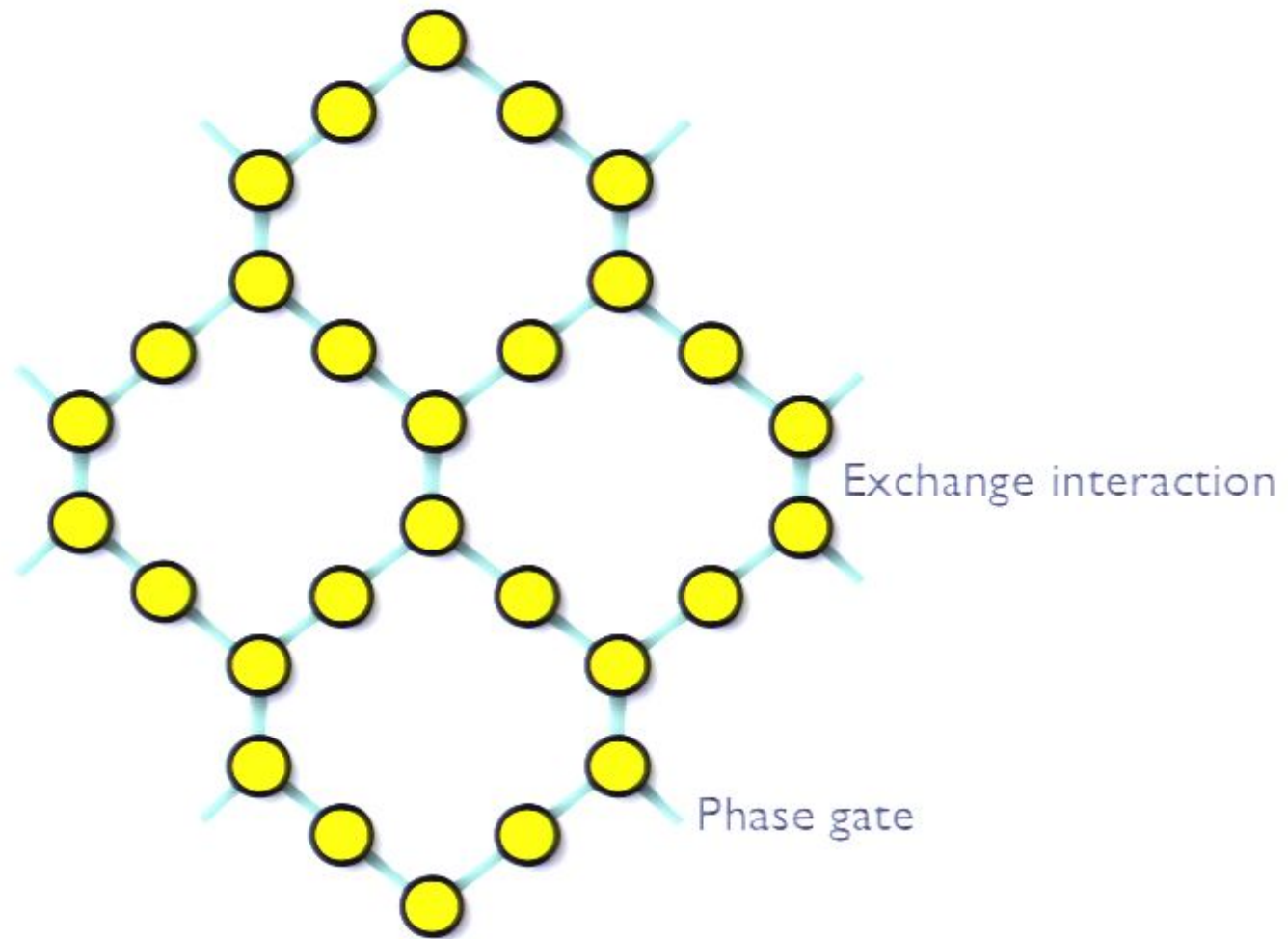


- Switch based on *exchange interaction*:



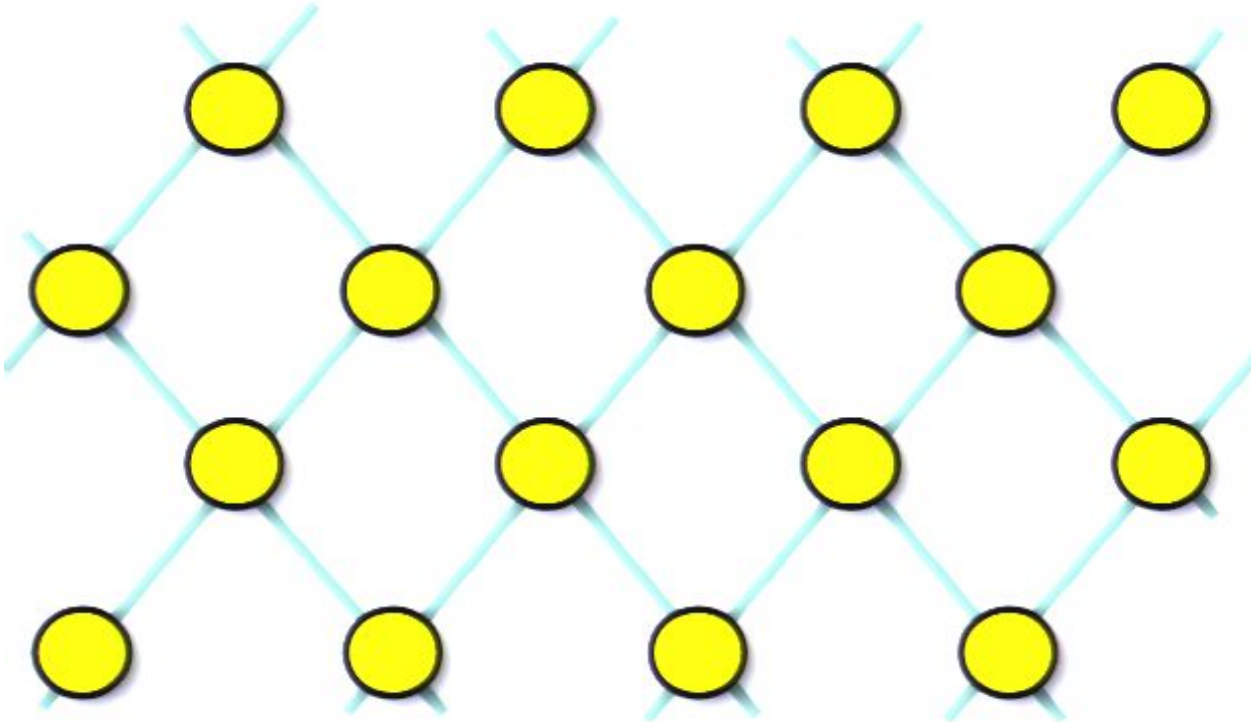
- “Gentle couplings” with *little entangling power*

- Gives rise to a toolbox of constructing new schemes, e.g.,



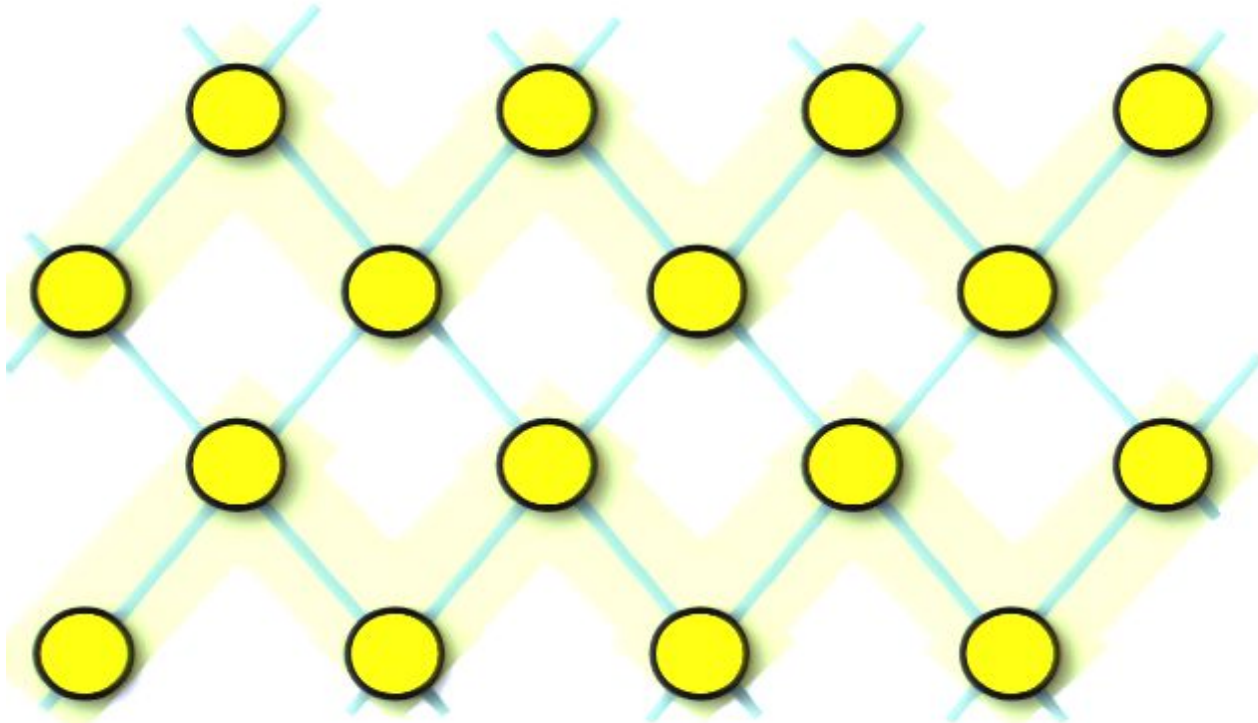
- Kitaev's planar code state and “computational phases”

## • Kitaev's toric code state:



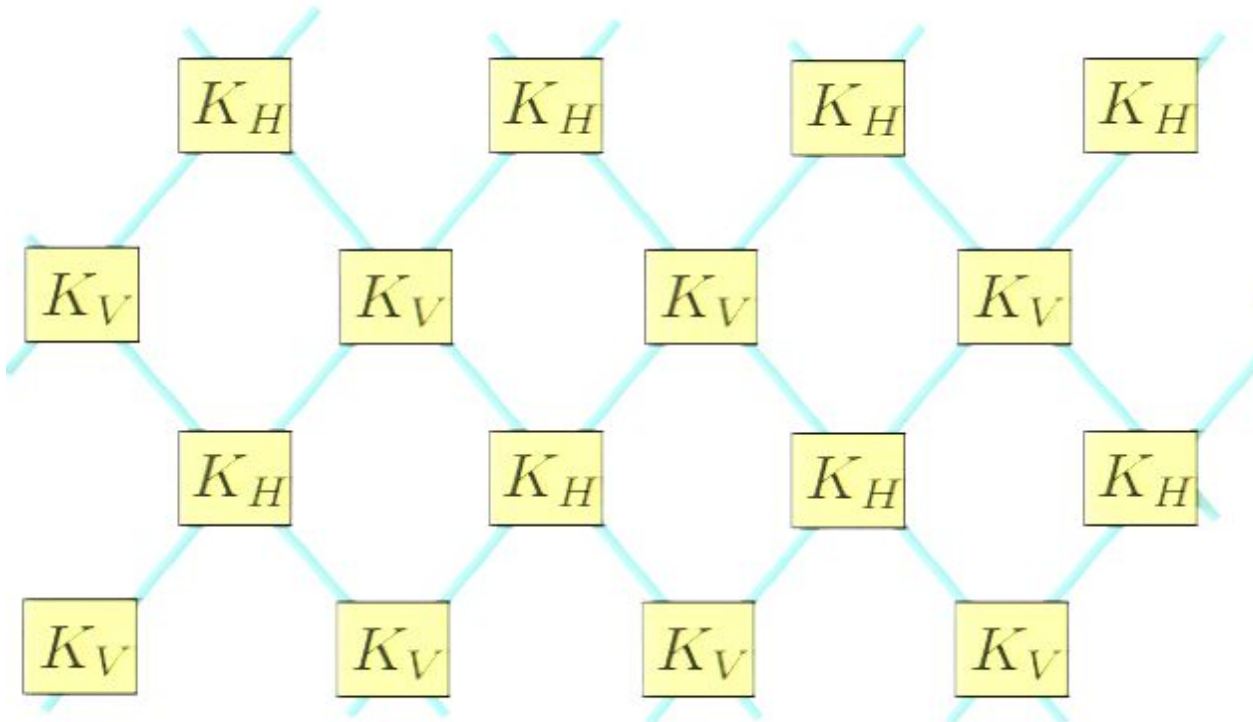
- State with non-trivial topological properties

• **Kitaev's toric code state:**





• **Kitaev's toric code state:**



- “Horizontal”:

$$K_H[s] = \begin{matrix} Z^s \\ Z^s \end{matrix}$$

- “Vertical”:

$$K_V[s] = Z^s \quad Z^s$$

- For any measurement in the Y-Z plane with angle

$$K_H[\phi] = \begin{pmatrix} 1 & & & \\ & e^{i\phi} & & \\ & & e^{i\phi} & \\ & & & 1 \end{pmatrix}$$

- Very nice, is *entangling* 2-qubit gate in correlation space!

- For any measurement in the Y-Z plane with angle

$$K_H[\phi] = \begin{pmatrix} 1 & & & \\ & e^{i\phi} & & \\ & & e^{i\phi} & \\ & & & 1 \end{pmatrix}$$

- Very nice, is *entangling* 2-qubit gate in correlation space!

- In fact, not universal: Measurements on the toric code state is **efficiently “simuleerbar”!**

- Modify slightly, essentially by phase gate

$$\tilde{K}_H[\phi] = K_H[\phi] \sqrt{ZH}$$

- Modify slightly, essentially by phase gate

$$\tilde{K}_H[\phi] = K_H[\phi] \sqrt{ZH}$$

- Can use **ancillae**, ZZ-controlled phase between logical qubit and ancilla like local Z-rotation (“rerouting”)

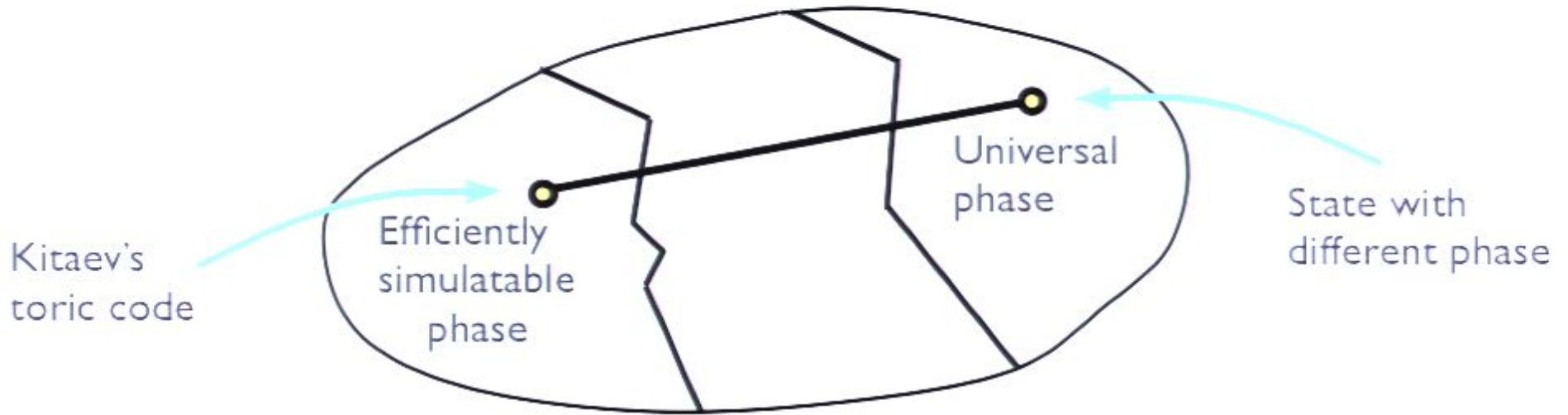
- **Single-qubit gate:**



- **Two-qubit gate:**



- **“Computational phases of matter”**





- A late interlude on three pages:

What I did not speak about, but might be worthwhile to have an informal discussion about

# Expander graphs and quantum expanders

- **Expander graphs:** “Highly connected graphs”



- $G = (L, E)$  is a  $d$ -regular graph with  $n$  vertices and adjacency matrix  $A$
- $\hat{A} = \frac{1}{d}A$  is then a transition matrix of a random walk

- **Random walks on expander graphs are extremely quickly mixing:**

$$\|\hat{A}^t p - u\|_1 \leq \sqrt{n} \alpha^t$$

where  $u = (1, \dots, 1)/n$  is uniform and  $\alpha$  a parameter,  $|\lambda_2(A)|, |\lambda_n(A)| \leq \alpha$



- **How to construct classical expanders? E.g., Margulis expander:**

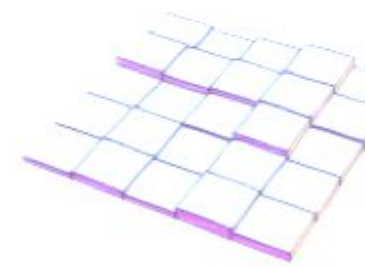
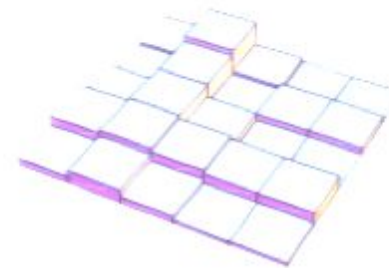
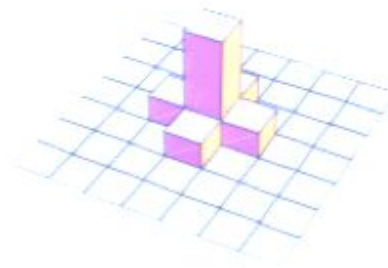
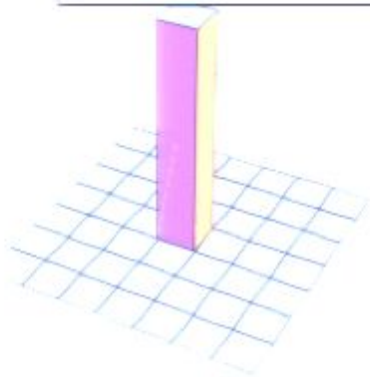
- Vertex set discrete  $n \times n$  plane, viewed as 2D vector space  $\mathbb{Z}_n \times \mathbb{Z}_n$

- Consider four affine mappings:

$$T_1 : v \mapsto \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} v \quad T_2 : v \mapsto \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} v + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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- How to construct **quantum expanders** - CPMs that are mixing on quantum systems in the same way as random walks on classical expanders?

- Take affine maps literally, acting in discrete phase space of quantum system. **Done!**
- Think of  $\mathbb{Z}_n \times \mathbb{Z}_n$  as **discrete phase space** with a **Wigner function**  $W_\rho$

		Action on Wigner function:
Shift:	$X :  x\rangle \mapsto  x + 1\rangle$ $Z :  x\rangle \mapsto e^{2\pi/nx}  x\rangle$	$W_{X\rho X^\dagger}(q, p) = W_\rho(q + 1, p)$
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- **Classical:**  $u \mapsto \frac{1}{8} \sum_i T_i(u)$     • **Quantum:**  $\rho \mapsto \frac{1}{8} \sum_i U(T_i)\rho U(T_i)^\dagger$

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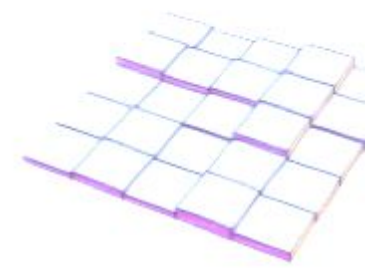
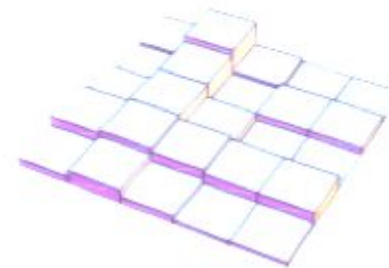
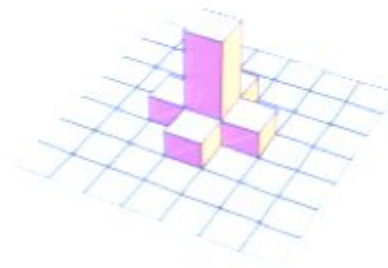
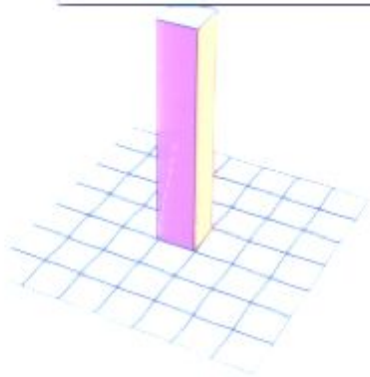
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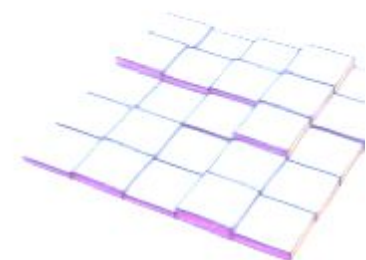
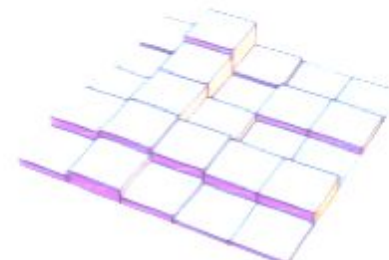
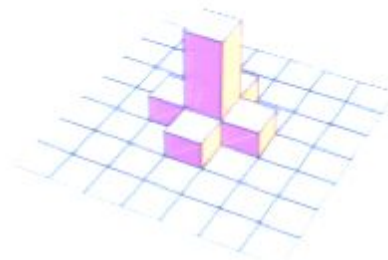
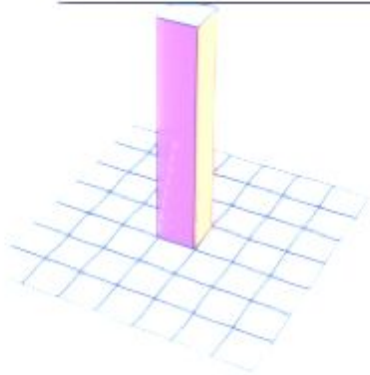
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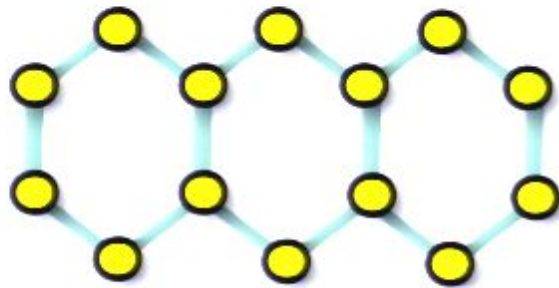
- Simplest known (efficient) quantum expander
- Similar ideas for a link of graph theory and quantum info?



- 
- So... summary and open questions

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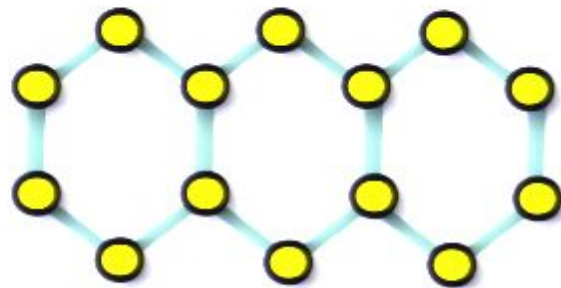
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- **Open questions:**

- What gapped ground states of nearest-neighbor Hamiltonian universal?

- Can turn round and learn about classical efficient simulatability?

- Full classification of all 2D resources?

- By-product angle  $\phi$



$$\phi \ll \pi$$



$$\phi = \pi$$

- By-product angle  $\phi$



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**Thanks for your attention!**